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On the Inferential Zigzag and Its Activation Towards Clarifying What It Is Commonsense Reasoning

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Abstract. This paper has a twofold goal: The first is to study how the inferential zigzag can be activated, even computationally, trying to analyse what kind of reasoning consists of, where its 'mechanism' is rooted, how it can be activated since without all this it can just seem a metaphysical idea. The second, not so deeply different - as it can be presumed at a first view - but complementary, is to explore the subject's link with the old thought on conjectures of the 15th Century Theologist and Philosopher Nicolaus Cusanus who was the first thinker consciously and extensively using conjectures.

AMS Subject Classification 2020: 03B65
Keywords and Phrases: Commonsense reasoning, Language at work, Inferential zigzag, 'Out of logic'.

1 Introduction

The present paper is written with a very specific target: the clarification of some aspects of the *inferential zigzag* introduced in [14, 19, 16] previously not analysed in detail. However, in order to make explicit and clear the motivations behind the technical results, the paper contains also a few general conceptual remarks as well as a final Section in which some considerations expressed by the Renaissance logician Nicolaus Cusanus are briefly surveyed in connection with what is presented in the technical side of the paper. The aim is to reach a better understanding of what the inferential zigzag seems to consist of, and of how it can be practically, specifically produced. That is, to explain how, at each statement p, a mixed inferential chain can start; to explain how the zigzag proceeds by inflexions either forward or backward and leading, finally, to another statement q, such that either $p < q$, or $q < p$ (with the symbol $<$ as a shorthand for the conditional statement If p, then q). In [14, 19, 16] attention was focused more on the concepts behind the proposal, than the practicality of the algorithmic path that could be followed. Without the clarifications in the present paper, the zigzag can be seen just as a more or less interesting, but purely theoretical idea. Let us clarify, however, that it is (and it always was) manifest from the beginning that the idea is constructive in nature, inherently lending itself to subsequent implementations. This is witnessed by the fact that a number of considerations on how to reduce the complexity of its possible implementations, which without any specific strategy appears to be exponential in time, were discussed [19]. So, the point in question has not to do with this general aspect, but with the possibility of suggesting a specific path to be followed that seems from a conceptual point of view peculiarly in synch with the theoretical aspects discussed in [14] and [19]. This opens the way for a further examination of possible strategies for implementation, based on already established mathematical and computational intelligence models that can be suitably matched to the essence

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of the problem at hand. Detailed implementation procedures will not be discussed here, and left for a more technical paper to follow and currently under preparation. Let us, moreover, observe that our proposed path, the zigzag, in itself is quite unusual in the field of Logic, not only among the standard approaches but also in those (occasionally called deviant, but far more common than it seems at first) cases in which uncertainty and imprecision are taken into account, in the same vein of other concepts that preceded its introduction, e.g. conjecturing and linked notions. This is the true reason why some clarifying conceptual remarks appear in the paper. A few general epistemological and historical comments seem necessary to understand the way the idea has developed and how it is connected to its historical antecedents. May the zigzag be a non classic approach, it is nonetheless well routed in the history of Logic, or, perhaps better, the study of reasoning. The paper is structured in the following way. The present Introduction is followed by three Sections all devoted to purely theoretical analyses which, respectively, *present a model of commonsense reasoning*, *introduce the Zigzag*, and *develop its first properties*. Section 5 provides some preliminary considerations about the computational costs of the process considered; finally, Section 6 surveys some remarks on Conjectures done by Nicolaus Cusanus, that are connected with the inferential zigzag.

2 Around a Model of Commonsense Reasoning

Let us remark that the language or, at least, the wording used in the following, is not the usual one found in the majority of logic papers. In fact, we are here trying just to approach some specific aspects of what we called Language at Work [19] without referring to the general setting of mathematical logic, of which we are in fact out. We shall then present a very simple model of Commonsense Reasoning, another name for the language at work, where the zigzag idea was born, and that has a very soft mathematical structure in which usual laws like those of Duality are often not valid. In this setting, in fact, some laws are not universal, but have only a local validity. Few laws seem to warrant the typical flexibility shown by both natural language and commonsense reasoning. It may be worth noting that since thinking is a natural phenomenon as breathing is, reasoning should also be seen and considered as a natural phenomenon.

2.1 Reasoning from a premise

The natural phenomenon of reasoning tries, departing from a given information, linguistically compacted in a statement p called the premise, to reach a previously unknown conclusion (concerning what is described by p), also compacted in a linguistic statement q, such that either the existing knowledge on p results increased, or diminished. In the first case, symbolized by $p < q$, q is a consequence of p, in the second symbolized by $q < p$, q is a hypothesis or explanation for p. In general, reasoning is seen, and defined, as the action of refuting or conjecturing q from p, under which p and q are linked by $p < q'$ in the first case, and by $p < /q'$ in the second. That is, q is a conjecture from p whenever $not - q(q')$ is not a consequence of p; it cannot be stated $p < q'$, $not - q$ cant be deduced from p. In different words, q is not a refutation of p, q does not refute or contradict p[17].

In short, reasoning from a premise p is but finding either a refutation, or a conjecture. The only condition the premise p is supposed to verify is: $p < /p'$; p is not self-refuting, self-contradictory or, in Aristotles ancient words of wisdom, p is an inferentially impossible statement. The premise should indicate something not impossible but possible, something sensate. Notice that for these first concepts only relation $<$ and negation not ($'$) are needed.

The (inferential) binary relation $<$ between any two statements such as p and q. $p < q$, translates the conditional statement If p, then q that, as it is well known, in language is not always understood in the same form. The conditional statement $p < q$ is sometimes taken as one of the unconditional statements not p or q, or not p or (p and q), or p and q, etc.

Understanding the conditional statement $p < q$ by an unconditional one like it is, for instance, p and q, is for describing it in a form that, in principle, can be submitted to some kind of verification. Reasoning is a kingdom in which the density of conditional statements is remarkable; reasoning requires a good management of conditional statements.

It is such non-uniqueness a motive to consider $<$ as a primitive, undefined relation, only submitted to be reflexive to be sure that it is never empty and that, at each situation and/or context, should be represented by translating how the conditional If/then is there understood; for instance, and respectively, representing $p < q$, by $p' + q$, $p' + p \cdot q$, $p \cdot q$, $q + p' \cdot q'$, etc., shortening and by a point (), and or by a cross (+), like not is shortened by a comma ('). It should be noticed that such operators are not supposed to be endowed with (usual) properties like idempotence $(p + p = p)$, commutativity $(p \cdot q = q \cdot p)$,associativity$(p \cdot (q \cdot r) = (p \cdot q) \cdot r)$, etc., considered, if existing, local properties that is, not holding in all the universe of statements but only in some part, or parts, of it.

2.2 Inferential situations for a conclusion

It should come to notice that between whatsoever statements p and q, it can just exist one of the four inferential situations:

1. $p < q$; I, e., q is a consequence of p.
2. $q < p$; I, e., q is a hypothesis or explanation for p.
3. Both $p < q$ and $q < p$, written $p \sim q$ or $q \sim p$, i.e., p and q are inferentially equivalent.
4. Neither $p < q$, nor $q < p$, written $p \perp q$ or $q \perp p$, i.e., p and q are inferentially not comparable, or orthogonal.

Observe that relation $\sim$ is not necessarily an algebraic equivalence since it is just reflexive and symmetric, but its transitivity is not always warranted unless $<$ enjoys it.

A forward chain of inference like $p < u$, $u < v$, $v < w$, $w < q$, usually written $p < u < v < w < q$ is called a *deductive* process, or a deduction, and a backward chain like $p > u > v > w > q$, or $q < w < v < u < p$, is called an abductive process, or an abduction. Of course, for concluding $p < q$ in the deductive chain, and $q < p$ in the abductive chain, $<$ has to be a transitive relation at least locally for the involved terms.

Hence, and with the exception of (3) allowing the indistinguishability of p and q from the inferential point of view, and then accepting substitution of p by q, or q by p anywhere, a conclusion q of p, a conjecture q of p, only can be either a consequence, or a hypotheses, or an orthogonal element to the premise, in which case it is said that q is a speculation, or guess, from p, and, depending on how it is $p < /q'$ verified, the speculation is weak (if $q' < p$), or strong (if $q' \perp p$) [17].

Thus, reasoning just consists in refuting, deducing, abducing and speculating or guessing, i.e. obtaining orthogonal conjectures from the premise, a process that can also be understood as inducing. Thus: induction can be identified with speculation, guessing with obtaining conjectures, statements inferentially orthogonal to p.

It is interesting to observe that, under local transitivity of $<$, if r is not self-contradictory $(r < /r')$ and refutes p $(p < r')$, it is $p \perp r$. In fact, were it $p < r$, since it is $r' < p'$, then $p < r'$ and the corresponding local transitivity, forces the contradictory $p < p'$. Analogously, were $r < p$ it will follow $r < r'$ and, hence, p and r are not comparable under $<$, are othogonal.

If, under transitivity, consequences can be obtained by going forwards with $<$, and hypotheses by going backwards with $<$, how can speculations be obtained? If deduction corresponds to the first, and abduction to the second, to which inferential mechanism can speculation, induction, correspond?

3 On the Inferential Zigzag

Notice that a consequence q of p, $p < q$, is not always obtained by just an immediate first step ahead but by, at least, two different possibilities:

- The first is when q is at the end of a chain of steps such as $p < u$, $u < v$, $v < w$, and $w < q$, requiring the transitivity of $<$ to conclude $p < q$.

- The second is through the property $p < p+p^a$, with p^a brings any opposite statement of p, allowing to define $q = p + p^a$, without requiring whatsoever additional property of $<$, and if this q is not self-contradictory.

It is obvious that instead of p^a it serves any statement q, and even the same p, but combining p and one of its opposites p^a helps to cover more knowledge than that offered by only p, or by a q that is totally disconnected from p. Notice that the statement $p+p'$ has the risk of being too large (remember that $(p+p')'$ is self-contradictory and, thus, in some lattices with maximum $p+p'$ can easily be such maximum).

It analogously happens with hypotheses, $h < p$ or $p > h$, reachable either by some steps $p > u$, $u > v$, $v > w$, $w > h$, allowing to conclude $p > h$ if $>$ is transitive in the set p, u, v, w, h, or through the property $p \cdot p^a < p$, allowing to define $h = p \cdot p^a$ and without requiring additional properties for $<$, and provided $p \cdot p^a$ is not self-contradictory.

It should be noticed that the first ways correspond to what is usually done for proving that a conjecture is either a consequence, or a hypothesis; and the second to what is done for finding either a still unknown consequence or an explanation. They serve, respectively, for proving and for finding; if the first can be seen as a technical way, the second is a dialectical way.

A reason for considering p^a instead of p' lies in the fact that, under the transitivity of $<$, $p \cdot p'$ is self-contradictory; in fact:

$p \cdot p' < p$ implies $p' < (p \cdot p')'$ that $p \cdot p' < p'$ conducts to $p \cdot p' < (p \cdot p')'$, q.e.d.

This Non-contradiction theorem, obviously valid for all statement and, in particular, for those s such that $s < p'$, can be easily and directly extended to these statement s such that $s < p'$ -statements referred by p - provided $<$ is transitive where convenient, and the conjunction is monotonic that is, verifies, $p < q =>$ $p \cdot r < q \cdot r$ and $r \cdot p < r \cdot q$ for all r. In fact, starting from $p < q => q' < p'$ and from the property $s < p'$, by monotony follows $s \cdot p < p' \cdot p$ that, with $p' \cdot p < (p' \cdot p)'$ implies $s \cdot p < (p' - p)'$;but, since from the first inequality follows $(p \cdot p')' < (s \cdot p)'$, it finally results $s \cdot p < (s \cdot p)'$. Thus also $s \cdot p$ is self-contradictory.

This theorem forces to avoid as s, when the two presumed laws do hold, all statements that are refuted by p and, in particular, both the negation and whatsoever antonym of p. It suggests taking a statement $s = s(p)$ depending on the premise p but different from the negation and any opposite.

It should be noticed that given a premise p, neither refutations, nor consequences, nor hypotheses, nor speculations, are unique. Usually, there are sets of them, not reducible to a singleton. Hence, either the same person at different moments, or two different persons. will not conjecture the same from a given premise. In the same vein, refutations do not usually coincide; different people can refuse the same statement by means of different refutations. This non-uniqueness of conjectures and refutations is, of course and in fact, a matter of common experience among people, and a testament to the power of human reasoning; what can be seen of some relevance is that the current model gives a first explanation of it.

It is noteworthy that the non-uniqueness of conjectures comes directly from the non-uniqueness of those statements s such that $s < p'$; from the possible hypotheses for p'. Actually and in particular, there are a lot of words for which more than one opposite term is used in language. Analogously, it is not sure that in a mixed chain of inference the inflections are always produced in the same places and in the same sense (backwards, or forwards), and the obtained speculation at the end of a zigzag strongly depends on this.

3.1 The inferential ZigZag as a mechanism for reasoning

Basic references [14, 19, 16] have overlooked such a development, and are limited to a hinting on how the inferential zigzag and, especially speculating, guessing or also inducing can be effectively done. This paper tries to fill this gap by giving a first hint on how the zigzag can be actually developed. It can be said that reasoning is done thanks to a mechanism consisting in activating an inferential zigzag.

4 Developing the Zigzag

Before continuing, lets see how weak speculations can be effectively reached. Since they are defined by $p \perp q$ and $q' < p$, it is clear that $not - q, q'$ is a hypothesis for p. Hence, in principle q' can be reached by abduction i.e. going backward from p up to find it, and provided $<$ is locally transitive around p. Thus, two questions are posed; when to stop for finding q as the searched speculation (a question whose answer is here avoided as it corresponds to looking for the meaning of a statement), and how, once q' is given, q can be actually reached; something that depends on the character the linguistic negation can show in q:

1. If negation is weak at q, or $q < (q')' = q''$, q will be found by negating q and moving backwards from q''.
2. If negation is Intuitionistic at q, or $q'' < q$, q will be found by negating q' and moving forwards from q''.
3. If negation is strong at q, or $q'' \sim q$, it suffices to negate q' to obtain q.
4. If negation is wild at q, or $q'' \perp q$, no one of the three former situations holds, and, since it is $p \perp q$, it is not sure if one of them, previously unknown, will appear. Actually, a priori nothing can be said in general.

Thus, with the exception of (4), a weak speculation is reached at the end of a forward or backward step after negating q'.

Notice that if q is a strong speculation from p, or, it is $p \perp q$ and $p \perp q$, no similar way to the formers can be immediately inferred as we have $q' \perp p$ instead of $q' < p$. In principle, it seems that there is no inferential way of mixing deductive and abductive movements that can be foreseen to reach q. It seems that q cant be reached by enchaining statements, and one can be tempted to hope in the help of some bizarre entity, akin to the old muses, mysteriously imbuing q into the thinker. It will be seen how such suppositions are unnecessary.

4.1 Advancing and retroceding

Nevertheless, it should be noticed that from p it is possible to advance inferentially by disjunction, and to retrocede by conjunction. For instance, $p \cdot u < p < p + v$ for all statements u and v, shows a recoil by conjunction, and an advancement by disjunction, both from p. In the same way it is possible to realize alternate movements backwards/forwards or forwards/backwards, like, $p > p \cdot u < u < u + w$, etc.

In this last case, and not presuming more laws than those of the skeleton, if with $q = u + w$ it is $p \perp q$, it will depend on q' if q is a speculation from p or, simply, an element inferentially orthogonal to p. It is obvious that such inferentially mixed forms can be followed by $u = p \cdot p^a$ and $v = p + p^a$; i.e. by only using what can be known, or supposed, on p, and avoiding $p \cdot p'$ and $p + p'$ due to what was formerly stated concerning their self-contradiction if $<$ is transitive.

4.2 An example of reasoning with speculations

Lets show a very simple example starting from p and arriving at a speculation by supposing that all the statements come from the disjunction of five of them, a, b, c, d, e, expressing the available initial information on something.

- Suppose $p = a + e$, is the premise and take $q = a + b + d$. Since it is $p \cdot q = a$, lets focus our attention on a. Then: $p > a < a + b + d = q$, with $p \perp q$ and, since $q' = c + e$, it is also $q' \perp p$. In this example, and provided $p' = b + c + d$, since it is not $q < p$, is not possible to suppose the coincidence of q and p^a. Thus, what can be supposed is that, pivoting on a, $q = a + b + d$ informs on p.

Analogously,

- if with the same premise is $q = b + d$, it is $p \perp q$, and since $q' = a + e = p$ means $q' < p$, as $<$ is reflexive, q is a weak speculation from p.

Thus it seems that in all the cases in which the statements are constructed as the disjunction of some pieces of basic information on something, or atoms of knowledge, as it happens frequently, both weak and strong speculations can be obtained in ways like the former and through inferential chains mixing forwards and backwards movements. That is, through the so called inferential zigzag under which reasoning from p can be seen as a kind of Inferential Brownian Movement around the premise.

Summing up and with just the skeletons laws, deciding if the next movement in q should be either forwards, or backwards, can be done by either considering the conjunction of opposites $q \cdot q^a$, or another conjunction $p \cdot q$ if q informs on p, that is, by means of all the (available) knowledge on p. Always with care on not being $p \cdot q$ self-contradictory.

5 The Cost of Zigzagging

We have already discussed in [19] the fact that in order to render the notion of the inferential zigzag computable when using atoms of information, an exhaustive search of the problem space is necessary, to take into account all possible combinations of the morsels themselves and determine their cumulative role in achieving unlimited speculation. Such an approach would require exponential time $O(2^n)$ due to the necessity of exploring the entire power set to be performed in full, and as such would be computationally unfeasible even for a small number of atoms. This compounds with the fact that while examples are presented with atoms in the unities for sake of clarity, it is to be expected that any meaningful reasoning will require orders of magnitude more, rendering factual the worry about computational attainability.

In [19] a number of strategies that are directed toward limiting complexity by reducing the size of the searching space have been already proposed, such as reducing the number of total clauses by a plausibility selection and weighing and thresholding, where each movement in the zigzag has an associated cost, proportional to parameters inferred by the reasoning structure itself, and exploration stops when a certain threshold is passed. Here is presented a simpler strategy of reduction that preserves the polynomial complexity of search depth and is easily applicable to the specific task of speculation.

The first pass of the strategy is to add to the simple system some information about the proximity between atoms. This is necessary as without any added information there is no way of implementing a reduction of the Hasse diagram representing the power set of atoms, which is necessary to lower complexity. This can be done either by prior experience or by evaluation. In the case of prior experience, we consider a number of tuples composed of atoms (such as a, b, e, a, c and so on) that are derived from previous knowledge, e.g. instances where such atoms appeared together in a previous successful speculation or in some premise. In evaluation we have to provide such a database of tuples directly, either by directly asking a panel of humans to evaluate the proximity of atoms, or by grading single atoms and then aggregating such information. Either way, a database of atoms proximity tuples is obtained. Such a database is then used to calculate what is called a frequent itemset, a set of items appearing together, listed in order of frequency. Such structure can be stored in a DB, or structured in a Markov chain. By setting a threshold and keeping in mind the necessity for polynomiality, the atoms that are frequently found together can be clustered, and the complexity of exploring the resulting set be reduced to $O(n)$, allowing effective computation for the generation of all possible inferential zigzags.

Classical algorithms such as Apriori [1] and other members of the Apriori-like family could be used for the task, but in order to compute a frequent itemset of order l, they must produce all the subsets, bringing exponential complexity again to the table. A more suitable choice is Max-Miner [2], which obtains the same information by computing at most $l+1$ passes over the original dataset. A number of newer algorithms claim to improve on Max-Miner, but due to its simplicity and the fact that in this context a clear explanation is worth more than fractional improvements in efficiency, the choice for a better implementation is left to a more technical forthcoming paper. This approach has a number of advantages: first and foremost, it reduces complexity allowing effective computation of the inferential zigzag; second, reduction in complexity does not come at the cost of reducing the expressivity of the original idea in terms its of cognitive approach. As the zigzag is a formal version of speculation, reducing by clustering has a cognitive resonance with analogy, a process often employed in order to make effective reasoning in presence of an abundance of information. In figure 1, an example of pruning a Hasse diagram for a five atoms reasoning search is shown.

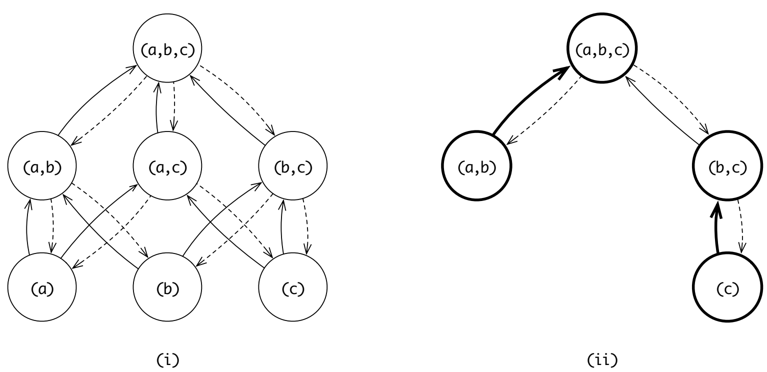

Figure 1: Example of pruning a Hasse diagram for a five atoms reasoning using Max-Miner, (i) the algorithms take as input: the starting Hasse diagram of the power set of a, b, c, d, e, which is used in [19] to explore all possible zigzag inferences when in possession of five atoms of information; a list of common atoms tuples, derived e.g. from previous reasoning on the same atoms, or by experience. (ii) the most popular couples of atoms are clumped together, and a new diagram with lower complexity is created. The threshold for clumping is chosen appropriately in order to attain effective computability.

6 Zigzagging Along the Centuries

Inferential zigzagging is something that helps to complete and extend the use of notions such as conjecturing. The general idea behind this paper without, of course, any reference to theorems was suggested to the authors by works [3, 4] of 15th Century German philosopher Nicolaus Cusanus (aka Nicholas of Cusa, 1401-1464). What specifically and concretely inspired this papers argumentation, in fact, is the continuous use made there of the methodology called unity of opposites (compositio oppositorum)[2].

The original idea behind the present paper can thus be informally traced in a few six hundred years old considerations. We already referred to old thinkers in a recent Essay [19]: what follows can be useful for outlining a general conceptual setting in which this as well as others remarks can be more adequately understood[3].

In dealing from a conceptual point of view with a few pages of Nicolaus Cusanus we shall (literally) zigzag a little around the XV Century. Since our eyes are turned to the foundational efforts done in mathematics and logic in the XX Century, we will be zigzagging through Centuries. What follows aims at drawing additional information for increasing the understanding of notions such as conjecturing[4]. These notions do not seem to play an explicit role in the bulk of present day mathematical logic, more interested in precision and accuracy. They are crucial instead in everyday language and reasoning, as well as in many facets of scientific investigations when cognition comes into play, such as AI.

In what follows we shall present:

a) a few quotations from Cusanus writings, to highlight why, in our opinion, his ideas are relevant to present day investigations

b) some remarks on the way in which the notion of *conjecture* is used by working mathematicians in the context of their daily work, and not when thinking about *foundational questions*

c) a number of reflections on the use of the term mathematical logic

6.1 Cusanus and reasoning

We feel necessary to recall some general remarks presented by Cusanus in the opening of his volume [3, 4]. Behind the veil of an old language (and notwithstanding it), they point to interesting connections with present day questions. We shall not scrutiny whether some other ideas and points of his analyses can be also and, perhaps, more incisively - useful for the same aim. The early motivations offered by Cusanus for having devoted space to a reflection precisely to the notion of conjecture are illuminating. As we shall see from the brief excerpts that follow, two points are crucial:

i. impossibility of reaching the precision of truth, and

ii. limitations of the human mind are such to imply that the conjectures of each person will be different

Cusano begins with the following statement of intents: "since a favourable opportunity to do so has now presented itself to me, I would like to illustrate my conception of conjecture". What follows is an interesting but admittedly contort presentation, due perhaps to the desire to prize the greatness of the person to which the book is sent:

> In the preceding books of the Learned Ignorance you have seen, even more profoundly and more clearly than I have done myself with all my efforts, that the accuracy of truth is unattainable. From this it follows that every positive assertion of man concerning the true is conjecture.

[2]This notion in different terms (and in a completely different context from Cusanos) has been analyzed by the third author in [18, 15].

[3]That the thought of Nicholas of Cusa can be useful for clarifying crucial aspects of problems and questions of present day relevance is also witnessed by a relatively new book [20] which collect the contributions presented at the first Congress on Cusanus to be held in Asia at the turn of the Millennium as well as a short monograph entirely devoted to the Art of Conjecture appeared in 2021 [5]. We came across both volumes when the paper was, in fact, finished and we are, then, here, simply acknowledging their existence. Their content will be very useful for further investigations along the present conceptual line, looking for stronger and less episodic connections with Cusanus suggestions as done in the present paper. Just to provide an indication a paper in the Conference volume explicitly deals with epistemology [6] and many others touch on topics of crucial present day interest.

[4]in the following we shall show how the conjunction of opposites can be connected to it

He then proceeds with a statement not easily understandable at a quick reading, but which forces a reflection in connection to present day questions of cognitive relevance: The unity of unattainable truth is therefore known through the otherness of conjecture, and the conjecture of otherness is known in the absolutely simple unity of truth. A sort of clarification follows:

> A created intelligence, which is endowed with a finite actuality, cannot but exist in one way in one individual and in another way in another individual, so that among all who formulate conjectures there is always a difference; consequently, it will always be absolutely certain that, with respect to the identity of the true which remains unattainable, the conjectures of different persons will differ in degree and yet remain without proportion to each other, so that no one will ever be able to understand perfectly what another means, although some may come closer to it than some other.

Subsequently he will explain his ways of approaching and defining conjectures, his secret being a careful and guided use of examples in a sort of maieutic or Socratic approach.

> For this reason, in order to make the secret of my conjectures clearer and easier to understand, I will first make use of a rational numerical progression, which is well known to all, and I will represent my thought by means of demonstrative examples, through which our discourse can arrive at the general art of conjecture.

Despite being useful for our general discussion, a deeper analysis is out of context here. We want instead to stress the general inspiration that can be offered by Cusanus to contemporary investigation in such new fields as information and cognitive sciences by his vision of science and logic. It is clear that, presently, we live in a very different cultural context. Not only Fregean revolution is more than one hundred and fifty years old, but also Gdel results are approaching a whole Century of life. The way in which Cusanus see the problem of truth is very different from ours, as we are acquainted with Tarksis approach. His comments about the subjectivity of conjectures would be considered, if not immediately dismissed, as opening (very interesting, maybe, but) *general* epistemological questions not something that could be of specific interest to a (traditional) working logician of our time. We shall come back to this point in 6.3.

We conclude by briefly discussing a paradigmatic, practical example, having to do with the *conjunction of opposites*, which could shed some light on this point.

With each statement p, one of its antonyms p^a (in just the former sense of being before p' respect to $<$, that is, refuted by p), by means of the linguistic conjunction and (), to obtain the statement p and p^a $(p \cdot p^a)$ jointly considering what p refers to and also what is referred to by an antonym or opposite of p, or in general by a statement refuted by p. A conjunction of opposites, in sum, with which to have a self-contradiction, an inferential impossibility, is not so immediate if it is not taken a statement refuted by p.

Before Hegel, Marx, Lenin, and all the Marxian thinkers, the conjunctio oppositorum methodology, known in English as the unity of opposites, was, formerly and systematically, managed for reasoning by a theologian and philosopher Bishop.

Notice that the disjunction $p + p^a$ represents much of what, in the universe of discourse, is specifically known on p, but without being all that is known, like with additional conditions $p + p'$ tries to give and gives effectively in Ortholattices for instance.

6.2 Cusanus as a contemporary thinker

And, however, it seems to us that from a suitable, although unusual, perspective Cusanus words are very modern, contemporary: in the sense of being able to contribute to clarify the questions (of logical nature) which are of crucial interest in topics of common sense reasoning and cognitive science and AI. More tuned, epistemologically, to them many technical papers in mathematical logic appeared in the last decades One reason for that is that they seem very direct and fresh, not burdened by the important but heavy general apparatus of mathematical logic as structured in the last century. An apparatus that, in many situations, is not destined to provide clarifications, when we are interested to investigate and scrutiny very specific and circumscribed problems. This induces us to go back to our original question. We asked if this reference to old thinkers is only casual or whether there is some deeper reason for it. We favour the second hypothesis, a position we will explain in the next subsection. Before that we shall briefly look at the question of how the notion of conjecture has been treated in math.

While the term conjecture is and always has been informally used in mathematics and considered part of the daily dialogue among mathematicians, it has very rarely been considered of crucial interest among (the very tiny tribes of) logicians. The notion has been central in Poppers reflections on the scientific method and, perhaps, this fact has contributed to consider it as important only from an epistemological point of view. Supported in this by the distinction between a logic of discovery from a logic of confirmation. Lets dwell a little bit more on this concept.

A conjecture is here understood as a proposition that is unproven (otherwise it would have been a theorem) but about which there is a sort of common consensus in the context of the already established results in the field. But there is also something else: an agreement that the conjecture could be experimentally tested and checked, in order to arrive at a proof, inside the received conceptual context.

No one would call conjecture a proposition: this wording would strongly depart from that of a traditional theorem. Many of Cantors ideas had not been considered conjectures. The same happened as well to some of his proved propositions, at least at the beginning. Similarly, at the moment of the appearance and presentation of a new conjecture, the common view is that its subsequent demonstration would not necessarily imply or, even better, require a change in the overall architecture of mathematics (at least in the specific chapter involved), especially for what regards the ontological assumptions.

It may, of course, happen (and, in fact, it does happen and did happen for the interesting ones) that proving a conjecture would force to re-discuss many general assumptions and provide also conceptual changes. In those instances, this happens along the way, not at the beginning. This is what happened with Hilberts Entscheidungsproblem or with Fermats last theorem: two crucial conjectures, although they were not, for exogenous reasons, called this way at the time of their formulation.

The former needed the creation of the completely new Theory of Computation, an ever-present notion in math that in centuries had not been in need of formalisation. The latter needed three centuries of development of new pieces of math. Both are historically akin to the inferential zigzag. The first looks like its deducting part, and the second its abductive part. Conjectures play and have played a very important role, but they have been seen as a sort of future theorems (when lucky) or statements to be refuted momentarily missing a reason for refutation. Is there a reason to be interested in the form and specificity of the logical features of conjectures? For decades starting with Frege and going on with the foundational debates at the beginning of XX Century Logic had other goals and other crucial problems to afford. It seems that no space was left for an autonomous investigation of such notions that have acquired visibility also in the development of the logical brand of AI but this is no paradox at all: such subtle results could not be achieved without sophisticated formal tools. An attitude that has only slightly modified over the decades [8], but abruptly changed when the need for studying Commonsense Reasoning, Language at Work, emerged from AI.

6.3 A fresh way of looking at Logic

Some useful suggestion on commonsense logic paradoxically comes from the general vision of sophisticated thinkers with much bigger aims, due to their theological and religious commitments. Despite that, it was clear to them that global projects as the one that in a distant future would have been envisaged by by Hilbert for math and by Lord Kelvin for physics at the end of the XIX Century were not tenable.

A general and usually tacit shared assumption in the received view of mathematical logic is that progress in the understanding of logical aspects of every facet (as well as in the nuances) of the empirical phenomenon of reasoning cannot but descend and be derived by further developments of the central bulk of logic as outlined in the Thirties of the last Century. If the subtleties and profundity of this approach successfully tackled such seminal questions, so more so this powerful edifice should be able to deal with apparently trivial matters. This can still be possible in principle, although what happened (and is happening) in AI[5] suggests some reflections.

This implicit assumption obscures and neglects the fact that many specific aspects of reasoning can be afforded by mathematical tools, in many cases of great simplicity, in a sort of Galilean approach, without referring to this magnificent but burdensome construction. We could also realize that the corpus of mathematical logic, with all its well-deserved authority, determines what is crucial and important and, in a sense, what is relevant, giving little space to what does not ontologically conform to its bases. Something that is common to all disciplines and that, usually, does not impede due to the open mindedness of the scientific way of approaching the questions that minor fields and topics are investigated and developed. Something that may lack is informal ideas and motivations for specific features of these subfields.

For many decades Logic has had other goals to look at than specific aspects, leaving them to minor applications of the big construction[6]. Due mainly to the profundity of its central results, the wonderful edifice of 20th Century Logic has tended to neglect that its main target has substantially been to put it bluntly the internal consistency of mathematics, and not a general theory of reasoning.

A more general way to express this is that classical logic has to do with specific properties of those forms of reasoning that consider clear cut situations in a static world. These instances represent but a very small percentage of human reasoning, which is dynamic par excellence, and more often than not based on incomplete and imprecise information. *Mathematical* Logic owes its name not only to the fact that it uses a mathematical *language* and mathematical *tools* but also to the fact that it is the logic of *mathematical* reasoning, but not of *Commonsense* Reasoning.

We can, perhaps, also add something more. Logic, as acutely observed by Jean van Heijenoort in Frege and Vagueness, excluded vagueness, from his horizon in its founding years (see [9]). This was a correct choice, at the beginning. One cannot consider vagaries when trying to establish a new theory: Galileo did that by forgetting friction, while constructing mechanics. But now, van Hejienoort states, some time has passed, and we must consider vagaries. Looking at vagaries and admitting vagueness into the realm of Logic imposes to look anew at many questions. Among them, the central notions of coherence and completeness. Vagueness opens the way to new motivations and the subtle analyses of old logicians provide useful inspiration, since they were thought in a period in which present day formal requirements were not required. This draws an unusual parallel with the present situation, in which we are urged to construct systems and models in which these same requirements are not strictly applicable.

When analysing questions and problems from Cognitive Science, for instance, not only the notion of conjecture is essential (perhaps with different nomenclature), but it is an everyday experience that the conjectures of each person will be different. And that is exactly what the model should consider and try to explain.

Specific aspects of reasoning can be looked at in a fresh way and not as particular cases of the big construction. In order to do so we need also to help ourselves with epistemological and conceptual reflections tuned with this approach. In this direction we found that many general remarks done by Cusanus are very stimulating and useful, which warrants the discussion of them in this paper.

[5]We refer to the well-known fact that the simpler facts affordable by humans looked the most difficult to tackle by automatic means, and also to the big steps forward obtained by brute force methods.

[6]This neither means nor implies that the sophisticated and powerful tools forged in the core of the crucial questions are not useful or cannot be applied to other conceptually very different questions [11]. The point is that they can be usefully applied and creatively used when they are specifically relevant for the problems in question, which should be looked at in their complexity and, in some cases, elusiveness.

And we can now come back again to the starting question of this Section. We think that one can affirm that the inspiration provided by ancient texts is not casual. It corresponds to a similarity (although, paradoxically, both in a very specific and vague sense) with the general conceptual framework. Once vanished the illusion of a unique, firm and stable foundation of the workings of scientific investigation along established lines, to be pursued in an automatic way [12, 13, 10], the sophisticated conceptual analyses of middle age and renaissance scholars can provide useful suggestions to be checked. Of course, through the language and methods of contemporary investigation.

6.4 The zigzag is not unique

Let us, finally, observe that, in this Section, nothing has been explicitly said about zigzagging, as formally described in Section 3. We will limit this here to a comment. Lets observe that since each person can follow a different zigzag, that could be the reason why each one can find a different conjecture as well as different proofs for either a consequence or a hypothesis. The idea of a personal, individual[7] approach to reasoning, that is so omnipresent in everyday life and often the cause of infinite discussions and diatribes and so evidently missing in ordinary logic, should (and could) finally be reconciled with implementable procedures.

7 Conclusion

In [14, 19, 16] the so called Formal Skeleton of ordinary/commonsense reasoning, was presented, using which actual reasoning can be developed through a sort of 'Brownian Movement' around a premise, called the inferential zigzag, with which refutations, consequences, hypotheses, and speculations are obtained. A process that, within conditions, is effectively realizable (i.e. programmable) [19, 17], and that can be usefully employed in a better implementation of cognitive reasoning [7]. Nevertheless a conceptual problem remained open: how such zigzag can be effectively developed. That is, if a (theoretic) automatism acting without requiring the help of any mysterious entity, but in a known and describable form, could be algorithmically implemented in at least some specific and limited cases. By acquiring total certainty on the not metaphysical character of induction through developing a mathematical theory on it, such a theoretical question is partially answered in this paper. What is here presented contributes to dissolve the old worries concerning the mystery of induction: induction, or guessing, was identified with speculation. Possibly such dissolution is not of great practical relevance, but it has, of course, a conceptual, theoretical, importance since it means but a view on how people themselves actually reason, and sometimes can quickly envisage an unexpected conjecture. In some sense at least in the context of the conceptual setting defined in [19] the problem concerning the scientific understanding of what is ordinary or commonsense reasoning has now one possible clarification. The present paper, in fact, provides an indication of how this conceptual problem can be effectively and practically solved. If reasoning is achieved by developing effectively inferential zigzags, we have shown how the forward/backward inflexions at each point in an inferential chain are produced.

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[7] To avoid a wrong interpretation in the direction of a sort of non-objectivity of reasoning: we are referring to the individual path that each person, in everyday reasoning, can follow and which can be very different from the one followed by other persons. This variety is irrelevant in a standard setting (complete information, no vagueness, no approximations). Everything changes in the setting of everyday life in which, moreover, also implicit (hidden) presuppositions play a role.

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