PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Special Issue dedicated to Stefano De Marchi on the occasion of his 60th birthday, Volume 15 · 2022 · Pages 35–44

An adaptive algorithm for determining the optimal degree of regression in constrained mock-Chebyshev least squares quadrature

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Abstract

In this paper we develop an adaptive algorithm for determining the optimal degree of regression in the constrained mock-Chebyshev least-squares interpolation of an analytic function to obtain quadrature formulas with high degree of exactness and accuracy from equispaced nodes. We numerically prove the effectiveness of the proposed algorithm by several examples.

1 Introduction

Let f be a continuous function in the interval $[-1,1]$ and let $w(x) \in L_1([-1,1])$ be a nonnegative weight function. A widespread problem in applied mathematics consists in the approximation of the weighted integral

$$I[f] := \int_{-1}^{1} f(x)w(x)dx, \tag{1}$$

by using a quadrature formula. If the function f is known in the whole interval $[-1,1]$, one can use a Gauss–Christoffel quadrature rule [10]

$$Q[f] := \sum_{k=1}^{m} w_k f(\xi_k), \tag{2}$$

where $\xi_1,\dots,\xi_m \in (-1,1)$ are the nodes and $w_1,\dots,w_m \geq 0$ are the weights of the quadrature formula, which has algebraic degree of exactness $2m-1$, that is it integrates exactly polynomials of degree up to $2m-1$. In many practical applications, however, the function f is not known on the whole interval $[-1,1]$, but only on a finite set of points

$$\mathcal{X}_n = \{x_0,\dots,x_n\},$$

which are often equispaced, that is

$$x_i = -1 + \frac{2}{n}i, \quad i = 0,\dots,n.$$

In these cases, composite trapezoidal or composite Simpson rules, of degree of exactness $1,3$, respectively, are widely used, since all Newton–Cotes rules of higher order (greater than 7 for $w(x)=1$) have weights which differ in sign and become rapidly unstable [14]. A possible approach to overcome this problem is based on the use of interpolation techniques which mitigate the Runge and the Gibbs phenomena and allow to obtain efficient quadrature formulas based on equidistant points (see [6, 8, 14, 15, 16] and references therein). Two interesting approaches to get quadrature formulas from equispaced nodes have been presented in [15, 16] and are based on the idea that it is possible to obtain quadrature rules by using Gauss–Christoffel formulas in combination with local polynomial interpolants or global rational interpolants, respectively. To describe the approach proposed by Majidian [16], we fix $s \in \mathbb{N}$ and we select the s—tubes

$$\mathcal{N}_k^s = \{x_{j_k},\dots,x_{j_k+s-1}\} \subset \mathcal{X}_n, \qquad k = 1,\dots,m,$$

such that $x_{j_k} \leq \xi_k \leq x_{j_k+s-1}$, $k=1,\dots,m$. The key point is to substitute the exact values $f(\xi_k)$ in (2) with the values, at ξ_k, of the Lagrange polynomial interpolants on the s—tube $\mathcal{N}_k^s$. Consequently, the quadrature formula (2) becomes

$$Q_s[f] = \sum_{k=1}^{m} \sum_{i=j_k}^{j_k+s-1} w_k f(x_i)\ell_i(\mathcal{N}_k^s,\xi_k), \tag{3}$$

where $\ell_i(\mathcal{N}_k^s,\cdot)$ is the Lagrange fundamental polynomial relative to the nodes in $\mathcal{N}_k^s$. Since the nodes are surrounding Christoffel abscissas ξ_k, accurate approximation of integral (1) by means of quadrature formula (3) are expected, at least for local polynomial interpolants of low degree. A different approach is proposed by De Marchi et al. in [6] based on the so-called mapped bases or

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fake nodes [5], which requires the definition of a bijective differentiable function $S:[-1,1]\to\mathbb{R}$ which maps the set $\mathcal{X}_n$ onto the set of Chebyshev–Lobatto nodes

$$X_n^{CL}=\left\{x_i^{CL}=-\cos\left(\frac{i\pi}{n}\right), i=0,\dots,n\right\}, \tag{4}$$

that is $S(x_i)=x_i^{CL}$, $i=0,\dots,n$. Therefore, by using the function S^{-1} as a change of variable, the integral (1) can be approximated by the well-known Gauss–Chebyshev quadrature formula. This method allows to mitigate the Gibbs phenomenon without resampling the given function.

Among all techniques known to defeat the Runge phenomenon, it is included the mock-Chebyshev subset interpolation, which produces a polynomial that interpolates f only on the proper subset $\mathcal{X}'_m$ of $\mathcal{X}_n$, constituted by nodes which best mimic the behavior of the Chebyshev–Lobatto nodes of order m. The accuracy of this interpolant can be improved by using the remaining nodes $\mathcal{X}_n\setminus\mathcal{X}'_m$ for a simultaneous regression [4, 7]. This is the idea of the constrained mock-Chebyshev least squares interpolant. More in details, by fixing $m=\lfloor\pi\sqrt{n}/\sqrt{2}\rfloor$, an integer r such that $m<r\le n$ and a polynomial basis $\mathcal{B}=\{u_0(x),\dots,u_r(x)\}$, the constrained mock-Chebyshev least squares interpolant is defined by

$$\hat{P}_{r,n}[f](x)=\sum_{i=0}^{r}\hat{a}_i u_i(x), \tag{5}$$

where $[\hat{a}_0,\dots,\hat{a}_r]^T$ is the solution of the Karush-Kuhn-Tucker linear system, or simply KKT system [7]. To take advantage of the good properties of the approximant (5), in [8] has been introduced a quadrature formula which makes use of the constrained mock-Chebyshev least squares interpolant of degree

$$r^\star=m+p,\quad p=\left\lfloor\frac{\pi}{\sqrt{2}}\sqrt{\frac{n}{6}}\right\rfloor,$$

in combination with the Gaussian-Christoffel quadrature formula. In particular, the exact values $f(\xi_j)$ $j=1,\dots,m$, are approximated by the evaluations of the constrained mock-Chebyshev least squares interpolant at the same points, that is

$$f(\xi_j)\approx\hat{P}_{r^\star,n}[f](\xi_j).$$

The constrained mock-Chebyshev least squares quadrature formula [8] is defined as follows

$$\hat{Q}_{r^\star,n}[f]=\sum_{j=1}^{m}w_j\hat{P}_{r^\star,n}[f](\xi_j). \tag{6}$$

By setting

$$\hat{w}_j=\sum_{i=1}^{m}w_i\hat{P}_{r^\star,n}[\ell_j](\xi_i),\quad j=0,\dots,n,$$

and

$$\ell_i(x)=\prod_{\substack{j=0\\ j\neq i}}^{n}\frac{x-x_j}{x_i-x_j},\quad i=0,\dots,n,\quad x\in[-1,1],$$

the formula (6) can be rewritten as

$$\hat{Q}_{r^\star,n}[f]=\sum_{i=0}^{n}\hat{w}_i f(x_i),$$

in order to make evident the dependence of this formula on the evaluations of the function f in all nodes of $\mathcal{X}_n$ [8]. The goal of this paper is to increase the accuracy of the constrained mock-Chebyshev least squares quadrature formula (6) by considering the family of quadrature formulas

$$\left\{\hat{Q}_{r,n}[f]:=\sum_{i=1}^{m}w_i\hat{P}_{r,n}[f](\xi_i),\quad r=m,\dots,2m-1\right\}. \tag{7}$$

More precisely, we propose an adaptive algorithm for numerically determining the "optimal" degree $r^\star_{opt}$ of the constrained mock-Chebyshev least-squares interpolation which produces a more accurate quadrature formula $\hat{Q}_{r^\star_{opt},n}[f]$.

The overview of the paper is the following. In Section 2 we develop the adaptive algorithm for determining the "optimal" degree of regression $r^\star_{opt}$ and we analyze its computational cost. In Section 3 we show how to generalize the algorithm for approximating integrals of bivariate analytic functions on the square $[-1,1]^2$. Finally, in Section 4, numerical experiments demonstrate the effectiveness of the proposed algorithm.

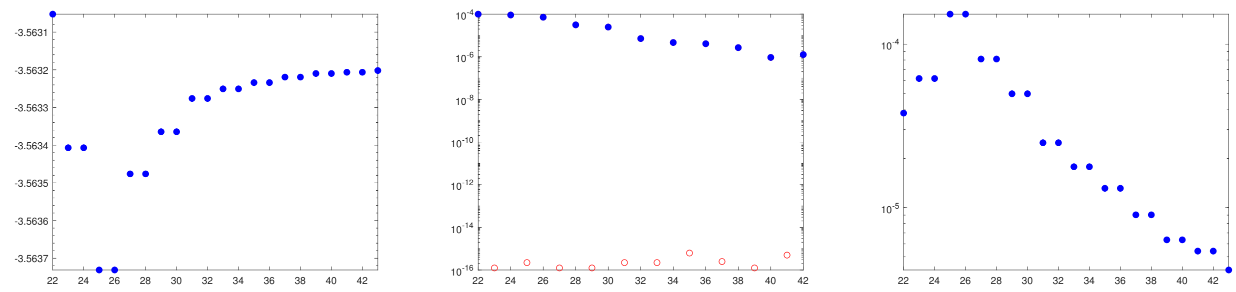

Figure 1: Sequence of approximations $\hat{Q}_{r,n}[f]$ (left) and sequence of approximate relative errors $\hat{E}_r[f]$ (center) versus sequence of exact relative errors $E_r[f]$ (right) with $f(x) = \frac{1}{x^2-1.1}$, $w(x) = 1$, $n = 100$ and $m = 22$.

2 Computing accurate quadrature formulas with high degree of exactness from equispaced nodes

The main goal of this section is the determination of a procedure for the choice of the "optimal" value of r which guarantees the "best approximation accuracy" of the quadrature formula $\hat{Q}_{r,n}[f]$, measured through the *exact relative error*

$$E_r[f] = \frac{\left|\hat{Q}_{r,n}[f] - I[f]\right|}{|I[f]|},$$

where we assume $|I[f]| > 0$. We denote this value by $r^\star_{opt} = r^\star_{opt}(f)$. To this aim, we analyze the trend of *approximate relative errors*

$$\hat{E}_r[f] = \frac{\left|\hat{Q}_{r+1,n}[f] - \hat{Q}_{r,n}[f]\right|}{\left|\hat{Q}_{r,n}[f]\right|}, \qquad m \le r \le 2m-2, \tag{8}$$

computed by using quadrature formulas of subsequent degrees up to the maximum degree of exactness $2m-1$. At first sight, it might be thought to choose $r^\star_{opt}$ as the value of $r \in \{m, m+1, \ldots, 2m-2\}$ which minimizes the approximate relative error $\hat{E}_r[f]$. Unfortunately, in general this choice could be misleading since, even for starting values of r, it could occur that two successive approximations $\hat{Q}_{r,n}[f]$ and $\hat{Q}_{r+1,n}[f]$ are so close to each other that the approximate relative error $\hat{E}_r[f]$ is very small, for example less than a tolerance tol, despite the exact relative error $E_r[f]$ is not, being much greater than tol. An example of this situation is well illustrated in Figure 1, where the sequence of approximate relative errors assumes values less than 10^{-14} despite all exact relative errors are not less than 10^{-6}. The approximate relative errors $\hat{E}_r[f]$ less than tol are then outliers and therefore they have to be discarded in the process of the determination of $r^\star_{opt}$. Instead of fixing a tolerance tol a priori, we distinguish outliers from valid values of relative errors $\hat{E}_r[f]$ by analyzing the sequence of consecutive triples

$$t_r = \left\{\hat{E}_r[f], \hat{E}_{r+1}[f], \hat{E}_{r+2}[f]\right\}, \qquad r = m, \ldots, 2m-3. \tag{9}$$

We call the triple t_r monotonic if and only if

$$\hat{E}_r[f] \ge \hat{E}_{r+1}[f] \ge \hat{E}_{r+2}[f] \qquad \text{or} \qquad \hat{E}_r[f] \le \hat{E}_{r+1}[f] \le \hat{E}_{r+2}[f],$$

otherwise we call the triple t_r non monotonic. Note that a triple t_r is monotonic if and only if

$$d_r d_{r+1} \ge 0,$$

where we set

$$d_r = \log_{10}\left(\hat{E}_{r+1}[f]\right) - \log_{10}\left(\hat{E}_r[f]\right), \qquad r = m, \ldots, 2m-2.$$

We fix a suitable constant $\delta > 0$ (for example $\delta = 0.5$) and we search the outliers among the elements of non monotonic triples by assuming that:

- if $d_r < -\delta$ and $d_{r+1} > \delta$, then $\hat{E}_{r+1}$ is outlier (see Figure 2 (left));
- if $d_r > \delta$ and $d_{r+1} < -\delta$, then $\hat{E}_r$ is outlier (see Figure 2 (right)).

The process of determining new outliers ends when $r = 2m-2$. The tolerance tol is determined adaptively by initializing it with the epsilon machine eps and by updating it by the rule

$$tol := \max\{tol, \hat{E}_r\} \tag{10}$$

as soon as we find a new outlier $\hat{E}_r$. To avoid under-estimation of the exact relative error we take into account the possible presence of outliers, by assuming as outliers all approximate relative errors $\hat{E}_r$ less than or equal to the computed tolerance tol. The Algorithm 1 computes the tolerance tol. The Algorithm 2 detects the outliers and remove them, providing in output the increasing sequence $\{r_1, \ldots, r_p\} \subset \{m, \ldots, 2m-2\}$ of degrees r such that $\hat{E}_r > tol$. In Figure 3 we display the effect of the Algorithm 2, in detecting outliers (left) and removing them (right).

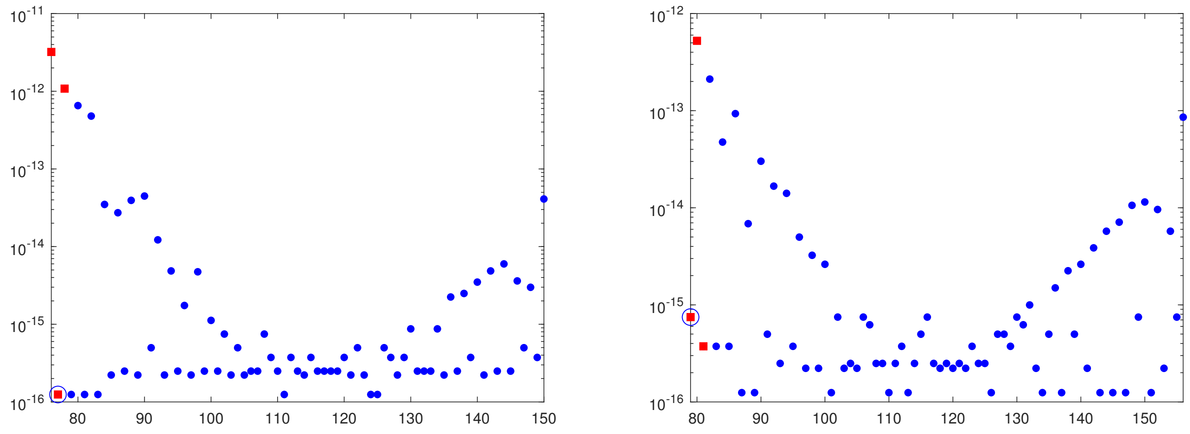

Figure 2: Approximate relative errors, with respect to the Gauss–Legendre scheme ($w(x)=1$), for the function $f(x)=\frac{1}{x^2-1.1}$ (•) with $n=1200$, $m=76$ (left) and $n=1280$, $m=79$ (right) with related first non monotonic triples (■) and corresponding outliers (○).

Algorithm 1 Tolerance determination

Require: $\hat{E}_m[f],\dots,\hat{E}_{2m-2}[f]$
Ensure: tol
  $tol \leftarrow eps$
  for $i=m,\dots,2m-3$ **do**
    $d_i \leftarrow \log_{10}(\hat{E}_{i+1}[f])-\log_{10}(\hat{E}_i[f])$
  end for
  while $j \le m-2$ **do**
    if $d_j \le -\delta$ and $d_{j+1} \ge \delta$ **then**
      $tol=\max\{tol,\hat{E}_{j+1}[f]\}$
      $j \leftarrow j+2$
    else if $d_j \ge \delta$ and $d_{j+1} \le -\delta$ **then**
      $tol=\max\{tol,\hat{E}_j[f]\}$
      $j \leftarrow j+1$
    else
      $j \leftarrow j+2$
    end if
  end while

Algorithm 2 Outlier detection

Require: $\hat{E}_m[f],\dots,\hat{E}_{2m-2}[f]$
Ensure: $r_1,\dots,r_p$
  $j \leftarrow 1$
  for $i=m,\dots,2m-2$ **do**
    if $\hat{E}_i[f] > tol$ **then**
      $r_j \leftarrow i$
      $j \leftarrow j+1$
    end if
  end for

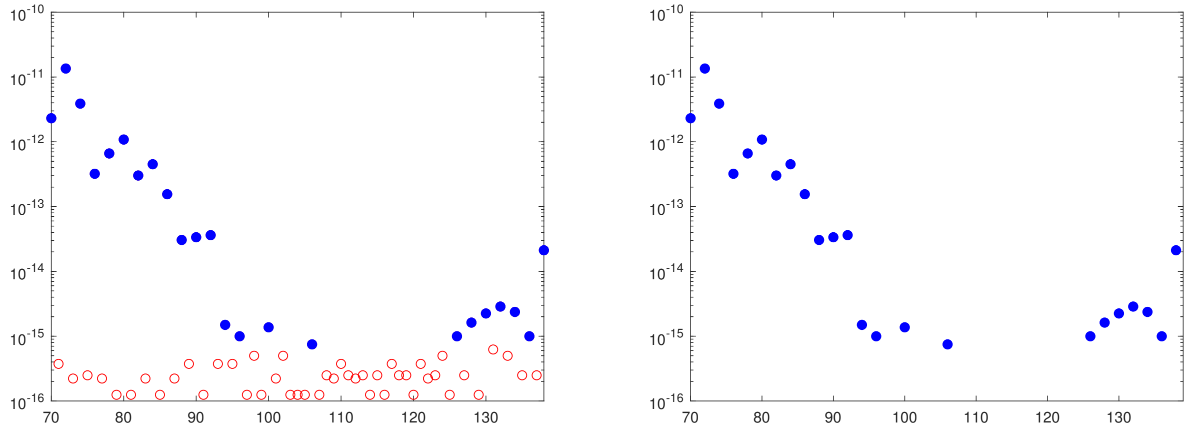

Figure 3: Significant approximate relative errors (•) and outliers (∘) for the function $f(x) = \frac{1}{x^2-1.1}$ $n = 1000$, $m = 70$ relative to the Gauss–Legendre scheme ($w(x) = 1$).

From now on we assume that all outliers in the sequence $\{(r, \hat{E}_r)\}_{r=m}^{2m-2}$ have been removed. We denote by $\{(r_j, \hat{E}_{r_j})\}_{j=1}^{p}$, $m \le r_1 < \cdots < r_p \le 2m-2$ the subset of significant data. We consider the sequence of intervals $\{I_j\}_{j=0}^{p}$, defined as follows

$$I_j = \begin{cases} [m, r_1), & \text{if } j = 0, \\ (r_j, r_{j+1}), & \text{if } j = 1, \dots, p-1, \\ (r_p, 2m-2], & \text{if } j = p. \end{cases} \tag{11}$$

By definition, all outliers belong to

$$I = \bigcup_{j=0}^{p} I_j.$$

Let $q \ge 0$ be the number of intervals I_j in (11) containing at least one outlier.

Case $q > 0$. We denote by I_{j_k}, $0 \le j_1 < j_2 < \cdots < j_q \le p$ the intervals containing at least one outlier and by N_{j_k} the number of outliers in I_{j_k}. We set

$$\mu = \frac{1}{q}\sum_{k=1}^{q} N_{j_k}, \quad \sigma = \sqrt{\frac{1}{q}\sum_{k=1}^{q}(N_{j_k} - \mu)^2}.$$

As well-known, the standard deviation σ tells us the typical amount by which the values $\{N_{j_\ell}\}$ deviate from their average value μ. We define

$$\mathcal{R} = \{r_{j_\ell} : \ell = 1, \dots, q \wedge N_{j_\ell} > \mu + \sigma\},$$

and we set

$$r^\star_{opt} = \begin{cases} \min \mathcal{R} & \text{if } \mathcal{R} \ne \emptyset, \\ 2m-2 & \text{if } \mathcal{R} = \emptyset. \end{cases}$$

By the nature of the constrained mock-Chebyshev least-squares interpolation [4, 7], the case $r^\star_{opt} < 2m-2$ frequently occurs as soon as the exact relative error $E_{r^\star_{opt}}[f]$ is near to the machine precision already for values of $r \ll 2m-2$. In such cases, by increasing the degree of the regression $r \ge r^\star_{opt}$, it is also possible that the exact relative error $E_r[f]$ became worse. In fact, as r approaches to n, the polynomial $\hat{P}_{r,n}[f](x)$ *tends* to the polynomial interpolant on the set of nodes $\mathcal{X}_n$, $\hat{P}_{n,n}[f](x)$, which in its turn, can suffer the Runge phenomenon. If $\mathcal{R} \ne \emptyset$, from the definition of $r^\star_{opt}$, we expect a not increasing trend of significant data $\hat{E}_{r_1}[f], \dots, \hat{E}_{r^\star_{opt}}[f]$ if $r^\star_{opt} > r_1$ or a non decreasing trend of the significant data $\hat{E}_{r^\star_{opt}}[f], \dots, \hat{E}_p[f]$ if $r^\star_{opt} = r_1$. If $\mathcal{R} = \emptyset$ nothing can be said on the trend of the significant data.

Case $q = 0$. We set $r^\star_{opt} = 2m-2$. In this case nothing can be said on the trend of the significant data.

We are now able to determine a value $r^\star_{opt}$ of the degree of regression which produces more accurate quadrature formulas. The accuracy of $Q_{r^\star_{opt},n}[f]$ will depend on the quality of approximation of the constrained mock-Chebyshev least-squares to the function f. We distinguish the following cases:

- if $r^\star_{opt} = r_1$, we use a linear regression $l(r)$ to model the trend of the data

$$\{\log_{10}(\hat{E}_{r^\star_{opt}}[f]), \dots, \log_{10}(\hat{E}_{r_p}[f])\}$$

  and we set

$$r^\star_{opt} = r_k,$$

  where

$$\log_{10}(\hat{E}_{r_k}) = \min_{s\in\{1,\dots,p\}} \{\log_{10}(\hat{E}_{r_s}) : \log_{10}(\hat{E}_{r_s}) - l(r_s) \geq 0\};$$

- if $r^\star_{opt} = r_j$ with $j = 2, \dots, p-1$, then if $\log_{10}(\hat{E}_{r_j}) - \log_{10}(\hat{E}_{r_{j+1}}) > \delta$, we set $r^\star_{opt} = r_{j+1}$, else we use a linear regression $l(r)$ to model the trend of the data

$$\{\log_{10}(\hat{E}_{r_1}[f]), \dots, \log_{10}(\hat{E}_{r^\star_{opt}}[f])\}$$

  and we set

$$r^\star_{opt} = r_k,$$

  where

$$\log_{10}(\hat{E}_{r_k}) = \min_{s\in\{1,\dots,j\}} \{\log_{10}(\hat{E}_{r_s}) : \log_{10}(\hat{E}_{r_s}) - l(r_s) \geq 0\};$$

- if $r^\star_{opt} = r_p$, we use a linear regression $l(r)$ to model the trend of the data

$$\{\log_{10}(\hat{E}_{r_1}[f]), \dots, \log_{10}(\hat{E}_{r^\star_{opt}}[f])\}$$

  and we set

$$r^\star_{opt} = r_k,$$

  where

$$\log_{10}(\hat{E}_{r_k}) = \min_{s\in\{1,\dots,j\}} \{\log_{10}(\hat{E}_{r_s}) : \log_{10}(\hat{E}_{r_s}) - l(r_s) \geq 0\}.$$

Algorithm 3 Adaptive algorithm for determining a quadrature formulas with high degree of exactness and accuracy from equispaced nodes

Require: $\mathcal{X}_n = [x_0, \dots, x_n]^T, f = [f_0, \dots, f_n]^T$
Ensure: $\hat{Q}_{r^\star_{opt},n}[f]$
Compute m
Compute the mock-Chebyshev subset $\mathcal{X}'_m$
Compute $\mathcal{X}''_{n-m} = \mathcal{X}_n \setminus \mathcal{X}'_m$
Set $\mathcal{X}_n = [\mathcal{X}'_m, \mathcal{X}''_{n-m}]$
Compute the Gauss–Christoffel nodes and weights of order m
for $r = m : 2m-1$ **do**
    Compute $\hat{P}_{r,n}[f]$
    Compute $\hat{Q}_{r,n}[f]$
end for
Compute the approximate relative errors $\hat{E}_m[f], \dots, \hat{E}_{2m-2}[f]$
Run Algorithm 1
Run Algorithm 2
Compute $r^\star_{opt}$

Computational cost

We determine the computational cost of the Algorithm 3 described above. The computational cost of $m = \lfloor \pi\sqrt{n}/\sqrt{2} \rfloor$ is negligible. By using the procedure proposed in [1], the selection of the mock-Chebyshev subset from the uniform grid $\mathcal{X}_n$ requires about $O(mn)$ flops. The reordering of the set $\mathcal{X}_n$ involves a searching algorithm for the computation of the set $\mathcal{X}''_{n-m}$, whose computational cost is $\mathcal{O}(m\log(n))$ flops. Gauss-Christoffel nodes and weights can be computed through the Chebfun package [13]. The function `legpts` to compute Legendre nodes and weights, used in the numerical experiments, requires $\mathcal{O}\left(\frac{m(\log m)^2}{\log(\log m)}\right)$ flops. For each $r \in [m, 2m-1]$, the computation of the coefficients of the polynomial $\hat{P}_{r,n}[f]$ through the Lagrange multipliers method [7] requires $\mathcal{O}(m^2 n)$ flops for the construction of the KKT matrix and $\mathcal{O}(m^3)$ for the solution of the linear system through the QR factorization [2]. In passing from r to $r+1$ the new KKT matrix can be obtained from the previous one by a negligible cost, and then the computational cost for computing $\hat{P}_{r,n}[f]$, $r = m, \dots, 2m-1$, is still $\mathcal{O}(m^2 n)$. Since $\hat{Q}_{r,n}[f]$ can be computed by $\mathcal{O}(m^2)$ flops, then the total cost of the for loop is $\mathcal{O}(m^2 n)$ flops. The tolerance is computed through the Algorithm 1 and the detection of the outliers is made by using the Algorithm 2, which both require $\mathcal{O}(m)$ flops. Finally, we determine the degree of regression $r^\star_{opt}$, which produces accurate quadrature formulas, by using the procedure described in the Section 2. In its turn this procedure requires $\mathcal{O}(n)$ flops. Since $m = \mathcal{O}(\sqrt{n})$, the computational cost of the Algorithm 3 is $\mathcal{O}(n^2)$ flops.

3 Computing accurate cubature formulas with high degree of exactness from regular grids of nodes

Let $f(x,y)$ be a sufficiently regular function in the square $[-1,1]^2$ and let $\mathcal{X}_{n_x} \times \mathcal{Y}_{n_y}$ be the grid of $(n_x+1)\times(n_y+1)$ equispaced points in the same domain. In analogy with the univariate case, we set $m_x = \lfloor \pi\sqrt{n_x}/\sqrt{2} \rfloor$, $m_y = \lfloor \pi\sqrt{n_y}/\sqrt{2} \rfloor$ and we fix $r_{x,y} = (r_x, r_y) \in \mathbb{N}\times\mathbb{N}$ such that $m_x \le r_x \le n_x$ and $m_y \le r_y \le n_y$. We denote by

$$\hat{P}_{r_{x,y}}[f] := \hat{P}_{(r_x,r_y),(n_x,n_y)}[f], \tag{12}$$

the tensor product extension of the polynominal $\hat{P}_{r,n}[f]$ and we consider the quadrature formulas

$$\begin{aligned} I[f] &:= \int_{-1}^{1}\int_{-1}^{1} w(s,t) f(s,t)\,ds\,dt \approx \sum_{i=1}^{m_x}\sum_{j=1}^{m_y} w_i \kappa_j f(\xi_i,\eta_j) \\ &\approx \sum_{i=1}^{m_x}\sum_{j=1}^{m_y} w_i\kappa_j \hat{P}_{r_{x,y}}(\xi_i,\eta_j) =: \hat{Q}_{r_{x,y}}[f], \end{aligned} \tag{13}$$

where $\xi_1,\dots,\xi_{m_x}$ and $\eta_1,\dots,\eta_{m_y}$ are nodes of a Gaussian quadrature formula with weights $w_1,\dots,w_{m_x}$ and $\kappa_1,\dots,\kappa_{m_y}$ of order m_x and m_y, respectively. For a matter of simplicity, we restrict to the case $n_x = n_y = n$, then $m_x = m_y = m$, and we consider the family of quadrature formulas

$$\hat{Q}_{(r,r)}[f], \quad m \le r \le 2m-1, \tag{14}$$

and the approximate relative errors

$$\hat{E}_r[f] = \frac{\left|\hat{Q}_{(r+1,r+1)}[f] - \hat{Q}_{(r,r)}[f]\right|}{\left|\hat{Q}_{(r,r)}[f]\right|}, \quad m \le r \le 2m-1. \tag{15}$$

In analogy to the univariate case, by using the Algorithm 3, it is possible to determine a value $r^\star_{opt}$ of the degree of regression which produces accurate quadrature formulas.

4 Numerical experiments

4.1 Univariate case

In this Section, we compute the approximation of the integral (1) by using the quadrature formula (7) with $r = r^\star_{opt}$, where the polynomial $\hat{P}_{r^\star_{opt},n}[f]$ is expressed in the Chebyshev basis B^C. To this aim, we consider the grid of 1001 equispaced nodes in $[-1,1]$, that is $n = 1000$, $m = 70$. The experiments are performed on the following functions

$$f_1(x) = \frac{1}{1+8x^2}, \qquad f_2(x) = \frac{1}{1+25x^2}, \qquad f_3(x) = \frac{1}{((x+1)^4 + (2/50)^2)},$$

$$f_4(x) = e^{-x^2}, \qquad f_5(x) = \frac{1}{x^4 + (\frac{\sqrt{26}}{5}-1)x^2 + (\frac{13}{50})^2}, \qquad f_6(x) = \frac{1}{x+1.01},$$

by using th Gauss–Legendre weight $w(x) = 1$.

In Table 1, from left to right, we compare the relative errors obtained by applying the trapezoidal composite rule (E_T), the Cavalieri–Simpson composite rule (E_{CS}), the quadrature formula proposed in [16] with $s = 6$ (E_M), the constrained mock-Chebyshev least squares quadrature formula proposed in [8] a (E_{MC}) and the proposed here quadrature formula ($E_{r^\star_{opt}}$). To appreciate the accuracy of the estimate of the exact relative error, obtained through the Algorithm 3, in the last column we report also $\hat{E}_{r^\star_{opt}}$. To show the efficacy of the Algorithm 3 in computing the optimal regression degree $r^\star_{opt}$, in Figures 4-9 we display the sequences of exact relative errors, approximate relative errors (with discarded outliers, in red) and the regression line of the significant data, for all test functions.

4.2 Bivariate case

We consider the grid of 151 × 151 equispaced nodes in $[-1,1]^2$, that is $n_x = n_y = 150$, $m_x = m_y = 27$. The experiments are perfomed by using the Gauss–Legendre weight and the well-known Franke's function

$$\begin{aligned} f(x,y) = {} & 0.75\exp\left(-\frac{(9(x+1)/2-2)^2}{4} - \frac{(9(y+1)/2-2)^2}{4}\right) \\ & +0.75\exp\left(-\frac{(9(x+1)/2+1)^2}{49} - \frac{(9(y+1)/2+1)}{10}\right) \\ & +0.5\exp\left(-\frac{(9(x+1)/2-7)^2}{4} - \frac{(9(y+1)/2-3)^2}{4}\right) \\ & -0.2\exp\left(-(9(x+1)/2-4)^2 - (9(y+1)/2-7)^2\right), \end{aligned}$$

	E_T	E_{CS}	E_M	E_{MC}	$E_{r^\star_{opt}}$	$\hat{E}_{r^\star_{opt}}$
$f_1(x)$	1.33e-04	8.88e-10	7.84e-13	2.55e-16	0	2.55e-16
$f_2(x)$	8.97e-08	1.51e-14	1.02e-12	1.39e-10	4.13e-12	1.98e-12
$f_3(x)$	3.00e-10	4.09e-16	1.06e-10	3.75e-13	1.59e-14	6.34e-15
$f_4(x)$	3.28e-07	8.77e-15	4.44e-16	5.94e-16	5.94e-16	2.97e-16
$f_5(x)$	1.44e-07	5.41e-14	1.24e-13	2.24e-16	7.86e-16	8.99e-16
$f_6(x)$	6.26e-04	6.14e-07	1.52e-06	1.67e-07	8.81e-09	9.26e-10

Table 1: Comparisons among the relative errors in trapezoidal composite rule (E_T), Cavalieri–Simpson composite rule (E_{CS}), quadrature formula proposed in [16] (E_M), quadrature formula through the constrained mock-Chebyshev interpolant with optimal degree ($E_{r^\star_{opt}}$) and the approximate relative error obtained through Algorithm 3 ($\hat{E}_{r^\star_{opt}}$).

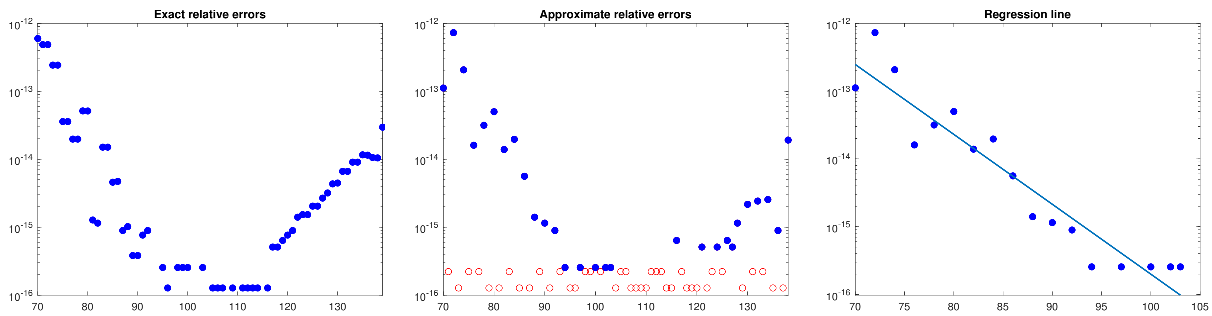

Figure 4: From left to right. Exact relative errors, approximate relative errors (•) with discarded outliers (∘) and regression line of the significant data for the function $f_1(x)$.

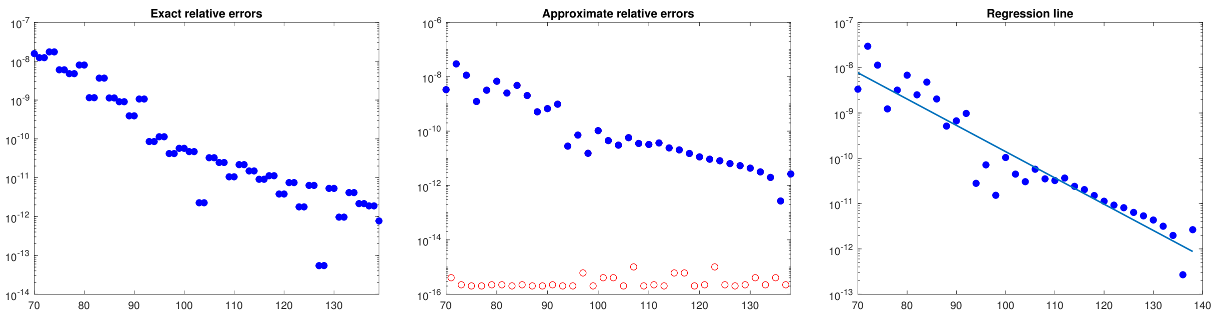

Figure 5: From left to right. Exact relative errors, approximate relative errors (•) with discarded outliers (∘) and regression line of the significant data for the function $f_2(x)$.

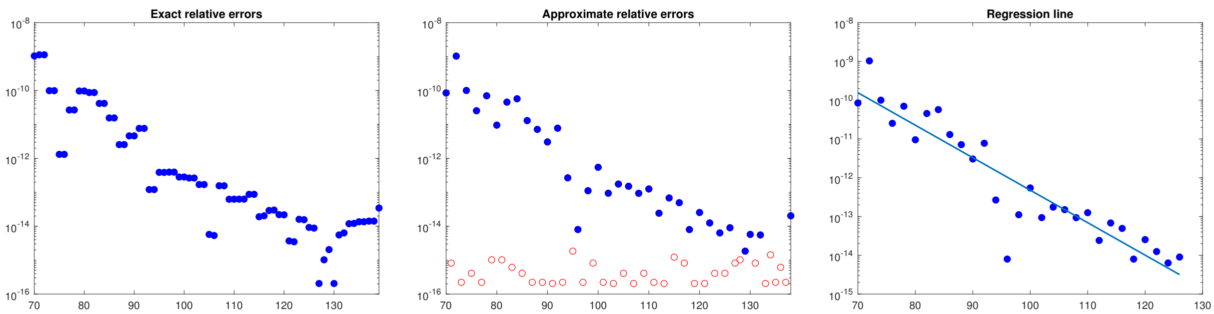

Figure 6: From left to right. Exact relative errors, approximate relative errors (•) with discarded outliers (∘) and regression line of the significant data for the function $f_3(x)$.

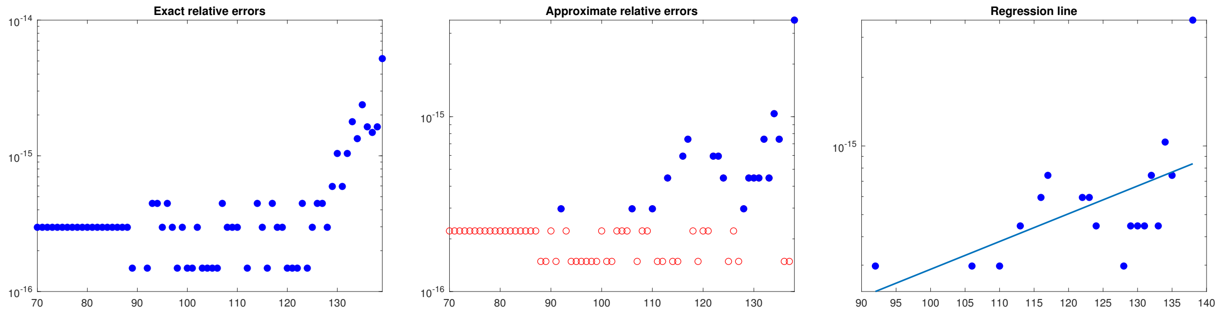

Figure 7: From left to right. Exact relative errors, approximate relative errors (●) with discarded outliers (○) and regression line of the significant data for the function $f_4(x)$.

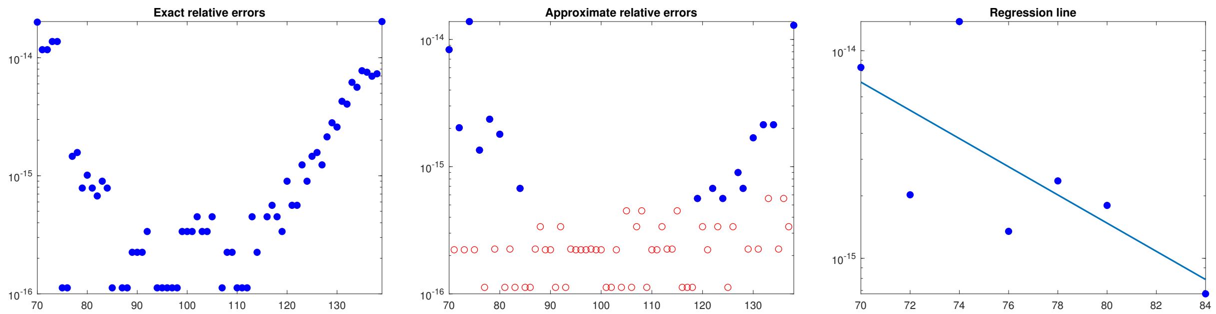

Figure 8: From left to right. Exact relative errors, approximate relative errors (●) with discarded outliers (○) and regression line of the significant data for the function $f_5(x)$.

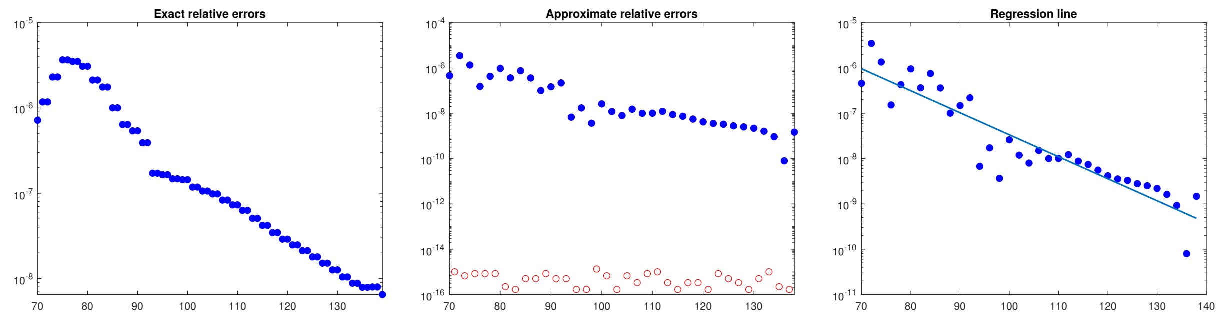

Figure 9: From left to right. Exact relative errors, approximate relative errors (●) with discarded outliers (○) and regression line of the significant data for the function $f_6(x)$.

	E_T	E_{CS}	$E_{r^\star_{opt}}$	$\hat{E}_{r^\star_{opt}}$
$f(x,y)$	2.30e-05	1.71e-10	4.69e-12	2.46e-11

Table 2: Comparisons among the relative errors in trapezoidal composite rule (E_T), Cavalieri–Simpson composite rule (E_{CS}), cubature formulas through the constrained mock-Chebyshev tensor product interpolant with optimal degree ($E_{r^\star_{opt}}$) and approximate relative error obtained through Algorithm 3 ($\hat{E}_{r^\star_{opt}}$).

where the polynomial $\hat{P}_{r_{x,y}}[f]$ is expressed in the tensor product Chebyshev basis $B^C \otimes B^C$. In Table 2, from left to right, we compare the relative errors obtained by applying the tensor product trapezoidal composite rule (E_T), the tensor product Cavalieri–Simpson composite rule (E_{CS}) and the proposed here quadrature formula ($E_{r^\star_{opt}}$). To appreciate the accuracy of the estimate of the exact relative error, obtained through the Algorithm 3, in the last column we report also $\hat{E}_{r^\star_{opt}}$.

5 Conclusions

In this paper, we have developed an adaptive algorithm for determining accurate quadrature formulas with high degree of exactness from $n+1$ equispaced nodes in the interval $[-1,1]$. Starting from the mock-Chebyshev interpolant of an analytic function $f(x)$, the increasing of the degree p of the simultaneous regression allows us to generate a family of quadrature formulas, with increasing degree of exactness $r = m + p$. Infact, we approximate, in a Gauss–Cristoffel quadrature rule with m nodes, the values of $f(x)$ at the Legendre nodes of order $m = \lfloor \pi\sqrt{n}/\sqrt{2} \rfloor$ through the constrained mock-Chebyshev least-squares approximant. A data cleanup strategy and a linear regression on the significant relative errors allow us to determine the "optimal" degree of regression to obtain quadrature formulas as much accurate as possible. The procedure is generalized to obtain cubature formulas on the square $[-1,1]^2$ through the constrained mock-Chebyshev least-squares tensor product interpolation.

Acknowledgments

This research has been achieved as part of RITA "Research ITalian network on Approximation" and as part of the UMI group "Teoria dell'Approssimazione e Applicazioni". The research was supported by GNCS-INdAM 2022 projects. The authors are members of the INdAM Research group GNCS.

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