CODIMENSIONS OF STAR-ALGEBRAS AND LOW EXPONENTIAL GROWTH

ANTONIO GIAMBRUNO AND DANIELA LA MATTINA

Abstract. In this paper we prove that if A is any algebra with involution $*$ satisfying a non trivial polynomial identity, then its sequence of $*$-codimensions is eventually non decreasing. Furthermore by making use of the $*$-exponent we reconstruct the only two $*$-algebras, up to T^*-equivalence, generating varieties of almost polynomial growth. As a third result we characterize the varieties of algebras with involution whose exponential growth is bounded by 2.

1. Introduction

Let A be an algebra with involution $*$ over a field F of characteristic zero. One associates to A, in a natural way, a numerical sequence $c_n^*(A)$, $n = 1, 2, \ldots$, called the sequence of $*$-codimensions of A which is the main tool for the quantitative investigation of the polynomial identities of the algebra A. Recall that $c_n^*(A)$, $n = 1, 2, \ldots$, is the dimension of the space of multilinear $*$-polynomials in n fixed variables in the corresponding relatively free algebra with involution of countable rank. Such sequence has been extensively studied (see [8, 15, 16, 17, 18, 19]) but it turns out that it can be explicitly computed only in very few cases. In case A is a PI-algebra, i.e, it satisfies a non trivial polynomial identity, it was proved in [9] that, as in the ordinary case, $c_n^*(A)$, $n = 1, 2, \ldots$, is exponentially bounded.

For this reason the interest focused in the computation of such asymptotics since they represent an invariant of the T*-ideal of the $*$-identities satisfied by A.

Recently in [1] the authors characterized the varieties of PI-algebras with involution by proving that any such variety is generated by the Grassmann envelope of a finite dimensional superalgebra with superinvolution. The major application of this result was obtained in [8] where it was shown that the exponent ($\exp^*(A)$) of a PI-algebra with involution exists and is an integer. More precisely, for general PI-algebras, it was proved that there exist constants $C_1 > 0, C_2, t, s$ such that

$$C_1 n^t \exp^*(A)^n \leq c_n^*(A) \leq C_2 n^s \exp^*(A)^n \tag{1}$$

for all $n \geq 1$.

Next step is to ask if the polynomial factor in (1) is uniquely determined, i.e., $t = s$, giving in this way a second invariant of a T*-ideal, after the $*$-exponent.

Recently in [6] the authors gave a positive answer to this question for the class of $*$-fundamental algebras.

More precisely they proved the following: let $A = \bar{A} + J$ be a $*$-fundamental algebra over an algebraically closed field where $\bar{A}$ is a $*$-semisimple subalgebra and

2010 *Mathematics Subject Classification.* Primary 16R10, 16R50; Secondary 16P90, 16W10.
Key words and phrases. polynomial identity, involution, growth.
Partially supported by GNSAGA of INdAM.

J is the Jacobson radical of A. Then

$$\lim_{n\to\infty} \log_n \frac{c_n^*(A)}{exp^*(A)^n} = -\frac{1}{2}(\dim(\bar{A})^- - r) + s,$$

where $J^s \neq 0$, $J^{s+1} = 0$, $(\bar{A})^-$ is the Lie algebra of skew elements of $\bar{A}$ and r is the number of $*$-simple algebras appearing in the decomposition of $\bar{A}$ which are not simple algebras.

Now, if $\mathcal{V}$ is a variety of $*$-algebras, the growth of $\mathcal{V}$ is the growth of the sequence of $*$-codimensions of a generating algebra.

Inspired by the above results here we are able to obtain further results on the growth of varieties of algebras with involution.

More precisely we shall prove that if A is any algebra with involution satisfying a non trivial polynomial identity, then its sequence of $*$-codimensions is eventually non decreasing.

Furthermore by making use of the $*$-exponent we shall reconstruct the only two $*$-algebras, up to T^*-equivalence, generating varieties of almost polynomial growth, i.e, such that they grow exponentially but any proper subvariety grows polynomially.

As a third result we shall characterize the varieties of algebras with involution whose exponential growth is bounded by 2.

2. Preliminaries

Throughout this paper F will denote a field of characteristic zero and A an associative F-algebra with involution $*$ (also called a $*$-algebra). Let us write $A = A^+ \oplus A^-$, where $A^+ = \{a \in A | \ a^* = a\}$ and $A^- = \{a \in A | \ a^* = -a\}$ denote the sets of symmetric and skew elements of A, respectively.

Let $X = \{x_1, x_2, \ldots\}$ be a countable set and let $F\langle X, *\rangle = F\langle x_1, x_1^*, x_2, x_2^*, \ldots\rangle$ be the free associative algebra with involution on X over F. In order to simplify the notation we shall write simply $f(x_1, \ldots, x_n)$ to indicate a $*$-polynomial of $F\langle X, *\rangle$ in which the variables $x_1, \ldots, x_n$ or their star appear.

Recall that $f(x_1, \ldots, x_n) \in F\langle X, *\rangle$ is a $*$-polynomial identity (or simply a $*$-identity) of A and we write $f \equiv 0$ if $f(a_1, \ldots, a_n) = 0$, for all $a_1, \ldots, a_n \in A$.

We denote by $\mathrm{Id}^*(A) = \{f \in F\langle X, *\rangle \mid f \equiv 0 \text{ on } A\}$ the set of $*$-polynomial identities of A. Clearly $\mathrm{Id}^*(A)$ is a T^*-ideal of $F\langle X, *\rangle$, i.e., an ideal invariant under all endomorphisms of the free algebra (commuting with the involution).

Recently it was proved in [1] that any PI-algebra with involution A over a field of characteristic zero, satisfies the same $*$-identities as the Grassmann envelope $G(B)$ of a finite dimensional superalgebra with superinvolution B. Let us recall the basic definitions in order to present such a result.

Let $B = B_0 \oplus B_1$ be an associative superalgebra over F endowed with a superinvolution $\sharp$. We shall call B an algebra with superinvolution. Recall that a superinvolution on B is a graded linear map $\sharp : B \longrightarrow B$ such that $(a^\sharp)^\sharp = a$ for all $a \in B$ and $(ab)^\sharp = (-1)^{(\deg a)(\deg b)} b^\sharp a^\sharp$ for any homogeneous elements $a, b \in B$. Here $\deg c$ denotes the homogeneous degree of $c \in B_0 \cup B_1$.

Since $\mathrm{char}\, F = 0$, we can write $B = B_0^+ \oplus B_0^- \oplus B_1^+ \oplus B_1^-$, where for $i = 0, 1$, $B_i^+ = \{a \in B_i \mid a^* = a\}$ and $B_i^- = \{a \in B_i \mid a^* = -a\}$ denote the sets of symmetric and skew elements of B_i, respectively. Notice that if B is a superalgebra with trivial grading, i.e., $B_1 = 0$, then the superinvolutions on B coincide with the involutions on B.

In a natural way one defines the free algebra with superinvolution $F\langle X, \sharp\rangle$, the ideal of identities with superinvolution $\mathrm{Id}^\sharp(B)$, etc.

Let G be the infinite dimensional Grassmann algebra over F, i.e., the algebra generated by the elements $1, e_1, e_2, \ldots$ subject to the relations $e_ie_j = -e_je_i$, for all $i, j \geq 1$. Recall that G has a natural $\mathbb{Z}_2$-grading $G = G_0 \oplus G_1$, where G_0 and G_1 are the spans of the monomials in the e_i's of even and odd length, respectively. One defines a superinvolution $\sharp$ on the Grassmann algebra $G = G_0 \oplus G_1$ by requiring that $e_i^\sharp = -e_i$, for $i \geq 1$. Hence $G^+ = G_0$ and $G^- = G_1$.

Now if $B = B_0 \oplus B_1$ is a superalgebra endowed with a superinvolution $\sharp$, it was proved in [1] that the Grassmann envelope of B, $G(B) = B_0 \otimes G_0 \oplus B_1 \otimes G_1$ has an induced involution $*$ by requiring that $(a \otimes g)^* = a^\sharp \otimes g^\sharp$, on all homogeneous elements $g \in G$ and $a \in B$. Notice that, if B is endowed with the trivial grading, the superinvolution on B is just an involution and $\mathrm{Id}^*(G(B)) = \mathrm{Id}^*(B)$.

The main property of such a Grassmann envelope is the following: if A is a PI-algebra with involution over a field of characteristic zero, then A satisfies the same $*$-identities as the Grassmann envelope $G(B)$ of a finite dimensional algebra with superinvolution B, i.e.,

$$\mathrm{Id}^*(A) = \mathrm{Id}^*(G(B)). \tag{2}$$

It is well known that in characteristic zero $\mathrm{Id}^*(A)$ is completely determined by its multilinear polynomials and we denote by

$$P_n^* = \mathrm{span}_F\{w_{\sigma(1)} \cdots w_{\sigma(n)} |\ \sigma \in S_n, w_i = x_i \text{ or } w_i = x_i^*, 1 \leq i \leq n\}$$

the space of multilinear $*$-polynomials of degree n in $x_1, \ldots, x_n$, i.e., for every $i = 1, \ldots, n$, either x_i or x_i^* appears in every monomial of P_n^* at degree 1 (but not both).

The study of $\mathrm{Id}^*(A)$ is equivalent to the study of $P_n^* \cap \mathrm{Id}^*(A)$ for all $n \geq 1$ and we denote by

$$c_n^*(A) = \dim_F \frac{P_n^*}{P_n^* \cap \mathrm{Id}^*(A)}, \quad n \geq 1,$$

the n-th $*$-codimension of A.

As a consequence of (2) we have that $c_n^*(A) = c_n^*(G(B))$, for all $n \geq 1$.

Such result allowed the authors in [8] to determine the exponential rate of growth of the $*$-codimensions of $G(B)$, and consequently of A. In order to state this result we make the following definition.

Let F be an algebraically closed field of characteristic zero and let B be a finite dimensional algebra with superinvolution. Then by [4] $B = \bar{B} + J$, where $\bar{B}$ is a maximal semisimple subalgebra with induced superinvolution and $J = J^\sharp$ is the Jacobson radical of B. Let $\bar{B} = B_1 \oplus \cdots \oplus B_q$ be a direct sum of simple algebras with superinvolution. We make the following.

Definition 2.1. *A subalgebra $C = C_1 \oplus \cdots \oplus C_t$ of B, where $C_1, \ldots, C_t$ are distinct subalgebras from the set $\{B_1, \ldots, B_q\}$ is called admissible if $C_1JC_2J \cdots JC_t \neq 0$. The subalgebra $C + J$ with induced superinvolution will be called reduced.*

The result in [8] reads as follows. If $B = B_1 \oplus \cdots \oplus B_q + J$ is defined as above, then there exist constants $C_1 > 0, C_2, t_1, t_2$ such that

$$C_1 n^{t_1} d^n \leq c_n^*(G(B)) \leq C_2 n^{t_2} d^n, \tag{3}$$

where d is the maximal dimension of an admissible subalgebra of B.

Since the codimensions do not change by extending the base field, by putting together the results in (2) and (3) the following result is clear.

Theorem 2.1. [8] *Let A be a PI-algebra with involution $*$ over a field of characteristic zero. Then there exist constants $C_1 > 0, C_2, t_1, t_2$ such that*

$$C_1 n^{t_1} d^n \leq c_n^*(A) \leq C_2 n^{t_2} d^n.$$

Hence $\lim_{n\to\infty} \sqrt[n]{c_n^*(A)} = \exp^*(A)$, *the $*$-exponent of A, exists and is an integer.*

Hence, in order to characterize the varieties of $*$-algebras of a given $*$-exponent t, a starting point is the study of the varieties of algebras with superinvolution generated by finite dimensional reduced algebras whose semisimple part is of dimension t.

3. Non decreasing sequences

In this section we prove that if A is an associative algebra with involution $*$ then the sequence of $*$-codimensions $c_n^*(A)$, $n = 1, 2, \ldots$, is eventually non-decreasing.

Theorem 3.1. *Let A be a PI-algebra with involution $*$. Then the sequence of $*$-codimensions $c_n^*(A)$, $n = 1, 2, \ldots$, is eventually non-decreasing, that is, $c_{n+1}^*(A) \geq c_n^*(A)$, for n large enough.*

Proof. Let $B = C + J$ be a finite dimensional algebra with superinvolution with $J^t = 0$, for some t, such that $\mathrm{Id}^*(A) = \mathrm{Id}^*(G(B))$. We shall prove that if $n \geq t$, $c_n^*(G(B)) \leq c_{n+1}^*(G(B))$.

If B is a nilpotent algebra, i.e., $C = 0$, then $c_n^*(G(B)) = 0$ for any $n \geq t$ and we are done.

Now assume that $C \neq 0$.

Given $n \geq t$ let $c_n^*(G(B)) = r$ and let $f_1, \ldots, f_r$ be $*$-polynomials of P_n^* in the variables $x_1, x_1^*, \ldots, x_n, x_n^*$ that are linearly independent modulo $P_n^* \cap \mathrm{Id}^*(G(B))$. For any $1 \leq i \leq r$, we construct the following $*$-polynomials:

$$h_i = h_i(x_1, ..., x_{n+1}) = \sum_{j=1}^{n} f_i(x_1, \ldots, x_{n+1}x_j + x_j x_{n+1}, \ldots, x_n) \in P_{n+1}^*,$$

where for any $j = 1, \ldots, n$, we have substituted in f_i the variable x_j with $x_{n+1}x_j + x_j x_{n+1}$.

We shall prove that $h_1, \ldots, h_r$ are linearly independent modulo $P_{n+1}^* \cap \mathrm{Id}^*(G(B))$.

Suppose by contradiction that $h = \sum_i \alpha_i h_i \equiv 0$ is a $*$-identity of $G(B)$ with some $\alpha_i \neq 0$. Since $f_1, \ldots, f_r$ are linearly independent modulo $P_n^* \cap \mathrm{Id}^*(G(B)$, we have that $f = \sum_i \alpha_i f_i$ is not a $*$- identity of $G(B)$.

Recall that $G(B) = B_0 \otimes G_0 + B_1 \otimes G_1$. Hence we can choose homogeneous elements $a_1, \ldots, a_n$ in a basis $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$ of B, where $\mathcal{B}_0 \subseteq C_0 \cup J_0$ and $\mathcal{B}_1 \subseteq C_1 \cup J_1$ and suitable $g_1, \ldots, g_n \in G_0 \cup G_1$ such that

$$f(a_1 \otimes g_1, \ldots, a_n \otimes g_n) \neq 0 \tag{4}$$

in $G(B)$.

Notice that, for any $i = 1, \ldots, r$, there exists a polynomial with superinvolution $p_i(x_1, \ldots, x_n, x_1^\sharp, \ldots, x_n^\sharp)$ such that

$$f_i(a_1 \otimes g_1, \ldots, a_n \otimes g_n) = p_i(a_1, \ldots, a_n, a_1^\sharp, \ldots, a_n^\sharp) \otimes g_1 \cdots g_n.$$

Hence, since the non-zero evaluation of f in (4) is equal to

$$\sum_{i=1}^{r} \alpha_i f_i(a_1 \otimes g_1, \dots, a_n \otimes g_n) = (\sum_{i=1}^{r} \alpha_i p_i(a_1, \dots, a_n, a_1^\sharp, \dots, a_n^\sharp)) \otimes g_1 \cdots g_n,$$

we must have that $\sum_{i=1}^{r} \alpha_i p_i(a_1, \dots, a_n, a_1^\sharp, \dots, a_n^\sharp) \neq 0$.

By using left and right multiplication by the unit element e of C we can decompose the Jacobson radical J of B into the direct sum of graded C-bimodules

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11},$$

where for $i \in \{0, 1\}$, J_{ik} is a left faithful module or a 0-left module according as $i = 1$ or $i = 0$, respectively. Similarly, J_{ik} is a right faithful module or a 0-right module according as $k = 1$ or $k = 0$, respectively. Moreover, for $i, k, l, m \in \{0, 1\}$, $J_{ik} J_{lm} \subseteq \delta_{kl} J_{im}$ where δ_{kl} is the Kronecker delta ([10, Lemma 2]).

Now, without loss of generality we may assume that if $a_i \in J$ then $a_i \in J_{kl}$, for some $k, l \in \{0, 1\}$. Take $g_0 \in G_0$ such that $g_0 g_1 \cdots g_n \neq 0$; then if $b \in C \cup J_{00} \cup J_{01} \cup J_{10} \cup J_{11}, g \in G$ we have:

$$(e \otimes g_0)(b \otimes g) + (b \otimes g)(e \otimes g_0) = \begin{cases} 2b \otimes g_0 g, & if\, b \in C \cup J_{11} \\ b \otimes g_0 g, & if\, b \in J_{10} \cup J_{01}, \\ 0, & if\, b \in J_{00}. \end{cases}$$

Hence since $n \geq t$, by (4) some a_j must lie in C and we have:

$$h_i(a_1 \otimes g_1, \dots, a_n \otimes g_n, e \otimes g_0) = \alpha p_i(a_1, \dots, a_n, a_1^\sharp, \dots, a_n^\sharp) \otimes g_0 g_1 \cdots g_n,$$

where α is a positive integer.

Thus

$$\sum_{i=1}^{r} \alpha_i h_i(a_1 \otimes g_1, \dots, a_n \otimes g_n, e \otimes g_0) = \alpha(\sum_{i=1}^{r} \alpha_i p_i(a_1, \dots, a_n, a_1^\sharp, \dots, a_n^\sharp)) \otimes g_0 g_1 \cdots g_n \neq 0,$$

contrary to our assumption. In conclusion the $*$-polynomials $h_1, \dots, h_r$ are linearly independent modulo $P_{n+1}^* \cap \mathrm{Id}^*(G(B))$ and the proof is complete. $\square$

4. Characterizing varieties of $*$-algebras of polynomial growth

In this section we shall give a characterization of the varieties of algebras with involution of polynomial growth. We recall that for a given variety of $*$-algebras $\mathcal{V}$ the growth of $\mathcal{V}$ is defined as the growth of the sequence of $*$-codimensions of any algebra A generating $\mathcal{V}$, i.e., $\mathcal{V} = \mathrm{var}^*(A)$. Then we say that $\mathcal{V}$ has polynomial growth if $c_n^*(\mathcal{V})$ is polynomially bounded.

In what follows it is useful to regard $F\langle X, *\rangle$ as generated by symmetric and skew variables: if for $i = 1, 2, \dots,$ we let $y_i = x_i + x_i^*$ and $z_i = x_i - x_i^*$, then $F\langle X, *\rangle = F\langle y_1, z_1, y_2, z_2, \dots\rangle$. Hence a $*$-identity of A is a polynomial $f(y_1, \dots, y_n, z_1, \dots, z_m) \in F\langle X, *\rangle$ such that $f(s_1, \dots, s_n, k_1, \dots, k_m) = 0$ for all $s_1, \dots, s_n \in A^+$, $k_1, \dots, k_m \in A^-$ and $P_n^* = span_F\big\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n, \quad w_i = y_i \quad or \quad w_i = z_i, \quad i = 1, \dots, n\big\}$ is the vector space of multilinear polynomials of degree n in the variables $y_1, z_1, \dots, y_n, z_n$. Hence for every $i = 1, \dots, n$, either y_i or z_i appears in every monomial of P_n^* at degree 1 (but not both).

Now let us focus on the algebra $UT_n = UT_n(F)$ of $n \times n$ upper triangular matrices over the field F. One can define an involution $*$ on UT_n in the following way: if

$a \in UT_n$, $a^* = ba^tb^{-1}$, where t denotes the usual transpose and b is the following permutation matrix:

$$b = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

Clearly a^* is the matrix obtained from a by reflecting a along its secondary diagonal. Hence, if $a = (a_{ij})$ then $a^* = (a^*_{ij})$ where $a^*_{ij} = a_{n+1-j,\ n+1-i}$. This involution on UT_n is called the (canonical) reflection involution. Now, if $A = A_0 \oplus A_1$ is a subalgebra of UT_n endowed with trivial grading, i.e, $A_1 = 0$, then as we remarked before, the reflection involution on A is also a superinvolution on A; we shall call it the reflection superinvolution and we shall denote it by $\sharp$.

Given polynomials $f_1, \ldots, f_n \in F\langle y_1, z_1, y_2, z_2, \ldots\rangle$ let us denote by $\langle f_1, \ldots, f_n\rangle_{T^*}$ the T^*-ideal generated by $f_1, \ldots, f_n$.

Next we consider the following two algebras with involution:

1) $F \oplus F$, a two dimensional algebra endowed with the exchange involution $(a, b)^* = (b, a)$;
2) $M = F(e_{11} + e_{44}) \oplus F(e_{22} + e_{33}) \oplus Fe_{12} \oplus Fe_{34}$, the subalgebra of UT_4 endowed with the reflection involution. Here the e_{ij}s are the usual matrix units.

Such algebras were extensively studied in [7] and [19]; in particular it was proved that $\mathrm{Id}^*(F \oplus F) = \langle [y_1, y_2], [y, z], [z_1, z_2]\rangle_{T^*}$ and $\mathrm{Id}^*(M) = \langle z_1z_2\rangle_{T^*}$.

We consider the above algebras also as algebras (with trivial grading) with superinvolution and, when no confusion arises, we shall adopt the same notation for both structures.

Next we consider a non-trivial $\mathbb{Z}_2$-grading on M: we denote by M^{sup} the algebra M with grading $M_0 = F(e_{11} + e_{44}) \oplus F(e_{22} + e_{33})$ and $M_1 = Fe_{12} \oplus Fe_{34}$. Notice that the reflection involution on M^{sup} is a superinvolution. Hence M^{sup} can be viewed as an algebra with superinvolution.

The above algebras characterize the varieties of algebras with superinvolution of polynomial growth ([5, 14]).

Recall that given two $*$-algebras A and B, we say that A is T^*-equivalent to B, and we write $A \sim_{T^*} B$, if $\mathrm{Id}^*(A) = \mathrm{Id}^*(B)$.

In the following proposition we prove that the Grassmann envelope of M^{sup} is T^*-equivalent to M.

Proposition 4.1. *The algebras with involution $G(M^{sup})$ and M satisfy the same $*$-identities.*

Proof. Notice that $G(M^{sup})^+ = M_0^+ \otimes G_0 \oplus M_1^- \otimes G_1$ and $G(M^{sup})^- = M_0^- \otimes G_0 \oplus M_1^+ \otimes G_1 = M_1^+ \otimes G_1$, where $M_0^- = 0$, $M_0^+ = span\{e_{11} + e_{44}, e_{22} + e_{33}\}$, $M_1^+ = span\{e_{12} + e_{34}\}$ and $M_1^- = span\{e_{12} - e_{34}\}$. Hence, it is immediate to see that $z_1z_2 \equiv 0$ is a $*$-identity for $G(M^{sup})$. Since $\mathrm{Id}^*(M) = \langle z_1z_2\rangle_{T_*}$ this says that $\mathrm{Id}^*(M) \subseteq \mathrm{Id}^*(G(M^{sup}))$.

Let $f \in \mathrm{Id}^*(G(M^{sup}))$ be a multilinear polynomial of degree n. By the Poincaré-Birkhoff-Witt theorem f can be written as a linear combination of products of the type

$$y_{j_1} \cdots y_{j_r} z_{k_1} \cdots z_{k_t} w_1 \cdots w_m,$$

where $w_1, \ldots, w_m$ are left normed commutators in the y_is and z_is, $j_1 < \cdots < j_r$ and $k_1 < \cdots < k_t$.

Because of $z_1 z_2$ ([18, Remark 8]), f is a linear combination of the polynomials

$$(5) \quad y_1 \cdots y_n,\ y_{i_1} \cdots y_{i_{t'}} z_l y_{j_1} \cdots y_{j_{s'}},\ y_{p_1} \cdots y_{p_t} [y_r, y_m] y_{q_1} \cdots y_{q_s} \ (\bmod \operatorname{Id}^*(M)),$$

where $i_1 < \ldots < i_{t'}$, $j_1 < \ldots < j_{s'}$, $p_1 < \ldots < p_t$, $r > m < q_1 < \ldots < q_s$.

Write

$$f = \delta y_1 \cdots y_n + \sum_{l,I,J} \alpha_{l,I,J} y_{i_1} \cdots y_{i_{t'}} z_l y_{j_1} \cdots y_{j_{s'}} + \sum_{r,P,Q} \beta_{r,P,Q} y_{p_1} \cdots y_{p_t} [y_r, y_m] y_{q_1} \cdots y_{q_s},$$

where for any fixed t' and t, $I = \{i_1, \ldots, i_{t'}\}$, $J = \{j_1, \ldots, j_{s'}\}$, $P = \{p_1, \ldots, p_t\}$, $Q = \{m, q_1, \ldots, q_s\}$ are such that $I \uplus J \uplus \{l\} = P \uplus Q \uplus \{r\} = \{1, \ldots, n\}$, and $i_1 < \cdots < i_{t'}, j_1 < \cdots < j_{s'}$, $p_1 < \ldots < p_t$, $r > m < q_1 < \ldots < q_s$.

First suppose that $\delta \neq 0$, then by making the evaluation $y_1 = \cdots = y_n = 1_M \otimes 1_G$ and $z_l = 0$ for all $l = 1, \ldots, n$, one gets $\delta 1_M \otimes 1_G = 0$ and so $\delta = 0$, a contradiction.

Suppose that there exists $\beta_{r,P,Q} \neq 0$ for some t, r, P and Q; then by making the evaluation $y_{p_1} = \cdots = y_{p_t} = (e_{11} + e_{44}) \otimes 1_G$, $y_r = (e_{12} - e_{34}) \otimes e_1$, $y_m = (e_{22} + e_{33}) \otimes 1_G, y_{q_1} = \cdots = y_{q_s} = (e_{22} + e_{33}) \otimes 1_G$ and $z_l = 0$ for all $l = 1, \ldots, n$, one gets $\beta_{r,P,Q} e_{12} \otimes e_1 + \beta_{r,Q,P} e_{34} \otimes e_1 = 0$. Thus $\beta_{r,P,Q} = \beta_{r,Q,P} = 0$, a contradiction.

Let now $\alpha_{l,I,J} \neq 0$ for some t', l, I and J. By making the evaluation $z_l = (e_{12} + e_{34}) \otimes e_1$, $y_{i_1} = \cdots = y_{i_{t'}} = (e_{11} + e_{44}) \otimes 1_G$ and $y_{j_1} = \cdots = y_{j_{s'}} = (e_{22} + e_{33}) \otimes 1_G$ one gets $\alpha_{l,I,J} e_{12} \otimes e_1 + \alpha_{l,J,I} e_{34} \otimes e_1 = 0$ and thus $\alpha_{l,I,J} = \alpha_{l,J,I} = 0$, a contradiction.

Therefore $f \in \operatorname{Id}^*(M)$ and, so, $\operatorname{Id}^*(G(M^{sup})) \subseteq \operatorname{Id}^*(M)$ and the proof is complete. □

In what follows B will denote a finite dimensional algebra with superinvolution. Hence $B = \bar{B} + J$ where $\bar{B} = B_1 \oplus \cdots \oplus B_q$ is a direct sum of simple algebras with superinvolution.

We recall the classification of the finite dimensional simple algebras with superinvolution.

We start by considering the following simple superalgebras:

- $M_{k,l}(F)$ is the superalgebra of $(k+l) \times (k+l)$ matrices with $\mathbb{Z}_2$-grading:

$$(M_{k,l}(F))_0 = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \mid X \in M_k(F), T \in M_l(F) \right\},$$

$$(M_{k,l}(F))_1 = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \mid Y \in M_{k \times l}(F), Z \in M_{l \times k}(F) \right\};$$

- $Q(n) = M_n(F) \oplus c M_n(F)$ is the superalgebra with grading $Q(n)_0 = M_n(F)$, $Q(n)_1 = c M_n(F)$ with $c^2 = 1$.

We have the following.

Theorem 4.1 ([2, 12, 20]). *Let B be a finite dimensional simple algebra with superinvolution over an algebraically closed field F of characteristic different from 2. Then B is isomorphic to one of the following:*

(1) *$M_{k,l}(F)$ with the orthosymplectic or transpose superinvolution,*
(2) *$M_{k,l}(F) \oplus M_{k,l}(F)^{sop}$ with the exchange superinvolution,*
(3) *$Q(n) \oplus Q(n)^{sop}$ with the exchange superinvolution.*

Recall that $M_{k,l}(F)$ has an orthosymplectic superinvolution osp, and we shall denote it by $(M_{k,l}(F), osp)$, if and only if $l = 2s$ and osp is defined as follows:

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{osp} = \begin{pmatrix} I_k & 0 \\ 0 & Q \end{pmatrix}^{-1} \begin{pmatrix} X & -Y \\ Z & T \end{pmatrix}^{t} \begin{pmatrix} I_k & 0 \\ 0 & Q \end{pmatrix},$$

where t denotes the usual transpose, $Q = \begin{pmatrix} 0 & I_s \\ -I_s & 0 \end{pmatrix}$ and I_k, I_s are the identity matrices of orders k and s, respectively.

Also $M_{k,l}(F)$ has a transpose superinvolution trp, and we shall denote it by $(M_{k,l}(F), trp)$, if and only if $k = l$ and trp is defined as follows:

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{trp} = \begin{pmatrix} T^t & -Y^t \\ Z^t & X^t \end{pmatrix}.$$

We recall that if A is a superalgebra then the superopposite algebra A^{sop} of A is the superalgebra which has the same graded vector space structure as A but the product in A^{sop} is given on homogenous elements a, b by

$$a \circ b = (-1)^{(\deg a)(\deg b)} ba.$$

The direct sum $R = A \oplus A^{sop}$ is a superalgebra with $R_0 = A_0 \oplus A_0^{sop}$ and $R_1 = A_1 \oplus A_1^{sop}$.

Let $(M_2(F), t)$ and $(M_2(F), s)$ denote the algebra of 2×2 matrices over F with transpose and symplectic involution, respectively, where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{t} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{s} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Notice that $(M_2(F), t)$ and $(M_2(F), s)$ can be viewed as algebras (with trivial grading) with superinvolution and we shall denote them with the same notation. We remark that $(M_{2,0}(F), osp) = (M_2(F), t)$ and $(M_{0,2}(F), osp) = (M_2(F), s)$.

In what follows we shall denote by $\mathrm{var}^*(A)$ the variety of algebras with involution generated by A and by $\mathrm{var}^\sharp(B)$ the variety of algebras with superinvolution generated by B.

We say that an algebra B is endowed with trivial superinvolution $\sharp$ if the grading on B is trivial and $a^\sharp = a$ for all $a \in B$.

Lemma 4.1. *Let $B = B_1 \oplus \cdots \oplus B_q + J$ be a finite dimensional algebra with superinvolution $\sharp$ over an algebraically closed field F. If there exist two simple components $B_i \cong B_j \cong F$, $i \neq j$, with trivial superinvolution $\sharp$ such that $B_i J B_j \neq 0$ then $M \in var^*(G(B))$.*

Proof. Since $B_i J B_j \neq 0$, if u_1 and u_2 denote the unit elements of B_i and B_j, respectively, we have that $u_1 J u_2 \neq 0$, with $u_1, u_2 \in B_0^+$ and $u_1 u_2 = u_2 u_1 = 0$. Let $m \geq 1$ be the greatest integer such that $u_1 J u_2 \subseteq J^m$ and consider $B' = B/J^m$, the quotient algebra of B with induced superinvolution. Notice that B' contains two orthogonal even symmetric idempotents u_1', u_2' such that $u_1' J' u_2' \neq 0$ where $J' = J(B')$ is the Jacobson radical of B' and $J' u_l' J' u_k' = u_l' J' u_k' J' = 0$ for every $l, k \in \{1, 2\}$. Now, since $B' \in \mathrm{var}^\sharp(B)$, without loss of generality, we may assume that in B we have that

$$u_1 J u_2 \neq 0 \text{ and } J u_l J u_k = u_l J u_k J = 0, \quad l, k \in \{1, 2\}.$$

Hence, there exists $j = j_0 + j_1$ with $j_0 \in J_0, j_1 \in J_1$ such that

$$u_1 j u_2 = u_1(j_0 + j_1,)u_2 \neq 0.$$

It follows that either $u_1 j_0 u_2 \neq 0$ or $u_1 j_1 u_2 \neq 0$.

Suppose first that $u_1 j_0 u_2 \neq 0$. Let D be the subalgebra of B, generated by the elements:

$$u_1, u_2, u_1 j_0 u_2, u_2 j_0^\sharp u_1.$$

Clearly D is a subalgebra of B with induced superinvolution (the $\mathbb{Z}_2$-grading is trivial). We claim that D is isomorphic as an algebra with superinvolution to M. In order to prove this it is enough to consider the isomorphism of algebras with superinvolution

$$\varphi : \ D \longrightarrow M$$

induced by setting $\varphi(u_1) = e_{11} + e_{44}$, $\varphi(u_2) = e_{22} + e_{33}$, $\varphi(u_1 j_0 u_2) = e_{12}$, and $\varphi(u_2 j_0^\sharp u_1) = e_{34}$.

This says that $M \cong D \in \mathrm{var}^\sharp(B)$ and, so, $G(M) = M \otimes G_0 \in \mathrm{var}^*(G(B))$. Hence, since $G(M) \sim_{T^*} M$ we get that $M \in \mathrm{var}^*(G(B))$.

Suppose now that $u_1 j_1 u_2 \neq 0$. We let D be the subalgebra of B generated by the elements

$$u_1, u_2, u_1 j_1 u_2, u_2 j_1^\sharp u_1.$$

Clearly D is a subalgebra of B with induced superinvolution ($D_0 = span\{u_1, u_2\}$ and $D_1 = span\{u_1 j_1 u_2, u_2 j_1^\sharp u_1\}$). Moreover it is easy to check that D is isomorphic as an algebra with superinvolution to M^{sup}. In fact, it is enough to consider the isomorphism of algebras with superinvolution

$$\varphi : \ D \longrightarrow M^{sup}$$

induced by setting $\varphi(u_1) = e_{11} + e_{44}$, $\varphi(u_2) = e_{22} + e_{33}$, $\varphi(u_1 j_1 u_2) = e_{12}$, and $\varphi(u_2 j_1^\sharp u_1) = e_{34}$.

This says that $M^{sup} \in \mathrm{var}^\sharp(B)$ and, so, $G(M^{sup}) \in \mathrm{var}^*(G(B))$. Since by Proposition 4.1, $G(M^{sup}) \sim_{T^*} M$ we get that $M \in \mathrm{var}^*(G(B))$ and we are done also in this case. $\square$

Lemma 4.2. *Let $B = B_1 \oplus \cdots \oplus B_q + J$ be a finite dimensional algebra with superinvolution $\sharp$ over an algebraically closed field F. If $F \oplus F \notin var^*(G(B))$ then $B_i \cong F$ with trivial superinvolution, for $i = 1, \dots, q$.*

Proof. Suppose first that for some i, $B_i = M_{k,2s}$ with orthosymplectic superinvolution. If $s > 0$, $C = Fe_{k+1,k+1} \oplus Fe_{k+s+1,k+s+1}$ is a subalgebra of B_i with induced superinvolution and $C \cong F \oplus F$ with exchange superinvolution. But then $G(C) \cong G(F \oplus F) = (F \oplus F) \otimes G_0 \in \mathrm{var}^*(G(B))$ and $(F \oplus F) \otimes G_0 \sim_{T^*} F \oplus F$ with exchange involution, a contradiction. Hence $s = 0$.

Suppose now that $k \geq 2$. In this case $F(e_{11} + e_{22}) \oplus F(e_{12} - e_{21})$ is a subalgebra of B_i with induced superinvolution isomorphic to $F \oplus F$ with exchange superinvolution. As above we get a contradiction. Thus $k = 1$ and $B_i = F$ with trivial superinvolution.

Next suppose that $B_i = M_{k,k}$ with transpose superinvolution. We consider the subalgebra $Fe_{11} \oplus Fe_{k+1,k+1} \cong F \oplus F$ and as above we get a contradiction.

Finally suppose that $B_i = A \oplus A^{sop}$ where $A = M_{k,l}(F)$ or $Q(n)$. In both cases we consider the subalgebra $Fe_{11} \oplus Fe_{11} \cong F \oplus F$ and we get a contradiction also in this case. $\square$

Now we are in a position to prove the following theorem characterizing the varieties of algebras with involution of polynomial growth.

Theorem 4.2. *Let A be a PI-algebra with involution $*$ over a field F of characteristic zero. Then the sequence $c_n^*(A), n = 1, 2, \dots,$ is polynomially bounded if and only if $M, F \oplus F \notin var^*(A)$.*

Proof. Since the algebras M and $F \oplus F$ generate varieties of exponential growth (see [11, Chapter 11]), if $c_n^*(A)$ is polynomially bounded, then $M, F \oplus F \notin \mathrm{var}^*(A)$. Conversely suppose that $M, F \oplus F \notin \mathrm{var}^*(A)$. Since we are dealing with codimensions that do not change by extending the base field, we may assume that the field F is algebraically closed. Moreover, $c_n^*(A) = c_n^*(G(B))$ where $B = B_1 \oplus \cdots \oplus B_q + J$ is a finite dimensional algebra with superinvolution. Since the conclusion holds if B is nilpotent, we may assume that $q \geq 1$. Hence, by Lemma 4.2, since $F \oplus F \notin \mathrm{var}^*(G(B))$ then $B_i \cong F$ with trivial superinvolution, for $i = 1, \dots, q$. Moreover, by Lemma 4.1, since $M \notin \mathrm{var}^*(G(B))$, $B_i J B_j = 0$, for all $1 \leq i, j \leq q$. By the above characterization of the $*$-exponent this says that $\exp^*(A) \leq 1$ and, so, $c_n^*(A)$ $n = 1, 2, \dots,$ is polynomially bounded. □

As a consequence we have the following corollary.

Corollary 4.1. *The algebras M and $F \oplus F$ are the only algebras, up to T^*-equivalence, generating varieties of almost polynomial growth.*

5. Characterizing varieties of $*$-exponent > 2

In this section we shall introduce some algebras with involution that will allow us to characterize the varieties of $*$-exponent > 2. We recall that, when dealing with $*$-identities, we may assume that a PI-algebra with involution is the Grassmann envelope of a finite dimensional algebra with superinvolution. Hence we start by constructing some finite dimensional algebras with superinvolution.

Recall that any $\mathbf{g} = (g_1, \dots, g_k) \in \mathbb{Z}_2^k$ induces an elementary $\mathbb{Z}_2$-grading on UT_k by setting

$$UT_k^{(0)} = \mathrm{span}\{e_{ij} \mid g_i + g_j = 0\} \text{ and } UT_k^{(1)} = \mathrm{span}\{e_{ij} \mid g_i + g_j = 1\}.$$

Next we introduce two subalgebras of UT_k with induced suitable elementary $\mathbb{Z}_2$-gradings. We let

$$A = Fe_{11} \oplus F(e_{22} + e_{33}) \oplus Fe_{44} \oplus Fe_{12} \oplus Fe_{34} \subseteq UT_4 \tag{6}$$

and

$$B = F(e_{11}+e_{66}) \oplus F(e_{22}+e_{55}) \oplus F(e_{33}+e_{44}) \oplus Fe_{12} \oplus Fe_{13} \oplus Fe_{23} \oplus Fe_{45} \oplus Fe_{46} \oplus Fe_{56} \subseteq UT_6. \tag{7}$$

We define different superinvolutions on A and B and we obtain the following algebras:

1) C_1 is the algebra A with trivial grading and reflection superinvolution $\sharp$;
2) C_2 is the algebra A with elementary grading induced by $\mathbf{g} = (0, 1, 1, 0)$ and reflection superinvolution $\sharp$;
3) C_3 is the algebra B with trivial grading and reflection superinvolution $\sharp$;
4) C_4 is the algebra B with elementary grading induced by $\mathbf{g} = (0, 0, 1, 1, 0, 0)$ and reflection superinvolution $\sharp$;

5) C_5 is the algebra B with elementary grading induced by $\mathbf{g} = (0,1,0,0,1,0)$ and superinvolution $\dagger$ defined on the matrix units by

$$e_{ij}^{\dagger} = \begin{cases} -e_{ij}^{\sharp} & \text{if } (i,j) \in \{(1,2),(5,6)\} \\ e_{ij}^{\sharp} & \text{otherwise} \end{cases}.$$

We also define the following algebras with superinvolution:

6) $C_6 = (M_2(F), s)$;
7) $C_7 = (M_2(F), t)$;
8) $C_8 = (M_{1,1}(F), trp)$;
9) $C_9 = Q(1) \oplus Q(1)^{sop}$.

In what follows often in order to simplify the notation we shall identify a simple algebra B with superinvolution with one of the algebras given in Theorem 4.1. The following remark is immediate.

Remark 5.1. *If B is a simple algebra with superinvolution over an algebraically closed field F and $\dim B > 2$ then either $B \supseteq Q(1) \oplus Q(1)^{sop}$ or $B \supseteq (M_2(F), t)$ or $B \supseteq (M_2(F), s)$ or $B \supseteq (M_{1,1}(F), trp)$.*

Lemma 5.1. *Let $B = \bar{B} + J$ be a reduced algebra with superinvolution $\sharp$ over an algebraically closed field F. If $\dim \bar{B} > 2$ then $var^{\sharp}(B)$ contains one of the algebras $C_1, \dots, C_9$.*

Proof. Let $\bar{B} = B_1 \oplus \cdots \oplus B_q$. If $q = 1$, by Remark 5.1, B contains one of the algebras $C_6, \dots, C_9$.

Hence we may assume that $q \geq 2$. Assume that B does not contain any of the algebras $C_6, \dots, C_9$. Then, by Remark 5.1, $\dim B_i \leq 2$ for all $i = 1, \dots, q$.

We start by analyzing the case $q = 2$. Then $B = B_1 \oplus B_2 + J$ and $B_1 J B_2 \neq 0$. Moreover, since $\dim \bar{B} = \dim(B_1 \oplus B_2) > 2$, we must have that either $B_1 = F$ and $B_2 = F \oplus F$ (the case $B_1 = F \oplus F$, $B_2 = F$ is easily reduced to this case by taking $\sharp$), or $B_1 = B_2 = F \oplus F$.

Suppose first that $B_1 = F$, $B_2 = F \oplus F$. Let u_1, u_2 be the unit elements of B_1 and B_2, respectively. We write $u_2 = u_3 + u_4$ where $u_3 = (1,0)$ and $u_4 = u_3^{\sharp}$. Clearly $u_1 J u_2 \neq 0$ and let m be the greatest integer such that $u_1 J u_2 \subseteq J^m$. By working inside the algebra B/J^{m+1} with induced superinvolution, we may assume that $J u_1 J u_2 = u_1 J u_2 J = 0$. Let $j \in J$ be a homogeneous element such that $u_1 j u_2 \neq 0$. Then we consider the subalgebra with induced superinvolution generated by $u_1, u_3, u_1 j u_2$. At this point one can construct the algebras C_1 and C_2 (see [3] and [13] for the details of the proof).

We remark that if $B_1 = B_2 = F \oplus F$ then B contains a subalgebra with induced superinvolution isomorphic to $F \oplus (F \oplus F) + J$ with $FJ(F \oplus F) \neq 0$ and we are back to the previous case.

Suppose now that $k \geq 3$. Then we may clearly assume that $B_1 J B_2 J B_3 \neq 0$ and $B_1 = B_2 = B_3 = F$. Let u_1, u_2, u_3 be the unit elements of B_1, B_2, B_3, respectively. Clearly $u_1 J u_2 J u_3 \neq 0$ and let m be the greatest integer such that $u_a J u_b J u_c \subseteq J^m$ for all $a, b, c \in \{1,2,3\}$. By passing to the algebra B/J^{m+1} with induced superinvolution, we may assume that $J u_a J u_b J u_c = u_a J u_b J u_c J = 0$ for all $a, b, c \in \{1,2,3\}$. Let I be the ideal of B generated by $u_a J u_b J u_a$ with $a, b \in \{1,2,3\}, a \neq b$.

Clearly I is stable under the superinvolution and by passing to B/I we may assume that $u_1Ju_2Ju_3 \neq 0$ and $u_aJu_bJu_a = 0$ with $a, b \in \{1, 2, 3\}, a \neq b$. It follows that there exist homogeneous elements $j_1, j_2 \in J$ such that $u_1j_1u_2j_2u_3 \neq 0$. At this stage one considers the subalgebra D with superinvolution generated by $u_1, u_2, u_3, u_1j_1u_2, u_2j_2u_3$. Notice that by the above a linear basis of D is given by the elements

$$u_1, u_2, u_3, u_1j_1u_2, u_2j_2u_3, u_1j_1u_2j_2u_3, u_2j_1^\sharp u_1, u_3j_2^\sharp u_2, u_3j_2^\sharp u_2j_1^\sharp u_1.$$

Depending on the homogeneous degree of j_1 and j_2, one can recover the algebras C_3, C_4 and C_5 by proving that D is isomorphic to one of them (see [3] and [13] for the details). □

Now we have all the ingredients to prove the main result of this section. To this end we list nine algebras that will play a basic role in what follows.

Consider the following algebras with involution:

1) $D_1 \subseteq UT_4$ is the algebra A in (6) with reflection involution $*$;
2) $D_2 = G_0e_{11} \oplus G_0(e_{22} + e_{33}) \oplus G_0e_{44} \oplus G_1e_{12} \oplus G_1e_{34} \subseteq UT_4(G)$ is the algebra with involution defined on a basis by
$$(ge_{ij})^\circ = \begin{cases} -ge_{ij}^* & \text{if } (i,j) \in \{(1,2),(3,4)\} \\ ge_{ij}^* & \text{otherwise} \end{cases},$$
where * denotes the reflection involution on $UT_4(G)$;
3) $D_3 \subseteq UT_6$ is the algebra B in (7) with reflection involution $*$;
4) $D_4 = G_0(e_{11} + e_{66}) \oplus G_0(e_{22} + e_{55}) \oplus G_0(e_{33} + e_{44}) \oplus G_0e_{12} \oplus G_1e_{13} \oplus G_1e_{23} \oplus G_1e_{45} \oplus G_1e_{46} \oplus G_0e_{56} \subseteq UT_6(G)$ is the algebra with involution defined on a basis by
$$(ge_{ij})^\circ = \begin{cases} -ge_{ij}^* & \text{if } (i,j) \in \{(1,3),(2,3),(4,5),(4,6)\} \\ ge_{ij}^* & \text{otherwise} \end{cases},$$
where * denotes the reflection involution on $UT_6(G)$;
5) $D_5 = G_0(e_{11} + e_{66}) \oplus G_0(e_{22} + e_{55}) \oplus G_0(e_{33} + e_{44}) \oplus G_1e_{12} \oplus G_0e_{13} \oplus G_1e_{23} \oplus G_1e_{45} \oplus G_0e_{46} \oplus G_1e_{56} \subseteq UT_6(G)$ is the algebra with involution defined on a basis by
$$(ge_{ij})^\circ = \begin{cases} -ge_{ij}^* & \text{if } (i,j) \in \{(2,3),(4,5)\} \\ ge_{ij}^* & \text{otherwise} \end{cases},$$
where * denotes the reflection involution on $UT_6(G)$;
6) $D_6 = (M_2(F), s)$;
7) $D_7 = (M_2(F), t)$;
8) $D_8 = M_{1,1}(G) = \begin{pmatrix} G_0 & G_1 \\ G_1 & G_0 \end{pmatrix}$ with involution: $\begin{pmatrix} g_0 & g_1 \\ g_1' & g_o' \end{pmatrix}^* = \begin{pmatrix} g_o' & g_1 \\ -g_1' & g_o \end{pmatrix}$;
9) $D_9 = G \oplus G^{op}$ with involution $(a, b)^* = ((-1)^{degb}b, (-1)^{dega}a)$, for $a, b \in G_0 \cup G_1$.

Theorem 5.1. *Let A be a PI-algebra with involution $*$ over a a field F of characteristic zero. Then $\exp^*(A) > 2$ if and only if $D_i \in var^*(A)$, for some $i \in \{1, \ldots, 9\}$.*

Proof. By the above $\mathrm{var}^*(A) = \mathrm{var}^*(G(B))$, where B is a finite dimensional algebra with superinvolution $\sharp$. Without loss of generality we may assume (see [11, Theorem 7.6.1]) that F is algebraically closed. Also $\exp^*(A)$ is the maximal dimension of an admissible subalgebra $\bar{B} = B_1 \oplus \cdots \oplus B_t$ of B with induced superinvolution. Suppose first that $\exp^*(A) > 2$. Then $\dim \bar{B} > 2$ and, by Lemma 5.1, $C_i \in \mathrm{var}^\sharp(\bar{B} + J(B)) \subseteq \mathrm{var}^\sharp(B)$, for some $i = 1, \ldots, 9$. Hence $G(C_i) \in \mathrm{var}^*(G(B))$, for some $i \in \{1, \ldots, 9\}$. Now, since $G(C_i) \sim_{T^*} D_i$, for all $i = 1, \ldots, 9$, we get the desired conclusion.

Conversely, suppose that $D_i \in \mathrm{var}^*(A)$. Now, since $\mathrm{var}^*(D_i) = \mathrm{var}^*(G(C_i))$ we get that $\exp^*(D_i)$ equals the maximal dimension of an admissible subalgebra of C_i and, it is easy to show that $\exp^*(D_i) = 3$, for $i = 1, \ldots, 5$ and $\exp^*(D_i) = 4$, for $i = 6, \ldots, 9$. Hence, it follows that $\exp^*(A) > 2$ and this completes the proof. $\square$

As a consequence of Theorems 4.2 and 5.1, we get the following result characterizing $*$-algebras with $*$-exponent equal to two.

Corollary 5.1. *Let A be a PI-algebra with involution $*$ over a field of characteristic zero Then $exp^*(A) = 2$ if and only if $D_i \notin var^*(A)$, for all $i \in \{1, \ldots, 9\}$ and either $F \oplus F$ or $M \in var^*(A)$.*

References

[1] E. Aljadeff, A. Giambruno, Y. Karasik, *Polynomial identities with involution, superinvolutions and the Grassmann envelope*, Proc. Amer. Math. Soc. **145** (2017), no. 5, 1843–1857.

[2] Yu Bahturin, M. Tvalavadze and T. Tvalavadze, *Group gradings on superinvolution simple superalgebras*, Linear Algebra Appl. **431** (2009), no. 5-7, 1054–1069.

[3] R. Bezerra dos Santos, *$*$-superalgebras and exponential growth*, J. Algebra **473** (2017), 283–306.

[4] A. Giambruno, A. Ioppolo and D. La Mattina, *Varieties of algebras with superinvolution of almost polynomial growth* Algebr. Represent. Theory **19** (2016), no. 3, 599–611

[5] A. Giambruno, A. Ioppolo and D. La Mattina, *Superalgebras with Involution or Superinvolution and Almost Polynomial Growth of the Codimensions*, Algebr. Represent. Theory, (2018) **22** (2019), no. 4, 961-976.

[6] A. Giambruno, D. La Mattina and C. Polcino Milies, *Star-fundamental algebras: polynomial identities and asymptotics*, Trans. Amer. Math. Soc. doi.org/10.1090/tran/8182.

[7] A. Giambruno and S. Mishchenko, *On star-varieties with almost polynomial growth*, Algebra Colloq. **8** (2001), no. 1, 33–42.

[8] A. Giambruno, C. Polcino Milies and A. Valenti, *Star-polynomial identities: computing the exponential growth of the codimensions*, J. Algebra **469** (2017), 302–322.

[9] A. Giambruno and A. Regev, *Wreath products and P.I. algebras*, J. Pure Appl. Algebra **35** (1985), no. 2, 133–149.

[10] A. Giambruno and M. Zaicev, *Asymptotics for the standard and the Capelli identities,* Israel J. Math. **135** (2003), 125-145.

[11] A. Giambruno and M. Zaicev, *Polynomial identities and asymptotic methods*, AMS, Math. Surv. Monogr. **122** (2005).

[12] C. Gomez-Ambrosi and I.P. Shestakov, *On the Lie structure of the skew-elements of a simple superalgebra with involution*, J. Algebra **208** (1998), 43–71.

[13] A. Ioppolo, *The exponent for superalgebras with superinvolution*, Linear Algebra Appl. **555** (2018), 1-20.

[14] A. Ioppolo and D. La Mattina, *Polynomial codimension growth of algebras with involutions and superinvolutions*, J. Algebra **472** (2017), 519–545.

[15] D. La Mattina, T. do Nascimento Silva and A.C. Vieira, *Minimal star-varieties of polynomial growth and bounded colength*, J. Pure Appl. Algebra **222** (2018), no. 7, 1765–1785.

[16] D. La Mattina and F. Martino, *Polynomial growth and star-varieties*, J. Pure Appl. Algebra **220** (2016), no. 1, 246–262.

[17] D. La Mattina, S. Mauceri and P. Misso, *Polynomial growth and identities of superalgebras and star-algebras*, J. Pure Appl. Algebra **213** (2009), no. 11, 2087–2094.

[18] D. La Mattina and P. Misso, *Algebras with involution with linear codimension growth,* J. Algebra **305** (2006), no. 1, 270–291.

[19] S. Mishchenko and A. Valenti, *A star-variety with almost polynomial growth*, J. Algebra **223** (2000), no. 1, 66–84.

[20] M.L. Racine, *Primitive superalgebras with superinvolution*, J. Algebra **206** (1998), no. 2, 588–614.

Dipartimento di Matematica e Informatica, Università di Palermo, Via Archirafi 34, 90123, Palermo, Italy

Email address: antonio.giambruno@unipa.it, antoniogiambr@gmail.com

Dipartimento di Matematica e Informatica, Università di Palermo, Via Archirafi 34, 90123, Palermo, Italy

Email address: daniela.lamattina@unipa.it