UPPER BOUNDS FOR THE TIGHTNESS OF THE G_δ-TOPOLOGY

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Abstract. We prove that if X is a regular space with no uncountable free sequences, then the tightness of its G_δ topology is at most continuum and if X is in addition Lindelöf then its G_δ topology contains no free sequences of length larger then the continuum. We also show that the higher cardinal generalization of our theorem does not hold, by constructing a regular space with no free sequences of length larger than ω_1, but whose G_δ topology can have arbitrarily large tightness.

1. Introduction

Given a space X, the *G_δ-modification of X* (or G_δ-topology on X), X_δ is defined as the topology on X which is generated by the G_δ-subsets of X. The problem of bounding the cardinal invariants of X_δ in terms of those of X is a well-studied one in set-theoretic topology. For example if c, s, L, t denote respectively the cellularity, the spread, the Lindelöf degree and the tightness of X, then $c(X_\delta) \leq 2^{c(X)}$ for every compact space X (see [7]), $s(X_\delta) \leq 2^{s(X)}$ for every Hausdorff space X (see [1]) and $L(X_\delta) \leq 2^{L(X)\cdot t(X)}$ for every Hausdorff space X (see [10]). This is nothing but a small sample of bounds for the G_δ topology that have been proved in the past; for more results and applications of the G_δ topology to homogeneous compacta we refer the reader to our paper [1] and its bibliography.

Note that we have not mentioned a bound for the tightness of the G_δ topology yet, and indeed finding such a bound seems to be particularly tricky. Answering a question posed in [1], Dow, Juhász, Soukup, Szentmiklóssy and Weiss [5] proved that the inequality $t(X_\delta) \leq 2^{t(X)}$ holds within the realm of regular Lindelöf spaces. The Lindelöf property is essential in their argument, and in fact the authors were able to construct a consistent example of a regular countably tight space X such that $t(X_\delta)$ can be as big as desired. They left open whether a countably

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tight space X such that $t(X_\delta) > 2^{\aleph_0}$ can be found in ZFC. This question was later solved in the positive by Usuba [12], who also found a bound on the tightness of the G_δ-modification of every countably tight space, modulo the consistency of a certain very large cardinal. More precisely, Usuba proved that if κ is an ω_1-strongly compact cardinal then $t(X_\delta) \leq \kappa$, for every countably tight space X. Chen and Szeptycki [3] managed to prove a very tight consistent bound for the special class of Fréchet α_1-spaces, namely $t(X_\delta) \leq \aleph_1$ if the Proper Forcing Axiom holds.

Exploiting the notion of a free sequence, we will give another bound on the tightness of the G_δ topology.

A free sequences is a special kind of discrete set that was introduced by Arhangel'skii and is one of the essential tools in his celebrated solution of the Alexandroff-Urysohn problem on the cardinality of first-countable compacta. Recall that the set $\{x_\alpha : \alpha < \kappa\} \subseteq X$ is a free sequence provided that $\overline{\{x_\beta : \beta < \alpha\}} \cap \overline{\{x_\beta : \alpha \leq \beta < \kappa\}} = \emptyset$ for each $\alpha < \kappa$. We define $F(X)$ to be the supremum of cardinalities of free-sequences in X. The cardinal functions $F(X)$ and $t(X)$ are intimately related. Indeed, $F(X) \leq L(X)t(X)$ for every space X and $t(X) = F(X)$, for every compact Hausdorff space X. However, the gap between $F(X)$ and $t(X)$ can be arbitrarily large even for a Lindelöf space X, as observed by Okunev [9].

We will prove a result about the tightness of the G_δ modification which has the Dow, Juhász, Soukup, Szentmiklóssy and Weiss bound as a consequence and also implies the following new bound: if X is a regular space such that $F(X) = \omega$, then $t(X_\delta) \leq 2^{\aleph_0}$. The higher cardinal generalization of this is not true, as we will construct, for every cardinal κ, a regular space X such that $F(X) = \omega_1 < \kappa = t(X_\delta)$. As a byproduct of our bound we will obtain that if X is a Lindelöf regular space such that $F(X) = \omega$ then $F(X_\delta) \leq 2^{\aleph_0}$.

Given a set S, we denote by $\mathcal{P}(S)$ the powerset of S and by $[S]^{\leq\kappa}$ the set of all subsets of S which have cardinality at most κ. For undefined notions see [6], but our notation regarding cardinal functions follows [8].

2. The tightness of the G_δ-modification

Let X be a space, let W be a subset of X and let κ be an infinite cardinal. We say that a collection $\mathcal{U}$ of subsets of X is a Cl_κ-cover of W provided that for any $C \in [W]^{\leq\kappa}$ there is $U_C \in \mathcal{U}$ such that $\overline{C} \subseteq U_C$.

We say that a space X is Cl_κ-Lindelöf if whenever W is a subset of X and $\mathcal{U}$ is an open Cl_κ-cover of W, then W is covered by countably many elements of $\mathcal{U}$.

Lemma 1. *Every Lindelöf space X is $Cl_{t(X)}$-Lindelöf.*

Proof. It suffices to observe that every open $Cl_{t(X)}$-cover of a set $W \subseteq X$ is actually a cover of $\overline{W}$. □

Lemma 2. *Every space X satisfying $F(X) = \omega$ is Cl_ω-Lindelöf.*

Proof. Let Wbe a subset of X and $\mathcal{U}$ be an open Cl_ω-cover of W. Assume by contradiction that no countable subfamily of $\mathcal{U}$ covers W. We will then construct a free sequence of cardinality ω_1 inside W.

Suppose that, for some $\beta < \omega_1$, we have chosen points $\{x_\tau : \tau < \beta\} \subset W$ and elements $U_\tau \in \mathcal{U}$ for every $\tau < \beta$ with the property that $\overline{\{x_\gamma : \gamma < \tau\}} \subseteq U_\tau$. Choose $U_\beta \in \mathcal{U}$ in such a way that $\overline{\{x_\alpha : \alpha < \beta\}} \subseteq U_\beta$. By our assumption, the family $\{U_\tau : \tau \leq \beta\}$ does not cover W, and therefore we can fix a point $x_\beta \in W \setminus \bigcup\{U_\tau : \tau \leq \beta\}$.

Eventually, $\{x_\tau : \tau < \omega_1\}$ is a free sequence of cardinality ω_1 in X, which is a contradiction. □

Theorem 3. *Let X be a regular space and let κ be an infinite cardinal. If X is Cl_κ-Lindelöf, then $t(X_\delta) \leq 2^\kappa$.*

Proof. Let A be any subset of X and fix a point p in the G_δ-closure of A.

Let $\mathcal{N}_\kappa(X) = \{C \in [X]^{\leq\kappa} : p \notin \overline{C}\}$. By the regularity of X, for every $C \in \mathcal{N}_\kappa(X)$, we can find disjoint open sets U_C and V_C such that $\overline{C} \subset U_C$ and $p \in V_C$.

Let ϕ be a choice function on $\mathcal{P}(X)$. We will build by induction an increasing family $\{W_\alpha : \alpha < \kappa^+\} \subset [A]^{2^\kappa}$.

Let W_0 be any subset of A of cardinality $\leq 2^\kappa$ and assume we have already defined $\{W_\beta : \beta < \alpha\}$. If α is a limit ordinal then put $W_\alpha = \bigcup\{W_\beta : \beta < \alpha\}$. If $\alpha = \gamma + 1$ then let:

$$W_\alpha = W_\gamma \cup \{\phi(A \cap \bigcap\{V_C : C \in \mathcal{C}\}) : \mathcal{C} \in [\mathcal{N}_\kappa(W_\gamma)]^{\leq\omega}\}$$

Note that $|W_\alpha| \leq 2^\kappa$.

Finally, let $W = \bigcup\{W_\alpha : \alpha < \kappa^+\}$. Since $|W| \leq 2^\kappa$, it suffices to show that p is in the G_δ-closure of W.

Indeed, let $\{O_n : n < \omega\}$ be a family of open neighbourhoods of p.

Claim. There is a countable family $\mathcal{C}_n \subset \mathcal{N}_\kappa(W \setminus O_n)$ such that $W \setminus O_n \subset \bigcup\{U_C : C \in \mathcal{C}_n\}$.

Proof of Claim. Since O_n is an open neighbourhood of p, we have that $\mathcal{N}_\kappa(W \setminus O_n) = [W \setminus O_n]^{\leq\kappa}$ and therefore U_C is defined for every $C \in [W \setminus O_n]^{\leq\kappa}$ and $\overline{C} \subset U_C$. In particular, $\mathcal{U} = \{U_C : C \in \mathcal{N}_\kappa(W \setminus O_n)\}$ is a Cl_κ-open cover of $W \setminus O_n$. Now, the statement of the claim follows from the fact that X is a Cl_κ-Lindelöf space. △

For every $n < \omega$, fix a family $\mathcal{C}_n$ satisfying the Claim and let $\mathcal{S} = \bigcup\{\mathcal{C}_n : n < \omega\}$ and $S = \bigcup \mathcal{S}$.

Since the set S has cardinality at most κ, there is an ordinal $\delta < \kappa^+$ such that $S \subset W_\delta$. It follows then that the point $q = \phi(\bigcap_{C \in \mathcal{S}} V_C \cap A)$ belongs to $W_{\delta+1} \subset W$.

Note that, for every $n < \omega$, $q \in \bigcap\{V_C : C \in \mathcal{C}_n\} \cap W \subset W \setminus \bigcup\{U_C : C \in \mathcal{C}_n\} \subset O_n \cap W$. Therefore $q \in \bigcap\{O_n : n < \omega\} \cap W$ and we are done. □

Corollary 4. *(Dow, Juhász, Soukup, Szentmiklóssy and Weiss* [5]*) If X is a Lindelöf regular space, then $t(X_\delta) \leq 2^{t(X)}$.*

Corollary 5. *If X is a regular space and $F(X) = \omega$, then $t(X_\delta) \leq 2^\omega$.*

It's natural to ask whether the higher cardinal version of Corollary 5 holds true. The following theorem shows that this is not the case.

Let θ be a regular uncountable cardinal. Recall that an elementary submodel M of $H(\theta)$ is said to be *ω-covering* if for every countable subset A of M there is a countable set $B \in M$ such that $A \subset B$. The union of any elementary chain of elementary submodels of length ω_1 is an ω-covering elementary submodel, so ω-covering submodels of cardinality ω_1 exist in ZFC (see [4]).

Theorem 6. *For every uncountable cardinal κ, there is a space Y such that $F(Y) = \omega_1 < \kappa = t(Y_\delta)$.*

Proof. Let $X = \Sigma(2^\kappa) = \{x \in 2^\kappa : |x^{-1}(1)| \leq \aleph_0\}$ and let $p \in 2^\kappa$ be the point defined by $p(\alpha) = 1$, for every $\alpha < \kappa$. We will prove that $Y = X \cup \{p\}$ with the topology inherited from 2^κ is the required example.

The following Claim was proved by the second author in [11] for the case $\kappa = \omega_2$, but the argument works for any uncountable cardinal κ without any modifications. We include it for the reader's convenience.

Claim. $L(X) = \aleph_1$.

Proof of Claim. Let $\mathcal{U}$ be an open cover of X. Without loss of generality we can assume that for every $U \in \mathcal{U}$, there is a finite partial function $\sigma : \kappa \to 2$ such that $U = \{x \in 2^\kappa : \sigma \subset x\}$. The domain of σ will then be called the *support of* U and we will write $supp(U) = dom(\sigma)$.

Let θ be a large enough regular cardinal and M be an ω-covering elementary submodel of $H(\theta)$ such that $X, \mathcal{U}, \kappa \in M$ and $|M| = \aleph_1$.

We claim that $\mathcal{U} \cap M$ covers X. Indeed, let $x \in X$ be any point and let $A \in M$ be a countable set such that $x^{-1}(1) \cap M \subset A$.

Let $Z = \{y \in X : (\forall \alpha \in \kappa \setminus A)(y(\alpha) = 0)\}$. Then $Z \in M$ and Z is a compact subspace of X. So there is a finite subfamily $\mathcal{V} \in M$ of $\mathcal{U}$ such that $Z \subset \bigcup \mathcal{V}$. Since $\mathcal{V}$ is finite, we have $\mathcal{V} \subset M$. It then follows that $\mathcal{U} \cap M$ covers Z.

Let a be the point such that $a(\alpha) = x(\alpha)$ for all $\alpha \in M \cap \kappa$ and $a(\alpha) = 0$ for all $\alpha \in \kappa \setminus M$. The fact that $x^{-1}(1) \cap M \subset A$ implies that $a \in Z$ and hence there is $U \in \mathcal{U} \cap M$ such that $a \in U$. Note that $supp(U)$ is a finite element of M and hence $supp(U) \subset M$. But since x and a coincide on M we then have that $x \in U$ as well, as we wanted.

This proves $L(X) \leq \aleph_1$, but we can't have $L(X) = \aleph_0$ because X is countably compact non-compact. Hence $L(X) = \aleph_1$.

$\triangle$

It is well known that X is Fréchet-Urysohn and hence X has countable tightness. Since $F(X) \leq L(X) \cdot t(X)$ we have $F(X) \leq \omega_1$, but then also $F(Y) \leq \omega_1$. It's easy to see that $t(p, Y_\delta) = \kappa$. $\square$

In [2] Carlson, Porter and Ridderbos proved the following improvement of the Pytkeev inequality $L(X_\delta) \leq 2^{L(X) \cdot t(X)}$ mentioned in the introduction.

Theorem 7. [2] *(Theorem 2.7) If X is a Hausdorff space, then $L(X_\delta) \leq 2^{L(X)F(X)}$.*

Putting together Corollary 5 and the above theorem we obtain:

Corollary 8. *Let X be a regular Lindelöf space such that $F(X) = \omega$. Then $F(X_\delta) \leq 2^{\aleph_0}$.*

We don't know whether the Lindelöf property can be removed from the above corollary.

Question 9. *Let X be a regular space satisfying $F(X) = \omega$. Is it true that $F(X_\delta) \leq 2^\omega$?*

It's reasonable to conjecture that the higher cardinal version of Corollary 8 holds, at least for Lindelöf spaces.

Question 10. *Let X be a regular (Lindelöf) space. Is it true that $F(X_\delta) \leq 2^{F(X)}$?*

Note that neither the consistent example from [5] of a regular countably tight space X such that $t(X_\delta)$ can be arbitrarily large nor the example from Theorem 6 work for the above question since $F(X) = |X|$ for the former and $F(X_\delta) \leq 2^{\aleph_0}$ for the latter.

We finish with two easy bounds for the tightness of the G_δ topology, by making using of the weight and the spread.

Proposition 11. *Let X be a regular space. Then:*

(1) $t(X_\delta) \leq 2^{d(X)}$.
(2) $t(X_\delta) \leq 2^{s(X)}$.

Proof. To prove (1) recall that $w(X) \leq 2^{d(X)}$ for every regular space X (see [8]). Now $t(X_\delta) \leq w(X_\delta) \leq w(X)^\omega \leq 2^{d(X)\cdot\omega} = 2^{d(X)}$.

To prove (2) recall that $nw(X) \leq 2^{s(X)}$ for every regular space X (see [8]) and proceed as before. □

Proposition 11, (1) is not true for Hausdorff spaces, as the following example shows.

Example 12. *There is a separable Hausdorff space X such that $t(X_\delta) > 2^{\aleph_0}$.*

Proof. Let Y be the Katětov extension of the integer. That is, if $\mathcal{U}$ is the set of all non-priincipal ultrafilters on ω then $Y = \omega \cup \mathcal{U}$, every point of ω is isolated and a basic neighbourhood of $p \in \mathcal{U}$ is a set of the form $\{p\} \cup A \setminus F$, where $A \in p$ and F is finite.

Let $X = Y \cup \{\infty\}$, where $\infty \notin Y$ and declare $V \subset X$ to be a neighbourhood of ∞ if and only if $|X \setminus V| \leq 2^{\aleph_0}$. It is easy to see that X is a separable Hausdorff space and $t(X_\delta) > 2^{\aleph_0}$. □

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