A singular (p, q)-equation with convection and a locally defined perturbation

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Abstract

We consider a parametric Dirichlet problem driven by the (p, q)-Laplacian and a reaction which is gradient dependent (convection) and the competing effects of two more terms, one a parametric singular term and a locally defined perturbation. We show that for all small values of the parameter the problem has a positive smooth solution.

Keywords: Convection, nonlinear regularity, nonlinear maximum principle, pseudomonotone operator, positive solution
2010 MSC: 35J60, 35J92, 35P30, 47H10

1. Introduction

In this paper we study the following singular parametric Dirichlet (p, q)-equation with $1 < q < p$:

$$-\Delta_p u(z) - \Delta_q u(z) = \lambda u(z)^{-\eta} + f(z, u(z)) + c|\nabla u(z)|^{p-1} \text{ in } \Omega,\ u\Big|_{\partial\Omega} = 0,\ u > 0,\ \lambda, c > 0,\ 0 < \eta < 1, \quad (P_\lambda)$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$. For every $r \in (1, +\infty)$ by Δ_r we denote the r-Laplace differential operator defined by $\Delta_r u = \text{div}\,(|\nabla u|^{r-2}\nabla u)$ for all $u \in W_0^{1,r}(\Omega)$. In (P_λ) we have the sum of two such operators. So, the differential operator in (P_λ) is not homogeneous. The reaction of (P_λ) depends on the gradient of u (convection) and in addition it involves the competing effects of two more terms. One is a parametric singular term $\lambda u^{-\eta}$ ($\lambda > 0$ being the parameter) and the other is a Carathéodory perturbation with $f(z, \cdot)$ only locally (near 0^+) defined. The presence of the gradient of u in the reaction makes (P_λ) nonvariational. So, we follow a topological approach based on truncation techniques and on the theory of nonlinear operators of monotone type. We show that for all small parameter $\lambda > 0$ values, the problem has a positive solution. Recently nonlinear singular problems with convection were studied by Bai-Gasiński-Papageorgiou [2] (Dirichlet problems), and by Papageorgiou-Rădulescu-Repovš [15] (Neumann problems). In both works the differential operator is the p-Laplacian and the reaction is globally defined. For other papers dealing with convection double phase problems, we refer to [7, 8, 19, 20]. Parametric problems without convection can be found in [1, 12, 17], where a variety of mathematical methods are combined in a synergic way. Finally, a weighted (p, q)-equation with Robin boundary condition is studied in [13].

2. Preliminaries - Hypotheses

The main spaces in the analysis of (P_λ) are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\}$. By $\|\cdot\|$ we denote the norm of $W_0^{1,p}(\Omega)$. On account of the Poincaré inequality we

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have $\|u\| = \|\nabla u\|_p$ for all $u \in W_0^{1,p}(\Omega)$. The space $C_0^1(\overline{\Omega})$ is an ordered Banach space with order (positive) cone $C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by $\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}\big|_{\partial\Omega} < 0\}$ with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. For $r \in (1, +\infty)$ by $A_r : W_0^{1,r}(\Omega) \to W_0^{1,r}(\Omega)^* = W^{-1,r'}(\Omega)$ ($\frac{1}{r} + \frac{1}{r'} = 1$) we denote the nonlinear operator defined by $\langle A_r(u), h\rangle = \int_\Omega |\nabla u|^{r-2}(\nabla u, \nabla h)_{\mathbb{R}^N} dz$ for all $u, h \in W_0^{1,r}(\Omega)$. This operator is continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$ (i.e., if $u_n \xrightarrow{w} u$ in $W_0^{1,r}(\Omega)$ and $\limsup_{n\to+\infty}\langle A_r(u_n), u_n - u\rangle \leq 0$, then $u_n \to u$ in $W_0^{1,r}(\Omega)$ (see [6], p. 279)). Given two measurable functions $u, v : \Omega \to \mathbb{R}$ such that $u(z) \leq v(z)$ for a.a. $z \in \Omega$, we introduce the following two items:

$$[u, v] = \{h \in W_0^{1,p}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega\},$$
$$\operatorname{int}_{C_0^1(\overline{\Omega})}[u, v] = \text{ the interior in } C_0^1(\overline{\Omega}) \text{ of } [u, v] \cap C_0^1(\overline{\Omega}).$$

If $u \in W_0^{1,p}(\Omega)$, we define $u^\pm = \max\{\pm u, 0\}$. We know that $u^\pm \in W_0^{1,p}(\Omega)$, $u = u^+ - u^-$, $|u| = u^+ + u^-$. By $\widehat{\lambda}_1(q)$ we denote the first eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$. We know that $\widehat{\lambda}_1(q) > 0$, it is simple and isolated and all the eigenfunctions corresponding to it, have fixed sign. By $\widehat{u}_1(q)$ we denote the positive, L^q-normalized (i.e., $\|\widehat{u}_1(q)\|_q = 1$) eigenfunction corresponding to $\widehat{\lambda}_1(q) > 0$. We know that $\widehat{u}_1(q) \in \operatorname{int} C_+$ and $\widehat{\lambda}_1(q)$ is the only eigenvalue with eigenfunctions of fixed sign (see [5], Section 6.2). Now we introduce our hypotheses on the perturbation $f(z, x)$:

H: $f : \Omega \times [0, b] \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$, and

- (i) $f(z, b) \leq -\widehat{c} < 0$ for a.a. $z \in \Omega$;
- (ii) there $\exists\ a_b \in L^\infty(\Omega)$ such that $|f(z, x)| \leq a_b(z)$ for a.a. $z \in \Omega$, all $0 \leq x \leq b$;
- (iii) there $\exists\ \delta \in (0, b)$ and $\eta \in L^\infty(\Omega) \setminus \{\widehat{\lambda}_1(q)\}$ such that $\widehat{\lambda}_1(q) \leq \eta(z)$ for a.a. $z \in \Omega$, $\eta(z)x^{q-1} \leq f(z, x)$ for a.a. $z \in \Omega$, all $0 \leq x \leq \delta$.
- (iv) there $\exists\ \widehat{\xi}_b > 0$ such that for a.a. $z \in \Omega$, the function $x \to f(z, x) + \widehat{\xi}_b x^{p-1}$ is nondecreasing on $[0, b]$.

Remark 1. We can always assume that $f(z, x) = 0$ for a.a. $z \in \Omega$, all $x \leq 0$. We stress that $f(z, \cdot)$ is defined only locally near zero.

In the next section we deal with an auxiliary Dirichlet problem, the solution of which will help us overcome the difficulty posed by the singular term.

3. Auxiliary Problem

Hypotheses $H\,(ii), (iii)$ imply that we can find $c_1 > 0$ such that

$$f(z, x) \geq \eta(z)x^{q-1} - c_1 x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0, b]. \tag{1}$$

Based on this unilateral growth condition of $f(z, \cdot)$ on $[0, b]$, we introduce the Carathéodory function $k : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$k(z, x) = \begin{cases} \eta(z)(x^+)^{q-1} - c_1(x^+)^{p-1} & \text{if } x \leq b, \\ \eta(z)b^{q-1} - c_1 b^{p-1} & \text{if } b < x. \end{cases} \tag{2}$$

We consider the following auxiliary Dirichlet problem

$$-\Delta_p u(z) - \Delta_q u(z) = k(z, u(z)) \text{ in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{3}$$

Proposition 1. *Problem* (3) *admits a unique solution* $\overline{u} \in \operatorname{int}_{C_0^1(\overline{\Omega})}[0, b]$.

Proof. Problem (3) is variational. So, let $K(z,x) = \int_0^x k(z,s)ds$ and consider the C^1-functional $\psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by $\psi(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega K(z,u)dz$ for all $u \in W_0^{1,p}(\Omega)$. It is clear from (2) that $\psi(\cdot)$ is coercive. Also, using the Sobolev embedding theorem, we see that $\psi(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $\overline{u} \in W_0^{1,p}(\Omega)$ such that

$$\psi(\overline{u}) = \min\left[\psi(u) \,:\, u \in W_0^{1,p}(\Omega)\right]. \tag{4}$$

Recall that $\widehat{u}_1 = \widehat{u}_1(q) \in \operatorname{int} C_+$. Let $t \in (0,1)$ small such that $t\widehat{u}_1(z) \le \delta$ for all $z \in \overline{\Omega}$. We have

$$\psi(t\widehat{u}_1) = \frac{t^p}{p}\|\nabla \widehat{u}_1\|_p^p - \frac{t^q}{q}\int_\Omega [\eta(z) - \widehat{\lambda}_1(q)]\widehat{u}_1^q dz + \frac{c_1 t^p}{p}\|\widehat{u}_1\|_p^p.$$

Note that $\gamma_0 = \int_\Omega [\eta(z) - \widehat{\lambda}_1(q)]\widehat{u}_1^q dz > 0$. Hence we can write $\psi(t\widehat{u}_1) \le c_2 t^p - c_3 t^q$ for some $c_2, c_3 > 0$. Since $t \in (0,1)$ and $q < p$, choosing $t \in (0,1)$ even smaller if necessary, we have $\psi(t\widehat{u}_1) < 0$, hence $\psi(\overline{u}) < 0 = \psi(0)$ (see (4)) and so $\overline{u} \neq 0$. From (4) we have

$$\psi'(\overline{u}) = 0 \quad \Rightarrow \quad \langle A_p(\overline{u}), h\rangle + \langle A_q(\overline{u}), h\rangle = \int_\Omega k(z,\overline{u})h dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{5}$$

In (5) first we choose $h = -\overline{u}^- \in W_0^{1,p}(\Omega)$ and obtain $\overline{u} \ge 0$, $\overline{u} \neq 0$. Then we use the test function $h = (\overline{u} - b)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned}\langle A_p(\overline{u}), (\overline{u}-b)^+\rangle + \langle A_q(\overline{u}), (\overline{u}-b)^+\rangle &= \int_\Omega [\eta(z)b^{q-1} - c_1 b^{p-1}](\overline{u}-b)^+ dz \quad \text{(see (2))} \\ &\le \int_\Omega f(z,b)(\overline{u}-b)^+ dz \le 0, \quad \text{(see (1))},\end{aligned}$$

which implies $\overline{u} \le b$. So, we have proved that $\overline{u} \in [0,b]$, $\overline{u} \neq 0$. It follows that

$$-\Delta_p \overline{u} - \Delta_q \overline{u} = \eta(z)\overline{u}^{q-1} - c_1\overline{u}^{p-1} \text{ in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{6}$$

From (6) and the nonlinear regularity theory of [11], we have $\overline{u} \in C_+ \setminus \{0\}$. Also from (6) we see that $\Delta_p \overline{u} + \Delta_q \overline{u} \le c_1 \overline{u}^{p-1}$ in Ω and this, by the nonlinear maximum principle of [18] (pp. 111, 120), implies that

$$\overline{u} \in [0,b] \cap \operatorname{int} C_+. \tag{7}$$

Let $\delta \in (0,1)$ and set $\overline{u}_\delta = \overline{u} + \delta$. With $\widehat{\xi}_b > 0$ as postulated by hypothesis $H\,(iv)$, we have

$$\begin{aligned}-\Delta_p \overline{u}_\delta - \Delta_q \overline{u}_\delta + \widehat{\xi}_b \overline{u}_\delta^{p-1} &\le -\Delta_p \overline{u} - \Delta_q \overline{u} + \widehat{\xi}_b \overline{u}^{p-1} + \chi(\delta) \quad \text{with } \chi(\delta) \to 0^+ \text{ as } \delta \to 0^+ \\ &= \eta(z)\overline{u}^{q-1} - c_1\overline{u}^{p-1} + \widehat{\xi}_b \overline{u}^{p-1} + \chi(\delta) \\ &\le f(z,\overline{u}) + \widehat{\xi}_b \overline{u}^{p-1} + \chi(\delta) \quad \text{(see (1) and (7))} \\ &\le f(z,b) + \widehat{\xi}_b b^{p-1} + \chi(\delta) \quad \text{(see (7) and hypothesis } H\,(iv)) \\ &\le -\widehat{c} + \widehat{\xi}_b b^{p-1} + \chi(\delta) \quad \text{(see hypothesis } H\,(i)) \\ &\le \widehat{\xi}_b b^{p-1} - \widehat{c}_0 \quad \text{for } \delta \in (0,1) \text{ small and some } \widehat{c}_0 > 0 \\ &< -\Delta_p b^{p-1} - \Delta_q b^{q-1} + \widehat{\xi}_b b^{p-1}.\end{aligned} \tag{8}$$

From (8) and [14, Proposition 2.10] we get $\overline{u}(z) < b$ for all $z \in \overline{\Omega}$ and so $\overline{u} \in \operatorname{int}_{C_0^1(\overline{\Omega})}[0,b]$. To show the uniqueness of this positive solution $\overline{u} \in \operatorname{int} C_+$, we use the functional $j : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p}\|\nabla u^{1/q}\|_p^p + \frac{1}{q}\|\nabla u^{1/q}\|_q^q & \text{if } u \ge 0,\ u^{1/q} \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.}\end{cases}$$

From Díaz-Saá [4], we know that $j(\cdot)$ is convex. Suppose that $\overline{v} \in W_0^{1,p}(\Omega)$ is another nontrivial solution of (3). Again we have $\overline{v} \in [0, b] \cap \operatorname{int} C_+$. On account of [16, Proposition 4.1.22], we have $\frac{\overline{u}}{\overline{v}}, \frac{\overline{v}}{\overline{u}} \in L^\infty(\Omega)$. If $\operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < \infty\}$ (the effective domain of $j(\cdot)$) and $h = \overline{u}^q - \overline{v}^q \in W_0^{1,p}(\Omega)$, then for $|t| < 1$ small, we have $\overline{u}^q + th, \overline{v}^q + th \in \operatorname{dom} j$. So, the convex functional $j(\cdot)$ is Gateaux differentiable at $\overline{u}^q$ and at $\overline{v}^q$ in the direction h. Using the chain rule and the nonlinear Green's identity (see [16], p. 35), we have

$$
\begin{aligned}
j'(\overline{u}^q)(h) &= \frac{1}{q} \int_\Omega \frac{-\Delta_p \overline{u} - \Delta_q \overline{u}}{\overline{u}^{q-1}} h dz = \frac{1}{q} \int_\Omega [\eta(z) - c_1 \overline{u}^{p-q}] h dz, \\
j'(\overline{v}^q)(h) &= \frac{1}{q} \int_\Omega \frac{-\Delta_p \overline{v} - \Delta_q \overline{v}}{\overline{v}^{q-1}} h dz = \frac{1}{q} \int_\Omega [\eta(z) - c_1 \overline{v}^{p-q}] h dz.
\end{aligned}
$$

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. Hence we have $0 \leq \int_\Omega c_1[\overline{v}^{p-q} - \overline{u}^{p-q}](\overline{u}^q - \overline{v}^q) dz \leq 0$, and so $\overline{u} = \overline{v}$. Therefore $\overline{u} \in \operatorname{int} C_+$ is the unique solution of (3). $\square$

Since $\overline{u} \in \operatorname{int} C_+$, from the proof of the Lemma of Lazer-McKenna [10], we have that $\overline{u}^{-\eta} \in L^1(\Omega)$ (since $0 < \eta < 1$). Also, Lemma 14.16, p. 355, of Gilbarg-Trudinger [9], implies that we can find $\delta_0 > 0$ such that, if $\Omega_{\delta_0} = \{z \in \overline{\Omega} : d(z, \partial\Omega) < \delta_0\}$, then $\widehat{d}(\cdot) = d(\cdot, \partial\Omega) \in C^2(\overline{\Omega}_{\delta_0})$. So, it follows that $\widehat{d} \in C_+ \setminus \{0\}$ and so using [16, Proposition 4.1.22] (p. 274), we can find $c_4 > 0$ such that $\widehat{d} \leq c_4 \overline{u}$. Then for all $h \in W_0^{1,p}(\Omega)$

$$
\begin{aligned}
\int_\Omega \frac{|h|}{\overline{u}^\eta} dz = \int_\Omega |\overline{u}|^{1-\eta} \frac{|h|}{\overline{u}} dz &\leq c_5 \int_\Omega \frac{|h|}{\overline{u}} dz \quad \text{for some } c_5 > 0 \\
&\leq c_6 \int_\Omega \frac{|h|}{\widehat{d}} dz \quad \text{for some } c_6 > 0 \\
&\leq c_7 \left\| \frac{h}{\widehat{d}} \right\|_p \quad \text{for some } c_7 > 0 \\
&\leq c_8 \left\| \nabla h \right\|_p \quad \text{for some } c_8 > 0 \quad \text{(by Hardy's inequality, see Brezis [3], p. 313)}, \\
\Rightarrow \quad \frac{h}{\overline{u}^\eta} &\in L^1(\Omega) \quad \text{for all } h \in W_0^{1,p}(\Omega). \qquad (9)
\end{aligned}
$$

4. Positive Solution

In this section using topological tools we prove the existence of a positive solution for problem (P_λ) when $\lambda > 0$ is small. So, consider the truncation at b function $p_b : \mathbb{R} \to \mathbb{R}$ defined by $p_b(x) = x$ if $x \leq b$ and $p_b(x) = b$ if $b < x$. This is a Lipschitz function. Therefore, if $u \in W_0^{1,p}(\Omega)$, we know that $u^+ \in W_0^{1,p}(\Omega)$ and then the chain rule for Sobolev functions (see [16], p. 22), says that $p_b(u^+(\cdot)) \in W_0^{1,p}(\Omega)$ and we have

$$
\nabla p_b(u^+) = p_b'(u^+) \nabla u^+ = \begin{cases} \nabla u^+ & \text{if } u^+(z) \leq b, \\ 0 & \text{if } b < u^+(z). \end{cases} \qquad (10)
$$

We introduce the Carathéodory function $\ell_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$
\ell_\lambda(z, x) = \begin{cases} \lambda \overline{u}(z)^{-\eta} + f(z, \overline{u}(z)) & \text{if } x \leq \overline{u}(z), \\ \lambda x^{-\eta} + f(z, p_b(x)) & \text{if } \overline{u}(z) < x. \end{cases} \qquad (11)
$$

Let N_{ℓ_λ} denote the Nemitsky operator corresponding to ℓ_λ, that is, $N_{\ell_\lambda}(u)(\cdot) = \ell_\lambda(\cdot, u(\cdot))$ for all $u \in W_0^{1,p}(\Omega)$. On account of (9) we see that $N_{\ell_\lambda}(u) \in W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$. Also, from Lemma 2.2.27, p. 141, of Gasiński-Papageorgiou [5], we have that $L^{p'}(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$ continuously and densely. Hence $|\nabla p_b(u^+(\cdot))|^{p-1} \in L^{p'}(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$. So, we can define the operator $K_0 : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ by setting $K_0(u) = A_p(u) + A_q(u) - N_{\ell_\lambda}(u) - c|\nabla p_b(u^+)|^{p-1}$.

Proposition 2. *If hypotheses H hold, then $K_0(\cdot)$ is surjective.*

Proof. First we show that $K_0(\cdot)$ is pseudomonotone. Since $K_0(\cdot)$ is everywhere defined and bounded, according to [5, Proposition 3.2.49] (p. 333), it suffices to show that $K_0(\cdot)$ is generalized pseudomonotone (see [5, Definition 3.2.45], p. 330). So, suppose that $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$, $K(u_n) \xrightarrow{w} u^*$ in $W^{-1,p'}(\Omega)$ and

$$\limsup_{n\to+\infty}\langle K_0(u_n), u_n - u\rangle \le 0. \tag{12}$$

From (12) we have

$$\limsup_{n\to+\infty}[\langle A_p(u_n), u_n-u\rangle + \langle A_q(u_n), u_n-u\rangle - \int_\Omega \ell_\lambda(z,u_n)(u_n-u)dz - \int_\Omega c|\nabla p_b(u_n^+)|^{p-1}(u_n-u)dz] \le 0. \tag{13}$$

Consider the Carathéodory function $\widehat{\ell}_\lambda(z,x)$ defined by

$$\widehat{\ell}_\lambda(z,x) = \begin{cases} \frac{\lambda}{\overline{u}(z)^\eta} & \text{if } x \le b, \\ \frac{\lambda}{x^\eta} & \text{if } b < x. \end{cases} \tag{14}$$

We have $\int_\Omega \widehat{\ell}_\lambda(z,u_n)u_n dz = \int_{\{u_n\le\overline{u}\}} \frac{\lambda(u_n-u)}{\overline{u}^\eta}dz + \int_{\{\overline{u}<u_n\}} \frac{\lambda(u_n-u)}{u_n^\eta}dz$ (see (14)). Note that on account of (9) and since $u_n \to u$ in $L^p(\Omega)$, we have $\int_{\{u_n\le\overline{u}\}} \frac{\lambda(u_n-u)}{\overline{u}^\eta}dz \to 0$ as $n \to +\infty$. Also we have

$$\left|\int_{\{\overline{u}<u_n\}} \frac{\lambda(u_n-u)}{u_n^\eta}dz\right| \le \int_{\{\overline{u}<u_n\}} \frac{\lambda|u_n-u|}{\overline{u}^\eta}dz \to 0 \quad \text{as } n\to+\infty.$$

Therefore

$$\int_\Omega \widehat{\ell}_\lambda(z,u_n)(u_n-u)dz \to 0 \quad \Rightarrow \quad \int_\Omega \ell_\lambda(z,u_n)(u_n-u)dz \to 0 \quad \text{(see (11))}. \tag{15}$$

Moreover, since $u_n \to u$ in $L^p(\Omega)$, it follows that

$$c\int_\Omega |\nabla p_b(u_n^+)|^{p-1}(u_n-u)dz \to 0. \tag{16}$$

Returning to (13) and using (15) and (16), we obtain

$$\begin{aligned} & \limsup_{n\to+\infty}\,[\langle A_p(u_n), u_n-u\rangle + \langle A_q(u_n), u_n-u\rangle] \le 0, \\ \Rightarrow\ & \limsup_{n\to+\infty}\,[\langle A_p(u_n), u_n-u\rangle + \langle A_q(u), u_n-u\rangle] \le 0, \quad \text{(since } A_q(\cdot) \text{ is monotone)}, \\ \Rightarrow\ & \limsup_{n\to+\infty}\langle A_p(u_n), u_n-u\rangle \le 0, \\ \Rightarrow\ & u_n \to u \text{ in } W_0^{1,p}(\Omega) \quad \text{(by the } (S)_+\text{-property)}. \end{aligned} \tag{17}$$

From (17) and the continuity of K_0, we have $u^* = K_0(u)$ and $\langle K_0(u_n), u_n\rangle \to \langle K_0(u), u\rangle$, and hence $K_0(\cdot)$ is generalized pseudomonotone, thus pseudomonotone. Also, we have $\langle K_0(u), u\rangle \ge \|u\|^p - c_9[\|u\| + 1]$ for some $c_9 > 0$, which implies that $K_0(\cdot)$ is coercive. But a pseudomonotone coercive operator is surjective (see [5], p. 336). So, we conclude that $K_0(\cdot)$ is surjective. □

Now we are ready to state and prove our existence theorem.

Theorem 1. *If hypotheses H hold, then for all $\lambda > 0$ small problem (P_λ) has a positive solution $u_0 \in \operatorname{int} C_+$, $u_0(z) < b$ for all $z \in \overline{\Omega}$.*

Proof. On account of Proposition 2, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that $K_0(u_0) = 0$. We have

$$A_p(u_0) + A_q(u_0) = N_{\ell_\lambda}(u_0) + c|\nabla p_b(u^+)|^{p-1} \quad \text{in } W^{-1,p'}(\Omega). \tag{18}$$

On (18) first we act with $(\overline{u}-u_0)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned}\langle A_p(u_0),(\overline{u}-u_0)^+\rangle + \langle A_q(u_0),(\overline{u}-u_0)^+\rangle &\geq \int_\Omega [\lambda \overline{u}^{-\eta} + f(z,\overline{u})](\overline{u}-u_0)^+ dz \ \text{(see (11) and recall } c>0)\\ &\geq \int_\Omega f(z,\overline{u})(\overline{u}-u_0)^+ dz\\ &\geq \int_\Omega [\eta(z)\overline{u}^{q-1} - c_1\overline{u}^{p-1}](\overline{u}-u_0)^+ dz \ \text{(see (1))}\\ &= \langle A_p(\overline{u}),(\overline{u}-u_0)^+\rangle + \langle A_q(\overline{u}),(\overline{u}-u_0)^+\rangle \ \text{(see Proposition 1)},\end{aligned} \tag{19}$$

which implies that $\overline{u} \leq u_0$. Next on (18) we act with $(u_0-b)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned}\langle A_p(u_0),(u_0-b)^+\rangle + \langle A_q(u_0),(u_0-b)^+\rangle &= \int_\Omega [\lambda u_0^{-\eta} + f(z,b)](u_0-b)^+ dz \quad \text{(see (10), (11))}\\ &\leq \int_\Omega [\lambda \overline{u}^{-\eta} - \widehat{c}](u_0-b)^+ dz \quad \text{(see (19) and hypothesis } H(i))\\ &\leq \int_\Omega [\lambda b^{-\eta} - \widehat{c}](u_0-b)^+ dz \quad \text{(see (7))}\\ &\leq 0 \quad \text{for } \lambda>0 \text{ small,}\end{aligned}$$

which implies that $u_0 \leq b$. So, we have proved that $u_0 \in [\overline{u}, b]$. Then we have

$$-\Delta_p u_0 - \Delta_q u_0 = \lambda u_0^{-\eta} + f(z,u_0) + c|\nabla u_0|^{p-1} \quad \text{in } \Omega,$$

and hence u_0 is a positive solution of (P_λ) for $\lambda > 0$ small. The nonlinear regularity theory of [11] implies that $u_0 \in \operatorname{int} C_+$. Suppose that for some $z_0 \in \Omega$, we have $u_0(z_0) = b$. Then $\nabla u_0(z_0) = 0$ and so we can find $\Omega_0 \subseteq \Omega$ open, $\overline{\Omega}_0 \subseteq \Omega$ with C^2-boundary $\partial\Omega_0$ such that $c|\nabla u_0(z)|^{p-1} \leq \varepsilon < \widehat{c}$ for all $z \in \overline{\Omega}_0$. We have

$$\begin{aligned}-\Delta_p u_0(z) - \Delta_q u_0(z) + \widehat{\xi}_b u_0(z)^{p-1} - \lambda u_0(z)^{-\eta} &= f(z,u_0(z)) + \widehat{\xi}_b u_0(z)^{p-1} + c|\nabla u_0(z)|^{p-1}\\ &\leq f(z,b) + \widehat{\xi}_b b^{p-1} + \varepsilon \quad \text{(see hypothesis } H(iv)).\end{aligned} \tag{20}$$

Taking $\lambda > 0$ even smaller if necessary, we will have that $\lambda b^{-\eta} + \varepsilon < \widehat{c}$. Then from (20) and $H(i)$ we have

$$-\Delta_p u_0(z) - \Delta_q u_0(z) + \widehat{\xi}_b u_0(z)^{p-1} - \lambda u_0(z)^{-\eta} \leq -\widehat{c} + \widehat{\xi}_b b^{p-1} \leq -\Delta_p b - \Delta_q b + \widehat{\xi}_b b^{p-1} - \lambda b^{-\eta} \text{ in } \Omega_0,$$

$\Rightarrow \quad u_0(z) < b$ for all $z \in \Omega_0$ (see [14, Proposition 2.10]).

This contradicts the fact that $u_0(z_0) = b$, $z_0 \in \Omega_0$. So, we conclude that $u_0(z) < b$ for all $z \in \overline{\Omega}$. □

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