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Temari Balls, Spheres, SphereHarmonic: From Japanese Folkcraft to Music

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Abstract: Temari balls are traditional Japanese toys and artworks. The variety of their geometries and tessellations can be investigated formally and computationally with the means of combinatorics. As a further step, we also propose a musical application of the core idea of Temari balls. In fact, inspired by the classical idea of *music of spheres* and by the *CubeHarmonic*, a musical application of the Rubik's cube, we present the concept of a new musical instrument, the *SphereHarmonic.* The mathematical (and musical) description of Temari balls lies in the wide background of interactions between art and combinatorics. Concerning the methods, we present the tools of permutations and tessellations we adopted here, and the core idea for the SphereHarmonic. As the results, we first describe a classification of structures according to the theory of groups. Then, we summarize the main passages implemented in our code, to make the SphereHarmonic play on a laptop. Our study explores an aspect of the deep connections between the mutually inspiring scientific and artistic thinking.

Keywords: geometry; Temari balls; music; sphere

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1. Introduction

Since Ancient Greece, the perfect geometrical forms of Platonic solids and spheres have been fascinating humankind. The fundamental proportions for musical intervals led Pythagoras to conjecture universal music as the organizational principle for the entire universe, the *harmony of spheres* or *musica universalis*, idea that influenced the philosophical debate till the Renaissance in Italy and Europe.

Already in the Middle Ages, Geometry and Music had been considered as parts of the same group of disciplines in the Quadrivium. According to some scholars [1], the geometry of the hypersphere, despite being a modern concept, can be used to explain some apparent contradictions in Dante's cosmology. Surfaces of different curvatures covered with tessellations inspired some well-known artworks by Escher and are the object of recent studies [2,3].

From the West to the Far East, the geometry of spheres met the colorful variety of tilings in Japanese Temari balls, see Figure 1. Temari balls are astonishing examples of geometric folkcraft made from embroidery. The patient art of Temari construction has been studied by mathematicians [4,5]. In particular, one of the initial steps of Temari preparation involves the marking of spherical Platonic solids [6,7].

In addition, following the ancient suggestions from geometry and music, we envisage a possible musical rendition of the basic geometries of Temari balls. Joining novelty and tradition, tools from algebra and computer sciences can help us investigate existing objects and create new ones.

To consider the variety of patterns in Temari balls and their possible musical rendition, we consider mathematical formalization provided by *combinatorics*. Leibniz defined as *ars* *combinatoria* the technique of representation and recombination of symbols and ideas, in the context of a map of concepts [8,9]. With the name of *ars magna*, it was already explored in the XIII century by Radmond Lull, in terms of a universal logic to prove the truth starting from and recombining a few elements [10,11]. Combinatorics has quite a tradition of musical applications [12], in particular, to generate chords and musical sequences, create new music [13], and formalize chord combinations in music theory [14], numbers of equivalence classes of musical objects in Western music theory [15]. In the pedagogical domain, rhythms have been used to explain permutations [16]. Of particular interest is joining the geometric [17,18] with the combinatorial approach [19].

In this research, we start from combinatorial geometries in Temari balls, we mathematically formalize them, and then we propose a musical application of their structure. Thus, we use formalization and computing as a bridge across visual arts and music. We conceptualize a new musical instrument having notes on each triangle, and where triangles can be moved, as in a spherical Rubik's cube. We call this idea *SphereHarmonic*, being inspired by the CubeHarmonic, a musical instrument ideated by M. Mannone, and realized as a physical device in collaboration with researchers led by Prof. Kitamura at the Tohoku University in Sendai, Japan [20].

Figure 1. A collection of Temari balls. Credits: Wikipedia, user Σ64, license CC BY 4.0.

This article is organized as follows. In Section 2, we present the formal tools to analyze Temari balls, followed by the key concept of musical *tonnetz*, helpful to turn geometries into music. In Section 3, we apply the presented tools to the actual computation of Temari ball patterns, and we discuss the possible structure of the SphereHarmonic. In Section 4, we discuss the implications and future developments of our research.

2. Materials and Methods

In this Section, we introduce the concept of *tonnetz* and *CubeHarmonic* before presenting the core idea of the *SphereHarmonic*.

2.1. Tonnetz in Music Theory, Triangles as Triads

In mathematical music theory, the *tonnetz* (network of tones) [21,22] is a lattice where each point indicates a musical note, and each segment indicates an interval. Thus, three points arranged as a triangle indicate a musical chord. When each point indicates a chord, each segment indicates a modulation. The tonnetz was first proposed by Euler [23]. Classic examples of tonnetz are unfolded in the geometry of the plane and of the torus, which can be obtained from the plane [24,25].

The CubeHarmonic joins the concept of tonnetz with permutations. Figure 2 shows the prototype. The general idea of permutations for music theory and music composition has been exploited by composers such as Mozart and Lully. Local transformations of the tonnetz through permutations are described in the "slot-machine" rotations. This is a metaphor to indicate a permutation in a cross-partition [26]. In a cross-partition, each column has numbers representing the notes of a chord. More details on slot-machine transformations are described in the article [27]. Examples of music played with the CubeHarmonic can be retrieved online (https://www.youtube.com/watch?v=r_wNpQnsWhg Accessed on 10 August 2022). In the CubeHarmonic, a portion of the plane of the tonnetz can be exploited, with cuts and glues adapting it to the surface of the cube. Some symmetries of the plane can be preserved through suitable rotations of the Rubik's cube. In the current prototypes of CubeHarmonic, however, the facets of each of the six faces contain the notes of just one chord, to avoid cacophony after only a few rotations.

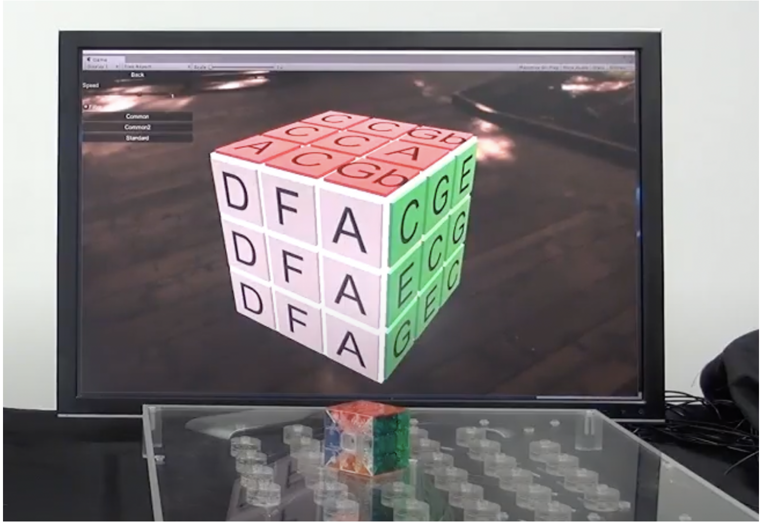

Figure 2. The prototype of *CubeHarmonic* developed at the Tohoku University, with the IM3D platform and a virtual-reality screen. Detail from a video's photogram. Video by M. Mannone with Kitamura Lab. CC-BY license for NIME conference.

On the sphere, we can still have portions of the plane adapted through cuts and glues, as the classic problem of globe and planisphere attests. The idea of triangles indicating chords can be preserved if there are buttons to press and play notes at each vertex. If each triangle indicates a musical note, then a group of three triangles pressed together plays a chord. If triangles can slide one along the other through suitable rotations, it is possible to "hear the sound of rotations" as is allowed by the CubeHarmonic. Of course, each triangle on the sphere would be an elliptic one. In this way, we can create a new musical instrument, called *SphereHarmonic*.

2.2. Basic Divisions of Temari

As the CubeHarmonic changes the combinations of tones by rotating various parts of the Rubik's cube, we make music out of the *SphereHarmonic* by rotating parts of a sphere. Furthermore, the division after the rotations must be equal to the initial division although the combinations of the colors (tones) are different. In order to satisfy this constraint, we introduce the basic divisions of Temari [7], the equal divisions of a spherical surface. The Temari basic divisions enable us to rotate some hemispherical parts while keeping the original division. For this reason, we describe the geometry of Temari basic divisions in the following:

Temari basic divisions are classified into two types: simple divisions and combination divisions. Figure 3a–d shows an example of simple divisions and combination divisions. These are the divisions of the spherical surface by spherical congruent triangles. The Temari artists classify the divisions by combining the types of divisions, S or C, and the number of spherical digons (A *digon* is a polygon with two sides (edges) and two vertices). obtained from the first stage of the dividing process. Therefore, the number used for the notation is not the number of the spherical triangles but the number of the spherical digons. The simple divisions are made by means of equal divisions of the spherical surface by n longitudes and an equator. A simple division is constructed by $2n$ spherical isosceles triangles and is referred to as "simple n division" denoted as Sn. For example, S8 is the division of the spherical surface by sixteen isosceles triangles of which interior angles are $\pi/4$, $\pi/2$, and $\pi/2$ as shown in Figure 3a. It is notable that n is an even number because the basic divisions use only great circles. The combination division, on the other hand, is a division by n longitudes and other great circles which contain spherical points defined by a so-called "magic number" R_n. The magic numbers help retrieve proportions in the process of building Temari balls. With these values, Temari artists approximate spherical icosahedral edge lengths as $(1/6 + 1/100)$ times the ball circumference [6,28], for example. There are three types of combination divisions: the combination 6 (C6), the combination 8 (C8), and the combination 10 (C10), as shown in Figure 3b–d.

Figure 3a′–d′ shows the spherical patterns designed by using the basic divisions shown in Figures 3a–d. We mapped a simple textured triangle shown in Figure 3e′ onto the spherical triangles consisting of the surfaces. These spherical patterns demonstrate how the basic divisions affect the Temari patterns.

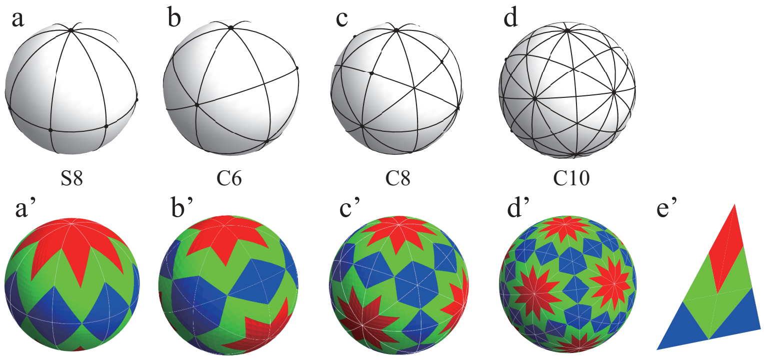

Figure 3. (**a**–**d**):Traditional basic divisions (S8, C6, C8, and C10, from left to right). Large marks on the top denote poles. (**a′**–**d′**): Spherical patterns corresponding to (**a**–**d**). (**e′**): A simple textured triangular unit used to construct spherical patterns (**a′**–**d′**).

Procedures for constructing the combination divisions are similar: (1) equal division by n longitudes ($n = 6$, 8, or 10), (2) marking of spherical points using a corresponding magic number R_n, and (3) drawing (stitching) great circles based on markers. The exact values of magic numbers are $R_6 = \cos^{-1}(1/3)/2\pi$, $R_8 = 1/8$, and $R_{10} = \cos^{-1}(1/\sqrt{5})/2\pi$, respectively. Although Temari artists use approximated values of them empirically, the errors are small enough for drawing the divisions.

From the viewpoint of tessellation, basic divisions are the divisions of a spherical surface into congruent triangles using only great circles. Coxeter [29] constructed a complete list of types of divisions on a spherical, Euclidean, and hyperbolic plane called Coxeter prints [2]. Each Coxeter print is characterized by three natural numbers $\{p, q, r\}$ which relate to the inner angles of the triangle. Figure 4 shows the definition of Coxeter print denoted as $\{p, q, r\}$. The triangle PQR (in white) and its flipped counterpart PR'Q (in gray) are placed alternatively to construct the divisions. The angles of the triangles used for the division are π/p, π/q, and π/r. Each vertex consists of equal angles: for example, vertex P in Figure 4 consists of $2p$ number of π/p angles. It is notable that Coxeter prints of a spherical surface use only great circles because all vertices are equally divided into even numbers. For spherical surfaces, $\{p, q, r\}$ must satisfy the inequality

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1 \tag{1}$$

and the results are $\{p, q, r\}$=$\{$m, 2, 2$\}$, $\{3, 3, 2\}$, $\{4, 2, 3\}$, and $\{5, 2, 3\}$, where m is an arbitrary natural number ($m \geq 2$). These solutions are a complete list of the divisions of a spherical surface into congruent triangles using only great circles.

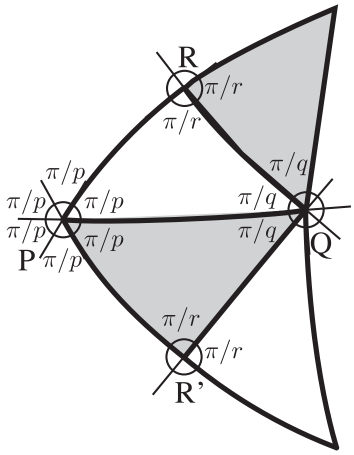

Figure 4. Schematic illustration of a part of a Coxeter print $\{p, q, r\}$.

The basic divisions have one-to-one correspondences to Coxeter prints. A simple division Sn corresponds to $\{n/2, 2, 2\}$ and combination divisions C6, C8, and C10 correspond to $\{3, 3, 2\}$, $\{4, 2, 3\}$, and $\{5, 2, 3\}$, respectively. We again note that n is an even number. Comparisons of basic divisions and Coxeter prints are shown in Figure 5. We can state, therefore, that Temari artists empirically know all types of Coxeter prints on a spherical surface.

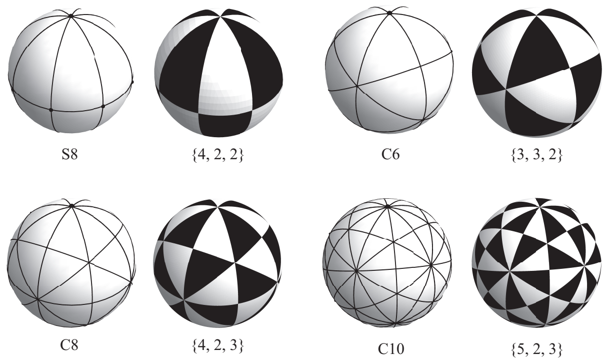

Figure 5. One-to-one correspondence between Coxeter prints and Temari basic divisions.

2.3. Concept and Visualization of SphereHarmonic

SphericalHarmonic is a musical instrument designed on a basis of Temari basic divisions. The device can be rotated along with each great circle that divides the surface. Each spherical triangle appearing on the surface represents a tone as each component consisting of a facet of *CubeHarmonic* does. The locations of the spherical triangles are changed by rotations of hemispheres divided by great circles. In other words, the combinations of the tones are changed by the rotations along with the great circles which separate the surface of the *SphericalHarmonic*.

We regard the vertices as the combinations of tones, while, in the *CubeHarmonic*, chords are given by faces with 9 little squares (the facets). This is because the vertices of the *SphereHarmonic* are the centers of the faces of the spherical polyhedrons as shown in Figure 6. A selection of a vertex corresponds to a selection of a combination of tones, a chord. And the duration time of the selection corresponds to the seconds a chord is sounding.

The numbers of rotational axes of Sn, C6, C8, and C10 are $n/2+1$, 6, 9, and 15, respectively. The spherical triangles are defined by the three great circles drawn according to the axes. The number of spherical triangles of Sn, C6, C8, and C10 are 2n, 24, 48, and 120, respectively. These numbers correspond to the possible maximum numbers of tones.

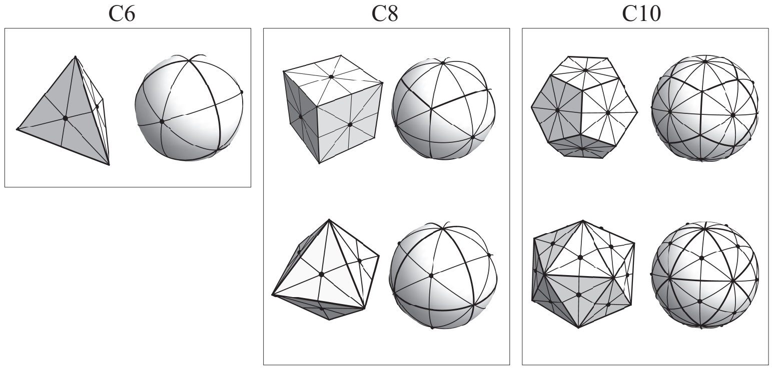

Figure 6. Correspondences between Platonic and spherical polyhedra.

To visualize the surface of *SphereHarmonic*, we use stereographic projections [30]. A schematic illustration of the stereographic projection of a spherical surface is shown in Figure 7. We set the center of the sphere as the origin of the Cartesian coordinates. We use the vector from the origin O to the north pole P as the z-axis, and the plane perpendicular to the z-axis containing the origin as the xy-plane. A stereographic projection is an intersection between the xy-plane and the line connecting the spherical point and the north pole. As illustrated in Figure 7, the spherical points Q, R, and S are projected to Q, R′, and S′, respectively. The points on the equator are projected to themselves and other points are projected to the inside of the equator. In the case of a unit sphere, the spherical point (x, y, z) (light gray surface of the lower hemisphere in Figure 7) is projected to $(x/(1-z), y/(1-z))$ (dark gray disk in Figure 7). Note that only the lower hemisphere ($z < 0$) is projected. Projections of the upper hemisphere are possible by setting the south pole as P. We also use the projections of the upper hemisphere in order to illustrate the whole surface. We presuppose that all the spheres in this article are unit spheres and their centers are at the origin of the Cartesian coordinates from hereafter.

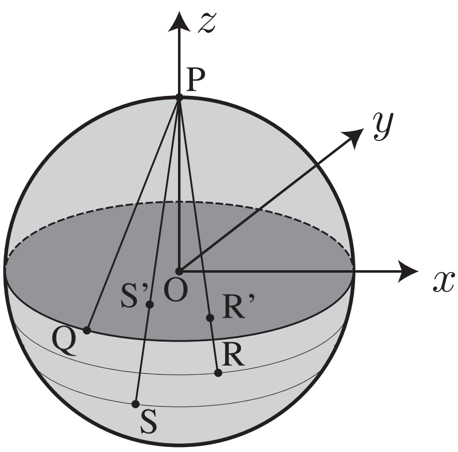

Figure 7. Schematic illustration of a stereographic projection of a spherical surface.

Figure 8 shows the stereographic projections of the basic Temari Ball divisions with axes of rotations. We illustrate two images if the roles of axes are different (Sn and C8). The great circles are characterized by the unit vectors perpendicular to the circles; we refer to the unit vectors as axes. For example, the outermost circles of the projections in Figure 8 represent great circles dividing the spheres into two hemispheres: the upper for $z > 0$ and the lower for $z \leq 0$, respectively. The coordinates of the axes are taken in order for the z-coordinate of all axes to be greater than or equal to 0, and its length is unity. Besides the description using upper and lower hemispheres, we also introduce another description of the hemispheres based on the axes. We refer to the volume close to an axis (the inner product of the axis vector and the position vector is positive) as the northern hemisphere of the axis and the rest as the southern hemisphere.

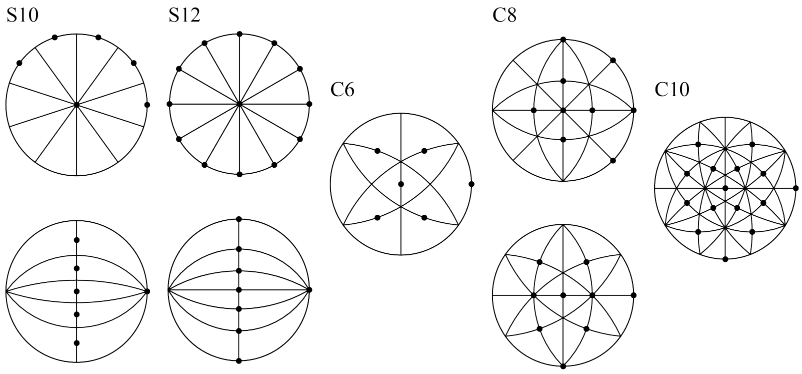

Figure 8. Stereographic projections of Temari basic divisions.

The black small circles in Figure 8 represent the location of axes of rotations. Some axes are not in accordance with the vertices for Sn (for $n = 4k \pm 2$) and C6. They are caused by their own symmetrical properties. The latter is caused by the asymmetry of the section of a regular tetrahedron by the plane containing an edge. The axes are characterized by the minimum amount of rotation because the divisions of the surface after the rotations of a hemisphere must be the same as before. The axes of combination divisions C6 and C10 are characterized as π. On the other hand, there are two types of axes for the divisions of Sn and C8; $2\pi/n$ and π for Sn, and $\pi/2$ and π for C8. Figure 8 illustrates the projection patterns obtained by setting all types of axes to (0, 0, 1).

Figure 9 illustrates the sequential rotations of the hemispheres of C8 from the initial state (Figure 9A) to the final state (Figure 9D) via the intermediate states (Figure 9B,C) as an example. The left and right projections for each state represent the lower and upper hemispheres, respectively. The congruent triangles and the axes of rotations are numbered for convenience. Two spherical triangles numbered 36 and 39 are filled with light and dark greys to clarify the rotations. The rotation of the northern hemisphere with axis 1 is illustrated as in Figure 9A,B. Similarly, the rotation of the southern hemisphere with axis 1 and that of the northern hemisphere with axis 9 is shown in Figure 9B,C and Figure 9C,D, respectively. Thick curves denote the great circles determined by the axes of rotations—for example, the curves containing axis 6 in Figures 9C,D represent the great circle perpendicular to axis 9. Note that the amount of rotation of axes 1 and 9 is different, $\pi/2$ for axis 1 and π for axis 9, because of the difference in the axis type.

For combination divisions, the triangles have parity. This means that black triangles cannot replace the white ones and vice versa in Figure 5. This is because the black and white triangles have different orders of angles, although they are congruent. These characteristics restrict the combination of triangles obtained from the rotations of the hemispheres. The triangles in Figure 9 were numbered accounting for the parity, and the locations of even/odd numbers were not changed after the rotations. Besides, the triangles of simple divisions do not have this property because all the triangles are isosceles. In the following calculations, we assume that all pairs of triangles having the same parity can be transposed by finite numbers of rotational manipulations. We have not proven this assumption; however, it seems to be acceptable intuitively.

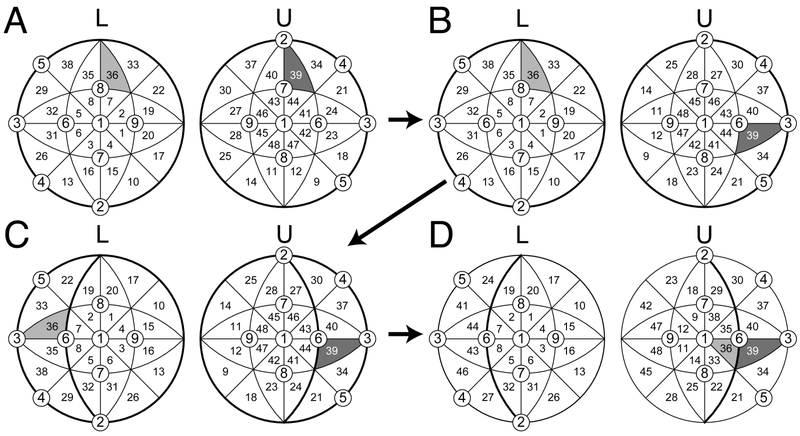

Figure 9. Representation of sequential rotations of the hemispheres of C8. L stands for "lower", and U for "upper".

3. Results

3.1. Computing Combinatorial Patterns of Temari Balls

The number of combinations or the variation of chords depends on the type of division. Tables 1 and 2 shows the basic properties of Temari-ball basic divisions. Table 1 shows the properties of faces and rotation axes, and Table 2 displays vertices' properties, respectively.

Both the total number of vertices and the number of each type of vertices are shown to exemplify the correspondence between the number of vertices and the degrees of vertices. Degrees of vertices correspond to the number of tones of a chord. The basic divisions have two or three types of vertices: 4 and n for Sn, 4 and 6 for C6, 4, 6 and 8 for C6, and 4, 6, and 10 for C10.

Table 1. Rotational properties of *SphereHarmonic* (triangles and rotations).

Divisions	Number of Triangles	Number of Axes		Amount of Rotations
Sn	$2n$	$1 + n/2$	1 $n/2$	$2\pi/n$ π
C6	24	6		π
C8	48	9	6 3	π $\pi/2$
C10	120	15		π

The numbers of congruent triangles are $2n$ for Sn, 24 for C6, 48 for C8, and 120 for C10, respectively.

Table 1 shows the rotational properties of *SphereHarmonic*. It should be noted again that the axes of rotations are not always the same as the vertices, as shown in Figure 8.

Numbers of vertex combinations were obtained, assuming that all triangles had different numbers/tones. In the case of the simple divisions, the numbers were straightforwardly obtained by considering the n-selection among $2n$ because of the lack of parity; they were $\binom{2n}{n}$ for n-degree vertices and $\binom{2n}{4}$ for degree 4, respectively. On the other hand, the numbers for the combination divisions were calculated considering the parity as the products of the numbers of combinations for white and black triangles. The results were $\binom{12}{2}^2 = 4356$ for 4-degree vertices and $\binom{12}{3}^2 = 48{,}400$ for degree 6 ones in the case of C6. Similar calculations were carried out in the cases of C8 and C10. The results are also summarized in Table 2.

Table 2. Basic properties of *SphereHarmonic* (vertices).

Divisions	Number of Vertices	Degree of Vertices	Number of Combinations
Sn	$n+2$	2 n	n 4	$\binom{2n}{n}$ $\binom{2n}{4}$
C6	10	6 4	4 6	4356 48,400
C8	26	12 8 6	4 6 8	76,176 4,096,576 112,911,876
C10	62	30 20 12	4 6 10	3,132,900 1,171,008,400 29,828,113,326,144

It was complicated to obtain the variations of configurations for the whole surface. For the simple divisions Sn, the estimated number was obtained by [the number of the selection of n tones among $2n$] times [the number of the circular permutation of the upper hemisphere] times [the number of the circular permutation of the lower hemisphere] times [number of rotations of upper hemisphere]= $\binom{2n}{n}(n-1)!(n-1)!n = (2n)!/n$. On the other hand, we could not succeed in estimating the numbers for combination divisions. We notice that, in the case with $(2n)!/n$ as the numbers of rotations of the upper hemisphere, the instrument could be complex. For our application, we focused on real results. However, a more complete approach should consider some musical renditions suggesting the passage toward complexity. In general, we have a complexity result, e.g., PSPACE-hard, that would require a polynomial time to be solved by a Turing machine.

Another simple problem is listing the configurations that cannot change the combinations of tones. The simplest but trivial solution is to allocate the same tones to all triangles. Another trivial solution consists in allocating one tone to black triangles and another to white ones. However, this solution does not match the simple divisions because of the lack of parity nature.

3.2. Prototype of C8

To devise a prototype of SphereHarmonic, we can choose the C8 because it contains two types of rotation axes. With this feature, the prototype allows us to examine rotations of different angles and several configuration variations. Furthermore, the number of triangles is enough for the variation of chords.

Figure 10 shows the image of the prototype. The prototype was developed using Wolfram Mathematica® and it can be found at https://github.com/medusamedusa/SphereHarmonic (accessed on 10 August 2022). The upper and lower circles represent the projections of the northern and southern hemispheres, respectively. The left circles are used for the selection of a chord and the right ones for the rotations. We allocated different notes (colors) to the triangles. As for the results, four octaves of tones were used because the number of triangles of C8 is forty-eight.

To select angles and vertices, we used the EventHandler function. We allocate different rotational actions to the circled numbers and perform the different actions to play corresponding chords through the vertices. The DynamicModule function enabled us to change the note distribution automatically when a circled number was clicked.

The rotations of the same numbers between the upper and lower hemispheres are different. We implemented the movements to rotate in the same direction as the hemisphere with the circled number. This implementation allows us to perform both clockwise and anti-clockwise rotations.

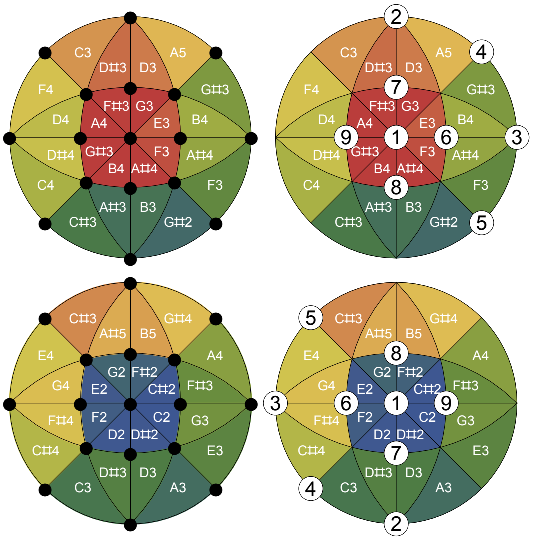

Figure 10. Image of the prototype of *SphereHarmonic* C8.

4. Discussion and Conclusions

In this article, we analyzed sphere tiling in Japanese Temari balls. In addition, being inspired by a Rubik's cube-based electronic musical instrument, the CubeHarmonic [20], we applied both permutations and chords to the geometries in Temari balls, conceptualizing a new musical instrument, the *SphereHarmonic*.

In particular, we devised a prototype of *SphereHarmonic* with C8-type symmetry. Although we are satisfied with the prototype of C8, those of other types seem necessary in order to compare the user experiences. Nevertheless, we estimate that the prototype of C10 could be repetitive because the amount of rotations of C10 is only π; the rotation is the change with opposite triangles.

Considering the suitable distributions of notes is an attractive problem for both mathematicians and musicians.

It is possible to design another type of *SphereHarmonic* as follows. Each triangle is a musical note rather than a triad. A group of three triangles played together gives a musical chord. Slipping a triangle from one place to another, the three adjacent triangles give a different triad. This structure is thought of in analogy with the correspondence one note-one facet adopted for the CubeHarmonic [20] and the HyperCubeHarmonic [31]. We can also consider a spherical surface tessellated with a mixture of triangles and squares, as in the Rubik's V-sphere. However, for a completely laptop-based application, no physical constraints apply.

Lifting the idea to one higher dimension, we can imagine the *HyperSphereHarmonic*, where one can hear the "sound of multiple dimensions", as done for the HyperCubeHarmonic. A computer simulation can allow the switch and the rotation between different spheres, adding more degrees of freedom to musical performance, as it happens with the HyperCubeHarmonic. As an additional feature, it is not necessary to solve the Rubik's cube to play the CubeHarmonic, thanks to the 'reset' button, which automatically restores the unscrambled configuration of pitches on the cube, even if the physical cube is still mixed. A similar function can be implemented for the SphereHarmonic.

As future developments of this research thread, we may also envisage studies mapping visual patterns obtained in the Rubik's cube [32] into musical patterns obtained with the CubeHarmonic. A similar approach can be exported to the SphereHarmonic.

Our study can shed light on future and further connections between combinatorics, music, and folkcrafts. Such an interdisciplinary endeavor may lead to new pedagogical applications and creative developments. And, once more, mathematics is paving the way to unsuspected connections between cultures and knowledge.

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Data Availability Statement: The notebook file of Mathematica® is available from https://github.com/medusamedusa/SphereHarmonic (accessed on 10 August 2022) .

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Peterson, M. Dante and the 3-Sphere. *Am. J. Phys.* **1979**, *47*, *1031*. Available online: https://mathinees-lacaniennes.net/images/stories/articles/dante.pdf (accessed on 8 August 2022). [CrossRef]
2. Fathauer, R. *Tessellations: Mathematics, Art, and Recreation*; Routledge: London, UK, 2021.
3. Ernst, B. *The Magic Mirror of M.C. Escher*; Random House Chapman & Hall Pure and Applied Mathematics: New York, NY, USA, 1976.
4. Lipschütz, H.; Skrodzki, M.; Reitebuch, U.; Polthier, K. Linked Knots from the *gyro* Operation on the Dodecahedron. In Proceedings of the Bridges 2022: Mathematics, Art, Music, Architecture, Culture, Helsinki and Espoo, Finland, 1–5 August 2022; pp. 175–182. Available online: https://archive.bridgesmathart.org/2022/bridges2022-175.pdf (accessed on 8 August 2022).
5. Yackel, C. Teaching Temari: Geometrically Embroidered Spheres in the Classroom. In Proceedings of the Bridges Conference 2011, Coimbra, Portugal, 27–31 July 2011; pp. 563–566. Available online: https://archive.bridgesmathart.org/2012/bridges2012-563.pdf (accessed on 8 August 2022).
6. Belcastro, S.M.; Yackel, C. *Crafting by Concepts: Fiber Arts and Mathematics*; AK Peters/CRC Recreational Mathematics Series: Boca Raton, FL, USA, 2011.
7. Suess, B.B. *Temari Techniques: A Visual Guide to Making Japanese Embroidered Thread Balls*; Breckling Press: Elmhurst, Australia, 2012.
8. Leibniz, G.W. De Arte Combinatoria. 1666. Available online: https://www.math.ucla.edu/~pak/hidden/papers/Quotes/Leibniz-Arte-Combinatoria.pdf (accessed on 8 August 2022).
9. Zweig, J. Ars Combinatoria. *Art J.* **2014**, *56*, 20–29. Available online: https://www.janetzweig.com/zweig.ars-combinatoria.pdf (accessed on 8 August 2022). [CrossRef]
10. *"Ars combinatoria."* Treccani, Online Encyclopedia. Available online: https://www.treccani.it/enciclopedia/ars-combinatoria/ (accessed on 8 August 2022).
11. "Lullo, Raimondo (Radmond Lull)." Treccani, Online Encyclopedia. Available online: https://www.treccani.it/enciclopedia/raimondo-lullo/ (accessed on 8 August 2022).
12. Brlek, S.; Chemillier, M.; Reutenauer, C. Music and combinatorics on words: A historical survey. *J. Math. Music* **2018**, *12*, 125–133. [CrossRef]
13. Allouche, J.-P.; Johnson, T. Combinatorics of words and morphisms in some pieces of Tom Johnson. *J. Math. Music* **2018**, *12*, 248–257. [CrossRef]
14. Hook, J. Why are there twenty-nine tetrachords? Music Theory Online. 2007. Available online: https://www.mtosmt.org/issues/mto.07.13.4/mto.07.13.4.hook.html (accessed on 8 August 2022).
15. Fripertinger, H. Enumeration in Musical Theory. Beiträge zur Elektronischen Musik 1. 1999. Available online: https://www.emis.de/journals/SLC/opapers/s26fripert.pdf (accessed on 8 August 2022).
16. Fenton, W.E. Teaching permutations through rhythm patterns. *J. Math. Arts* **2008**, *3*, 143–146. [CrossRef]
17. Amiot, E.; Baroin, G. Old and new isometries between pc sets in the planet-4D model. Music Theory Online. 2015. Available online: https://mtosmt.org/issues/mto.15.21.3/mto.15.21.3.amiot-baroin.html (accessed on 8 August 2022).
18. Baroin, G. The planet-4D model: An original hypersymmetric music space based on graph theory. In *Mathematics and Computation in Music, Proceedings of the Third International Conference, MCM 2011, Paris, France, 15–17 June 2011*; Agon, C., Andreatta, M., Assayag, G., Amiot, E., Bresson, J., Mandereau, J., Eds.; Springer: Berlin/Heidelberg, Germany, 2011; pp. 326–329. [CrossRef]
19. Peck, R. Combinatorial Spaces. In *Mathematics and Computation in Music, Proceedings of the 8th International Conference, MCM 2022, Atlanta, GA, USA, 21–24 June 2022*; Montiel, M., Agustín-Aquino, O.A., Gómez, F., Kastine, J., Lluis-Puebla, E., Milam, B., Eds.; Lecture Notes in Computer Science, 13267; Springer: Cham, Germany, 2022; pp. 48–60. [CrossRef]
20. Mannone, M.; Kitamura, E.; Huang, J.; Sugawara, R.; Chiu, P.; Kitamura, Y. CubeHarmonic: A New Musical Instrument Based on Rubik's Cube with Embedded Motion Sensor. In Proceedings of the ACM SIGGRAPH '19, Los Angeles, CA, USA, 28 July–1 August 2019; pp. 53:1–53:2. [CrossRef]
21. Yust, J. Generalized Tonnetze and Zeitnetze, and the topology of music concepts. *J. Math. Music* **2020**, *14*, 170–203. [CrossRef]

22. Jedrzejewski, F. Non-Contextual JQZ Transformations. In *Mathematics and Computation in Music, Proceedings of the 7th International Conference, MCM 2019, Madrid, Spain, 18–21 June 2019*; Editors, M., Montiel, F., Gomez-Martin, O., Agustin-Aquino, O.A., Eds.; Springer: Cham, Germany, 2019; pp. 149–160.
23. Cannas, S.; Polo, M. Euler's "Tentamen": Historical and Mathematical Aspects on the Consonance Theory. In *Mathematics and Computation in Music, Proceedings of the 8th International Conference, MCM 2022, Atlanta, GA, USA, 21–24 June 2022*; Montiel, M., Agustín-Aquino, O.A., Gómez, F., Kastine, J., Lluis-Puebla, E., Milam, B., Eds.; Lecture Notes in Computer Science; Springer: Cham, Germany, 2022; Volume 13267.
24. Baroin, G.; de Gérando, S. When Virtual Reality Helps Fathom Mathemusical Hyperdimensional Models. In *Mathematics and Computation in Music, Proceedings of the 8th International Conference, MCM 2022, Atlanta, GA, USA, 21–24 June 2022*; Montiel, M., Agustín-Aquino, O.A., Gómez, F., Kastine, J., Lluis-Puebla, E., Milam, B., Eds.; Springer: Cham, Germany, 2022; pp. 86–98.
25. Crans, A.; Fiore, T.; Satyendra, R. Musical Actions of Dihedral Groups. *Am. Math. Mon.* **2009**, *116*, 479–495. [CrossRef]
26. Alegant, B.; Mead, A. Having the last word: Schoenberg and the ultimate cadenza. *Music. Theory Spectr.* **2012**, *34*, 107–136. [CrossRef]
27. Mannone, M.; Kitamura, E.; Huang, J.; Sugawara, R.; Kitamura, Y. Musical Combinatorics, Tonnetz, and the CubeHarmonic. *Zb. Rad. Akad. Umet.* **2018**, *5*, 104–116. Available online: https://scindeks-clanci.ceon.rs/data/pdf/2334-8666/2018/2334-86661806104M.pdf (accessed on 8 August 2022). [CrossRef]
28. Diamond, A. *The Temari Book: Techniques and Patterns for Making Japanese Thread Balls*; Lars Books: Asheville, NC, USA, 1999.
29. Coxeter, H.S.M. Regular Compound Tessellations of the Hyperbolic Plane. *Proc. R. Soc. Lond. Math. Phys. Eng. Sci.* **1964**, *1373*, 147–167.
30. Feeman, T.G. *Portraits of the Earth: A Mathematician Looks at Maps*; American Mathematical Society: Providence, RI, USA, 2002.
31. Mannone, M.; Yoshino, T.; Chiu, P.; Kitamura, Y. Hypercube + Rubik's Cube + Music = HyperCubeHarmonic. In *Mathematics and Computation in Music, Proceedings of the 8th International Conference, MCM 2022, Atlanta, GA, USA, 21–24 June 2022*; Montiel, M., Agustín-Aquino, O.A., Gómez, F., Kastine, J., Lluis-Puebla, E., Milam, B., Eds.; Lecture Notes in Computer Science; Springer: Cham, Germany, 2022; Volume 13267, pp. 240–252. Available online: https://link.springer.com/chapter/10.1007/978-3-031-07015-0_20 (accessed on 8 August 2022).
32. Plaxco, D. Photogenic Knot Projections on n×n×n Rubik's Cubes. In Proceedings of the Bridges 2022: Mathematics, Art, Music, Architecture, Culture, Helsinki and Espoo, Finland, 1–5 August 2022; pp. 331–334. Available online: https://archive.bridgesmathart.org/2022/bridges2022-331.pdf (accessed on 8 August 2022).