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Multigraded Syzygies of Monomial Ideals

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"La vita è teatro, ma non sono ammesse le prove"

Anton Čechov, Il giardino dei ciliegi

UNIVERSITY OF PALERMO

Abstract

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Multigraded Syzygies of Monomial Ideals

by Antonino Ficarra

In this Thesis, we broadly consider homogeneous ideals, with a special focus on monomial ideals and ideals with (componentwise) linear powers. Three main topics are treated. Firstly, the concept of t-spread monomial ideal introduced by Ene, Herzog and Qureshi is generalized to vector-spread monomial ideal. The algebraic and homological property of this class are investigated. In particular, the Koszul cycles are determined and the algebraic shifting theory is extended for this class. Secondly, we consider the concept of homological shift ideals $\text{HS}_i(I)$ of a monomial ideal I. Roughly speaking, $\text{HS}_i(I)$ is the monomial ideal whose generators correspond to the multigraded pieces of the ith syzygy module of I. In particular $\text{HS}_0(I) = I$. The natural broad question we ask is: what are the properties of I shared by all $\text{HS}_i(I)$? A driving motivation for studying homological shift ideals comes from the Bandari–Bayati–Herzog conjecture, that predicts that for a polymatroidal ideal I, $\text{HS}_i(I)$ is polymatroidal, for all i. We show that the conjecture is true for $\text{HS}_1(I)$ and for all polymatroidal ideals generated in degree 2. Homological shift ideals of edge ideals and cover ideals of finite simple graphs are also considered. We show that for the fundamental class of Cohen–Macaulay very well–covered graphs G, $\text{HS}_i(J(G))$ has linear quotients for all i, and we conjecture that the same is true for all powers of $J(G)$. This conjecture is settled for bipartite graphs by using the theory of Hibi ideals, and for whisker graphs by using Rees algebras methods. Finally, our third topic is the study of the asymptotic behaviour of the v-number of homogeneous ideals. This important invariant was introduced by Cooper et all and further studied by Grisalde, Reyes and Villarreal, in connection with the theory of projective Reed–Muller–type codes and Algebraic Geometry. The v-number of I is the smallest degree d of a homogeneous polynomial f such that $(I : f)$ is an associated prime of I. Inspired by Brodmann theorem on the asymptotic stability of the set associated primes of the powers I^k of an ideal in a Noetherian ring, and also by the Cutkosky–Herzog–Trung and Kodiyalam results on the asymptotic linearity of the Castelnuovo–Mumford regularity $\text{reg}(I^k)$ of powers of a homogeneous ideal I in a polynomial ring, we ask if the function $\text{v}(I^k)$ is linear for $k \gg 0$. We call the function $\text{v}(I^k)$, the v-function of I. It measures the asymptotic homogeneous growth of the primary decomposition of I. Under reasonable assumptions, we show that $\text{v}(I^k)$ is indeed linear, and we conjecture that the same is true in general. The proof of these results uses Rees algebra methods and linear programming arguments. For monomial ideals in two variables, edge ideals of cochordal graphs, polymatroidal ideals and Hibi ideals, we compute explicitly the v-function. Finally we conjecture that for an ideal I with linear powers, we have $\text{v}(I^k) = \alpha(I)k-1$ for all $k \geq 1$, where $\alpha(I)$ is the initial degree of I. This conjecture is settled for edge ideals with linear powers, polymatroidal ideals and Hibi ideals.

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Contents

List of Symbols

$\lvert A\rvert$	cardinality of A
$\mathbf{a}$	integral vector $(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$
$[a, b]$	the interval of integers $\{a, a+1, \ldots, b\}$
A^d	cartesian product $A \times A \times \cdots \times A$ (d times)
$\alpha(F; i)$	sign rule $\lvert\{j \in F : j < i\}\rvert$
$a(G)$	maximum size of a set of pairwise 3-disjoint edges of G
$\binom{A}{i}$	subsets of size i of A
$\alpha(M)$	initial degree od M
$\mathrm{Ass}(I)$	the set of associated primes of R/I
$\mathrm{Ass}^\infty(I)$	the set of stable associated primes of R/I
$\mathrm{astab}(I)$	the least integer k_0 with $\mathrm{Ass}(I^k) = \mathrm{Ass}^\infty(I)$ for all $k \geq k_0$
$\mathrm{bideg}(f)$	the bidegree of f
$\beta_i(I)$	total Betti number of I
$\beta_{i,j}(I)$	graded Betti number of I
$\beta_{i,\mathbf{a}}(I)$	multigraded Betti number of I
$B(u)$	principal Borel ideal of u
$B_{\mathbf{t}}(u)$	$\mathbf{t}$-spread principal Borel ideal of u
$B(u_1, \ldots, u_m)$	the smallest strongly stable ideal containing $u_1, \ldots, u_m$
$\mathbb{C}$	the set of complex numbers
$\mathcal{C}(G)$	minimal vertex covers of the graph G
$\mathrm{char}(K)$	characteristic of the field K
$\mathrm{CM\text{-}type}(M)$	the Cohen–Macaulay type of M
coker	cokernel
$C(\psi)$	mapping cone of ψ
$C_{\mathbb{X}}(d)$	Reed–Muller–type code of $\mathbb{X}$
$\deg(f)$	degree of a homogeneous element f
$\mathbf{deg}(I)$	bounding multidegree of I
$\deg_{x_i}(I)$	the ith component of $\mathbf{deg}(I)$
$\deg_{x_i}(u)$	x_i-degree of a monomial u
Δ	simplicial complex
$\Delta^\vee$	Alexander dual of Δ
Δ_W	restriction of Δ on W
$\mathrm{depth}(I)$	depth of I
δ_I	minimum distance function of I
$\dim(\Delta)$	dimension of Δ
$\dim_K(\square)$	dimension of a K-vector space
$\dim(R)$	Krull dimension of R
$\bigoplus$	direct sum
$\mathrm{dstab}(I)$	the depth stability of I
$d(u, v)$	the distance between the monomials u and v
ε	the canonical map $\varepsilon : S \to S/I$
$\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_n$	standard basis vectors of $\mathbb{Q}^n$

e_F	the wedge product $\bigwedge_{i\in F} e_i$
$\mathbf{e}_F$	basis element
$E(G)$	edge set of the graph G
$e(M)$	multiplicity of a module M
$\varnothing$	the empty set
Ext	Ext functor
$\overline{f}$	residue class of f modulo an ideal I
$\mathbb{F}$	free resolution
$\mathbb{F}_{<i}$	restriction of the complex $\mathbb{F}$
$f(\Delta)$	f-vector of Δ
$f_i(\Delta)$	ith component of the f-vector of Δ
$\mathcal{F}(\Delta)$	the set of facets of Δ
$\mathcal{F}(I)$	the fiber cone of I, $\bigoplus_k(I^k/\mathfrak{m}I^k)$
$\mathcal{F}_{\mathfrak{p}}(I)$	the ring $\bigoplus_k(I^k/\mathfrak{p}I^k)$
$f(u;\sigma)$	admissible symbol
$\mathbb{G}$	free resolution
G^*	whisker graph of G
G^c	complementary graph of G
gcd	greatest common divisor
G_I	intersection graph of a transversal polymatroidal ideal I
$G(I)$	minimal monomial generating set of I
$G(I)_j$	jth graded piece of $G(I)$
$\mathrm{Gin}(I)$	generic initial ideal of I
$\mathrm{gr}_I(R)$	associated graded ring of I
$\mathrm{height}(I)$	height of the ideal I
$\mathrm{Hilb}_M(t)$	Hilbert function of M
$\widetilde{H}_i(\Delta;K)$	ith reduced simplicial homology module of Δ
$\widetilde{H}^i(\Delta;K)$	ith reduced simplicial cohomology module of Δ
$H_i(\mathbf{x};S/I)$	ith Koszul homology module of $\mathbf{x}$ with respect to S/I
H_P	Hibi ideal of the poset P
$\mathrm{HS}_i(I)$	the ith homological shift ideal of I
K	a field
K_n	complete graph on n vertices
$K_{n,m}$	complete bipartite graph
$K[\Delta]$	Stanley–Reisner ring of Δ
Ker	kernel
$K[I]$	the toric ring of the monomial ideal I
$K_\bullet(\mathbf{x};S/I)$	Koszul complex of $\mathbf{x}$ with respect to S/I
$K_i(\mathbf{x};S/I)$	ith module of the Koszul complex of $\mathbf{x}$ with respect to S/I
$J(G)$	cover ideal of the graph G
$\mathcal{J}(P)$	distributive lattice of the poset ideals of P
$\sqrt{I}$	the radical of I
$I_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$	LP–polymatroidal ideal $\mathfrak{p}_{[\alpha_1,\beta_1]}\cdot\mathfrak{p}_{[\alpha_2,\beta_2]}\cdots\mathfrak{p}_{[\alpha_t,\beta_t]}$
$I_{\mathbf{a},\mathbf{b}}$	the ideal $I=(x^{a_1}y^{b_1},x^{a_2}y^{b_2},\ldots,x^{a_m}y^{b_m})$
$I_{(\mathbf{a},\mathbf{b}\mid\boldsymbol{\alpha},\boldsymbol{\beta})}$	PLP–polymatroidal ideal of type $(\mathbf{a},\mathbf{b}\mid\boldsymbol{\alpha},\boldsymbol{\beta})$
$I_{\mathbf{b},n,d}$	ideal of Veronese type $(\mathbf{b},d)$
id	identity function $A\to A$
I_Δ	Stanley–Reisner ideal of Δ
$I(G)$	edge ideal of the graph G
$I^{(k)}$	kth symbolic power of I

$I_{>j}$	the ideal with $G(I_{>j}) = \{u \in G(I) : \lvert\mathrm{supp}(u)\rvert > j\}$
$I_{\leq j}$	the ideal $I \cap S_{\leq j}$
Im	image
$I_{n,d,t}$	t-spread Veronese ideal of degree d
$I_{n,d,\mathbf{t}}$	$\mathbf{t}$-spread Veronese ideal of degree d
$I^{s,\mathbf{t}}$	the ideal $(\mathrm{Gin}(I))^{\sigma_{\mathbf{0},\mathbf{t}}}$
$I^{\sigma_{\mathbf{t},\mathbf{s}}}$	the ideal with $G(I^{\sigma_{\mathbf{t},\mathbf{s}}}) = \{\sigma_{\mathbf{t},\mathbf{s}}(u) : u \in G(I)\}$
I_P	localization of I at P
$I^{\wp}$	polarization of I
$I^{\vee}$	Alexander dual of I
$I(\mathbb{X})$	vanishing ideal of $\mathbb{X}$
$\ell(I)$	analytic spread of I
lcm	least common multiple
$\mathrm{length}(M)$	length of a module M
$>_{\mathrm{lex}}$	lexicographic order
$\lim\limits_{k\to\infty} \mathrm{depth}(R/I^k)$	limit depth of R/I^k
$\succeq$	partial order, or monomial order
$\mathfrak{m}$	the graded maximal ideal $(x_1, \ldots, x_n)$ of $K[x_1, \ldots, x_n]$
$\mathcal{M}$	the category of finitely generated graded S-modules
$M_{\geq \ell}$	the module $\bigoplus_{d\geq \ell} M_d$
$M_{(*,k)}$	the module $\bigoplus_d M_{d,k}$
$\mathrm{Max}(I)$	the set of maximal associated primes of R/I
$\mathrm{Max}^{\infty}(I)$	the set of stable maximal associated primes of R/I
$\max(u)$	maximum of a monomial u
$\min(u)$	minimum of a monomial u
$M(I)$	the set of monomials belonging to I
$\mu(I)$	the minimal number of generators of I
$M_{n,d,t}$	the set of t-spread monomials of degree d
$M_{n,d,\mathbf{t}}$	the set of $\mathbf{t}$-spread monomials of degree d
$\mathrm{Mon}(S)$	the set of all monomials of S
$\mathrm{Mon}(S; \mathbf{t})$	the set of all $\mathbf{t}$-spread monomials of S
$\mathrm{Mon}_d(S)$	the set of all monomials of S of degree d
$\mathrm{Mon}_d(S; \mathbf{t})$	the set of all $\mathbf{t}$-spread monomials of S of degree d
$\mathfrak{m}_P$	the maximal ideal of the local ring R_P
$[n]$	the set of integers $\{1, 2, \ldots, n\}$
$\mathbb{N}$	the set of non-negative integers
$N_G(i)$	open neighbourhood of $i \in V(G)$
$N_G[i]$	closed neighbourhood of $i \in V(G)$
$\mathbf{1}$	the vector $(1, 1, \ldots, 1)$
$\mathfrak{p}$	a prime ideal
$\mathfrak{p}_x$	the ideal (x)
$\mathfrak{p}_y$	the ideal (y)
$\mathfrak{p}_A$	monomial prime ideal $(x_i : i \in A)$
$\mathfrak{p}_{[p,q]}$	monomial prime ideal $(x_i : p \leq i \leq q)$
$(P, \succeq)$	partially ordered set
$\mathrm{pd}(I)$	projective dimension of I
$P_M(t)$	the graded Poincaré of M
$P_M(t, s)$	the bigraded Poincaré of M
$\mathbb{P}^{s-1}$	projective space
$\mathfrak{q}$	a prime ideal

$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathrm{reg}(I)$	Castelnuovo–Mumford regularity of I
$>_{\mathrm{revlex}}$	reverse lexicographic order
$\mathcal{R}(I)$, $R[It]$	Rees algebra of I
R_P	the localization of R at P
$S = K[x_1, \ldots, x_n]$	polynomial ring with coefficients in K
$S_{\leq j}$	the polynomial ring $K[x_1, \ldots, x_j]$
$\mathrm{set}(u_i)$	the set of integers i such that $x_i \in (u_1, \ldots, u_{i-1}) : (u_i)$
$\sigma(F; i)$	sign rule $\lvert\{j \in F : j < i\}\rvert$
$\sigma_{\mathbf{t},\mathbf{s}}$	shifting operator
$\mathrm{soc}(I)$	the socle of an ideal I with linear resolution
$\mathrm{Soc}(I)$	the socle module of I, $\bigoplus_k (I^k : \mathfrak{m})/I^k$
$\mathrm{Soc}_{\mathfrak{p}}(I)$	the module $\bigoplus_k (I^k : \mathfrak{p})/I^k$
$\mathrm{Spec}(R)$	the spectrum of R
$\mathrm{supp}(f)$	support of a polynomial f
$\mathrm{supp}(I)$	support of a monomial ideal I
$\mathrm{supp}(u)$	support of a monomial u
$\mathrm{supp}_{\mathbf{t}}(u)$	$\mathbf{t}$-spread support of a monomial u
$\otimes$	tensor product
Tor	Tor functor
$\mathbb{T}(u_1, \ldots, u_m)$	Taylor complex of the sequence $\{u_1, \ldots, u_m\}$
u'	the monomial $u/x_{\max(u)}$
$u_{\mathcal{I}}$	the monomial $(\prod_{p_i \in \mathcal{I}} x_i)(\prod_{p_i \in P \setminus \mathcal{I}} y_i)$
$u^{\wp}$	polarization of the monomial u
$u : v$	the monomial $\mathrm{lcm}(u, v)/v = u/\gcd(u, v)$
$V(G)$	vertex set of the graph G
$V(I)$	the set of prime ideals containing I
$\mathrm{v}(I)$	the v-number of I
$\mathrm{v}(I^k)$	the v-function of I
$\mathrm{v}_{\mathfrak{p}}(I)$	the $\mathrm{v}_{\mathfrak{p}}$-number of I
$\mathrm{v}_{\mathfrak{p}}(I^k)$	the $\mathrm{v}_{\mathfrak{p}}$-function of I
$\bigwedge$	wedge product
$\omega(M)$	the number $\max\{d : (M/\mathfrak{m}M)_d \neq 0\}$
$\mathbb{X}$	a finite set of general points in $\mathbb{P}^{s-1}$
$\mathbf{x}$	regular sequence $x_1, x_2, \ldots, x_n$
$\mathbf{x}_j$	regular sequence $x_j, x_{j+1}, \ldots, x_n$
$\mathbf{x}^{\mathbf{a}}$	monomial $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$
$[\mathbf{x}^{\mathbf{a}}]$	the multigraded S-module $S(-\mathbf{a})$
$\mathbf{x}_{\varnothing}$	the monomial 1
$\mathbf{x}_F$	squarefree monomial $\prod_{i \in F} x_i$
$\mathbf{y}_F$	squarefree monomial $\prod_{i \in F} y_i$
$\mathbf{0}$	the vector $(0, 0, \ldots, 0)$
$[z]$	homology class of $z \in K_{\bullet}(\mathbf{x}; S/I)$
$\mathbb{Z}$	the set of integers
$\mathbb{Z}_{\geq 1}$	the set of positive integers

Chapter 1

Introduction

The general motif of this dissertation is the study of homological and algebraic properties of homogeneous ideals, with a particular focus on monomial ideals and ideals with (componentwise) linear powers. Monomial ideals gradually came into fashion after the pioneering work of Hochster [105] and after Stanley [144, 145] successfully solved the upper bound conjecture for spheres, by using some of the results of Hochster's student, Reisner. This was an important conjecture belonging to the realm of topology and combinatorics. So, back then, it came as a surprise that its solution involved Commutative Algebra and, in particular, the theory of Cohen–Macaulay rings.

Nowadays, many of the current trends in Commutative Algebra are deeply intertwined with Combinatorics and Monomial Ideals. In particular, in this dissertation, we mainly discuss three topics: vector–spread strongly stable ideals together with an extension of algebraic shifting theory, homological shift ideals with a particular attention to polymatroidal ideals, and edge and cover ideals, and finally the v-number of homogeneous ideals and the asymptotic behaviour of the v-number of powers of homogeneous ideals. Next, we describe the structure of the thesis.

In Chapter 2, we summarize some basic facts from Commutative Algebra. In particular, we define graded rings, minimal free resolutions, graded Betti numbers, Cohen–Macaulay, Gorenstein and complete intersection rings. In Section 2.4, we survey what is known in the literature about the minimal free resolutions of monomial ideals. We include the Taylor resolution, the Eliahou–Kervaire resolution, Koszul cycles, ideals with linear quotients and Betti splittings.

Combinatorial aspects of monomial ideals are discussed in Chapter 3, mainly in the framework of Stanley–Reisner rings and Alexander duality. The important technique of polarization (Section 3.1) allows us to concentrate our attention on squarefree monomial ideals. Hochster's formula is a powerful tool that relates the graded Betti numbers of a squarefree monomial ideal to simplicial (co)homology. Edge ideals and polymatroidal ideals are investigated in Sections 3.3 and 3.4.

Hereafter, unless otherwise stated, I is a graded ideal of the standard graded polynomial ring $S = K[x_1, \dots, x_n]$ with coefficients in a field K.

The materials from Chapter 4 are taken from [63]. We introduce vector–spread strongly stable monomial ideals. This class is a vast generalization of (strongly) stable ideals, whose various extensions are considered by a huge number of researchers. Our generalization is that of t-spread strongly stable ideal introduced by Ene, Herzog and Qureshi [63]. We compute the Koszul cycles of vector–spread strongly stable ideals and therefore we construct the minimal free resolution.

Theorem 4.2.8 (Ficarra, 2023 [63, Theorem 3.8]). *Let $I \subset S$ be a $\mathbf{t}$-spread strongly stable ideal. Then, for all $i \geq 1$, the K-vector space $H_i(\mathbf{x}; S/I)$ has as a basis the homology classes of the Koszul cycles*

$$e(u;\sigma) \quad \textit{such that} \quad u \in G(I), \quad \sigma \subseteq [\max(u) - 1] \setminus \operatorname{supp}_{\mathbf{t}}(u), \quad |\sigma| = i - 1.$$

As a consequence, we can compute the graded Betti numbers.

Corollary 4.4.2 *Let I be a* $\mathbf{t}$*-spread strongly stable ideal of S. Then,*

$$\beta_{i,i+j}(I)=\sum_{u\in G(I)_j}\binom{\max(u)-1-\sum_{\ell=1}^{j-1}t_\ell}{i},\quad \text{for all}\ \ i,j\geq 0.$$

In particular, the graded Betti numbers of a vector-spread strongly stable ideal $I\subset S$ do not depend upon the characteristic of the field K.

We extend the classical Algebraic Shifting theory developed by Kalai and others in this general frame (Section 4.4). This theory was recently used to generalize the Bigatti–Hulett theorem [89, Theorem 7.3.1] in the context of vector–spread ideals [37].

In Chapter 5, we consider homological shift ideals [101]. Let $I\subset S$ be a monomial ideal, then the ith *homological shift ideal* of I is the monomial ideal

$$\mathrm{HS}_i(I)\ =\ (\mathbf{x}^{\mathbf{a}}\ :\ \beta_{i,\mathbf{a}}(I)\neq 0).$$

Here, if $\mathbf{a}=(a_1,\dots,a_n)\in\mathbb{Z}_{\geq 0}^n$ we set $\mathbf{x}^{\mathbf{a}}=x_1^{a_1}\cdots x_n^{a_n}$, and $\beta_{i,\mathbf{a}}(I)$ denotes a *multigraded Betti number* of I. This concept that was introduced by Herzog, Moradi, Rahimbeigi and Zhu in [101], is motivated by a longstanding conjecture about polymatroidal ideals:

Conjecture 6.1.1 (Bandari–Bayati–Herzog [16, 101]). *Let $I\subset S$ be a polymatroidal ideal. Then all homological shift ideals $\mathrm{HS}_j(I)$ are again polymatroidal, for all $j\geq 0$.*

One of the main result we prove is

Theorem 6.1.2 (Ficarra, 2022 [60, Theorem 2.2]). *Let $I\subset S$ be a polymatroidal ideal. Then $\mathrm{HS}_1(I)$ is polymatroidal.*

The conjecture is also true in the following cases:

(i) for squarefree polymatroidal ideals, proved by Bayati [16] (Corollary 6.1.5),

(ii) for polymatroidal ideals that satisfy the strong exchange property, proved by Herzog, Moradi, Rahimbeigi and Zhu [101] (Corollary 6.1.6),

(iii) for all polymatroidal ideals generated in degree 2, proved by Ficarra and Herzog [66] (Theorem 7.3.3).

More generally, we have also the following result.

Theorem 5.1.4 (Ficarra–Herzog, 2023 [66, Theorem 1.3]). *Let $I\subset S$ be an equigenerated monomial ideal having linear quotients. Then $\mathrm{HS}_1(I)$ has linear quotients.*

This result is the best possible for ideals with linear quotients. Indeed, if I is equigenerated with linear quotients, then $\mathrm{HS}_2(I)$ may fail to have linear quotients, not even linear resolution (Example 7.1.1). Whereas, if I has linear quotients but it is generated in more than one degree, then $\mathrm{HS}_1(I)$ need not to have linear quotients, not even componentwise linear (Example 5.1.5). However, we expect that if I has a linear resolution, and not necessarily linear quotients, then $\mathrm{HS}_1(I)$ has linear quotients.

Let $I\subset S$ be a monomial ideal with d-linear resolution. Then

$$\mathrm{HS}_{n-1}(I)=x_1\cdots x_n\cdot\mathrm{soc}(I),$$

where $\mathrm{soc}(I)=(I:\mathfrak{m})_{\langle d-1\rangle}$ is the *socle* of I and $\mathfrak{m}=(x_1,\dots,x_n)$ (Corollary 5.3.2).

By symmetry, one would expect that $\mathrm{HS}_{n-1}(I)$ has liner resolution (or linear quotients). However, this needs not to be the case. We provide such an example at pages 139–140. On the other hand, if I is a polymatroidal ideal, Conjecture 6.1.1 in particular would imply that $\mathrm{HS}_{n-1}(I)$ and thus $\mathrm{soc}(I)$ is polymatroidal.

Conjecture 6.2.1 (Bandari–Herzog, 2013 [13], Chu–Herzog–Lu, 2021 [27]). *Let $I \subset S$ be a polymatroidal ideal. Then $\mathrm{soc}(I)$ is polymatroidal.*

In this case we have only partial results, but we know that this latter conjecture holds in the following cases: $n \le 3$, I is generated in at most degree 2, I is a squarefree polymatroidal ideal, I is a principal Borel ideal, I is a PLP–polymatroidal ideal, I is a LP–polymatroidal ideal.

In Chapters 7, 8 and 9 we study the homological shifts of edge ideals and cover ideals of finite simple graphs [66, 35, 36].

For an edge ideal $I(G)$ with linear resolution, it is not necessarily true that $\mathrm{HS}_i(I(G))$ has linear resolution as well, for all i (Example 7.1.1). On the other hand, we show that if the complementary graph G^c of G is a proper interval graph or a forest, then $\mathrm{HS}_i(I(G))$ has linear resolution, indeed even linear quotients, for all i (Theorems 7.1.2 and 7.2.1).

As a conclusion of Chapter 7, we prove the following.

Theorem 7.3.3 (Ficarra–Herzog, 2023 [66, Theorem 4.5]). *Let $I \subset S$ be a polymatroidal ideal generated in degree two. Then, $\mathrm{HS}_k(I)$ is polymatroidal, for all $k \ge 0$.*

Next, we consider cover ideals $J(G)$ of finite simple graphs G. By Alexander duality, $J(G)$ has a linear resolution if and only if $I(G)$ is Cohen–Macaulay. In particular, $I(G)$ is height–unmixed. By Gitler and Valencia [76, Corollary 3.4] we have

$$2\,\mathrm{height}(I(G)) \ge |V(G)|.$$

When equality holds, G is called a *very well–covered graph*. We say that the graph G is Cohen–Macaulay if $I(G)$ is a Cohen–Macaulay ideal. All Cohen–Macaulay very well–covered graphs have been characterized by Crupi, Rinaldo and Terai [40].

We consider the problem of determining all graphs G such that $\mathrm{HS}_i(J(G))$ has linear resolution, or even linear quotients, for all i. From the considerations above, the first case to study is when G is a Cohen–Macaulay very well–covered graph. In [35], we posed the following conjecture.

Conjecture 8.4.4 (Crupi–Ficarra, 2023 [35, Conjecture 4.4]). *Let G be a Cohen–Macaulay very well–covered graph with $2n$ vertices. Then $\mathrm{HS}_k((J(G))^\ell)$ has linear quotients with respect to the lexicographic order induced by $x_n > y_n > x_{n-1} > y_{n-1} > \cdots > x_1 > y_1$, for all $k \ge 0$, and all $\ell \ge 1$.*

This conjecture is widely open. Note that it would imply that $J(G)$ has linear powers. However in [35] and [36] we were able prove the following.

(i) The conjecture holds for $\ell = 1$ (Theorem 8.4.1).

(ii) The conjecture holds for all Cohen–Macaulay bipartite graphs (Corollary 8.4.11).

(iii) The conjecture holds for all whisker graphs (Theorem 9.3.8).

For whisker graphs, we have more results concerning the powers of $J(G)$. Indeed, we proved that these ideals satisfy the ℓ-exchange property (Theorem 9.3.5), and we computed the analytic spread, the limit depth and a bound for the depth stability (Theorem 9.3.9).

The last topic we study is the v-number of homogeneous ideals. Let $I \subset S$ be a homogeneous ideal. Then, the graded version of the primary decomposition theorem for Noetherian rings says that for each associated prime $\mathfrak{p} \in \operatorname{Ass}(I)$ there exists a homogeneous element $f \in S$ such that $(I : f) = \mathfrak{p}$. Thus, it is natural to define the following invariants. Denote by S_d the dth graded component of S. The v-*number of* I *at* $\mathfrak{p}$, denoted by $\mathrm{v}_{\mathfrak{p}}(I)$, is defined as the least integer d such that there exists $f \in S_d$ with $(I : f) = \mathfrak{p}$. Whereas, the v-*number of* I is defined as

$$\mathrm{v}(I) \;=\; \min\{d \;:\; \textit{there exists } f \in S_d \textit{ such that } (I : f) \in \operatorname{Ass}(I)\}.$$

The concept of v-number first appeared in the work of Cooper *et all* [31], in connection to Algebraic Geometry, projective Reed–Muller–type codes and minimum distance functions. It was further studied in [6, 29, 83, 112, 111, 140, 137].

Inspired by results of Brodmann [21, 22], Ratliff [139], Cutkosky, Herzog, Trung [41], Kodiyalam [118], and many others, about the asymptotic behaviour of powers of homogeneous ideals, we investigate the asymptotic behaviour of the function $\mathrm{v}(I^k)$.

By a classical result due to Brodmann, we have $\operatorname{Ass}(I^k) = \operatorname{Ass}(I^{k+1})$ for all $k \gg 0$, and we call this common set of associated primes the stable set of associated primes of I, denoted by $\operatorname{Ass}^\infty(I)$. For a finitely generated graded S-module $M = \bigoplus_d M_d \neq 0$, we set $\alpha(M) = \min\{d : M_d \neq 0\}$ and $\omega(M) = \max\{d : (M/\mathfrak{m}M)_d \neq 0\}$.

Theorem 10.2.1 (Ficarra–Sgroi, 2023 [71]). *Let $I \subset S = K[x_1, \dots, x_n]$ be a graded ideal, and let $\mathfrak{p} \in \operatorname{Ass}^\infty(I)$. Then, the following holds.*

(a) *For all $k \geq 1$, we have*

$$\alpha((I^k : \mathfrak{p})/I^k) \;\leq\; \mathrm{v}_{\mathfrak{p}}(I^k) \;\leq\; \omega((I^k : \mathfrak{p})/I^k).$$

(b) *The functions $\alpha((I^k : \mathfrak{p})/I^k)$, $\omega((I^k : \mathfrak{p})/I^k)$ are linear in k for $k \gg 0$.*

(c) *There exist eventually linear functions $f(k)$ and $g(k)$ such that*

$$f(k) \leq \mathrm{v}(I^k) \leq g(k), \quad \textit{for all} \quad k \gg 0.$$

As a second main result, we prove that

Theorem 10.2.6 (Ficarra–Sgroi, 2023 [71]). *Let $I \subset S = K[x_1, \dots, x_n]$ be a graded ideal. Suppose that*

(a) *either $\operatorname{Ass}^\infty(I) = \operatorname{Max}^\infty(I)$ or*

(b) *for all $\mathfrak{p} \in \operatorname{Ass}^\infty(I)$ and all $k \gg 0$, $(I^k : \mathfrak{p})/I^k$ is generated in a single degree.*

Then, $\mathrm{v}_{\mathfrak{p}}(I^k)$, for all $\mathfrak{p} \in \operatorname{Ass}^\infty(I)$, and $\mathrm{v}(I^k)$ are linear functions in k for $k \gg 0$.

We expect that, in general, the functions $\mathrm{v}_{\mathfrak{p}}(I^k)$ ($\mathfrak{p} \in \operatorname{Ass}(I)$) and $\mathrm{v}(I^k)$ are linear in k for $k \gg 0$. We also study such functions for monomial ideals in two variables, edge ideals with linear resolution, polymatroidal ideals and Hibi ideals.

During the PhD program, I was lucky enough to write several papers [3, 4, 5, 32, 33, 34, 35, 36, 37, 38, 39, 55, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73] with many wonderful collaborators and colleagues. I could not discuss all these articles in this dissertation. However, some of the topic studied include: t-spread lexsegment ideals, characterizations of extremal Betti numbers, squarefree powers, canonical traces, nearly Gorenstein rings, determinantal rings, binomial edge ideals, exterior algebras, asymptotic componentwise linearity, toric algebras of 1-dimensional simplicial complexes, Simon conjecture, and Fitting ideals, among others.

Chapter 2

Multigraded structures and monomial ideals

In this first chapter, we summarize the basic facts from commutative algebra and combinatorics needed in the later parts of the thesis. We begin with a quick review on graded algebras and graded Betti numbers. Cohen–Macaulayness, gorensteiness and complete intersections algebras are discussed. Monomial ideals are discussed from the algebraic viewpoint. The problem of constructing the minimal (multi)graded free resolution was raised by Kaplansky in sixties. Besides the Taylor complex [146], which is in general a non minimal resolution for a monomial ideal, historically the first important class of monomial ideals whose minimal resolution was successfully constructed is that of stable ideals [50]. This resolution is nowadays called the *Eliahou–Kervaire resolution*. The method of Koszul cycles was used by Aramova and Herzog [10], who recovered in a more simple fashion the Eliahou–Kervaire resolution and later with Hibi constructed a similar resolution for squarefree stable ideals. Later on, Charalambous and Evans realized that both the Taylor resolution and the Eliahou–Kervaire resolution can be constructed by iterated mapping cones [26]. Inspired by this discovery, Herzog and Takayama introduced *monomial ideals with linear quotients* [100]. This notion is the algebraic counterpart of the concept of *shellability* from simplicial complexes theory. Stable, squarefree stable and polymatroidal ideals are examples of ideals with linear quotients. More recently, Francisco, Há and Van Tuyl, by introducing the concept of *Betti splitting*, characterized when the mapping cone construction produces a minimal free resolution of a monomial ideal [74]. Ideals with linear quotients are a special case of a Betti splitting.

2.1 Graded algebras and Betti numbers

Definition 2.1.1 A commutative ring R is called (*positively*) *graded* if there exists a family $(R_n)_{n\geq 0}$ of additive subgroups of R, called a *graduation* of R, such that

(i) $R = \bigoplus_{d\geq 0} R_d$;

(ii) For all $d, h \geq 0$ we have that $R_d R_h \subseteq R_{d+h}$. That is, for all $x \in R_d$ and all $y \in R_h$, then $xy \in R_{d+h}$.

If $u \in R_d$, we say that u is a *homogeneous element* of degree d and we set $\deg(u) = d$.

The polynomial ring $S = K[x_1, \ldots, x_n]$ with coefficients in a field K is graded if we put $\deg(x_i) = 1$ for all i. In this case, we say that S is the *standard graded* polynomial ring. Note that S_d has a K-basis consisting of all monomials of S of degree d.

We denote by Mon(S) the set of all monomials of S. In particular, any polynomial $f \in S$ can be uniquely written as a sum of monomials:

$$f = \sum_{u \in \mathrm{Mon}(S)} a_u u,$$

with only finitely many $a_u \in K$ non zero, and with $u \in \mathrm{Mon}(S)$. The *support of* f, supp(f), is the set of monomials $u \in \mathrm{Mon}(S)$ such that $a_u \neq 0$.

For instance, if $S = K[x, y, z]$, then

$$S_3 = x^3K \oplus x^2yK \oplus x^2zK \oplus xy^2K \oplus xyzK \oplus xz^2K \oplus y^3K \oplus y^2zK \oplus yz^2K \oplus z^3K.$$

If n is an integer, set $[n] = \{1, 2, \ldots, n\}$. For a monomial $u = x_1^{a_1} \cdots x_n^{a_n} \in S$, the integral vector $\mathbf{a} = (a_1, \ldots, a_n)$ is called the *multidegree* of u. We also write $u = \mathbf{x}^{\mathbf{a}}$, in particular, for $\mathbf{a} = \mathbf{0} = (0, \ldots, 0)$, $\mathbf{x}^{\mathbf{0}} = 1$; whereas $\deg(u) = a_1 + \cdots + a_n$ is the *degree* of u. It is customary to identify the multidegree $\mathbf{a}$ with the monomial $\mathbf{x}^{\mathbf{a}}$.

Note that S is $\mathbb{Z}^n$-graded, that is, *multigraded*. Indeed $S = \bigoplus_{\mathbf{a}} S_{\mathbf{a}}$, where $S_{\mathbf{a}}$ is the K-vector space with unique generator $\mathbf{x}^{\mathbf{a}}$ and $S_{\mathbf{a}} S_{\mathbf{b}} \subseteq S_{\mathbf{a}+\mathbf{b}}$.

Definition 2.1.2 Let R be a commutative ring and A a R-algebra. A is called (*positively*) *graded* if there exists a family $(A_d)_{d \geq 0}$ of R-submodules of A, such that

(i) $A = \bigoplus_{d \geq 0} A_d$;

(ii) For all $d, h \geq 0$ we have $A_d A_h \subseteq A_{d+h}$, that is, for all $x \in A_d$ and all $y \in A_h$ then $xy \in A_{d+h}$.

Definition 2.1.3 Let A be a graded R-algebra and I a two-sided ideal of A. The ideal I is called a *graded ideal*, or also a *homogeneous ideal*, if

$$I = \bigoplus_{d \geq 0} I_d,$$

with $I_d = I \cap A_d$, for all $d \geq 0$, where A_d is the dth graded piece of A.

Observe that $I_d I_h \subseteq I_{d+h}$, thus $(I_d)_{d \geq 0}$ is a graduation of I and I is graded as a R-subalgebra of A.

Proposition 2.1.4 *Let A be a graded R-algebra, and let I be a graded ideal of A. Then A/I is a graded R-algebra, whose graded pieces are, for all $d \geq 0$,*

$$(A/I)_d = (A_d + I)/I.$$

Proposition 2.1.5 *Let A be a graded R-algebra. A two-sided ideal I of A is graded if and only if is generated by homogeneous elements.*

We concentrate our attention on finitely generated graded K-algebras, where K is a field. Despite the apparent generality of this notion, these algebras admit a familiar presentation. Indeed, one has

Proposition 2.1.6 *Let A be a finitely generated graded K-algebra, with K a field. Then, there exist a polynomial ring $S = K[x_1, \ldots, x_n]$ and a homogeneous ideal I of S such that $A \cong S/I$ as K-algebras.*

From now on, let $S = K[x_1, \ldots, x_n]$ be the standard graded polynomial ring with K a field. The *graded maximal ideal*, or *irrelevant ideal*, of S is $\mathfrak{m} = (x_1, \ldots, x_n)$. The definition of graded S-modules is analogous to that of graded ring.

Finitely generated graded K-algebras S/I with irrelevant ideal $\mathfrak{m}$ and local rings $(R, \mathfrak{m})$ share many similarities. The role of $\mathfrak{m}$ is analogous in both settings. Indeed, as in the case of local rings, a *graded version of Nakayama Lemma* holds.

Proposition 2.1.7 (Nakayama Lemma, graded version). *Let M a finitely generated graded S-module and let $m_1, \ldots, m_r \in M$. If $m_1 + \mathfrak{m}M, \ldots, m_r + \mathfrak{m}M$ generate $M/\mathfrak{m}M$, then $m_1, \ldots, m_r$ generate M.*

Definition 2.1.8 Let M, N be graded S-modules. A S-map $\varphi : M \to N$ is called a *graded homomorphism* if $\varphi(M_i) \subseteq N_i$, for all i.

We fix the category $\mathcal{M}$ whose objects are finitely generated graded S-modules, and whose morphism are graded homomorphisms. As in the case of a local ring, we are going to associate to any object $M \in \mathcal{M}$ some algebraic invariants, called the *graded Betti numbers*: $\beta_{i,j}(M)$.

An arbitrary S-map $f : M \to N$, with $M, N \in \mathcal{M}$ is in general non graded. Sometimes, it is convenient to change the grading on M in order to make f into a graded homomorphism. We can *shift the degrees by an integer* d as follows. If $M = \bigoplus_{i \geq 0} M_i$, the corresponding module obtained by shifting the degrees by d is

$$M(d) = \bigoplus_{i \geq 0} M(d)_i,$$

with $M(d)_i = M_{i+d}$. Observe that if $x \in M_i$, then x is a homogeneous element of degree $i - d$ in $M(d)$. Indeed, $x \in M(d)_{i-d} = M_{i-d+d} = M_i$.

Given an object of $M \in \mathcal{M}$, we can construct a *graded free resolution* of M, that is an exact complex

$$\mathbb{G} : \cdots \xrightarrow{h_{i+1}} G_i \xrightarrow{h_i} \cdots \xrightarrow{h_2} G_1 \xrightarrow{h_1} G_0 \xrightarrow{h_0} M \to 0$$

such that

(i) $G_i \in \mathcal{M}$, that is, the G_i's are finitely generated graded S-modules;

(ii) the morphism h_i are graded.

Graded free resolutions exist. Let $M \in \mathcal{M}$, and let $m_1, \ldots, m_r$ homogeneous generators of M with $\deg(m_i) = a_i$, for all i. Let $F_0 = \bigoplus_{i=1}^r Se_i$ with $\deg(e_i) = a_i$. As F_0 is a free S-module, we can define the graded surjective S-map $d_0 : F_0 \to M$ by setting $d_0(e_i) = m_i$. Observe that $F_0 \cong \bigoplus_{i=1}^r S(-a_i)$, thus $F_0 \in \mathcal{M}$. Let $K_0 = \ker(d_0)$, then $K_0 \in \mathcal{M}$. Indeed K_0 is graded and finitely generated, since S is noetherian and F_0 is finitely generated. We have the short exact sequence

$$0 \to K_0 \xrightarrow{i_0} F_0 \cong \bigoplus_{i=1}^{r} S(-a_i) \xrightarrow{d_0} M \to 0,$$

with i_0 the graded S-map given by inclusion. We can repeat this construction for K_0. Iterating the construction for all successive kernels K_i, $i \geq 0$, we get a graded free resolution of M

$$\mathbb{F} : \cdots \to F_{i+1} \xrightarrow{d_{i+1}} F_i \to \cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0.$$

Clearly, many (non isomorphic) graded free resolution exist.

Definition 2.1.9 Let R be a ring and M a finitely generated R-module. The elements $m_1, \ldots, m_r \in M$ form a *minimal generating set* for M if $m_1, \ldots, m_r$ generate M and no proper subset of them generate M.

Next lemma is pivotal and shows how to detect a minimal generating set.

Lemma 2.1.10 *Let $m_1, \ldots, m_r$ generators of a finitely generated graded S-module M. Let $F_0 = \bigoplus_{i=1}^r Se_i$ and $\varepsilon : F_0 \to M$ a surjective S-map with $\varepsilon(e_i) = m_i$, for all i. The following facts are equivalent:*

(i) *$m_1, \ldots, m_r$ form a minimal generating set of M;*

(ii) $\mathrm{Ker}(\varepsilon) \subseteq \mathfrak{m}F_0$.

Definition 2.1.11 Let $M \in \mathcal{M}$. A *minimal graded free resolution* of M is a graded free resolution

$$\mathbb{F} : \cdots \to F_{i+1} \xrightarrow{d_{i+1}} F_i \to \cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

such that $\mathrm{Im}(d_{i+1}) \subseteq \mathfrak{m}F_i$, for all $i \geq 0$.

A minimal graded free resolution of $M \in \mathcal{M}$ exists. Just repeat the construction of a graded free resolution of M, choosing at each step a minimal generating set of the kernels K_i. By Lemma 2.1.10, this is equivalent to $\mathrm{Im}(d_{i+1}) \subseteq \mathfrak{m}F_i$, for all $i \geq 0$. In particular, this is equivalent to the fact that the matrices describing the differentials d_i have entries in $\mathfrak{m}$.

Theorem 2.1.12 *Let M be a finitely generated graded S-module and let $\mathbb{F}$ be a minimal free resolution of M, with $F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$. Then, for all i, j,*

$$\beta_{i,j} = \dim_K \mathrm{Tor}_i^S(K, M)_j.$$

Thus, the following algebraic invariants are well defined,

Definition 2.1.13 Let $M \in \mathcal{M}$. The *graded Betti numbers* of M are defined as

$$\beta_{i,j}(M) = \dim_K \mathrm{Tor}_i^S(K, M)_j.$$

The ith *total Betti number* of M is $\beta_i(M) = \sum_j \beta_{i,j}(M)$.

Observe that if $M \cong N$ are isomorphic objects of $\mathcal{M}$, then $\beta_{i,j}(M) = \beta_{i,j}(N)$, for all $i, j \geq 0$. Indeed, $\mathrm{Tor}_i^S(K, M)_j \cong \mathrm{Tor}_i^S(K, N)_j$, for all $i, j \geq 0$..

Not only the graded Betti numbers of $M \in \mathcal{M}$ are unique, but also

Proposition 2.1.14 *Let M a finitely generated graded S-module and let $\mathbb{F}$, $\mathbb{G}$ be two minimal graded free resolutions of M. Then, $\mathbb{F}$, $\mathbb{G}$ are isomorphic. That is, for all i there exist graded S-isomorphisms $\alpha_i : F_i \to G_i$ such that the following diagram is commutative*

$$\begin{tikzcd}
\mathbb{F} : \cdots \arrow[r] & F_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "\alpha_{i+1}"'] & F_i \arrow[r] \arrow[d, "\alpha_i"] & \cdots \arrow[r] & F_2 \arrow[r, "d_2"] \arrow[d, "\alpha_2"'] & F_1 \arrow[r, "d_1"] \arrow[d, "\alpha_1"'] & F_0 \arrow[r, "d_0"] \arrow[d, "\alpha_0"'] & M \arrow[r] \arrow[d, "\mathrm{id}_M"] & 0 \\
\mathbb{G} : \cdots \arrow[r] & G_{i+1} \arrow[r, "\partial_{i+1}"'] & G_i \arrow[r] & \cdots \arrow[r] & G_2 \arrow[r, "\partial_2"'] & G_1 \arrow[r, "\partial_1"'] & G_0 \arrow[r, "\partial_0"'] & M \arrow[r] & 0
\end{tikzcd}$$

If $M \in \mathcal{M}$, the *initial degree* of M is $\alpha(M) = \min\{d : M_d \neq 0\}$.

Proposition 2.1.15 *Let M a finitely generated graded S-module. Then,*

$$\beta_{i,\ell}(M) = 0, \quad \textit{for all } i \geq 0 \textit{ and all } \ell < i + \alpha(M).$$

Proof. Let $d = \alpha(M)$ and let $\mathbb{F}$ be a minimal graded free resolution of M.

We proceed by induction on $i \geq 0$. For $i = 0$, $\beta_{0,j}(M)$ is the number of homogeneous generators of degree j of a minimal generating set of M. Thus, $\beta_{0,j}(M) = 0$ whenever $j < i + d = d$. Assume that the statement holds for $i > 0$, that is

$$\beta_{i,\ell}(M) = 0, \quad \text{for all } \ell < i + d.$$

By inductive hypothesis, $\beta_{i,\ell}(M) = 0$, for all $\ell < i + d$. Thus, the degrees of the elements of a minimal generating set of F_i are $\geq i + d$. By the minimality of $\mathbb{F}$, $\operatorname{Im}(d_{i+1}) \subseteq \mathfrak{m}F_i$, and as any element of $\mathfrak{m} = (x_1, \ldots, x_n)$ has degree ≥ 1, the degrees of the elements of $\operatorname{Im}(d_{i+1})$ are $\geq i + 1 + d$. Since d_{i+1} is a homogeneous map, the preimages of the generators of $\operatorname{Im}(d_{i+1})$ are generators of F_{i+1} with degrees $\geq i+1+d$. Thus, $\beta_{i+1,\ell}(M) = 0$, for all $\ell < (i + 1) + d$, as desired. □

Corollary 2.1.16 *Let M a finitely generated graded S-module. Then,*

$$\beta_{i,\ell}(M) = 0, \quad \textit{for all } i \geq 0 \textit{ and all } \ell < i.$$

For $\ell < i$, the Betti number $\beta_{i,\ell}(M)$ is zero. So, it is natural to collocate the graded Betti numbers in a table whose (i, j)th entry is the graded Betti number $\beta_{i,i+j}(M)$. We call this diagram, the *Betti table* or *Betti diagram* of M,

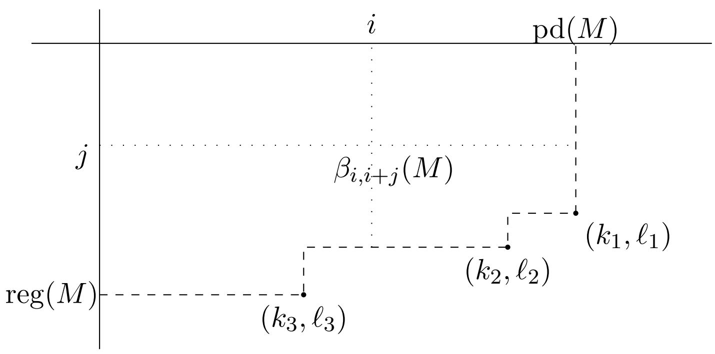


FIGURE 2.1: Betti table of the module $M \in \mathcal{M}$.

The Betti numbers outside the area bounded by the axis and the dashed line are zero. The Betti numbers in the entries $(k_1, \ell_1), (k_2, \ell_2), (k_3, \ell_3), \ldots$ are important invariants of M. Precisely,

Definition 2.1.17 (Bayer–Charalambous–Popescu, 1999 [18]). Let $M \in \mathcal{M}$ a graded finitely generated S-module. A graded Betti number $\beta_{k,k+\ell}(M)$ is called *extremal* if

(a) $\beta_{k,k+\ell}(M) \neq 0$,

(b) $\beta_{i,i+j}(M) = 0$ for all (i, j) such that $i \geq k$, $j \geq \ell$ and $(i, j) \neq (k, \ell)$.

In such a case, the pair (k, ℓ) is called a *corner* of the Betti table of M, or more simply a corner of M. The extremal Betti numbers of M can be ordered canonically

$$\beta_{k_1,k_1+\ell_1}(M), \quad \beta_{k_2,k_2+\ell_2}(M), \quad \ldots, \quad \beta_{k_r,k_r+\ell_r}(M), \tag{2.1}$$

with $k_1 > k_2 > \cdots > k_r$ and $\ell_1 < \ell_2 < \cdots < \ell_r$.

Let $M \in \mathcal{M}$. Then, the following algebraic invariants of M are defined.

- The *projective dimension* of M is the integer

$$\mathrm{pd}(M) \;=\; \max\{i \;:\; \beta_{i,j}(M) \neq 0, \; \textit{for some } j\}.$$

  By *Hilbert syzygies's Theorem* we have $\mathrm{pd}(M) \leq n$.

- The (*Castelnuovo-Mumford*) *regularity* of M is the integer

$$\mathrm{reg}(M) \;=\; \max\{j \;:\; \beta_{i,i+j}(M) \neq 0, \; \textit{for some } i\}.$$

  Note that $\mathrm{reg}(M) \geq \alpha(M)$.

  If M is generated in a single degree and $\mathrm{reg}(M) = \alpha(M)$, we say that M has a *d-linear resolution*. Let $M_{\langle d \rangle}$ be the S-submodule of M generated by M_d. We say that M is *componentwise linear* if $M_{\langle d \rangle}$ has a linear resolution for all d.

- The *depth* of M is the common length of any maximal regular sequence on M. Recall that $\mathbf{f} = f_1, \ldots, f_m$ is a *regular sequence* on M if the multiplication map by f_i, $M/(f_1, \ldots, f_{i-1})M \to M/(f_1, \ldots, f_{i-1})M$, is injective for all i, and $M/(\mathbf{f})M \neq 0$. By the famous *Auslander–Buchsbaum formula* we have that $\mathrm{depth}(M) + \mathrm{pd}(M) = n$. Thus,

$$\mathrm{depth}(M) \;=\; \min\{i \;:\; \beta_{n-i,j}(M) \neq 0, \; \textit{for some } j\}.$$

Remark 2.1.18 To know the minimal graded free resolution of a homogeneous ideal I of S is equivalent to know that of S/I. Indeed, $\beta_{i,j}(I) = \beta_{i+1,j}(S/I)$ for all i and j. In particular, $\mathrm{pd}(S/I) = \mathrm{pd}(I)+1$, $\mathrm{depth}(S/I) = \mathrm{depth}(I)-1$ and $\mathrm{reg}(S/I) = \mathrm{reg}(I)-1$.

Let $\dim(M)$ be the *Krull dimension* of $M \in \mathcal{M}$. We always have

$$\mathrm{depth}(M) \leq \dim(M).$$

Definition 2.1.19 Let $M \in \mathcal{M}$.

(a) We say that M is *Cohen–Macaulay* if $\mathrm{depth}(M) = \dim(M)$.

(b) Suppose that M is Cohen–Macaulay. Then, the *Cohen–Macaulay type* of M is the number

$$\text{CM-type}(M) \;=\; \beta_{\mathrm{pd}(M)}(M).$$

We say that M is *Gorenstein* if M is Cohen–Macaulay and $\text{CM-type}(M) = 1$.

(c) We say that M is *complete intersection* if $M \cong S/I$ where $I = (\mathbf{f})$ is an ideal generated by a homogeneous regular sequence $\mathbf{f}$ on S.

The following hierarchy holds:

$$\text{complete intersection} \;\Rightarrow\; \text{Gorenstein} \;\Rightarrow\; \text{Cohen–Macaulay}.$$

2.2 The Koszul complex

To compute the graded Betti numbers one can use the *Koszul complex* [89]. Let $\mathbf{f} = f_1, \ldots, f_m$ be a sequence of elements of S. The Koszul complex $K_{\bullet}(\mathbf{f}; S)$ attached to the sequence $\mathbf{f}$ is defined as follows: let F be the free S-module with basis $e_1, \ldots, e_m$.

- We let $K_i(\mathbf{f}; S) = \bigwedge^i F$, for all $i = 0, \ldots, m$. A basis of the free S-module $K_i(\mathbf{f}; S)$ is given by the wedge products $e_\tau = e_{k_1} \wedge e_{k_2} \wedge \cdots \wedge e_{k_i}$, where $\tau = \{k_1 < k_2 < \cdots < k_i\} \subseteq [m]$, with $\deg(u_\tau) = |\tau| = i$.

- We define the differential $\partial_i : K_i(\mathbf{f}; S) \to K_{i-1}(\mathbf{f}; S)$, $i = 1, \ldots, m-1$ by

$$\partial_i(e_\tau) \;=\; \sum_{\ell=1}^{i} (-1)^{\sigma(\tau;k_\ell)} f_{k_\ell} e_{\tau \setminus \{k_\ell\}},$$

  where $\sigma(\tau; k_\ell) = \{j : 1 \leq j < \ell\}$.

We order the wedge products *lexicographically*, as follows:
Let $\sigma = \{k_1 < k_2 < \cdots < k_p\}, \tau = \{\ell_1 < \ell_2 < \cdots < \ell_q\} \subseteq [m]$, we define $\sigma > \tau$ if $p = q$ and for some $j \in [p]$ one has

$$k_1 = \ell_1, \quad k_2 = \ell_2, \quad \ldots, \quad k_{j-1} = \ell_{j-1} \;\text{ and }\; k_j < \ell_j.$$

If $\sigma > \tau$, then we set $e_\sigma > e_\tau$. For example,

$$e_1 \wedge e_2 \;>\; e_1 \wedge e_3 \;>\; e_1 \wedge e_4 \;>\; e_2 \wedge e_3 \;>\; e_2 \wedge e_4 \;>\; e_3 \wedge e_4.$$

Let $I \subset S$ be an ideal, and let $\varepsilon : S \to S/I$ be the canonical epimorphism. We set $\mathbf{x} = x_1, \ldots, x_n$. Koszul homology allows us to calculate the graded Betti numbers.
Indeed, if $M \in \mathcal{M}$, we define

$$K_{\bullet}(\mathbf{f}; M) \;=\; K_{\bullet}(\mathbf{f}; S) \otimes_S M.$$

Then, we have the isomorphism $H_i(\mathbf{x}; M)_j \cong \operatorname{Tor}_i^S(K, M)_j$ [89, Corollary A.3.5] and consequently we have $\beta_{i,j}(M) = \dim_K H_i(\mathbf{x}; M)_j$ for all i and j.
Moreover, for a homogeneous ideal $I \subset S$, we have

$$\beta_{i-1,j}(I) = \beta_{i,j}(S/I) = \dim_K H_i(\mathbf{x}; S/I)_j \;\text{ for all }\; i \geq 1, j \geq 0. \tag{2.2}$$

Again, let $M \in \mathcal{M}$. In particular, we have

- $\operatorname{pd}(M) = \max\{i : H_i(\mathbf{x}; M)_j \neq 0,\ \textit{for some } j\} = \max\{i : H_i(\mathbf{x}; M) \neq 0\}$.

- $\operatorname{depth}(M) = n - \max\{i : H_i(\mathbf{x}; M) \neq 0\}$.

- $\operatorname{reg}(M) = \max\{j : H_i(\mathbf{x}; M)_{i+j} \neq 0,\ \textit{for some } i\}$.

The following algebraic characterization holds.

Theorem 2.2.1 *Let $\mathbf{f} = f_1, \ldots, f_m$ be a sequence of elements of S and let $I = (\mathbf{f})$. Then, the following conditions are equivalent.*

(a) *$\mathbf{f}$ is a regular sequence on S.*

(b) *$K_{\bullet}(\mathbf{f}; S/I)$ is the minimal free resolution of S/I.*

2.3 Monomial Ideals

Let $S = K[x_1, \dots, x_n]$ be the standard graded polynomial ring in n indeterminates, with K a field. Let $u = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} \in S$, $u \neq 1$, be a monomial. Then:

- $\deg(u) = a_1 + a_2 + \cdots + a_n$ is called the *degree* of u,
- $\deg_{x_i}(u) = \max\{j : x_i^j \text{ divides } u\} = a_i$ is called the x_i*-degree* of u,
- $\mathrm{supp}(u) = \{i : x_i \text{ divides } u\}$ is called the *support* of u.
- $\max(u) = \max(\mathrm{supp}(u)) = \max\{i : x_i \text{ divides } u\}$ is called the *maximum* of u,
- $\min(u) = \min(\mathrm{supp}(u)) = \min\{i : x_i \text{ divides } u\}$ is called the *minimum* of u.

For convenience we set $\min(1) = \max(1) = n$, for the monomial $1 \in S$.

By $\mathrm{Mon}(S)$ we denote the set of all monomials of S. Whereas, $\mathrm{Mon}_\ell(S)$ denotes the set of all monomial of S having degree ℓ.

An ideal I of S is called a *monomial ideal*, if I can be generated by monomials. We denote by $G(I)$ the unique minimal set of monomial generators of I. Whereas, for an integer $\ell \geq 0$, we set $G(I)_\ell = \{u \in G(I) : \deg(u) = \ell\}$.

Let $I \subset S$ be a proper monomial ideal. The *support* of I is the set

$$\mathrm{supp}(I) \;=\; \bigcup_{u \in G(I)} \mathrm{supp}(u) \;=\; \{i \;:\; x_i \text{ divides } u, \text{ for some } u \in G(I)\}.$$

We say that I is *fully supported* if $\mathrm{supp}(I) = \{1, 2, \dots, n\}$.

Note that a monomial ideal is a multigraded S-module. Thus, any monomial ideal I of S has a unique minimal (multi)graded free resolution

$$\mathbb{F} : 0 \to \bigoplus_j S(-j)^{\beta_{p,j}(I)} \to \bigoplus_j S(-j)^{\beta_{p-1,j}(I)} \to \cdots \to \bigoplus_j S(-j)^{\beta_{0,j}(I)} \to I \to 0,$$

where $S(-j)$ is the free S-module obtained by shifting the degrees of S by j. For all $i, j \geq 0$, the numbers $\beta_{i,j} = \beta_{i,j}(I) = \dim_K \mathrm{Tor}_i^S(K, I)_j$ are called the *graded Betti numbers* of I, and $\beta_i(I) = \sum_j \beta_{i,j}(I)$ is the ith *total Betti number* of I.

In particular, for all i, we have $= \bigoplus_j S(-j)^{\beta_{i,j}(I)} = \bigoplus_{\mathbf{a}} S(-\mathbf{a})^{\beta_{i,\mathbf{a}}(I)}$. We call $\beta_{i,\mathbf{a}}(I) = \dim_K \mathrm{Tor}_i^S(K, I)_{\mathbf{a}}$ a *multigraded Betti number* of I.

2.4 A tour through resolutions of monomial ideals

In this section, we collect several constructions that allow to determine partially or wholly the minimal graded free resolutions of monomial ideals.

The problem to determine the minimal graded free resolution of a monomial ideal goes back to *Kaplansky*. It was the first one to propose, in the sixties, the systematic study of monomial ideals and their minimal resolutions. Despite their simple structure, this problem is incredibly hard, and open. The general consensus is the following: no explicit complex exists that gives a *minimal* free resolution for all monomial ideals. On the other hand, many special constructions are known and for wide classes of monomial ideals Kaplansky problem can be solved. Next, we survey what is known in this direction.

Before starting our marathon of algebraic techniques, we state the following result.

Theorem 2.4.1 *Let M be a finitely generated graded S-module, $\mathbb{F}$ the minimal graded free resolution of M and let $\mathbb{G}$ any graded free resolution of M,*

$$\mathbb{G} : \cdots \to G_i \to \cdots \to G_1 \to G_0 \to M \to 0,$$

with $G_i = \bigoplus_{j\geq 0} S(-j)^{b_{i,j}}$ for all $i, j \geq 0$. Then, $\mathbb{F}$ is a subcomplex of $\mathbb{G}$, and

$$\beta_{i,j}(M) \leq b_{i,j}, \quad \textit{for all} \quad i, j \geq 0.$$

Proof. See [51, Theorem 4.26]. □

2.4.1 Kaplansky problem and Taylor resolution

The first mathematician to address Kaplansky problem was his student Diana Taylor. In 1966 [146], she constructed for any monomial ideal $I \subset S$ a lcm-complex that provides a (multi)graded free resolution of I, although non minimal in general. Nowadays this complex is called the *Taylor resolution.*

Let $I \subset S = K[x_1, \ldots, x_n]$ be a proper monomial ideal, and let $G(I) = \{u_1, \ldots, u_m\}$. The *Taylor resolution* of I is defined as follows. Following the same notations as in [89], to the sequence $\{u_1, \ldots, u_m\}$ of monomial generators of I, we associate a complex $\mathbb{T} = \mathbb{T}(u_1, \ldots, u_m)$ of free S-modules defined as follows: let T_1 be a free S-module with basis $\{e_1, \ldots, e_m\}$. Then

- $T_i = \bigwedge^i T_1$. More precisely, T_i is a free S-module with basis the elements

$$\mathbf{e}_F = e_{j_1} \wedge e_{j_2} \cdots \wedge e_{j_i}, \quad F = \{j_1 < j_2 < \cdots < j_i\} \subseteq [m];$$

- the differentials $\partial_i : T_i \to T_{i-1}$, for $i = 1, \ldots, m$, are defined by

$$\partial_i(\mathbf{e}_F) = \sum_{i\in F} (-1)^{\sigma(F;i)} \frac{\operatorname{lcm}(u_j : j \in F)}{\operatorname{lcm}(u_j : j \in F \setminus \{i\})} \mathbf{e}_{F\setminus\{i\}},$$

  where $\sigma(F; i) = \left|\{j \in F : j < i\}\right|$.

To each basis element $\mathbf{e}_F$, with $F = \{j_1 < j_2 < \cdots < j_i\} \subseteq [m]$, we assign (multi)degree

$$\deg(\mathbf{e}_F) = \deg(\operatorname{lcm}(u_j : j \in F)).$$

As a special case of Theorem 2.4.1 we have

Theorem 2.4.2 *The complex $\mathbb{T}(u_1, \ldots, u_m)$ is a graded free resolution of S/I, in general non minimal. In particular,*

$$\beta_i(S/I) \leq \binom{|G(I)|}{i}, \quad \textit{for all } 1 \leq i \leq |G(I)|.$$

The reader may see the resemblance with the Koszul complex $K_\bullet(\mathbf{u}; S/I)$. Indeed, the Taylor resolution is a modification of the Koszul complex. The problem of the latter complex is that it is not sensitive to the grading. Indeed, even if it is a resolution of I, such a resolution is non graded, unless $\mathbf{u} = u_1, \ldots, u_m$ is a (monomial) regular sequence, in which case $K_\bullet(\mathbf{u}; S/I)$ and $\mathbb{T}(u_1, \ldots, u_m)$ are isomorphic.

The Taylor complex was later generalized by Herzog in [85]. As a consequence of his construction, we have the following remarkable inequalities.

Theorem 2.4.3 (Herzog, 2007 [85]). *Let I and J be proper monomial ideals of S. Then*

(a) $\beta_i(S/(I+J)) \leq \sum_{j=0}^{i} \beta_j(S/I)\beta_{i-j}(S/J)$ *for all i.*

(b) $\mathrm{pd}(I+J) \leq \mathrm{pd}(I) + \mathrm{pd}(J) + 1$.

(c) $\mathrm{reg}(I+J) \leq \mathrm{reg}(I) + \mathrm{reg}(J) - 1$.

(d) $\mathrm{pd}(I \cap J) \leq \mathrm{pd}(I) + \mathrm{pd}(J)$.

(e) $\mathrm{reg}(I \cap J) \leq \mathrm{reg}(I) + \mathrm{reg}(J)$.

All the previous inequalities are equalities if $\mathrm{supp}(I) \cap \mathrm{supp}(J) = \varnothing$.

2.4.2 The Eliahou–Kervaire resolution

Hereafter, for a monomial $u \in S$, $u \neq 1$, we set $u' = u/x_{\max(u)}$.

A monomial ideal I of $S = K[x_1, \dots, x_n]$ is called *stable* if for any monomial $u \in I$ and any i, we have that $x_i(u/x_{\max(u)}) = x_i u' \in I$. The monomial ideal I is called *strongly stable* if for any monomial $u \in I$ and any $i < j$ with $j \in \mathrm{supp}(u)$, we have that $x_i(u/x_j) \in I$. Any strongly stable ideal is stable.

The class of stable ideals was introduced by Eliahou and Kervaire in 1990. In characteristic zero, $\mathrm{char}(K) = 0$, strongly stable ideals are the Borel–fixed ideals. In turn, the generic initial ideal $\mathrm{Gin}(I)$ (with respect to the reverse lexicographic order $>_{\mathrm{revlex}}$) of any graded ideal is Borel–fixed [89]. Therefore, strongly stable ideals play a fundamental role in the theory of monomial ideals.

For a monomial ideal $I \subset S$, we define $M(I)$ to be the set of all monomials of S belonging to I. The next pivotal lemma was proved by Eliahou and Kervaire.

Lemma 2.4.4 (Eliahou–Kervaire, 1990 [50, Lemma 1.1]). *Let $I \subset S$ be a stable ideal. Then, for all $w \in M(I)$ there exist unique monomials $u \in G(I)$ and $v \in \mathrm{Mon}(S)$ such that $w = uv$ and $\max(u) \leq \min(v)$. For any $w \in M(I)$, we set $g(w) = u$ and such position defines a map $g : M(I) \to G(I)$ which we call the decomposition map of I.*

Let $u \in G(I)$ be a minimal generator of I and let σ be a subset of $[\max(u) - 1]$. Following [50], we call $f(u; \sigma)$ an *admissible symbol*. To avoid unnecessary distinctions, we set $f(u; \sigma) = 0$ if $\sigma \not\subseteq [\max(u) - 1]$.

Theorem 2.4.5 (Eliahou–Kervaire, 1990 [50]). *Let $I \subset S$ be a stable ideal, and let $\mathbb{F}$ be the minimal free resolution of S/I.*

(a) *For all i, F_i has as a basis the admissible symbols $f(u; \sigma)$ with $u \in G(I)$, $\sigma \subseteq [\max(u) - 1]$ and $|\sigma| = i$.*

(b) *For all $i > 0$, the ith differential $d_i : F_i \to F_{i-1}$ of $\mathbb{F}$ is defined by*

$$d_i(f(u;\sigma)) = \sum_{k \in \sigma} (-1)^{\alpha(\sigma;k)} \Big(-x_k f(u; \sigma \setminus k) + \frac{x_k u}{g(x_k u)} f(g(x_k u); \sigma \setminus k)\Big),$$

where $\alpha(\sigma; k) = |\{j \in \sigma : j < k\}|$ and $g : M(I) \to G(I)$ is the decomposition map of I (Lemma 2.4.4).

(c) *For all i and j we have*

$$\beta_{i,i+j}(I) = \sum_{u \in G(I)_j} \binom{\max(u) - 1}{i}.$$

2.4.3 Koszul cycles

In this subsection, we consider *Koszul cycles*. Their computation allows, in principle, to calculate and construct the minimal free resolution of any ideal $I \subset S$. Sometimes, this is a much simpler method than the usual computation of syzygies via Gröbner basis. We illustrate this by recovering the Eliahou–Kervaire resolution of stable ideals in a simpler fashion, as shown by Aramova and Herzog.

Let I be a monomial ideal of S, and let $\varepsilon : S \to S/I$ be the canonical epimorphism. For all $1 \leq j \leq n$, we let $\mathbf{x}_j$ be the regular sequence $\mathbf{x}_j = x_j, x_{j+1}, \ldots, x_n$. In particular, $\mathbf{x}_1 = \mathbf{x} = x_1, \ldots, x_n$. One can define the complex $K_\bullet(\mathbf{x}_j; S/I) = K_\bullet(\mathbf{x}_j; S) \otimes S/I$. We set $\varepsilon(f)e_\sigma > \varepsilon(g)e_\tau$ if $e_\sigma > e_\tau$. We denote by $H_i(\mathbf{x}_j; S/I) = H_i(K_\bullet(\mathbf{x}_j; S/I))$ the ith homology module of $\mathbf{x}_j$ with respect to S/I. These modules are graded. If $z \in K_i(\mathbf{x}_j; S/I)$ is a *Koszul cycle*, *i.e.*, $\partial_i(z) = 0$, the symbol $[z]$ denotes the *homology class* of z in $H_i(\mathbf{x}_j; S/I)$.

We recall the following rule of multiplication: $\partial(a \wedge b) = \partial(a) \wedge b + (-1)^{\deg a} a \wedge \partial(b)$ for $a, b \in K_\bullet(\mathbf{x}; S/I)$, with a homogeneous.

To simplify the notation, we set $K_i(\mathbf{x}_j) = K_i(\mathbf{x}_j; S/I)$ and $H_i(\mathbf{x}_j) = H_i(\mathbf{x}_j; S/I)$, for all i and all $j = 1, \ldots, n$. Each module $H_i(\mathbf{x}_j)$ is a $S = K[x_1, \ldots, x_n]$-module, so it is in particular a $S/(\mathbf{x}_j) \cong K[x_1, \ldots, x_{j-1}]$-module, and for $j = 1$, a K-vector space.

Let $1 \leq j \leq n-1$. For all i, let $\alpha_i : K_i(\mathbf{x}_{j+1}) \to K_i(\mathbf{x}_j)$ be the inclusion. We define also a homomorphism $\beta_i : K_i(\mathbf{x}_j) \to K_{i-1}(\mathbf{x}_{j+1})$ as follows: Any element $a \in K_i(\mathbf{x}_j)$ can be written uniquely as $a = e_j \wedge b + c$, for unique $b \in K_{i-1}(\mathbf{x}_{j+1})$ and $c \in K_i(\mathbf{x}_{j+1})$, we set $\beta_i(a) = b$. One immediately verifies that $\beta_i \circ \alpha_i = 0$.

We can construct the following short exact sequence of complexes

$$0 \to K_\bullet(\mathbf{x}_{j+1}; S/I) \xrightarrow{\alpha} K_\bullet(\mathbf{x}_j; S/I) \xrightarrow{\beta} K_\bullet(\mathbf{x}_{j+1}; S/I)[-1] \to 0, \tag{2.3}$$

where $[-1]$ denotes the shifting of the homological degree by -1; indeed the following diagram is commutative with exact rows

$$\begin{tikzcd}
 & \vdots \arrow[d, "\partial_{i+2}"'] & \vdots \arrow[d, "\partial_{i+2}"'] & \vdots \arrow[d, "\partial_{i+1}"'] & \\
0 \arrow[r] & K_{i+1}(\mathbf{x}_{j+1}) \arrow[r, "\alpha_{i+1}"] \arrow[d, "\partial_{i+1}"'] & K_{i+1}(\mathbf{x}_j) \arrow[r, "\beta_{i+1}"] \arrow[d, "\partial_{i+1}"'] & K_i(\mathbf{x}_{j+1}) \arrow[r] \arrow[d, "\partial_i"'] & 0 \\
0 \arrow[r] & K_i(\mathbf{x}_{j+1}) \arrow[r, "\alpha_i"] \arrow[d, "\partial_i"'] & K_i(\mathbf{x}_j) \arrow[r, "\beta_i"] \arrow[d, "\partial_i"'] & K_{i-1}(\mathbf{x}_{j+1}) \arrow[r] \arrow[d, "\partial_{i-1}"'] & 0 \\
0 \arrow[r] & K_{i-1}(\mathbf{x}_{j+1}) \arrow[r, "\alpha_{i-1}"] \arrow[d, "\partial_{i-1}"'] & K_{i-1}(\mathbf{x}_j) \arrow[r, "\beta_{i-1}"] \arrow[d, "\partial_{i-1}"'] & K_{i-2}(\mathbf{x}_{j+1}) \arrow[r] \arrow[d, "\partial_{i-2}"'] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Therefore, we can apply the homology functor H to the short exact sequence of complexes (2.3). By abuse of notations, we denote the induced maps $H_i(\alpha), H_i(\beta)$

again by α_i, β_i, for all i. So, we have the long exact sequence

$$\begin{array}{l} \cdots \xrightarrow{\delta_{i+1}} H_{i+1}(\mathbf{x}_{j+1}) \xrightarrow{\alpha_{i+1}} H_{i+1}(\mathbf{x}_j) \xrightarrow{\beta_{i+1}} H_i(\mathbf{x}_{j+1}) \xrightarrow{\delta_i} \\ \xrightarrow{\delta_i} H_i(\mathbf{x}_{j+1}) \xrightarrow{\alpha_i} H_i(\mathbf{x}_j) \xrightarrow{\beta_i} H_{i-1}(\mathbf{x}_{j+1}) \xrightarrow{\delta_{i-1}} \\ \xrightarrow{\delta_{i-1}} H_{i-1}(\mathbf{x}_{j+1}) \xrightarrow{\alpha_{i-1}} \quad \cdots \quad \xrightarrow{\alpha_0} H_0(\mathbf{x}_j) \longrightarrow 0, \end{array} \tag{2.4}$$

where the maps δ_i are the *connecting homomorphisms*. It can be verified that δ_i is *multiplication by* $\pm x_j$, *i.e.*, $\delta_i([a]) = \pm x_j[a]$, see [89, Theorem A.3.3].

Theorem 2.4.6 (Aramova–Herzog, 1996 [10]). *Let $I \subset S$ be a stable monomial ideal. Then, the following statements hold.*

(a) *For all $i \geq 1$, the K-vector space $H_i(\mathbf{x}; S/I)$ has as a basis the homology classes of the Koszul cycles*

$$\varepsilon(u')e_\sigma \wedge e_{\max(u)} \quad \textit{with} \quad u \in G(I), \quad \sigma \subseteq [\max(u) - 1], \quad |\sigma| = i - 1.$$

For any such element, we set $f(u;\sigma) = 1 \otimes (-1)^{(i-1)(i-2)/2}[\varepsilon(u')e_\sigma \wedge e_{\max(u)}]$. If σ does not satisfy the above side conditions, we set $f(u;\sigma) = 0$.

(b) *Let $\mathbb{F}$ be the minimal free resolution of S/I. Then F_i has as a basis the elements $f(u;\sigma)$ given in* (a). *Moreover, the ith differential $d_i : F_i \to F_{i-1}$ is defined by*

$$d_i(f(u;\sigma)) = \sum_{k\in\sigma} (-1)^{\alpha(\sigma;k)}(-x_k f(u;\sigma\setminus k) + \frac{x_k u}{g(x_k u)} f(g(x_k u);\sigma\setminus k)),$$

where $g : M(I) \to G(I)$ is the decomposition map of I (Lemma 2.4.4).

(c) *For all i and j we have*

$$\beta_{i,i+j}(I) = \sum_{u\in G(I)_j} \binom{\max(u)-1}{i}.$$

A squarefree monomial ideal $I \subset S$ is called *squarefree stable* if for all squarefree monomials $u \in I$ and all i such that $i \notin \operatorname{supp}(u)$, we have $x_i(u/x_{\max(u)}) = x_i u' \in I$. The ideal I is called *squarefree strongly stable* if for all squarefree monomials $u \in I$ and all $i < j$ such that $i \notin \operatorname{supp}(u)$ and $j \in \operatorname{supp}(u)$, we have $x_i(u/x_j) \in I$. Any squarefree strongly stable ideal is squarefree stable.

Similarly to Lemma 2.4.4 we have

Lemma 2.4.7 *Let $I \subset S$ be a squarefree stable ideal. Then, for all $w \in M(I)$ there exist unique monomials $u \in G(I)$ and $v \in \operatorname{Mon}(S)$ such that $w = uv$ and $\max(u) \leq \min(v)$. For any $w \in M(I)$, we set $g(w) = u$ and such position defines a map $g : M(I) \to G(I)$ which we call the decomposition map of I.*

As a consequence of this lemma, the squarefree version of Theorem 2.4.6 holds.

Theorem 2.4.8 (Aramova–Herzog–Hibi, 2000 [12]). *Let $I \subset S$ be a squarefree stable monomial ideal. Then, the following statements hold.*

(a) *For all $i \geq 1$, the K-vector space $H_i(\mathbf{x}; S/I)$ has as a basis the homology classes of the Koszul cycles*

$$\varepsilon(u')e_\sigma \wedge e_{\max(u)} \quad \textit{with} \quad u \in G(I), \quad \sigma \subseteq [\max(u)-1] \setminus \operatorname{supp}(u), \quad |\sigma| = i-1.$$

For any such element, we set $f(u;\sigma) = 1 \otimes (-1)^{(i-1)(i-2)/2}[\varepsilon(u')e_\sigma \wedge e_{\max(u)}]$. If σ does not satisfy the above side conditions, we set $f(u;\sigma) = 0$.

(b) *Let $\mathbb{F}$ be the minimal free resolution of S/I. Then F_i has as a basis the elements $f(u;\sigma)$ given in* (a)*. Moreover, the ith differential $d_i : F_i \to F_{i-1}$ is defined by*

$$d_i(f(u;\sigma)) = \sum_{k\in\sigma}(-1)^{\alpha(\sigma;k)}(-x_k f(u;\sigma\setminus k) + \frac{x_k u}{g(x_k u)} f(g(x_k u);\sigma \setminus k)),$$

where $g : M(I) \to G(I)$ is the decomposition map of I (Lemma 2.4.7).

(c) *For all i and j we have*

$$\beta_{i,i+j}(I) = \sum_{u\in G(I)_j} \binom{\max(u)-j}{i}.$$

2.4.4 Linear quotients

Monomial ideals with linear quotients were introduced by Herzog and Takayama [100]. Let $I \subset S$ be a monomial ideal. We say that I has *linear quotients* if for some order $u_1, \ldots, u_m$ of its minimal generating set $G(I)$, all colon ideals $(u_1, \ldots, u_{\ell-1}) : (u_\ell)$, $\ell = 2, \ldots, m$, are generated by a subset of the set of variables $\{x_1, \ldots, x_n\}$.

In such a case $u_1, u_2, \ldots, u_m$ is called an *admissible order* of I, and it is called *non-increasing* if $\deg(u_1) \leq \cdots \leq \deg(u_m)$. By [110, Lemma 2.1], an ideal with linear quotients always has a non-increasing admissible order. So, from now, we consider only non-increasing admissible orders. Furthermore, we define

$$\operatorname{set}(u_k) \;=\; \{i : x_i \in (u_1, \ldots, u_{k-1}) : u_k\}.$$

By [89, Proposition 1.2.2], $(u_1, \ldots, u_{\ell-1}) : (u_\ell)$ is generated by the monomials

$$\frac{u_j}{\gcd(u_j, u_\ell)} = \frac{\operatorname{lcm}(u_j, u_\ell)}{u_\ell}, \qquad j = 1, \ldots, \ell-1.$$

Thus, by [89, Lemma 8.2.3], $u_1, \ldots, u_m$ is an admissible order of I if and only if for all $j < \ell$ there exist an integer $k < i$ and an integer p such that

$$\frac{\operatorname{lcm}(u_k, u_\ell)}{u_\ell} = x_p \quad \text{and} \quad x_p \quad \text{divides} \quad \frac{\operatorname{lcm}(u_j, u_\ell)}{u_\ell}.$$

Famous classes of monomial ideals with linear quotients include:

- stable ideals,
- squarefree stable ideals,
- edge ideals of cochordal graphs (Theorem 3.3.6),
- polymatroidal ideals (Theorem 3.4.2).

We define the map $g : M(I) \to G(I)$ as follows. If $w \in M(I)$, then $g(w) = u_j$, where j is the smallest integer such that $w \in (u_1, \dots, u_j)$. We call g the *decomposition map* of I. We say that g is *regular* if

$$\operatorname{set}(g(x_s u)) \subseteq \operatorname{set}(u), \quad \text{for all } u \in G(I) \text{ and all } s \in \operatorname{set}(u).$$

We set $\mathbf{x}_\varnothing = 1$ and $\mathbf{x}_F = \prod_{i \in F} x_i$ if $F \subseteq [n]$ is non empty.

Theorem 2.4.9 (Herzog–Takayama, 2002 [100]). *Let $I \subset S$ be a monomial ideal with linear quotients and admissible order $u_1, \dots, u_m$, and let $\mathbb{F}$ be the minimal free resolution of S/I. Then*

(a) *The ith free module of $\mathbb{F}$, F_i, has as a basis the symbols $f(u;\sigma)$ with $u \in G(I)$, $\sigma \subseteq \operatorname{set}(u)$ and $|\sigma| = i - 1$. Furthermore, $f(u;\sigma)$ has multidegree $\mathbf{x}_\sigma u$.*

(b) *Suppose that the decomposition map g is regular. Then, for all $i > 0$, the ith differential $d_i : F_i \to F_{i-1}$ of $\mathbb{F}$ is defined by*

$$d_i(f(u;\sigma)) = \sum_{k \in \sigma} (-1)^{\alpha(\sigma;k)} (-x_k f(u; \sigma \setminus k) + \frac{x_k u}{g(x_k u)} f(g(x_k u); \sigma \setminus k)).$$

(c) *For all i and j we have*

$$\beta_{i,i+j}(I) = \sum_{u \in G(I)_j} \binom{|\operatorname{set}(u)|}{i}. \tag{2.5}$$

(d) *An ideal with linear quotients is componentwise linear.*

This result generalizes Theorems 2.4.5, 2.4.6 and 2.4.8. Indeed the decomposition map of (squarefree) stable ideals is regular as noted in [100].

If I is generated in the same degree and it has linear quotients, then I has a linear resolution. This fact is extremely useful in combinatorial commutative algebra. Indeed, if we can prove that an equigenerated monomial ideal I and all its powers I^k ($k \geq 2$) have linear quotients, then it will follow that I has *linear powers*, *i.e.*, I and all its powers have linear resolutions.

2.4.5 Betti splittings

In this section, we survey the basics of Betti splittings theory. The techniques developed in this section include as particular cases the Eliahou–Kervaire resolution and the class of monomial ideals with linear quotients.

Let $G(I)$ be the unique minimal generating set of I. In [74], the authors pointed out that to compute a minimal graded free resolution of I, one can "split" the ideal I into "smaller" ideals I_1, I_2, *i.e.*, $I = I_1 + I_2$ with $G(I)$ the disjoint union of $G(I_1)$ and $G(I_2)$. Hence, to get the minimal free resolution of I, one can use the minimal free resolutions of I_1 and I_2 together with that of $I_1 \cap I_2$.

Let us consider the short exact sequence

$$0 \to I_1 \cap I_2 \xrightarrow{\psi_{-1}} I_1 \oplus I_2 \xrightarrow{\varphi} I = I_1 + I_2 \to 0, \tag{2.6}$$

where $\psi_{-1}(w) = (w, -w)$ and $\varphi((w,z)) = w + z$. Let $\mathbb{A}$ be the minimal free resolution of $I_1 \cap I_2$ and let $\mathbb{B}$ be the minimal free resolution of $I_1 \oplus I_2$. Note that $\mathbb{B}$ is the direct sum of the minimal free resolutions of I_1 and I_2. Since all S-modules A_i are free, thus projective, there exists a complex homomorphism $\psi : \mathbb{A} \to \mathbb{B}$ *lifting* the map ψ_{-1}, that means a sequence of maps $\psi_i : A_i \to B_i$ $(i \geq 0)$, called the *comparison maps*, making the following diagram

$$\begin{tikzcd}
\mathbb{A} : \cdots \arrow[r] & A_2 \arrow[r, "d_2^A"] \arrow[d, "\psi_2"'] & A_1 \arrow[r, "d_1^A"] \arrow[d, "\psi_1"'] & A_0 \arrow[r, "d_0^A"] \arrow[d, "\psi_0"'] & I_1 \cap I_2 \arrow[r] \arrow[d, "\psi_{-1}"] & 0 \\
\mathbb{B} : \cdots \arrow[r] & B_2 \arrow[r, "d_2^B"'] & B_1 \arrow[r, "d_1^B"'] & B_0 \arrow[r, "d_0^B"'] & I_1 \oplus I_2 \arrow[r] & 0
\end{tikzcd} \tag{2.7}$$

commutative.

It is well known that ψ gives rise to an acyclic complex $C(\psi)$ whose 0th homology module is $H_0(C(\psi)) = \operatorname{coker}(\psi_{-1}) = (I_1 \oplus I_2)/\operatorname{Im}(\psi_{-1}) \cong (I_1 \oplus I_2)/(I_1 \cap I_2) \cong I_1 + I_2 = I$, *i.e.*, $C(\psi)$ is a free resolution of I (see, for instance, [47, Appendix A3.12]).

The complex $C(\psi)$ is defined as follows:

(i) let $C_0 = B_0$, and $C_i = A_{i-1} \oplus B_i$, for $i > 0$;

(ii) let $d_0 = \varphi \circ d_0^B$, $d_1 = (0, \psi_0 + d_1^B)$, and $d_i = (-d_{i-1}^A, \psi_{i-1} + d_i^B)$, for $i > 1$.

This procedure, known as the *mapping cone*, may be visualized as follows:

$$\begin{tikzcd}
\mathbb{A}[-1] : \cdots \arrow[r] & A_2 \arrow[r, "d_2^A"] \arrow[drr, "\psi_2"] & & A_1 \arrow[r, "d_1^A"] \arrow[drr, "\psi_1"] & & A_0 \arrow[r, "d_0^A"] \arrow[drr, "\psi_0"] & I_1 \cap I_2 \arrow[r] & 0 & & \\
& \oplus & & \oplus & & \oplus & & & & \\
\mathbb{B} : \cdots \arrow[r] & B_3 \arrow[rr, "d_3^B"'] & & B_2 \arrow[rr, "d_2^B"'] & & B_1 \arrow[rr, "d_1^B"'] & & B_0 \arrow[r, "d_0^B"'] & I_1 \oplus I_2 \arrow[r] & 0
\end{tikzcd}$$

Here $\mathbb{A}[-1]$ is the complex $\mathbb{A}$ *homologically shifted* by -1.

Unfortunately, the free resolution $C(\psi)$ is not always minimal. The next theorem proved by Francisco, Há and Van Tuyl [74] characterizes when $C(\psi)$ is a minimal free resolution of I.

Theorem 2.4.10 (Francisco–Há–Van Tuyl, 2009 [74, Proposition 2.1]). *Let I, I_1, I_2 be monomial ideals of S such that $I = I_1 + I_2$ and $G(I)$ is the disjoint union of $G(I_1)$ and $G(I_2)$. Then the following conditions are equivalent:*

(a) *For all i and j, we have*

$$\beta_{i,j}(I) = \beta_{i,j}(I_1) + \beta_{i,j}(I_2) + \beta_{i-1,j}(I_1 \cap I_2). \tag{2.8}$$

(b) *For all i and j, the map*

$$\operatorname{Tor}_i^S(K, I_1 \cap I_2)_j \to \operatorname{Tor}_i^S(K, I_1)_j \oplus \operatorname{Tor}_i^S(K, I_2)_j$$

in the long exact sequence in Tor *induced from* (2.6) *is the zero map.*

(c) *Applying the mapping cone to* (2.6) *gives a minimal free resolution of I.*

Furthermore, if any of the equivalent conditions (a)–(b)–(c) *holds, then*

$$\begin{aligned}
\operatorname{pd}(I) &= \max\{\operatorname{pd}(I_1), \operatorname{pd}(I_2), \operatorname{pd}(I_1 \cap I_2) + 1\}, \\
\operatorname{reg}(I) &= \max\{\operatorname{reg}(I_1), \operatorname{reg}(I_2), \operatorname{reg}(I_1 \cap I_2) - 1\}.
\end{aligned}$$

Hence, we can give the next definition.

Definition 2.4.11 Let I, I_1, I_2 be monomial ideals of S such that $I = I_1 + I_2$ and $G(I)$ is the disjoint union of $G(I_1)$ and $G(I_2)$. We say that $I = I_1 + I_2$ is a *Betti splitting* if any of the previous equivalent conditions (a)–(b)–(c) is satisfied.

Next, we describe a special case of Betti splittings.

Definition 2.4.12 Let I be a monomial ideal of S. Let I_1 be the ideal generated by all elements of $G(I)$ divisible by x_i and let I_2 be the ideal generated by all other elements of $G(I)$. We call $I = I_1 + I_2$ a *x_i-partition* of I. If $I = I_1 + I_2$ is also a Betti splitting, we call $I = I_1 + I_2$ a *x_i-splitting*.

The following two results will be needed later.

Proposition 2.4.13 ([74, Corollary 2.7]). *Let $I = I_1 + I_2$ be a x_i-partition of I and I_1 be the ideal generated by all elements of $G(I)$ divisible by x_i. If I_1 has a linear resolution, then $I = I_1 + I_2$ is a Betti splitting.*

Proposition 2.4.14 ([20, Proposition 3.1]). *Let $I \subset S$ be a monomial ideal with a d-linear resolution, I_1, I_2 monomial ideals such that $I = I_1 + I_2$, $G(I) = G(I_1) \cup G(I_2)$ and $G(I_1) \cap G(I_2) = \varnothing$. Then the following facts are equivalent:*

(i) *$I = I_1 + I_2$ is a Betti splitting of I;*

(ii) *I_1 and I_2 have d-linear resolutions.*

If this is the case, then $I_1 \cap I_2$ has a $(d+1)$-linear resolution.

Monomial ideals with linear quotients are a special case of Betti splittings.

Proposition 2.4.15 *Let $I \subset S$ be a monomial ideal with linear quotients and admissible order $u_1, \dots, u_m$. Then $(u_1, \dots, u_{j-1}) + (u_j)$ is a Betti splitting, for $j = 2, \dots, m$.*

This proposition is a special case of the next result due to Bolognini [20].

Criterion 2.4.16 (Bolognini, 2016 [20, Theorem 3.3]). *Let I, I_1, I_2 be monomial ideals of S such that $G(I)$ is the disjoint union of $G(I_1)$ and $G(I_2)$. Suppose that I_1 and I_2 are componentwise linear. Then $I = I_1 + I_2$ is a Betti splitting.*

Let $j \in \{2, \dots, m\}$. Then, $(u_1, \dots, u_{j-1})$ has linear quotients and so it is componentwise linear (Theorem 2.4.9(d)). The ideal (u_j) is componentwise linear as well, for it is a principal ideal. Thus, $(u_1, \dots, u_{j-1}) + (u_j)$ is a Betti splitting.

Chapter 3

Combinatorics on monomial ideals

In this second chapter, we survey the combinatorial and algebraic methods, developed in the last fifty years, to manage monomial ideals. The first systematic study of monomial ideals was carried out by Hochster [105], who linked squarefree monomial ideals and their free resolutions to simplicial (co)homology of simplicial complexes.

Even thought squarefree monomial ideals are a special class of monomial ideals, via polarization one can always reduce to such a class. Indeed, polarization is a deformation technique that allows to transform a monomial ideal into a squarefree monomial ideal and preserves the graded Betti numbers, the Hilbert function, the height, the Cohen-Macaulayness and the Gorensteiness property.

Thus, if interested in such invariants of a monomial ideal I, one may assume from the very beginning that I is squarefree. In this case, there is a one-to-one correspondence between squarefree monomial ideals and simplicial complexes. This theory was developed by Hochster in the seventies, and was successfully employed by Stanley and Reisner who proved the upper bound conjecture for simplicial spheres. This breakthrough of the theory opened up a new research trend in Commutative Algebra, namely Combinatorial Commutative Algebra.

In Section 3.2 we discuss Alexander duality theory. Using this duality theory, one can characterize Cohen–Macaulay squarefree monomial ideals.

Perhaps, the most fashionable topic in Combinatorial Commutative Algebra is that of *edge ideals*. Edge ideals were introduced in 1990 by Villarreal [148]. Let G be a finite simple graph on vertex set $V(G) = \{1, 2, \dots, n\}$ and edge set $E(G)$. Then, the edge ideal of G, is the squarefree monomial ideal $I(G)$ generated by all monomials $x_i x_j$ such that $\{i, j\} \in E(G)$. After, recalling some useful notions from graph theory, we use Alexander duality to characterize when $I(G)$ is height–unmixed. It turns out that is equivalent to saying that G is unmixed or well–covered. That is, all minimal vertex covers of G have the same size. One can interpret the Alexander dual $I(G)^\vee$ of $I(G)$ as the cover ideal $J(G)$ of G. Bounds for the projective dimension and the Castelnuovo–Mumford regularity of (powers of) $I(G)$ are discussed. The classification of edge ideals with linear powers is explained in subsection 3.3.1.

Finally, in Section 3.4, we discuss discrete polymatroids and polymatroidal ideals.

3.1 The polarization technique

Let us recall the technique of *polarization* which is an operation that transforms a monomial ideal into a squarefree monomial ideal in a larger polynomial ring.

Let $u = x_1^{b_1} \cdots x_n^{b_n} \in \mathrm{Mon}(S)$. Then, the *polarization* of u is the monomial

$$u^\wp = \prod_{i=1}^{n} \Big(\prod_{j=1}^{b_i} x_{i,j}\Big) = \prod_{\substack{i=1,\dots,n\\ b_i>0}} x_{i,1} x_{i,2} \cdots x_{i,b_i}$$

in the polynomial ring $K[x_{i,j} : i = 1, \dots, n, j = 1, \dots, b_i]$.

Let $I \subset S$ be a monomial ideal, and set $a_i = \max\{\deg_{x_i}(u) : u \in G(I)\}$, for $i = 1, \dots, n$. Let
$$R = K[x_{i,j} : i = 1, \dots, n, j = 1, \dots, a_i]$$
be the polynomial ring in the variables $x_{i,j}$, $i = 1, \dots, n, j = 1, \dots, a_i$. The *polarization* of the monomial ideal I is defined to be the squarefree monomial ideal $I^{\wp}$ of R with minimal generating set $G(I^{\wp}) = \{u^{\wp} : u \in G(I)\}$.

Let $M \in \mathcal{M}$, recall that $\mathrm{Hilb}_M(t) = \sum_d \dim_K(M_d)t^d$ is the *Hilbert series* of M.

Polarization preserves the main algebraic and homological properties of I.

Theorem 3.1.1 ([89, Corollary 1.6.3]). *Let $I \subset S$ be a monomial ideal, and let $I^{\wp} \subset R$ be its polarization. Then, the following hold:*

(a) $\beta_{i,j}(I) = \beta_{i,j}(I^{\wp})$ *for all i and j.*

(b) $\mathrm{Hilb}_{S/I}(t) = (1-t)^d \, \mathrm{Hilb}_{R/I^{\wp}}(t)$, *where $d = \dim R - \dim S$.*

(c) $\mathrm{height}(I) = \mathrm{height}(I^{\wp})$.

(d) $\mathrm{pd}(S/I) = \mathrm{pd}(R/I^{\wp})$ *and* $\mathrm{reg}(S/I) = \mathrm{reg}(R/I^{\wp})$.

(e) *S/I is Cohen–Macaulay (Gorenstein) if and only if $R/I^{\wp}$ is Cohen–Macaulay (Gorenstein).*

The following key lemma proved by Olteanu [135, Proposition 5.3] follows by a more general result stated by Jahan [109, Lemma 3.3].

Lemma 3.1.2 *Let $I \subset S$ be a monomial ideal. Then I has linear quotients with admissible order $u_1, u_2, \dots, u_m$ of $G(I)$ if and only if $I^{\wp}$ has linear quotients with admissible order $u_1^{\wp}, u_2^{\wp}, \dots, u_m^{\wp}$ of $G(I^{\wp})$.*

3.2 Simplicial complexes

A *simplicial complex* Δ on the vertex set $[n]$ is a family of subsets of $[n]$, called the *faces* of Δ, such that

- $\{i\} \in \Delta$ for all $i \in [n]$, and
- if $F \subseteq \Delta$, $G \subseteq F$, we have $G \in \Delta$.

The dimension of $F \in \Delta$ is the number $|F| - 1$. The dimension of Δ is the number $d = \max\{|F| - 1 : F \in \Delta\}$. A face $F \in \Delta$ maximal with respect to the inclusion, is called a *facet*. Let $F_1, \dots, F_m$ be the facets of Δ, then we write $\Delta = \langle F_1, \dots, F_m \rangle$. We denote by $\mathcal{F}(\Delta)$ the set of facets of Δ.

Let Δ be a simplicial complex on $[n]$ and let $F \in \Delta$, we set $\mathbf{x}_F = \prod_{i \in F} x_i$ if F is non empty and we set $\mathbf{x}_{\varnothing} = 1$ otherwise. The *Stanley-Reisner ideal* of Δ is the following squarefree monomial ideal of $S = K[x_1, \dots, x_n]$,
$$I_\Delta = (\mathbf{x}_F \; : \; F \notin \Delta).$$

Whereas, the Stanley-Reisner ring of Δ is the factor ring $K[\Delta] = S/I_\Delta$.

Let $d = \dim(\Delta)$. For all $-1 \leq i \leq d$, we set $f_i(\Delta) = |\{F \in \Delta : |F| = i\}|$. We call $f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \dots, f_d(\Delta))$ the *f-vector* of Δ.

Proposition 3.2.1 ([51, Theorem 5.9]). *Let Δ be a simplicial complex on $[n]$ of dimension $d-1$ and with f-vector $f(\Delta) = (f_{-1}, f_0, f_1, \ldots, f_{d-1})$. Then*

$$\mathrm{Hilb}_{K[\Delta]}(t) = \frac{\sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i}}{(1-t)^d}.$$

In particular, $\dim K[\Delta] = \dim(\Delta) + 1$.

Let $S = K[x_1, \ldots, x_n]$ with K a field. We have the following bijection,

$$\{\textit{simplicial complexes on } [n]\} \longleftrightarrow \{\textit{squarefree monomial ideals of } S\}.$$

3.2.1 Hochster formula

Let Δ be a simplicial complex on the vertex set $[n]$, and let I_Δ be its Stanley-Reisner ideal. By Hochster's formula [89, Theorem 8.1.1] we have

$$\beta_i(S/I_\Delta) = \sum_{W \subseteq [n]} \dim_K \widetilde{H}^{|W|-i-1}(\Delta_W; K),$$

where $\widetilde{H}^j(\Delta_W; K)$ is the jth *reduced simplicial cohomology module* of the simplicial complex $\Delta_W = \{F \in \Delta : F \subseteq W\}$.

Hochster's formula allows us also to compute the graded Betti numbers.

Theorem 3.2.2 *Let $I = I_\Delta \subset S$ be a squarefree monomial ideal. Then*

$$\beta_{i,j}(I_\Delta) \;=\; \sum_{W \subseteq [n],\ |W|=j} \dim_K \widetilde{H}^{j-i-2}(\Delta_W; K).$$

Next, we record some nice consequences of Hochster formula. Let Δ be a simplicial complex on $[n]$. A vector $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$ is called *squarefree* if $a_i \in \{0, 1\}$ for all i. The following facts hold.

- Δ is connected if and only if $\widetilde{H}^0(\Delta; K) = 0$.
- $\beta_{i,\mathbf{a}}(I_\Delta) = 0$, if $\mathbf{a}$ is non squarefree.

3.2.2 Alexander duality

Let Δ be a simplicial complex on $[n]$. The *Alexander dual* of Δ is

$$\Delta^\vee \;=\; \{[n] \setminus F : F \notin \Delta\}.$$

One has $(\Delta^\vee)^\vee = \Delta$. We denote $I_{\Delta^\vee}$ by $I^\vee$. Thus, if $I \subset S$ is a squarefree monomial ideal, then $(I^\vee)^\vee = I$. For a subset F of $[n]$, let $\mathfrak{p}_F = (x_i : i \in F)$.

Proposition 3.2.3 *Let Δ be a simplicial complex on $[n]$. Then,*

(a) $I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} \mathfrak{p}_{[n]\setminus F}$ *is the minimal primary decomposition of* I_Δ.

(b) *Let $I_\Delta = \mathfrak{p}_{G_1} \cap \mathfrak{p}_{G_2} \cap \cdots \cap \mathfrak{p}_{G_m}$ be the minimal primary decomposition of I_Δ. Then $G(I_{\Delta^\vee}) = \{\mathbf{x}_{G_1}, \mathbf{x}_{G_2}, \ldots, \mathbf{x}_{G_m}\}$.*

The next fundamental result was proved by Eagon and Reiner [89, Theorem 8.1.9].

Criterion 3.2.4 (Eagon–Reiner 1998, [46]). *Let $I \subset S$ be a squarefree monomial ideal. Then I is Cohen–Macaulay if and only if its Alexander dual $I^\vee$ has a linear resolution.*

3.3 Edge ideals

Edge ideals have been introduced by Villarreal in 1990 [148]. Firstly, let us summarize some basic facts from *graph theory.*

A *simple graph* G is an ordered pair of disjoint finite sets $(V(G), E(G))$ such that $E(G)$ is a subset of the set of unordered pairs of $V(G)$. The set $V(G)$ is the set of *vertices* and the set $E(G)$ is called the set of *edges.*

If $e = \{u, v\}$ is an edge of G one says that the vertices u and v are *adjacent.*

A *walk* of length n in G is an alternating sequence of vertices and edges, written as $w = \{v_0, z_1, v_1, \dots, v_{n-1}, z_n, v_n\}$, where $z_i = \{v_{i-1}, v_i\}$ is the edge joining the vertices v_{i-1} and v_i. A walk may also be written $\{v_0, \dots, v_n\}$ with the edges understood, or $\{z_1, z_2, \dots, z_n\}$ with the vertices understood. If $v_0 = v_n$, the walk w is called a *closed walk.* A *path* is a walk with all its vertices distinct. A *cycle* of length n is a closed path $\{v_0, \dots, v_n\}$ in which $n \geq 3$. A *forest* is an acyclic graph.

To each simple graph G on the vertex set $V(G) = [n]$ we can associate a squarefree ideal $I(G)$ of the polynomial ring $S = K[x_1, \dots, x_n]$, called the *edge ideal* associated to G, defined as follows [149]:

$$I(G) = (x_i x_j \ : \ i \textit{ is adjacent to } j) = (x_i x_j \ : \ \{i, j\} \in E(G)).$$

Let G be a graph with *vertex set* $V(G) = [n]$ and with *edge set* $E(G)$. Let $i \in V(G)$ be a vertex of G. The *open neighborhood* of i is the set

$$N_G(i) = \{j \in V(G) : \{i, j\} \in E(G)\},$$

whereas the *closed neighborhood* of i is the set defined as follows

$$N_G[i] = \{j \in V(G) : \{i, j\} \in E(G)\} \cup \{i\}.$$

When the context is clear, we drop the subscript G.

If $W \subseteq V(G)$, we denote by $G \setminus W$ the subgraph of G with the vertices of W and their incident edges deleted.

A subset W of $V(G)$ is called a *vertex cover* if every edge of G is incident with at least one vertex in W. A vertex cover W is called a *minimal vertex cover* if there is no proper subset of W which is a vertex cover of G. The set of all minimal vertex cover of G is denote by $\mathcal{C}(G)$.

Attached to G, we have the *cover ideal* of G:

$$J(G) \ = \ (x_{i_1} \cdots x_{i_s} \ : \ W = \{i_1, \dots, i_s\} \textit{ is a minimal vertex cover of } G).$$

The following fundamental observation holds.

Proposition 3.3.1 *The Alexander dual of the edge ideal of G is the cover ideal of G,*

$$I(G)^{\vee} \ = \ J(G).$$

A graph G is *unmixed* or *well–covered* if all the minimal vertex covers have the same cardinality. In particular, all the associated primes of $I(G)$ have the same height. A graph G is called *Cohen–Macaulay* over the field K if $S/I(G)$ is a Cohen–Macaulay ring. It is clear that a Cohen–Macaulay graph is unmixed [149, Proposition 7.2.9].

A pairing off of all the vertices of a graph G is called a *perfect matching.* Thus G has a perfect matching if and only if G has an even number of vertices and there is a set of independent edges covering all the vertices, where for a set of independent edges we mean a set of pairwise disjoint edges [149].

In the case of edge ideals, the Eagon–Reiner criterion (Criterion 3.2.4) can be restated as follows.

Theorem 3.3.2 *Let G be a finite simple graph. Then $I(G)$ is Cohen–Macaulay if and only if $J(G)$ has a linear resolution.*

Hence, we also have

Corollary 3.3.3 *Let G be a finite simple graph. Then $J(G)$ is Cohen–Macaulay if and only if $I(G)$ has a linear resolution.*

From Theorem 3.3.2, we can observe that for $I(G)$ to be Cohen–Macaulay, it follows that $J(G)$ must be generated in a single degree, that is, G must be unmixed, as we noted before.

3.3.1 The Dirac–Fröberg Theorem

In this subsection, we address the problem of characterizing all graphs G such that $I(G)$ has a linear resolution. Via Alexander duality, this problem is equivalent to the classification of all graphs G such that $J(G)$ is a Cohen–Macaulay ideal.

A graph G is called *chordal* if it has no induced cycles of length bigger than three. Recall that a *perfect elimination order* of G is an ordering $v_1, \dots, v_n$ of its vertex set $V(G)$ such that $N_{G_i}(v_i)$ induces a complete subgraph on G_i, where G_i is the induced subgraph of G on the vertex set $\{i, i+1, \dots, n\}$. Hereafter, if $1, 2, \dots, n$ is a perfect elimination order of G, we denote it by $x_1 > x_2 > \cdots > x_n$.

Theorem 3.3.4 (Dirac, 1961 [44]). *A finite simple graph G is chordal if and only if G admits a perfect elimination order.*

The *complementary graph* G^c of G is the graph with vertex set $V(G^c) = V(G)$ and where $\{i, j\}$ is an edge of G^c if and only if $\{i, j\} \notin E(G)$. A graph G is called *cochordal* if and only if G^c is chordal.

Theorem 3.3.5 (Fröberg, 1988 [75]). *Let G be a finite simple graph. Then, $I(G)$ has a linear resolution if and only if G is cochordal.*

It is known by [89, Theorem 10.2.6] that $I(G)$ has linear resolution if and only if it has linear quotients. The theorems of Dirac and Fröberg classify all edge ideals with linear quotients. Furthermore if $x_1 > x_2 > \cdots > x_n$ is a perfect elimination order of G^c, then $I(G)$ has linear quotients with respect to the lexicographic order $>_{\text{lex}}$ induced by $x_1 > x_2 > \cdots > x_n$.

One can ask when $I(G)$ has linear powers. This problem has been solved by Herzog, Hibi and Zheng.

Theorem 3.3.6 (Herzog–Hibi–Zheng, 2004 [94]). *Let G be a finite simple graph. Then, the following conditions are equivalent.*

(a) *G is cochordal.*

(b) *$I(G)^k$ has linear resolution, for all $k \geq 1$.*

(c) *$I(G)^k$ has linear quotients, for all $k \geq 1$.*

3.4 Discrete Polymatroids

For a monomial $u \in S$, recall that the x_i-*degree* of u is the integer defined as

$$\deg_{x_i}(u) \;=\; \max\{j \geq 0 \;:\; x_i^j \textit{ divides } u\}.$$

A *polymatroidal ideal* $I \subset S = K[x_1, \dots, x_n]$ is a monomial ideal I generated in a single degree verifying the following *exchange property*: for all $u, v \in G(I)$ with $u \neq v$ and all i such that $\deg_{x_i}(u) > \deg_{x_i}(v)$, there exists j such that $\deg_{x_j}(u) < \deg_{x_j}(v)$ and $x_j(u/x_i) \in G(I)$.

The name polymatroidal ideal is justified by the fact that their minimal generating set corresponds to the set of bases of a *discrete polymatroid*. Discrete polymatroids were first introduced by Herzog and Hibi in 2002 [87]. A squarefree polymatroidal ideal is called *matroidal*.

This class of monomial ideals is very rich. Indeed, it includes

(i) *Graphic matroids.* They are the ideals generated by the monomials $\prod_{i \in F} x_i$, for all spanning forests F of a finite simple graph G on n vertices.

(ii) *Transversal polymatroidal ideals.* They are of the form $I = \mathfrak{p}_{A_1}\mathfrak{p}_{A_2} \cdots \mathfrak{p}_{A_r}$, for some finite collection $A_1, \dots, A_r$ of arbitrary nonempty subsets of $[n]$.

(iii) *Ideals of Veronese type.* Let $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}^n$ be a vector with non negative entries. Then, the ideal of *Veronese type* $(\mathbf{b}, d)$ is defined as

$$I_{\mathbf{b},n,d} \;=\; \big(x_1^{a_1} \cdots x_n^{a_n} \in S \;:\; \sum_{i=1}^{n} a_i = d, \quad a_i \leq b_i, \textit{ for } i \in [n]\big).$$

Any polymatroidal ideal also satisfy a dual version of the exchange property.

Lemma 3.4.1 (Herzog–Hibi, 2006 [86, Lemma 2.1]). *Let $I \subset S$ be a polymatroidal ideal. Then, for all $u, v \in G(I)$ and all i such that $\deg_{x_i}(u) > \deg_{x_i}(v)$, there exists j such that $\deg_{x_j}(u) < \deg_{x_j}(v)$ and $x_i(v/x_j) \in G(I)$.*

There are many useful characterization of polymatroidal ideals. The following one is due Bandari and Rahmati-Asghar.

Theorem 3.4.2 (Bandari–Rahmati, 2019 [14, Theorem 2.4]). *Let $I \subset S$ be a monomial ideal generated in a single degree. Then, I is polymatroidal if and only if I has linear quotients with respect to the lexicographic order induced by any ordering of the variables.*

Chapter 4

Vector-spread monomial ideals

Algebraic shifting is one of the most powerful techniques in Combinatorial Commutative Algebra [89, Chapter 11]. It is based on the idea of shifting and spreading the variables of the generators of a monomial ideal in a coherent way. The origins of this theory date back to a famous article of Erdös, Ko and Rado, *Intersection theorems for systems of finite sets* [54], and made its way into Commutative Algebra through the work of Gil Kalai [113]. Lately, algebraic shifting theory and monomial ideals arising from *shifting operators* [89] have seen a resurgence. The *t-spread monomial ideals* were introduced in 2019 by Ene, Herzog and Qureshi [53]. The homological and combinatorial properties of these and related classes of ideals are the subject of a large body of research [1, 2, 3, 4, 5, 7, 8, 9, 28, 32, 42, 43, 53, 34, 65, 102, 120, 132, 151].

Our purpose is to investigate the more general possible class of such ideals. Let $S = K[x_1, \dots, x_n]$ be a polynomial ring, with K a field. Following [63], given a vector $\mathbf{t} = (t_1, t_2, \dots, t_{d-1}) \in \mathbb{Z}_{\geq 0}^{d-1}$, $d \geq 2$, of non negative integers, we say that a monomial $u = x_{j_1} x_{j_2} \cdots x_{j_\ell} \in S$, with $j_1 \leq j_2 \leq \cdots \leq j_\ell$ and $\ell \leq d$, is a *vector-spread monomial of type* $\mathbf{t}$ or simply a $\mathbf{t}$*-spread monomial* if

$$j_{i+1} - j_i \geq t_i, \quad \text{for all } i = 1, \dots, \ell - 1.$$

For instance, $u = x_1^3 x_2 x_4$ is $(0, 0, 1, 2)$-spread, but not $(1, 0, 1, 2)$-spread. A monomial ideal $I \subseteq S$ is a $\mathbf{t}$*-spread monomial ideal* if it is generated by $\mathbf{t}$-spread monomials. If $t_i = t$, for all i, a $\mathbf{t}$-spread monomial is called an *ordinary* or *uniform* t-spread monomial, see [53]. A $\mathbf{1} = (1, 1, \dots, 1)$-spread monomial ideal is in particular *squarefree*.

Let $\mathbf{t} \in \mathbb{Z}_{\geq 0}^{d-1}$, $d \geq 2$. We say that a $\mathbf{t}$-spread monomial ideal $I \subset S = K[x_1, \dots, x_n]$ is a $\mathbf{t}$*-spread strongly stable ideal* if for any $\mathbf{t}$-spread monomial $u \in I$, and all $j < i$ such that x_i divides u and $x_j(u/x_i)$ is $\mathbf{t}$-spread, then $x_j(u/x_i) \in I$. For $\mathbf{t} = \mathbf{0} = (0, 0, \dots, 0)$ ($\mathbf{t} = \mathbf{1} = (1, 1, \dots, 1)$) one obtains the classical notion of strongly stable (squarefree strongly stable) ideals [89]. On the other hand, strongly stable ideals have a central role in Commutative Algebra. Indeed, for a field K of characteristic zero, they appear as *generic initial ideals* [89]. Eliahou and Kervaire constructed their minimal free resolutions [50]. Bigatti and Hulett showed that among all homogeneous ideals with the same Hilbert function, the lexicographic ideals (which are also strongly stable) have the biggest Betti numbers [19, 106]. Using shifting theory [53, 89, 113], Aramova, Herzog and Hibi extended these results to squarefree ideals [11, 12].

In [53], it was shown that ordinary t-spread strongly stable ideals have *linear quotients* [89, 100]. For ideals with linear quotients, the graded Betti numbers can be easily computed. Moreover, if the so–called *decomposition function* of I is *regular* [100], then the minimal free resolution can be explicitly described. Unfortunately, even ordinary 2-spread strongly stable ideals do not have *regular decomposition functions*, as noted in [53]. Hence, the minimal free resolution of ordinary t-spread strongly stable ideals could not be determined and has remained elusive ever since.

In this chapter, we construct the minimal free resolution of vector-spread strongly stable ideals generalizing the *Eliahou–Kervaire resolution* [50], and extend *algebraic shifting theory* to vector-spread strongly stable ideals [113].

This chapter is organized as follows. After we introduce the concept of vector-spread monomials and ideals in Section 4.1, in Sections 4.2 and 4.3 we construct the minimal free resolution of I a $\mathbf{t}$-spread strongly stable ideal of S. As pointed out before, the method of linear quotients is unavailable to us, as the decomposition functions of vector-spread strongly stable ideals are, in general, non regular. Thus we use *Koszul homology* as developed by Aramova and Herzog in [10]. Let $H_i(\mathbf{x};S/I)$ the ith homology module of $\mathbf{x}=x_1,\dots,x_n$ with respect to S/I. Due to the isomorphism $\text{Tor}_i^S(K,S/I)_j\cong H_i(\mathbf{x};S/I)_j$, one can calculate the graded Betti numbers of S/I as $\beta_{i,j}(S/I)=\dim_K H_i(\mathbf{x};S/I)_j$. Thus one has to determine a basis of this K-vector space. To do so we compute the *Koszul cycles* of S/I. As many examples indicate, Koszul cycles of arbitrary $\mathbf{t}$-spread strongly stable ideals do not have a nice expression as in the cases $\mathbf{t}=\mathbf{0},\mathbf{1}$ [10, 12] (see Theorems 2.4.6(a), 2.4.8(a), and Remark 4.2.10). To compute them we introduce the following notion (Definition 4.2.1). If $u=x_{j_1}x_{j_2}\cdots x_{j_\ell}$ is a $\mathbf{t}$-spread monomial, the $\mathbf{t}$*-spread support* of u is the set

$$\text{supp}_{\mathbf{t}}(u)\ =\ \bigcup_{i=1}^{\ell-1}\big[j_i,j_i+(t_i-1)\big],$$

where $[a,b]=\{c:a\le c\le b\}$, for $a,b\in\mathbb{Z}_{\ge1}$. Let $G(I)$ be the unique minimal set of monomial generators of I. The main result of Section 4.2 is

Theorem 4.2.8. *Let $I\subset S$ be a $\mathbf{t}$-spread strongly stable ideal. Then, for all $i\ge1$, the K-vector space $H_i(\mathbf{x};S/I)$ has as a basis the homology classes of the Koszul cycles*

$$e(u;\sigma)\quad\textit{such that}\quad u\in G(I),\quad \sigma\subseteq[\max(u)-1]\setminus\text{supp}_{\mathbf{t}}(u),\quad |\sigma|=i-1.$$

Firstly, we show that the elements $e(u;\sigma)$ (Definition 4.2.3) are Koszul cycles. We employ an inductive argument, as a direct proof is rather tedious (Remark 4.2.5). Then, we inductively determine the basis for the Koszul homologies of S/I on partial sequences of $\mathbf{x}$. Note that for $\mathbf{t}=\mathbf{0}$ ($\mathbf{t}=\mathbf{1}$) the conditions that σ must satisfy in Theorem 4.2.8 are the same as in [10, Proposition 2.1] ([12, Proposition 2.2]). So, we get a formula for the graded Betti numbers (Corollary 4.4.2) independent from the characteristic of the field K, generalizing the known results in [10, 12, 50, 53].

In Section 4.3, we introduce the $\mathbf{t}$*-spread decomposition function* (Definition 4.3.1). As a consequence, the differentials of the minimal free resolution of S/I are explicitly described (Theorem 4.3.2). Examples 4.2.9, 4.2.11, 4.3.3 illustrate our methods.

Finally, Section 4.4 is devoted to a generalization of the algebraic shifting theory. Classically, a *simplicial complex* Δ on the vertex set $[n]$ is called *shifted* if for all $F\in\Delta$, all $i\in F$, $j\in[n]$, $j>i$, then $(F\setminus\{i\})\cup\{j\}\in\Delta$ [89, 113]. Note that Δ is shifted if and only if the Stanley–Reisner ideal of Δ, I_Δ, is an ordinary squarefree (1-spread) strongly stable ideal [89]. The usefulness of *Combinatorial shifting* comes from the fact that a simplicial complex shares the same f*-vector* of its shifted simplicial complex [89], and moreover the f-vector of the shifted complex is easier to compute.

From the algebraic point of view, *Algebraic shifting* is defined as follows. Let K be a field of characteristic zero. Let $\text{Gin}(I)$ the *generic initial ideal* of $I\subset S$ with respect to the reverse lexicographic order [89], in particular $\text{Gin}(I)$ is ($\mathbf{0}$-spread) strongly stable. One defines $I^s = (\mathrm{Gin}(I))^\sigma$, where σ is the *squarefree operator* that assign to each monomial $u = x_{i_1}x_{i_2}\cdots x_{i_d}$ the monomial $\sigma(u) = x_{i_1}x_{i_2+1}\cdots x_{i_d+(d-1)}$ and to each monomial ideal I, the monomial ideal I^σ with minimal generating set $G(I^\sigma) = \{\sigma(u) : u \in G(I)\}$ [89, 113]. Then, the following properties hold

(Shift$_1$) I^s is a squarefree strongly stable monomial ideal;

(Shift$_2$) $I^s = I$ if I is a squarefree strongly stable ideal;

(Shift$_3$) I and I^s have the same Hilbert function;

(Shift$_4$) If $I \subseteq J$, then $I^s \subseteq J^s$.

We are mainly interested in the algebraic side of this construction. We introduce an analogous "$\mathbf{t}$-spread" algebraic shifting by the assignment $I^{s,\mathbf{t}} = (\mathrm{Gin}(I))^{\sigma_{\mathbf{0},\mathbf{t}}}$, where $\sigma_{\mathbf{0},\mathbf{t}}$ will be a suitable shifting operator. The $\mathbf{t}$-spread versions of (Shift$_1$)-(Shift$_4$) will be established. For $\mathbf{t} = \mathbf{1} = (1, 1, \dots, 1)$, our construction returns the classical one. In particular, $(\mathrm{Gin}(I))^{\sigma_{\mathbf{0},\mathbf{t}}} = I$, if I is a $\mathbf{t}$-spread strongly stable ideal (Theorem 4.4.5).

4.1 Basic concepts

From now on, $\mathbf{t} = (t_1, \dots, t_{d-1})$ is a vector of non negative integers and $d \geq 2$.

Definition 4.1.1 Let $u = x_{j_1}x_{j_2}\cdots x_{j_\ell} \in S$ be a monomial of degree $\ell \leq d$, with $1 \leq j_1 \leq j_2 \leq \cdots \leq j_\ell \leq n$. We say that u is a *vector-spread monomial of type* $\mathbf{t}$, or simply a $\mathbf{t}$-*spread monomial*, if

$$j_{i+1} - j_i \geq t_i, \quad \text{for all } i = 1, \dots, \ell - 1.$$

In particular, any variable x_j is $\mathbf{t}$-spread. We assume that $u = 1$ is $\mathbf{t}$-spread too. Whereas, we say that a monomial ideal $I \subseteq S$ is a *vector-spread monomial ideal of type* $\mathbf{t}$, or simply a $\mathbf{t}$-*spread monomial ideal*, if all monomials $u \in G(I)$ are $\mathbf{t}$-spread.

For instance, the monomial $u = x_1^3x_2x_4x_5$ is $(0, 0, 1, 2, 1)$-spread, but it is not $(1, 0, 1, 2, 1)$-spread. For a $\mathbf{t}$-spread monomial ideal $I \subset S$, it is $G(I)_k = \varnothing$ for $k > d$. Let $\mathbf{0} = (0, 0, \dots, 0)$ be the null vector with $d - 1$ components. All monomials of degree $\ell \leq d$ are $\mathbf{0}$-spread. If $t_i \geq 1$ for all i, a $\mathbf{t}$-spread monomial is *squarefree* [89].

In our context, if $t_i = t \geq 0$, for all $i = 1, \dots, d - 1$, we say that a $\mathbf{t}$-spread monomial (ideal) $u \in \mathrm{Mon}(S)$, ($I \subseteq S$), is an *uniform* or *ordinary* $\mathbf{t}$-*spread* monomial (ideal). Such definition agrees with that given in [53]. In this case, we drop the bold character "$\mathbf{t}$" and we simply speak of t-spread monomial ideals.

Let $T = K[x_1, x_2, \dots, x_n, \dots]$ be the polynomial ring in infinitely many variables. We let $\mathrm{Mon}(T; \mathbf{t})$ to be the set of all $\mathbf{t}$-spread monomials of T. Analogously, $\mathrm{Mon}(S; \mathbf{t})$ denotes the set of all $\mathbf{t}$-spread monomials of S. Furthermore, for all $0 \leq \ell \leq d$, we define the following sets

$$\mathrm{Mon}_\ell(T; \mathbf{t}) = \big\{u \in \mathrm{Mon}(T; \mathbf{t}) : \deg(u) = \ell\big\},$$
$$\mathrm{Mon}_\ell(S; \mathbf{t}) = \big\{u \in \mathrm{Mon}(S; \mathbf{t}) : \deg(u) = \ell\big\}.$$

Note that $\mathrm{Mon}_\ell(S; \mathbf{t}) = \mathrm{Mon}_\ell(T; \mathbf{t}) \cap S$, $\mathrm{Mon}_\ell(S; \mathbf{t}) = \varnothing$ for $\ell > d$, and $\mathrm{Mon}(S; \mathbf{t})$ is the disjoint union of the sets $\mathrm{Mon}_\ell(S; \mathbf{t})$, $\ell = 0, \dots, d$.

Sometimes, we may use the abbreviation $M_{n,\ell,\mathbf{t}}$ for $\mathrm{Mon}_\ell(S; \mathbf{t})$. For instance,

$$M_{5,4,(1,0,2)} = \{x_1x_2^2x_4, x_1x_2^2x_5, x_1x_2x_3x_5, x_1x_3^2x_5, x_2x_3^2x_5\}.$$

In order to compute the cardinality of the sets $M_{n,\ell,\mathbf{t}}$, we introduce a new *shifting operator*, see [11]. Let $\mathbf{0} = (0, 0, , \dots, 0)$ be the null vector with $d-1$ components, we define the map $\sigma_{\mathbf{0},\mathbf{t}} : \mathrm{Mon}(T; \mathbf{0}) \to \mathrm{Mon}(T; \mathbf{t})$, by setting $\sigma_{\mathbf{0},\mathbf{t}}(1) = 1$, $\sigma_{\mathbf{0},\mathbf{t}}(x_i) = x_i$ and for all monomials $u = x_{j_1}x_{j_2}\cdots x_{j_\ell} \in \mathrm{Mon}(T; \mathbf{0})$ with $j_1 \le j_2 \le \cdots \le j_\ell$, $2 \le \ell \le d$,

$$\sigma_{\mathbf{0},\mathbf{t}}(x_{j_1}x_{j_2}\cdots x_{j_\ell}) = \prod_{k=1}^{\ell} x_{j_k + \sum_{s=1}^{k-1} t_s}.$$

Whereas, $\sigma_{\mathbf{t},\mathbf{t}} : \mathrm{Mon}(T; \mathbf{t}) \to \mathrm{Mon}(T; \mathbf{t})$ denotes the identity function of $\mathrm{Mon}(T; \mathbf{t})$.

Lemma 4.1.2 *The map $\sigma_{\mathbf{0},\mathbf{t}}$ is a bijection.*

Proof. We define the map $\sigma_{\mathbf{t},\mathbf{0}} : \mathrm{Mon}(T; \mathbf{t}) \to \mathrm{Mon}(T; \mathbf{0})$, by setting $\sigma_{\mathbf{t},\mathbf{0}}(1) = 1$, $\sigma_{\mathbf{t},\mathbf{0}}(x_i) = x_i$, for all $i \in \mathbb{N}$, and for all monomials $u = x_{j_1}x_{j_2}\cdots x_{j_\ell} \in \mathrm{Mon}(T; \mathbf{t})$ with $1 \le j_1 \le j_2 \le \cdots \le j_\ell$, and $2 \le \ell \le d$,

$$\sigma_{\mathbf{t},\mathbf{0}}(x_{j_1}x_{j_2}\cdots x_{j_\ell}) = \prod_{k=1}^{\ell} x_{j_k - \sum_{s=1}^{k-1} t_s}.$$

One immediately verifies that $\sigma_{\mathbf{0},\mathbf{t}} \circ \sigma_{\mathbf{t},\mathbf{0}} = \sigma_{\mathbf{t},\mathbf{t}}$ and $\sigma_{\mathbf{t},\mathbf{0}} \circ \sigma_{\mathbf{0},\mathbf{t}} = \sigma_{\mathbf{0},\mathbf{0}}$. □

In particular, the restriction $\sigma_{\mathbf{t},\mathbf{0}}|_{M_{n,\ell,\mathbf{t}}}$ is a injective map whose image is the set $M_{n-(t_1+t_2+\ldots+t_{\ell-1}),\ell,\mathbf{0}} = \mathrm{Mon}_\ell(K[x_1, \dots, x_{n-(t_1+t_2+\ldots+t_{\ell-1})}])$. Thus,

Corollary 4.1.3 *For all $0 \le \ell \le d$,*

$$|M_{n,\ell,\mathbf{t}}| = \binom{n + (\ell-1) - \sum_{j=1}^{\ell-1} t_j}{\ell}. \tag{4.1}$$

Now, we introduce three fundamental classes of $\mathbf{t}$-spread ideals.

Definition 4.1.4 Let U be a non empty subset of $M_{n,\ell,\mathbf{t}}$, $\ell \le d$. We say that

- U is a $\mathbf{t}$-*spread stable set*, if for all $u \in U$, and $j < \max(u)$ such that $x_j(u/x_{\max(u)})$ is $\mathbf{t}$-spread, then $x_j(u/x_{\max(u)}) \in U$;
- U is a $\mathbf{t}$-*spread strongly stable set*, if for all $u \in U$, and all $j < i$ such that x_i divides u and $x_j(u/x_i)$ is $\mathbf{t}$-spread, then $x_j(u/x_i) \in U$;
- U is a $\mathbf{t}$-*spread lexicographic set*, if for all $u \in U$, $v \in M_{n,\ell,\mathbf{t}}$ such that $v \ge_{\mathrm{lex}} u$, then $v \in U$, where $\ge_{\mathrm{lex}}$ is the lexicographic order with $x_1 > x_2 > \cdots > x_n$ [89].

We assume the empty set $\varnothing$ to be a $\mathbf{t}$-spread stable, strongly stable and lexicographic set. Whereas, for I a $\mathbf{t}$-spread ideal of S, we say that I is a $\mathbf{t}$-*spread stable, strongly stable, lexicographic ideal*, if $U_\ell = I \cap M_{n,\ell,\mathbf{t}}$ is a $\mathbf{t}$-spread stable, strongly stable, lexicographic set, respectively, for all $\ell = 0, \dots, d$.

For $\mathbf{t} = \mathbf{0} = (0, 0, \dots, 0)$ we obtain the classical notions of stable, strongly stable and lexicographic sets and ideals [89]. For $\mathbf{t} = \mathbf{1} = (1, 1, \dots, 1)$, the *squarefree* analogues [12]. Finally, if $\mathbf{t} = (t, t, \dots, t)$ we have the ordinary t-spread stable, strongly stable and lexicographic sets and ideals, as in [53].

The following hierarchy of $\mathbf{t}$-spread monomial ideals of S holds

$$\begin{aligned}\mathbf{t}\text{-spread lexicographic ideals} &\Rightarrow \mathbf{t}\text{-spread strongly stable ideals}\\ &\Rightarrow \mathbf{t}\text{-spread stable ideals.}\end{aligned}$$

The next lemma provides the existence of a *standard decomposition* for all $\mathbf{t}$-spread monomials belonging to a $\mathbf{t}$-spread strongly stable ideal.

Lemma 4.1.5 *Let I be a $\mathbf{t}$-spread strongly stable ideal of S, and $w \in I$ a $\mathbf{t}$-spread monomial. Then, there exist unique monomials $u \in G(I)$ and $v \in \mathrm{Mon}(S)$ such that $w = uv$ and $\max(u) \le \min(v)$.*

Proof. The statement holds when $w \in G(I)$. In such a case, $w = w \cdot 1$, with $w \in G(I)$, $1 \in \mathrm{Mon}(S)$ and $\max(w) \le n = \min(1)$. Otherwise, there exists a $\mathbf{t}$-spread monomial $u \in G(I)$ such that u divides w and $\deg(u) < \deg(w)$. We choose u to be of minimal degree. Then $w = uv$, for a suitable monomial $v \in \mathrm{Mon}(S)$. Write $w = x_{j_1}x_{j_2}\cdots x_{j_\ell}$, then $u = x_{j_{k_1}}x_{j_{k_2}}\cdots x_{j_{k_s}}$ for $1 \le k_1 < k_2 < \cdots < k_s < \ell$. Now, $u_1 = x_{j_1}x_{j_2}\cdots x_{j_s} \in I$, as u_1 is $\mathbf{t}$-spread and I is $\mathbf{t}$-spread strongly stable. Moreover, u_1 divides w and it is a minimal generator. Otherwise, if there exists $u_2 \in G(I)$ such that u_2 divides u_1 and $\deg(u_2) < \deg(u_1)$, then $\deg(u_2) < \deg(u)$ and u_2 divides w, an absurd for the choice of u. Hence $w = u_1v_1$ with $u_1 = x_{j_1}\cdots x_{j_s} \in G(I)$ and $v_1 = x_{j_{s+1}}\cdots x_{j_\ell} \in \mathrm{Mon}(S)$. Clearly, the monomials u_1 and v_1 satisfying the statement are unique. ☐

As a consequence we have the following

Corollary 4.1.6 *Let I be a $\mathbf{t}$-spread monomial ideal of S. Then, the following conditions are equivalent:*

(i) *I is a $\mathbf{t}$-spread strongly stable ideal;*

(ii) *for all $u \in G(I)$, $i \in \mathrm{supp}(u)$, $j < i$ such that $x_j(u/x_i)$ is a $\mathbf{t}$-spread monomial, then $x_j(u/x_i) \in I$.*

Proof. (i) $\Longrightarrow$ (ii) is obvious. For the converse, let $w \in I$ be a $\mathbf{t}$-spread monomial, $i \in \mathrm{supp}(w)$ and $j < i$ such that $w_1 = x_j(w/x_i)$ is $\mathbf{t}$-spread, we need to prove that $w_1 \in I$. Write $w = uv$, with u and v as in Lemma 4.1.5. If $i \notin \mathrm{supp}(u)$, then u divides w_1, and so $w_1 \in I$. Otherwise, if $i \in \mathrm{supp}(u)$, then $j < i \le \max(u)$ and so $j \notin \mathrm{supp}(v)$. Thus, $w_1 = x_j(w/x_i) = x_j(u/x_i)v = u_1v$ with $u_1 = x_j(u/x_i)$, and u_1 is $\mathbf{t}$-spread, as w_1 is. By (ii), $u_1 \in I$. Hence u_1 divides w_1 and so $w_1 \in I$. ☐

4.2 Koszul cycles of vector-spread strongly stable ideals

Our main computational tool is Theorem 4.2.8. It allows to calculate a basis of the homology modules of the Koszul complex $K_\bullet(\mathbf{x}; S/I)$, where $\mathbf{x} = x_1, \dots, x_n$ and I is a $\mathbf{t}$-spread strongly stable ideal of S.

The symbol $[n]$ denotes the set $\{1, 2, \dots, n\}$, where $n \in \mathbb{Z}_{\ge 1}$. If $j, k \ge 1$ are integers, we set $[j, k] = \{\ell \in \mathbb{N} : j \le \ell \le k\}$, and $[j, k] \ne \varnothing$ if and only if $j \le k$. If $j = k = 0$, we set $[0, 0] = [0] = \varnothing$. For a monomial $u \in S$, $u \ne 1$, we set $u' = u/x_{\max(u)}$.

The next combinatorial tool will be fundamental for our aim.

Definition 4.2.1 Let $u = x_{j_1}x_{j_2}\cdots x_{j_\ell} \in \mathrm{Mon}(S; \mathbf{t})$ a $\mathbf{t}$-spread monomial of S, with $1 \le j_1 \le \cdots \le j_\ell \le n$. The $\mathbf{t}$*-spread support* of u is the following subset of $[n]$:

$$\mathrm{supp}_{\mathbf{t}}(u) = \bigcup_{i=1}^{\ell-1} \big[j_i, j_i + (t_i - 1)\big].$$

Note that $\mathrm{supp}_{\mathbf{0}}(u) = \varnothing$, and if u is squarefree, $\mathrm{supp}_{\mathbf{1}}(u) = \mathrm{supp}(u/x_{\max(u)}) = \{j_1, j_2, \dots, j_{\ell-1}\}$, where $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{Z}_{\geq 0}^{d-1}$.

Let us explain now the *combinatorial meaning* of the vector-spread support. Let $u = x_{j_1}x_{j_2}\cdots x_{j_\ell} \in \mathrm{Mon}(S;\mathbf{t})$, $u \neq 1$. For any $k \in [\max(u)-1] \setminus \mathrm{supp}_{\mathbf{t}}(u)$, we define $k^{(u)} = \min\{j \in \mathrm{supp}(u) : j > k\}$. We note that $k^{(u)}$ always exists as $k < \max(u)$ and $\max(u) \in \mathrm{supp}(u)$. An easy calculation shows that $w = x_k(u/x_{k^{(u)}})$ *is again a* $\mathbf{t}$*-spread monomial*, and moreover, if I is a $\mathbf{t}$-spread strongly stable ideal and $u \in I$, then $w \in I$ also, by definition. This property will be crucial in order to construct our Koszul cycles. For instance,

Example 4.2.2 Let $u = x_1^2x_2x_4x_6x_8 \in \mathrm{Mon}(S;(0,0,1,2,1))$, $S = K[x_1,\dots,x_8]$. We have $\mathrm{supp}_{(0,0,1,2,1)}(x_1x_1x_2x_4x_6x_8) = \{2,4,5,6\}$, and $[\max(u)-1]\setminus\mathrm{supp}_{\mathbf{t}}(u) = \{1,3,7\}$. For $k = 1$, $k^{(u)} = 2$, for $k = 3$, $k^{(u)} = 4$ and for $k = 7$, $k^{(u)} = \max(u) = 8$. Note that $x_1(u/x_2) = x_1^3x_4x_6x_8$, $x_3(u/x_4) = x_1^2x_2x_3x_6x_8$ and $x_7(u/x_8) = x_1^2x_2x_4x_6x_7$ are all $(0,0,1,2,1)$-spread monomials.

Let I be a $\mathbf{t}$-spread strongly stable ideal of S. We are going to construct suitable cycles of $K_i(\mathbf{x};S/I) = K_i(\mathbf{x})$.

We shall make the following conventions. For a non empty subset $A \subseteq [n]$, we set $\mathbf{x}_A = \prod_{i\in A} x_i$ and $e_A = \bigwedge_{i\in A} e_i$, whereas for $A = \varnothing$, $\mathbf{x}_\varnothing = 1$ and $e_\tau \wedge e_\varnothing = e_\varnothing \wedge e_\tau = e_\tau$ for any non empty subset $\tau \subseteq [n]$. We take account of repetitions. For example, if $A = \{1,1,2,3\}$, then $\mathbf{x}_A = x_1^2x_2x_3$ and $e_A = e_1 \wedge e_1 \wedge e_2 \wedge e_3 = 0$.

Let $u \in S$ be a $\mathbf{t}$-spread monomial and $\sigma = \{k_1 < k_2 < \cdots < k_{i-1}\} \subseteq [\max(u)-1]$ with $|\sigma| = i-1$, $i \geq 1$. For each $\ell = 1,\dots,i-1$, we define

$$k_\ell^{(u)} = j_\ell = \min\{j \in \mathrm{supp}(u) : j > k_\ell\}.$$

Clearly, $j_1 \leq j_2 \leq \cdots \leq j_{i-1} \leq \max(u)$. If $F = \{k_{s_1},\dots,k_{s_m}\} \subseteq \sigma$, we set

$$F^{(u)} = \{j_{s_1} \leq j_{s_2} \leq \cdots \leq j_{s_m}\} = \{\min\{j \in \mathrm{supp}(u) : j > k_{s_\ell}\} : \ell \in [m]\}.$$

Definition 4.2.3 Let $u \in S$ be a $\mathbf{t}$-spread monomial. We set $u' = u/x_{\max(u)}$. Let $\sigma = \{k_1 < k_2 < \cdots < k_{i-1}\} \subseteq [\max(u)-1] \setminus \mathrm{supp}_{\mathbf{t}}(u)$ with $|\sigma| = i-1$, $i \geq 1$. Let I be any $\mathbf{t}$-spread strongly stable ideal of S such that $u \in I$ and let $\varepsilon : S \to S/I$ be the canonical map. We define the following element of $K_i(\mathbf{x};S/I)$:

$$\begin{aligned} e(u;\sigma) &= \sum_{F\subseteq\sigma}(-1)^{u(\sigma;F)}\varepsilon(\mathbf{x}_F(u'/\mathbf{x}_{F^{(u)}}))e_{\sigma\setminus F}\wedge e_{F^{(u)}}\wedge e_{\max(u)} \qquad (4.2)\\ &= \varepsilon(u')e_\sigma \wedge e_{\max(u)} + \sum_{\varnothing\neq F\subseteq\sigma}(-1)^{u(\sigma;F)}\varepsilon(\mathbf{x}_F(u'/\mathbf{x}_{F^{(u)}}))e_{\sigma\setminus F}\wedge e_{F^{(u)}}\wedge e_{\max(u)}, \end{aligned}$$

with $u(\sigma;\varnothing) = 0$ and for $F \neq \varnothing$, $F \subseteq \sigma$, $u(\sigma;F)$ is defined recursively as follows:

- if $\max(\sigma) = k_{i-1} \notin F$, then $u(\sigma;F) = u(\sigma \setminus \{k_{i-1}\};F) + |F|$;
- if $\max(\sigma) = k_{i-1} \in F$, then

$$\begin{aligned} u(\sigma;F) &= u(\sigma \setminus \{k_r \in \sigma : j_r = j_{i-1}\}; F \setminus \{k_r \in F : j_r = j_{i-1}\})\\ &\quad + (|\{k_r \in \sigma : j_r = j_{i-1}\}| - 1)(|F|+1) + 1. \end{aligned}$$

The definition of the $u(\sigma;F)$'s will became clear in the proof of Proposition 4.2.4.

The element $e(u;\sigma)\in K_i(\mathbf{x})$ is well defined. Indeed, if for some $k_p<k_q$, $k_p,k_q\in\sigma$ we have $j_p=j_q=j$, then x_j^2 may not divide u, however, in such case the wedge product $e_{\sigma\setminus F}\wedge e_{F^{(u)}}\wedge e_{\max(u)}$ is zero, as $j_p,j_q\in F^{(u)}$ and $e_{j_p}\wedge e_{j_q}=e_j\wedge e_j=0$. The same reasoning applies if $j_p=\max(u)$, for some p. Moreover, $\varepsilon(u')e_\sigma\wedge e_{\max(u)}$ is the biggest summand of $e(u;\sigma)$ with respect to the order on the wedge products we have defined in Section 2.2.

As noted before, for $u\in I$ a $\mathbf{t}$-spread monomial and $k_\ell\in[\max(u)-1]\setminus\operatorname{supp}_{\mathbf{t}}(u)$, then $x_{k_\ell}(u/x_{j_\ell})\in I$, *i.e.*, $\varepsilon(x_{k_\ell}(u/x_{j_\ell}))=0$, as I is a $\mathbf{t}$-spread strongly stable ideal.

Proposition 4.2.4 *Let $I\subseteq S$ be a $\mathbf{t}$-spread strongly stable ideal. For all $i\ge1$, the elements*

$$e(u;\sigma)\ \ \textit{such that}\ \ u\in G(I),\quad \sigma\subseteq[\max(u)-1]\setminus\operatorname{supp}_{\mathbf{t}}(u),\quad |\sigma|=i-1,$$

are cycles of $K_i(\mathbf{x};S/I)$.

Remark 4.2.5 Proving the above proposition in the cases $i=1,2,3$ is straightforward. For the general case, $e(u;\sigma)$ is a sum of 2^{i-1} terms, if $|\sigma|=i-1$. A direct verification of the equation $\partial_i(e(u;\sigma))=0$ is nasty. Therefore, we employ an inductive argument. After verifying two base cases ($i=1,2$), we assume that $e(u;\vartheta)$ is a cycle for all proper subsets $\vartheta\subset\sigma$. Depending on some cases, we suitably write $e(u;\sigma)$ in terms of the $e(u;\vartheta)$'s, equations (4.3) and (4.4). It is from these equations that we obtained our coefficients $u(\sigma;F)$, $F\subseteq\sigma$, by observing that we must change sign each time we exchange two consecutive basis elements e_k,e_ℓ in a non zero wedge product involving them. Finally, using the rule of multiplication,

$$\partial(a\wedge b)=\partial(a)\wedge b+(-1)^{\deg a}a\wedge\partial(b)$$

for a $a,b\in K_{\bullet}(\mathbf{x};S/I)$, with a homogeneous, we complete our proof.

We begin by giving the *decomposition* for the $e(u;\sigma)$'s mentioned above.

Lemma 4.2.6 *Let $u\in S$ be a $\mathbf{t}$-spread monomial, and let $\sigma=\{k_1<\dots<k_{i-1}\}\subseteq[\max(u)-1]\setminus\operatorname{supp}_{\mathbf{t}}(u)$, $|\sigma|=i-1$, $i\ge2$. Then the following hold:*

(a) *If $j_{i-1}=\max(u)$, setting $\tau=\sigma\setminus\{k_{i-1}\}$, then*

$$e(u;\sigma)=-e(u;\tau)\wedge e_{k_{i-1}}. \tag{4.3}$$

(b) *If $j_{i-1}\neq\max(u)$, setting $\ell=\min\{\ell\in[i-1]:j_\ell=j_{i-1}\}$, $v=x_{k_{i-1}}u/x_{j_{i-1}}$ and $\rho=\sigma\setminus\{k_\ell,k_{\ell+1},\dots,k_{i-2},k_{i-1}\}$, then*

$$e(u;\sigma)=-e(u;\tau)\wedge e_{k_{i-1}}+(-1)^{i-1-\ell}e(v;\rho)\wedge e_{k_\ell}\wedge e_{k_{\ell+1}}\wedge\dots\wedge e_{k_{i-2}}\wedge e_{j_{i-1}}. \tag{4.4}$$

Proof. (a) Suppose that $j_{i-1}=\max(u)$. In such a case, for all $F\subseteq\sigma$ with $k_{i-1}\in F$, the corresponding term of $e(u;\sigma)$ is zero, as $e_{j_{i-1}}\wedge e_{\max(u)}=e_{\max(u)}\wedge e_{\max(u)}=0$. Moreover, for all $F\subseteq\sigma$ such that $k_{i-1}=\max(\sigma)\notin F$, that is $F\subseteq\tau$, we have $u(\sigma;F)=u(\tau;F)+|F|$, hence $(-1)^{u(\sigma;F)}(-1)^{|F|+1}=-(-1)^{u(\tau;F)+2|F|}=-(-1)^{u(\tau;F)}$.

So, we obtain the desired formula (4.3),

$$\begin{aligned}
e(u;\sigma) = &\quad \sum_{F\subseteq\tau}(-1)^{u(\sigma;F)}\varepsilon(\mathbf{x}_F(u'/\mathbf{x}_{F^{(u)}}))e_{\sigma\setminus F}\wedge e_{F^{(u)}}\wedge e_{\max(u)}\\
= &\quad \sum_{F\subseteq\tau}(-1)^{u(\sigma;F)}(-1)^{|F|+1}\varepsilon(\mathbf{x}_F(u'/\mathbf{x}_{F^{(u)}}))e_{\tau\setminus F}\wedge e_{F^{(u)}}\wedge e_{\max(u)}\wedge e_{k_{i-1}}\\
= &-(\sum_{F\subseteq\tau}(-1)^{u(\tau;F)}\varepsilon(\mathbf{x}_F(u'/\mathbf{x}_{F^{(u)}}))e_{\tau\setminus F}\wedge e_{F^{(u)}}\wedge e_{\max(u)})\wedge e_{k_{i-1}}\\
= &-\ e(u;\tau)\wedge e_{k_{i-1}}.
\end{aligned}$$

(b) Suppose $j_{i-1}\neq\max(u)$. Note that $v=x_{k_{i-1}}(u/x_{j_{i-1}})\in I$, as $u\in G(I)$, v is $\mathbf{t}$-spread, $k_{i-1}<j_{i-1}$ and I is $\mathbf{t}$-spread strongly stable. Moreover $\max(v)=\max(u)$, $\ell\le i-1$ and $\rho=\{k_1,k_2,\dots,k_{\ell-1}\}\subseteq\tau\subseteq[\max(v)-1]\setminus\operatorname{supp}_{\mathbf{t}}(v)$. Hence, we can consider the element

$$e(v;\rho)=\sum_{G\subseteq\rho}(-1)^{v(\rho;G)}\varepsilon(\mathbf{x}_G(v'/\mathbf{x}_{G^{(v)}}))e_{\rho\setminus G}\wedge e_{G^{(v)}}\wedge e_{\max(u)} \tag{4.5}$$

where $G^{(v)}=\{\min\{s\in\operatorname{supp}(v):s>g\}:g\in G\}$. For $k_r\in G$, $r<\ell$, so $s_r=\min\{s\in\operatorname{supp}(v):s>k_r\}=j_r$, as $j_r\in\operatorname{supp}(u)\setminus\{j_{i-1}\}$, and $k_{i-1}>j_r$, lest $k_{i-1}\le j_r<j_{i-1}$ would imply that $j_r=j_{i-1}$, an absurd. So, for all $G\subseteq\rho$, we have $G^{(u)}=G^{(v)}$. This implies, by the definition of the coefficients, that $v(\rho;G)=u(\rho;G)$ for all $G\subseteq\rho$.

Let $F\subseteq\sigma$ with $F\neq\varnothing$ such that the corresponding term of $e(u;\sigma)$ is non zero.

If $k_{i-1}\notin F$, then $F\subseteq\tau$ and in (a) we have already shown that the corresponding terms of $e(u;\sigma)$ and $-e(u;\tau)\wedge e_{k_{i-1}}$ are equal.

Suppose now that $k_{i-1}\in F$. Set $D=\{k_\ell,\dots,k_{i-2}\}$. We assume that $D\cap F$ is empty, otherwise $e_{F^{(u)}}=0$, as $j_r=j_{i-1}$ for some $k_r\in D\cap F$. So, we can write $F=G\cup\{k_{i-1}\}$ for a unique $G\subseteq\rho$. Thus, the relevant sum T of terms of $e(u;\sigma)$ indexed by $\{F=G\cup\{k_{i-1}\}:G\subseteq\rho\}$ is, as $\max(u)=\max(v)$ and $x_{k_{i-1}}(u'/x_{j_{i-1}})=v'$,

$$\begin{aligned}
T=&\sum_{\substack{G\cup\{k_{i-1}\}\\ G\subseteq\rho}}(-1)^{u(\sigma;G\cup\{k_{i-1}\})}\varepsilon(\mathbf{x}_Gx_{k_{i-1}}(u'/(x_{j_{i-1}}\mathbf{x}_{G^{(u)}})))e_{\sigma\setminus(G\cup\{k_{i-1}\})}\wedge e_{G^{(u)}}\wedge e_{j_{i-1}}\wedge e_{\max(u)}\\
=&\sum_{G\subseteq\rho}(-1)^{u(\sigma;G\cup\{k_{i-1}\})}\varepsilon(\mathbf{x}_G(v'/\mathbf{x}_{G^{(v)}}))e_{\rho\setminus G}\wedge e_D\wedge e_{G^{(v)}}\wedge e_{j_{i-1}}\wedge e_{\max(v)}\\
=&\sum_{G\subseteq\rho}(-1)^{u(\sigma;G\cup\{k_{i-1}\})+1+|D|(|G|+1)}\varepsilon(\mathbf{x}_G(v'/\mathbf{x}_{G^{(v)}}))e_{\rho\setminus G}\wedge e_{G^{(v)}}\wedge e_{\max(v)}\wedge e_D\wedge e_{j_{i-1}}.
\end{aligned}$$

Now, for all $F=G\cup\{k_{i-1}\}$, with $G\subseteq\rho$, we have

$$\begin{aligned}
u(\sigma;F)&=u(\sigma\setminus\{k_\ell,\dots,k_{i-1}\};F\setminus\{k_\ell,\dots,k_{i-1}\})+(|\{k_\ell,\dots,k_{i-1}\}|-1)(|F|+1)+1\\
&=u(\sigma\setminus\{k_\ell,\dots,k_{i-1}\};G)+(|\{k_\ell,\dots,k_{i-1}\}|-1)(|F|+1)+1\\
&=u(\rho;G)+|D|(|F|+1)+1.
\end{aligned}$$

Therefore, as $|G|+1=|F|$ and $|D|=i-1-\ell$, we have

$$(-1)^{u(\sigma;F)+1+|D|(|G|+1)}=(-1)^{u(\rho;G)+2+2|D||F|+|D|}=(-1)^{i-1-\ell}(-1)^{u(\rho;G)}.$$

So, we have $e(u;\sigma)=-e(u;\tau)\wedge e_{k_{i-1}}+T$, with, as $u(\rho;G)=v(\rho;G)$ for all $G\subseteq\rho$,

$$\begin{aligned}T&=\Big(\sum_{G\subseteq\rho}(-1)^{i-1-\ell}(-1)^{u(\rho;G)}\varepsilon(\mathbf{x}_G(v'/\mathbf{x}_{G^{(v)}}))e_{\rho\setminus G}\wedge e_{G^{(v)}}\wedge e_{\max(v)}\Big)\wedge e_D\wedge e_{j_{i-1}}\\&=(-1)^{i-1-\ell}\Big(\sum_{G\subseteq\rho}(-1)^{v(\rho;G)}\varepsilon(\mathbf{x}_G(v'/\mathbf{x}_{G^{(v)}}))e_{\rho\setminus G}\wedge e_{G^{(v)}}\wedge e_{\max(v)}\Big)\wedge e_D\wedge e_{j_{i-1}}\\&=(-1)^{i-1-\ell}e(v;\rho)\wedge e_{k_\ell}\wedge e_{k_{\ell+1}}\wedge\cdots\wedge e_{k_{i-2}}\wedge e_{j_{i-1}},\end{aligned}$$

and equation (4.4) holds. □

Finally we can prove Proposition 4.2.4.

Proof of Proposition 4.2.4. For $i=1$ we have $|\sigma|=0$, so $\sigma=\varnothing$, $u(\varnothing;\varnothing)=0$ by definition, and the element $e(u;\varnothing)=\varepsilon(u')e_{\max(u)}=\varepsilon(u/x_{\max(u)})e_{\max(u)}$ is clearly a cycle of $K_1(\mathbf{x})$. Let $i\geq 2$. We proceed by induction on $i\geq 2$.

For $i=2$, $\sigma=\{k_1\}$. Let $j_1=\min\{j\in\operatorname{supp}(u):j>k_1\}$. We have $u(\{k_1\};\varnothing)=0$ and $u(\{k_1\},\{k_1\})=u(\{k_1\}\setminus\{k_1\};\{k_1\}\setminus\{k_1\})+(|\{k_1\}|-1)(|\{k_1\}|+1)+1=u(\varnothing;\varnothing)+1=1$, so

$$e(u;\sigma)=\varepsilon(u')e_{k_1}\wedge e_{\max(u)}-\varepsilon(x_{k_1}u'/x_{j_1})e_{j_1}\wedge e_{\max(u)}.$$

If $j_1=\max(u)$, then $e(u;\sigma)=\varepsilon(u')e_{k_1}\wedge e_{\max(u)}$. In such a case,

$$\partial_2(e(u;\sigma))=\varepsilon(x_{k_1}u')e_{\max(u)}-\varepsilon(x_{\max(u)}u')e_{k_1}=0,$$

as $x_{k_1}u',x_{\max(u)}u'=u\in I$. Otherwise, if $j_1<\max(u)$, then

$$\partial_2(e(u;\sigma))=\varepsilon(x_{k_1}u')e_{\max(u)}-\varepsilon(x_{\max(u)}u')e_{k_1}-\varepsilon(x_{k_1}u')e_{\max(u)}+\varepsilon(x_{k_1}(u/x_{j_1}))e_{j_1}=0,$$

as the first and third terms cancel each other, and $x_{\max(u)}u'=u,x_{k_1}(u/x_{j_1})\in I$.

Suppose now $i>2$. Let $\sigma=\{k_1<\cdots<k_{i-2}<k_{i-1}\}$, and $\tau=\{k_1<\cdots<k_{i-2}\}$. We distinguish two cases.

(a) Suppose $j_{i-1}=\max(u)$. By induction $e(u;\tau)$ is a cycle. By Lemma 4.2.6 (a),

$$\begin{aligned}\partial_i(e(u;\sigma))&=\partial_i(-e(u;\tau)\wedge e_{k_{i-1}})\\&=-\partial_{i-1}(e(u;\tau))\wedge e_{k_{i-1}}-(-1)^{\deg(e(u;\tau))}x_{k_{i-1}}e(u;\tau)\\&=-(-1)^{\deg(e(u;\tau))}x_{k_{i-1}}e(u;\tau)=0.\end{aligned}$$

Indeed, $x_{k_{i-1}}u/x_{j_{i-1}}=x_{k_{i-1}}u/x_{\max(u)}=x_{k_{i-1}}u'\in I$. Thus, each non zero term of $x_{k_{i-1}}e(u;\tau)$ vanish, as it has coefficient $\varepsilon(\mathbf{x}_Fx_{k_{i-1}}u'/\mathbf{x}_{F^{(u)}})$, and $\mathbf{x}_Fx_{k_{i-1}}u'/\mathbf{x}_{F^{(u)}}\in I$ as I is a $\mathbf{t}$-spread strongly stable ideal. In such a case, $e(u;\sigma)$ is a cycle, as desired.

(b) Suppose $j_{i-1}\neq\max(u)$. Set $\ell=\min\{\ell\in[i-1]:j_\ell=j_{i-1}\}$, $v=x_{k_{i-1}}u/x_{j_{i-1}}$, $\rho=\sigma\setminus\{k_\ell,\dots,k_{i-2},k_{i-1}\}$ and $D=\{k_\ell,\dots,k_{i-2}\}$. By Lemma 4.2.6 (b),

$$e(u;\sigma)=-e(u;\tau)\wedge e_{k_{i-1}}+(-1)^{i-1-\ell}e(v;\rho)\wedge e_D\wedge e_{j_{i-1}}.\tag{4.6}$$

Let J be the smallest $\mathbf{t}$-spread strongly stable ideal of S that contains v. J is generated only in one degree $\deg(v)=\deg(u)$, and $J\subseteq I$. By inductive hypothesis, as $|\rho|<|\sigma|$, $e(v;\rho)$ is a cycle of $K_i(\mathbf{x};S/J)$. So, it is also a cycle of $K_i(\mathbf{x};S/I)=K_i(\mathbf{x})$.

By inductive hypothesis $\partial_{i-1}(e(u;\tau)) = \partial_\ell(e(v;\rho)) = 0$. Since $\deg(e(u;\tau)) = |\tau|$ and $\deg(e(v;\rho)) = |\rho|$, by equation (4.6) we have

$$\begin{aligned}\partial_i(e(u;\sigma)) &= -\partial_{i-1}(e(u;\tau)) \wedge e_{k_{i-1}} - (-1)^{\deg(e(u;\tau))} x_{k_{i-1}} e(u;\tau)\\ &\quad +(-1)^{i-1-\ell}[\partial_\ell(e(v;\rho)) \wedge e_D \wedge e_{j_{i-1}} + (-1)^{\deg(e(v;\rho))} e(v;\rho) \wedge \partial_{i-\ell}(e_D \wedge e_{j_{i-1}})]\\ &= -(-1)^{|\tau|}(x_{k_{i-1}} e(u;\tau) - (-1)^{|\rho|-|\tau|}(-1)^{i-1-\ell} e(v;\rho) \wedge \partial_{i-\ell}(e_D \wedge e_{j_{i-1}})).\end{aligned}$$

We have $i-1-\ell+|\rho|-|\tau| = i-1-\ell+\ell-1-(i-2) = 0$. So $(-1)^{|\rho|-|\tau|}(-1)^{i-1-\ell} = 1$.

Set

$$f = x_{k_{i-1}} e(u;\tau) - e(v;\rho) \wedge \partial_{i-\ell}(e_{k_\ell} \wedge e_{k_{\ell+1}} \wedge \cdots \wedge e_{k_{i-2}} \wedge e_{j_{i-1}}).$$

To show that $\partial_i(e(u;\sigma)) = 0$, it suffices to prove that f is zero. Let $F \subseteq \tau$. The set $D \cap F = \{k_\ell, \dots, k_{i-2}\} \cap F$ can have at most one element, otherwise $e_{F^{(u)}} = 0$, as shown before. Therefore, $F = G$ or $F = G \cup \{k_r\}$ for a unique $G \subseteq \rho$ and $r \in \{\ell, \dots, i-2\}$. By construction we have $G^{(u)} = G^{(v)}$ and $u(\rho;G) = v(\rho;G)$ for all $G \subseteq \rho$, as already observed in Lemma 4.2.6.

Suppose $F = G$, then the corresponding term of f is $a - b$, where

$$a = (-1)^{u(\tau;G)} \varepsilon(x_{k_{i-1}} \mathbf{x}_G (u'/\mathbf{x}_{G^{(u)}})) e_{\tau \setminus G} \wedge e_{G^{(u)}} \wedge e_{\max(u)},$$

and, as $i-1-\ell = |D|$,

$$\begin{aligned}b &= (-1)^{i-1-\ell}(-1)^{v(\rho;G)} \varepsilon(x_{j_{i-1}} \mathbf{x}_G x_{k_{i-1}} u'/(x_{j_{i-1}} \mathbf{x}_{G^{(v)}})) e_{\rho \setminus G} \wedge e_{G^{(v)}} \wedge e_{\max(v)} \wedge e_D\\ &= (-1)^{u(\rho;G)+|D|}(-1)^{|D|(|G|+1)} \varepsilon(x_{k_{i-1}} \mathbf{x}_G (u'/\mathbf{x}_{G^{(u)}})) e_{\rho \setminus G} \wedge e_D \wedge e_{G^{(u)}} \wedge e_{\max(u)}\\ &= (-1)^{u(\rho;G)+|D|(|G|+2)} \varepsilon(x_{k_{i-1}} \mathbf{x}_G (u'/\mathbf{x}_{G^{(u)}})) e_{\tau \setminus G} \wedge e_{G^{(u)}} \wedge e_{\max(u)}.\end{aligned}$$

We have, as $i-1-\ell = |D|$,

$$\begin{aligned}u(\tau;G) &= u(\tau \setminus \{k_{i-2}\};G) + |G|\\ &= u(\tau \setminus \{k_{i-3}, k_{i-2}\};G) + 2|G|\\ &\ \ \vdots\\ &= u(\tau \setminus \{k_\ell, \dots, k_{i-2}\};G) + (i-1-\ell)|G|\\ &= u(\rho;G) + |D| \cdot |G|.\end{aligned}$$

Thus, $(-1)^{u(\rho;G)+|D|(|G|+2)} = (-1)^{u(\tau;G)}$ and $a - b = 0$, in this case.

Otherwise, if $F = G \cup \{k_r\}$, the corresponding term of f is $a - b$, where

$$\begin{aligned}a &= (-1)^{u(\tau;G\cup\{k_r\})} \varepsilon(x_{k_{i-1}} x_{k_r} \mathbf{x}_G u'/(x_{j_{i-1}} \mathbf{x}_{G^{(u)}})) e_{\tau \setminus (G \cup \{k_r\})} \wedge e_{G^{(u)}} \wedge e_{j_{i-1}} \wedge e_{\max(u)}\\ &= (-1)^{u(\tau;G\cup\{k_r\})} \varepsilon(x_{k_{i-1}} x_{k_r} \mathbf{x}_G u'/(x_{j_{i-1}} \mathbf{x}_{G^{(u)}})) e_{\tau \setminus F} \wedge e_{G^{(u)}} \wedge e_{j_{i-1}} \wedge e_{\max(u)},\end{aligned}$$

and, setting $c = v(\rho;G) + r - \ell$,

$$\begin{aligned}b &= (-1)^{r-\ell}(-1)^{v(\rho;G)} \varepsilon(x_{k_r} \mathbf{x}_G v'/\mathbf{x}_{G^{(v)}}) e_{\rho \setminus G} \wedge e_{G^{(v)}} \wedge e_{\max(v)} \wedge e_{D \setminus \{k_r\}} \wedge e_{j_{i-1}}\\ &= (-1)^{c+(|D|-1)(|G|+1)+1} \varepsilon(x_{k_{i-1}} x_{k_r} \mathbf{x}_G u'/(\mathbf{x}_{G^{(u)}} x_{j_{i-1}})) e_{\rho \setminus G} \wedge e_{D \setminus \{k_r\}} \wedge e_{F^{(u)}} \wedge e_{\max(u)}\\ &= (-1)^{c+(|D|-1)(|G|+1)+1} \varepsilon(x_{k_{i-1}} x_{k_r} \mathbf{x}_G u'/(\mathbf{x}_{G^{(u)}} x_{j_{i-1}})) e_{\tau \setminus F} \wedge e_{G^{(u)}} \wedge e_{j_{i-1}} \wedge e_{\max(u)}.\end{aligned}$$

Now,

$$\begin{aligned}
u(\tau;G\cup\{k_r\}) &= u(\tau\setminus\{k_{i-2}\};G\cup\{k_r\})+(|G|+1)\\
&= u(\tau\setminus\{k_{i-3},k_{i-2}\};G\cup\{k_r\})+2(|G|+1)\\
&\ \ \vdots\\
&= u(\tau\setminus\{k_{r+1},\dots,k_{i-2}\};G\cup\{k_r\})+(i-r)(|G|+1)\\
&= u(\tau\setminus\{k_\ell,\dots,k_{i-2}\};G)+(|\{k_\ell,\dots,k_r\}|-1)(|G|+2)+1+(i-r)(|G|+1)\\
&= u(\rho;G)+(r-\ell+i-r)(|G|+1)+(r-\ell)+1\\
&= v(\rho;G)+(i-\ell)(|G|+1)+(r-\ell)+1.
\end{aligned}$$

Hence, since $(-1)^{(i-\ell)(|G|+1)}=(-1)^{(|D|+1)(|G|+1)}=(-1)^{(|D|-1)(|G|+1)}$, we have

$$(-1)^{u(\tau;G\cup\{k_r\})}=(-1)^{v(\rho;G)+(i-\ell)(|G|+1)+(r-\ell)+1}=(-1)^{c+(|D|-1)(|G|+1)+1}$$

and $a-b=0$. Therefore, $f=0$ and $e(u;\sigma)$ is a cycle, as desired. $\square$

Remark 4.2.7 Let $\sigma=\{k_1<k_2<\cdots<k_{i-1}\}\subseteq[\max(u)-1]\setminus\operatorname{supp}_{\mathbf{t}}(u)$, $i>2$. Set

$$r(u;\sigma)=\sum_{\substack{F\subseteq\sigma\\ k_1\in F}}(-1)^{u(\sigma;F)}\varepsilon(\mathbf{x}_F(u'/\mathbf{x}_{F(u)}))e_{\sigma\setminus F}\wedge e_{F(u)}\wedge e_{\max(u)}.$$

We show that $e(u;\sigma)=e_{k_1}\wedge e(u;\sigma\setminus\{k_1\})+r(u;\sigma)$. For this aim, it is enough to prove that, for all $F\subseteq\sigma$ such that $k_1\notin F$, we have $(-1)^{u(\sigma;F)}=(-1)^{u(\sigma\setminus\{k_1\};F)}$. For $\sigma=\{k_1\}$ this is clear. Let $|\sigma|=i-1\ge 2$. We distinguish two cases.

Case 1. Suppose $\max(\sigma)\notin F$, then $u(\sigma;F)=u(\sigma\setminus\{k_{i-1}\};F)+|F|$. Moreover $\max(\sigma\setminus\{k_1\})=\max(\sigma)$ as $|\sigma|\ge 2$. Therefore $u(\sigma\setminus\{k_1\};F)=u(\sigma\setminus\{k_1,k_{i-1}\};F)+|F|$. By induction on $|\sigma|$, $(-1)^{u(\sigma\setminus\{k_1,k_{i-1}\};F)}=(-1)^{u(\sigma\setminus\{k_{i-1}\};F)}$, and the desired conclusion follows in such a case.

Case 2. Suppose $\max(\sigma)\in F$. Set $D=\{k_r\in F:j_r=j_{i-1}\}$ and $d=|\{k_r\in\sigma: j_r=j_{i-1}\}|$. Observe that, as $k_1\notin F$, $d=|\{k_r\in\sigma\setminus\{k_1\}:j_r=j_{i-1}\}|$. Therefore, by the definition of the coefficients, we have

$$\begin{aligned}
u(\sigma;F) &= u(\sigma\setminus D;F\setminus\{k_{i-1}\})+(d-1)(|F|+1)+1,\\
u(\sigma\setminus\{k_1\};F) &= u(\sigma\setminus(\{k_1\}\cup D);F\setminus\{k_{i-1}\})+(d-1)(|F|+1)+1.
\end{aligned}$$

As $k_{i-1}\in D$ so $D\neq\varnothing$, we have $|\sigma\setminus D|<|\sigma|$. So by inductive hypothesis, $(-1)^{u(\sigma\setminus D;F\setminus\{k_{i-1}\})}=(-1)^{u(\sigma\setminus(\{k_1\}\cup D);F\setminus\{k_{i-1}\})}$, and the desired conclusion follows.

Note that e_{k_1} doesn't appear in $r(u;\sigma)$. Hence, we have the useful decomposition $e(u;\sigma)=e_{k_1}\wedge e(u;\sigma\setminus\{k_1\})+r(u;\sigma)$. Moreover, equations (4.3) and (4.4) give us recurrence relations for our Koszul cycles.

We are in position to state and prove the main result of this section.

Theorem 4.2.8 *Let $I\subset S$ be a $\mathbf{t}$-spread strongly stable ideal. Then, for all $i\ge 1$, the K-vector space $H_i(\mathbf{x};S/I)$ has as a basis the homology classes of the Koszul cycles*

$$e(u;\sigma)\quad\text{such that}\quad u\in G(I),\quad \sigma\subseteq[\max(u)-1]\setminus\operatorname{supp}_{\mathbf{t}}(u),\quad |\sigma|=i-1.\tag{4.7}$$

Proof. Let us prove the following more general statement, Claim 1. *For all $i \geq 1$ and all $j = 1, \dots, n$, a minimal generating set for $H_i(\mathbf{x}_j)$, as a $S/(\mathbf{x}_j)$-module, is given by the homology classes of the Koszul cycles*

$$e(u;\sigma) \text{ such that } u \in G(I), \quad \sigma \subseteq ([\max(u)-1] \setminus \mathrm{supp}_{\mathbf{t}}(u)) \cap [j,n], \quad |\sigma| = i-1.$$

We proceed by induction on $n-j \geq 0$. For the base case, let $n-j=0$. We only have to consider $H_1(\mathbf{x}_n)$. Indeed, for $i \geq 2$, $H_i(\mathbf{x}_n) = 0$, since $K_\bullet(x_n; S/I)$ has length one. $H_1(\mathbf{x}_n)$ is generated by the elements $[e(u;\varnothing)] = [\varepsilon(u/x_{\max(u)})e_{\max(u)}]$ with $u \in G(I)$ and $\max(u) = n$. Moreover $(\mathbf{x}_n) = (x_n)$ clearly annihilates these elements, so they form a minimal generating set of $H_1(\mathbf{x}_n)$ as a $S/(\mathbf{x}_n)$-module.

For the inductive step, suppose $n-j > 0$ and that the thesis holds for $j+1$. First, we consider the case $i=1$. By the sequence (2.4), we have the exact sequence

$$H_1(\mathbf{x}_{j+1}) \xrightarrow{\alpha_1} H_1(\mathbf{x}_j) \xrightarrow{\beta_1} H_0(\mathbf{x}_{j+1}) \xrightarrow{\delta_0} H_0(\mathbf{x}_{j+1}). \tag{4.8}$$

By the third isomorphism theorem for commutative rings,

$$H_0(\mathbf{x}_{j+1}) \cong \frac{S/I}{(\mathbf{x}_{j+1}, I)/I} \cong \frac{S}{(x_{j+1}, x_{j+2}, \dots, x_n, I)} \cong S_{\leq j}/I_{\leq j},$$

where $S_{\leq j} = K[x_1, \dots, x_j]$ and $I_{\leq j} = I \cap S_{\leq j}$. We observe that $I_{\leq j}$ is a monomial ideal of $S_{\leq j}$ with minimal generating set $G(I_{\leq j}) = \{u \in G(I) : \max(u) \leq j\}$.

Let $\mathrm{Ker}(\delta_0) = \mathrm{Im}(\beta_1)$ be the kernel of the rightmost non zero map of sequence (4.8), we obtain the short exact sequence of $S/(\mathbf{x}_{j+1})$-modules,

$$0 \to \mathrm{Im}(\alpha_1) \xrightarrow{\alpha_1} H_1(\mathbf{x}_j) \xrightarrow{\beta_1} \mathrm{Ker}(\delta_0) \to 0. \tag{4.9}$$

By inductive hypothesis, $H_1(\mathbf{x}_{j+1})$ is generated by the homology classes of the elements

$$e(u;\varnothing) = \varepsilon(u/x_{\max(u)})e_{\max(u)},$$

such that $\max(u) \geq j+1$ and $u \in G(I)$. These elements also generate $\mathrm{Im}(\alpha_1)$, as α_1 sends these homology classes to the corresponding homology classes in $H_1(\mathbf{x}_j)$. Whilst, $\mathrm{Ker}(\delta_0)$ has as a basis the elements $\varepsilon(u/x_{\max(u)})$ with $\max(u) = j$ and $u \in G(I)$. Each of these elements is pulled back in $H_1(\mathbf{x}_j)$ to the homology class of the element $e(u;\varnothing)$, with $\max(u) = j$ and $u \in G(I)$. Moreover $(\mathbf{x}_j)$ annihilates $H_1(\mathbf{x}_j)$. Indeed, consider $x_\ell[e(u;\varnothing)]$, $\ell \in [j,n]$. If $\ell = \max(u)$, then $x_\ell[e(u;\varnothing)] = [0]$. If $\ell \neq \max(u)$, then $\partial_2(e_\ell \wedge e(u;\varnothing)) = x_\ell e(u;\varnothing)$, so $x_\ell[e(u;\varnothing)] = [0]$. Therefore, we see that a generating set for $H_1(\mathbf{x}_j)$ as a $S/(\mathbf{x}_j)$-module is as given in Claim 1.

Now, let $i > 1$. By (2.4), we have the short exact sequence of $S/(\mathbf{x}_{j+1})$-modules,

$$0 \to \mathrm{Im}(\alpha_i) \xrightarrow{\alpha_i} H_i(\mathbf{x}_j) \xrightarrow{\beta_i} \mathrm{Ker}(\delta_{i-1}) \to 0. \tag{4.10}$$

By inductive hypothesis, a minimal generating set of the $S/(\mathbf{x}_{j+1})$-module $H_{i-1}(\mathbf{x}_{j+1})$ is given by the homology classes of the Koszul cycles

$$e(u;\sigma) \text{ such that } u \in G(I), \quad \sigma \subseteq ([\max(u)-1] \setminus \mathrm{supp}_{\mathbf{t}}(u)) \cap [j+1,n], \quad |\sigma| = i-2.$$

The map δ_{i-1} is multiplication by $\pm x_j$. We show that the minimal generating set of $\mathrm{Ker}(\delta_{i-1})$ is given by those elements $[e(u;\sigma)]$ of $H_{i-1}(\mathbf{x}_{j+1})$ such that $j \notin \mathrm{supp}_{\mathbf{t}}(u)$.

Let $u \in G(I)$ and let $[e(u;\sigma)]$ be an element of $H_{i-1}(\mathbf{x}_{j+1})$ as in Claim 1.

CASE 1. Suppose that $j \in \mathrm{supp}_{\mathbf{t}}(u)$, then $x_j(u/x_{\max(u)}) \notin I$. So, $\pm x_j e(u;\sigma) \neq 0$, as $\pm x_j\varepsilon(u')e_\sigma \wedge e_{\max(u)} \neq 0$. If for absurd $[e(u;\sigma)] \in \mathrm{Ker}(\delta_{i-1})$, then $\delta_{i-1}([e(u;\sigma)]) = \pm x_j[e(u;\sigma)] = [0]$. Since $\pm x_j e(u;\sigma) \neq 0$, there exists $a \in K_i(\mathbf{x}_{j+1})$, $a = \sum \varepsilon(u_\gamma)e_\gamma$, for some $\gamma \subseteq [j+1,n]$, $|\gamma| = i$, $u_\gamma \in S$, such that $\partial_i(a) = \pm x_j e(u;\sigma)$. Hence,

$$\begin{aligned} x_j e(u;\sigma) &= \varepsilon(x_j u/x_{\max(u)})e_\sigma \wedge e_{\max(u)} + (\text{smaller terms}) = \partial_i(a) \\ &= \partial_i\big(\sum \varepsilon(u_\gamma)e_\gamma\big) = \sum_{\gamma\,:\,\gamma\setminus\{\ell\}=\sigma\cup\{\max(u)\}} \varepsilon(x_\ell u_\gamma)e_\sigma \wedge e_{\max(u)} + R, \end{aligned}$$

where R is a sum of other terms not involving $e_\sigma \wedge e_{\max(u)}$. We have $x_j(u/x_{\max(u)}) \notin I$, *i.e.*, $\varepsilon(x_j(u/x_{\max(u)})) \neq 0$. Hence, for some e_{γ_0} occurring in a and some $\ell_0 \in \gamma_0$ such that $\gamma_0 \setminus \{\ell_0\} = \sigma \cup \{\max(u)\}$, we must have $x_j(u/x_{\max(u)}) = x_{\ell_0}u_{\gamma_0}$. We have $\ell_0 \neq \max(u)$, and since $x_j(u/x_{\max(u)}) = x_{\ell_0}u_{\gamma_0}$ and $j < j+1 \leq \max(u)$, we also have $\ell_0 < \max(u)$. Now, $\max(u) \in \gamma_0$ and the term $\pm\varepsilon(x_{\max(u)}u_{\gamma_0})e_{\gamma_0\setminus\{\max(u)\}}$ appears in R. The inequality $\ell_0 < \max(u)$ implies that $\gamma_0 \setminus \{\max(u)\} > \gamma_0 \setminus \{\ell_0\} = \sigma \cup \{\max(u)\}$, and since each wedge product appearing in $x_j e(u;\sigma)$ is smaller than $e_\sigma \wedge e_{\max(u)}$, we must have either $\varepsilon(x_{\max(u)}u_{\gamma_0}) = 0$ or there exist $\gamma_1 \subseteq [j+1,n]$ and an integer $\ell_1 \in \gamma_1$ such that the term $\pm\varepsilon(x_{\ell_1}u_{\gamma_1})e_{\gamma_1\setminus\{\ell_1\}}$ appears in R and cancels with $\pm\varepsilon(x_{\max(u)}u_{\gamma_0})e_{\gamma_0\setminus\{\max(u)\}}$.

SUBCASE 1.1. We have $x_{\max(u)}u_{\gamma_0} \in I$. Hence,

$$x_{\max(u)}u_{\gamma_0} \;=\; x_{\max(u)}x_j(u/x_{\max(u)})/x_{\ell_0} \;=\; x_j(u/x_{\ell_0}) \in I.$$

Therefore, $x_j(u/x_{\ell_0}) \in I$, absurd. Indeed, write $u = x_{j_1}x_{j_2}\cdots x_{j_d}$, then $j \in \mathrm{supp}_{\mathbf{t}}(u)$ implies that $j = j_p + r$, with $0 \leq r \leq t_p - 1$, $t_p \geq 1$. Moreover, $\ell_0 > j$, so $\ell_0 = j_q$, $q > p$. If $x_j(u/x_{\ell_0}) = x_{j_1}\cdots x_{j_p}x_{j_p+r}x_{j_{p+1}}\cdots x_{j_{q-1}}x_{j_{q+1}}\cdots x_{j_d} \in I$, then for some $v \in G(I)$, v divides $x_j(u/x_{\ell_0})$. As v is $\mathbf{t}$-spread but $x_j(u/x_{\ell_0})$ is not, we have $v \neq x_j(u/x_{\ell_0})$, hence $\deg(v) < \deg(x_j(u/x_{\ell_0})) = \deg(u)$. So, v must divide $x_j(u/x_{\ell_0}))/x_j = u/x_{\ell_0}$, and $u/x_{\ell_0} \in I$, absurd as $u \in G(I)$.

SUBCASE 1.2. We have $\gamma_1 \setminus \{\ell_1\} = \gamma_0 \setminus \{\max(u)\} = \sigma \cup \{\ell_0\}$ and $x_{\max(u)}u_{\gamma_0} = x_{\ell_1}u_{\gamma_1}$. Therefore, $\gamma_1 = \sigma \cup \{\ell_0, \ell_1\}$. The term $\pm\varepsilon(x_{\ell_0}u_{\gamma_1})e_{\sigma\cup\{\ell_1\}}$ appears in R and it is bigger than $\varepsilon(u')e_\sigma \wedge e_{\max(u)}$. So, we have two cases to consider. As before, in the first case, $\varepsilon(x_{\ell_0}u_{\gamma_1}) = 0$, and recalling that $x_j(u/x_{\max(u)}) = x_{\ell_0}u_{\gamma_0}$, we have

$$\begin{aligned} x_{\ell_0}u_{\gamma_1} &= x_{\ell_0}(x_{\max(u)}u_{\gamma_0})/x_{\ell_1} \;=\; x_{\max(u)}(x_{\ell_0}u_{\gamma_0})/x_{\ell_1} \;=\; x_{\max(u)}x_j(u/x_{\max(u)})/x_{\ell_1} \\ &= x_j(u/x_{\ell_1}) \in I, \end{aligned}$$

with $\ell_1 > j$. Arguing as in SUBCASE 1.1 we obtain an absurd. Otherwise there exist $\gamma_2 \subseteq [j+1,n]$ and an integer $\ell_2 \in \gamma_2$ such that the term $\pm\varepsilon(x_{\ell_2}u_{\gamma_2})e_{\gamma_2\setminus\{\ell_2\}}$ appears in R and cancels with $\pm\varepsilon(x_{\ell_0}u_{\gamma_1})e_{\sigma\cup\{\ell_1\}}$. We have $\gamma_2 = \sigma \cup \{\ell_1, \ell_2\}$ and consider the term arising from $\gamma_2 \setminus \{\ell_1\}$. We can distinguish two cases as before. After a finite number of steps s, we have $x_j(u/x_{\ell_s}) \in I$ for some $\ell_s > j$, obtaining an absurd. Hence, $\pm x_j e(u;\sigma) \notin \mathrm{Im}(\partial_i)$, and $\pm x_j[e(u;\sigma)] \notin \mathrm{Ker}(\delta_{i-1})$.

CASE 2. Suppose now $j \notin \mathrm{supp}_{\mathbf{t}}(u)$. By Remark 4.2.7,

$$e(u;\sigma \cup \{j\}) = e_j \wedge e(u;\sigma) + r(u;\sigma).$$

Recalling the map $\beta_i : K_i(\mathbf{x}_j) \to K_{i-1}(\mathbf{x}_{j+1})$, we have that $\beta_i(e(u;\sigma\cup\{j\})) = e(u;\sigma)$. By Proposition 4.2.4, $e(u;\sigma \cup \{j\})$ is a cycle. We prove that $[e(u;\sigma \cup \{j\})] \neq [0]$ in $H_i(\mathbf{x}_j)$. Suppose on the contrary that there exists $a \in K_{i+1}(\mathbf{x}_j)$ such that $\partial_{i+1}(a) =$

$e(u;\sigma\cup\{j\})$. Now, $a=\sum\varepsilon(u_\gamma)e_\gamma$, for some $\gamma\subseteq[j,n]$, $|\gamma|=i+1$ and $u_\gamma\in S$. So,

$$\begin{aligned} e(u;\sigma\cup\{j\}) &= \varepsilon(u/x_{\max(u)})e_{\sigma\cup\{j\}}\wedge e_{\max(u)}+(\text{smaller terms})=\partial_i(a)\\ &= \partial_i(\sum\varepsilon(u_\gamma)e_\gamma)=\sum_{\gamma\,:\,\gamma\setminus\{\ell\}=\sigma\cup\{j,\max(u)\}}\varepsilon(x_\ell u_\gamma)e_{\sigma\cup\{j\}}\wedge e_{\max(u)}+R, \end{aligned}$$

where R is a sum of other terms not involving $e_{\sigma\cup\{j\}}\wedge e_{\max(u)}$. For some e_{γ_0} occurring in a and some $\ell_0\in\gamma_0$ such that $\gamma_0\setminus\{\ell_0\}=\sigma\cup\{j,\max(u)\}$, we must have $u/x_{\max(u)}=x_{\ell_0}u_{\gamma_0}$. We have $\ell_0\neq\max(u)$ and $j<j+1\le\max(u)$, so $\ell_0<\max(u)$. Therefore, $\gamma_0=\sigma\cup\{j,\ell_0,\max(u)\}$ and $\pm\varepsilon(x_{\max(u)}u_{\gamma_0})e_{\gamma_0\setminus\{\max(u)\}}=\pm\varepsilon(x_{\max(u)}u_{\gamma_0})e_{\sigma\cup\{j,\ell_0\}}$ appears in R. Now $\ell_0<\max(u)$ implies $\gamma_0\setminus\{\max(u)\}>\gamma_0\setminus\{\ell_0\}=\sigma\cup\{j,\max(u)\}$, and since each wedge product appearing in $e(u;\sigma\cup\{j\})$ is smaller than $e_{\sigma\cup\{j\}}\wedge e_{\max(u)}$, we must have either $\varepsilon(x_{\max(u)}u_{\gamma_0})=0$ or there exist $\gamma_1\subseteq[j,n]$ and $\ell_1\in\gamma_1$ such that the term $\pm\varepsilon(x_{\ell_1}u_{\gamma_1})e_{\gamma_1\setminus\{\ell_1\}}$ appears in R and cancels with $\pm\varepsilon(x_{\max(u)}u_{\gamma_0})e_{\gamma_0\setminus\{\max(u)\}}$.

Subcase 2.1. We have $x_{\max(u)}u_{\gamma_0}\in I$. So, $x_{\max(u)}u_{\gamma_0}=x_{\max(u)}(u/x_{\max(u)})/x_{\ell_0}=u/x_{\ell_0}\in I$, but this is absurd, as u is a minimal monomial generator of I.

Subcase 2.2. We have $\gamma_1\setminus\{\ell_1\}=\gamma_0\setminus\{\max(u)\}=\sigma\cup\{j,\ell_0\}$ and $x_{\max(u)}u_{\gamma_0}=x_{\ell_1}u_{\gamma_1}$. Therefore, $\gamma_1=\sigma\cup\{j,\ell_0,\ell_1\}$. The term $\pm\varepsilon(x_{\ell_0}u_{\gamma_1})e_{\sigma\cup\{j,\ell_1\}}$ appears in R and it is bigger than $\varepsilon(u')e_{\sigma\cup\{j\}}\wedge e_{\max(u)}$. So, we have two cases to consider. As before, in the first case, $\varepsilon(x_{\ell_0}u_{\gamma_1})=0$, and recalling that $u/x_{\max(u)}=x_{\ell_0}u_{\gamma_0}$, we have

$$\begin{aligned} x_{\ell_0}u_{\gamma_1} &= x_{\ell_0}(x_{\max(u)}u_{\gamma_0})/x_{\ell_1} = x_{\max(u)}(x_{\ell_0}u_{\gamma_0})/x_{\ell_1} = x_{\max(u)}(u/x_{\max(u)})/x_{\ell_1}\\ &= u/x_{\ell_1}\in I, \end{aligned}$$

an absurd, as $u\in G(I)$. Otherwise, we iterate the reasoning. After a finite number of steps s, we have $u/x_{\ell_s}\in I$, for some ℓ_s, an absurd. Hence $e(u;\sigma\cup\{j\})\notin\operatorname{Im}(\partial_{i+1})$, and $[e(u;\sigma\cup\{j\})]\neq[0]$ in $H_i(\mathbf{x}_j)$. Therefore, $\beta_i([e(u;\sigma\cup\{j\})])=[e(u;\sigma)]$, and $[e(u;\sigma)]\in\operatorname{Im}(\beta_i)=\operatorname{Ker}(\delta_{i-1})$, as desired.

Finally, a basis for $\beta_i^{-1}(\operatorname{Ker}(\delta_{i-1}))$ is given by all the elements as in Claim 1 such that $j\in\sigma$. By inductive hypothesis, we know a basis for $H_i(\mathbf{x}_{j+1})$, and as α_i sends these homology classes to the corresponding homology classes of $H_i(\mathbf{x}_j)$, a minimal generating set for $\operatorname{Im}(\alpha_i)$ is given by all the elements as in Claim 1 such that $j\notin\sigma$.

We observe that $(\mathbf{x}_j)$ annihilates these elements. Indeed, the elements $[e(u;\sigma)]$ as in Claim 1 minimally generate $H_i(\mathbf{x}_j)$ as a $S/(\mathbf{x}_{j+1})$-module. So $(\mathbf{x}_{j+1})$ annihilates all $[e(u;\sigma)]$. It remains to prove that x_j annihilates all elements $[e(u;\sigma)]$. If $j\notin\sigma$, then by definition of $e(u;\sigma)$, e_j doesn't appear in the first term $\varepsilon(u/x_{\max(u)})e_\sigma\wedge e_{\max(u)}$ of $e(u;\sigma)$. We have

$$\partial_{i+1}(e_j\wedge e(u;\sigma))=x_je(u;\sigma)+e_j\wedge(-1)^{\deg(e_j)}\partial_i(e(u;\sigma))=x_je(u;\sigma),$$

so $x_j[e(u;\sigma)]=[0]$.

Suppose now $j\in\sigma$. Then $\beta_i(x_j[e(u;\sigma)])=x_j[e(u;\sigma\setminus\{j\})]=[0]$, because $[e(u;\sigma\setminus\{j\})]\in\operatorname{Ker}(\delta_{i-1})$. Hence, $x_j[e(u;\sigma)]\in\operatorname{Ker}(\beta_i)=\operatorname{Im}(\alpha_i)$. By Remark 4.2.7,

$$x_j[e(u;\sigma)] = x_j[e_j\wedge e(u;\sigma\setminus\{j\})+r(u;\sigma)] = x_j[r(u;\sigma)]\in\operatorname{Im}(\alpha_i)\subseteq H_i(\mathbf{x}_{j+1}),$$

the first summand vanishes, as $e_j\notin H_i(\mathbf{x}_{j+1})$. If we set $a=r(u;\sigma)$, $x_j[a]$ is a cycle, and we have $\partial_{i+1}(e_j\wedge a)=-x_ja$, so $x_j\big[r(u;\sigma)\big]=[0]$ and $x_j\big[e(u;\sigma)\big]=[0]$, as desired. So, a minimal generating set for the $S/(\mathbf{x}_j)$-module $H_i(\mathbf{x}_j)$ is as in Claim 1.

Finally for $j=1$, $S/(\mathbf{x}_1)=S/(x_1,\dots,x_n)\cong K$, and Claim 1 implies the result, as a minimal generating set of a K-vector space is a basis. $\square$

We provide an example that demonstrate our methods.

Example 4.2.9 Let $\mathbf{t} = (1,0,2)$, and let $I = (x_1, x_2x_3^2, x_2x_3x_4x_6, x_2x_4^2x_6)$. We set $w_1 = x_1, w_2 = x_2x_3^2, w_3 = x_2x_3x_4x_6, w_4 = x_2x_4^2x_6$. The ideal $I \subseteq S = K[x_1,\dots,x_6]$ is a $\mathbf{t}$-spread strongly stable ideal with minimal generating set $G(I) = \{w_1,w_2,w_3,w_4\}$. Let $\mathbf{x} = x_1,x_2,\dots,x_6$. The basis for the Koszul homologies of S/I are:

$$
\begin{array}{lrcl}
H_1(\mathbf{x};S/I)\text{: } & e(w;\varnothing) & = & \varepsilon(w/x_{\max(w)})\ e_{\max(w)}, \quad \text{for } w \in G(I);\\
H_2(\mathbf{x};S/I)\text{: } & \multicolumn{3}{l}{w_1 \ \text{ gives no rise to any basis element,}}\\
w_2 \text{ gives} & e(w_2;\{1\}) & = & \varepsilon(x_2x_3)\ e_1\wedge e_3,\\
w_3 \text{ gives} & e(w_3;\{1\}) & = & \varepsilon(x_2x_3x_4)\ e_1\wedge e_6,\\
 & e(w_3;\{3\}) & = & \varepsilon(x_2x_3x_4)\ e_3\wedge e_6,\\
w_4 \text{ gives} & e(w_4;\{1\}) & = & \varepsilon(x_2x_4^2)\ e_1\wedge e_6,\\
 & e(w_4;\{3\}) & = & \varepsilon(x_2x_4^2)\ e_3\wedge e_6 - \varepsilon(x_2x_3x_4)\ e_4\wedge e_6;\\
H_3(\mathbf{x};S/I)\text{: } & \multicolumn{3}{l}{w_1, w_2 \text{ give no rise to any basis element,}}\\
w_3 \text{ gives} & e(w_3;\{1,3\}) & = & \varepsilon(x_2x_3x_4)\ e_1\wedge e_3\wedge e_6,\\
w_4 \text{ gives} & e(w_4;\{1,3\}) & = & \varepsilon(x_2x_4^2)\ e_1\wedge e_3\wedge e_6 - \varepsilon(x_2x_3x_4)\ e_1\wedge e_4\wedge e_6;\\
H_j(\mathbf{x};S/I)\text{: } & \multicolumn{3}{l}{\varnothing, \quad \text{for all } j \geq 4.}
\end{array}
$$

For instance, consider $e(w_4;\{1,3\}) \in K_3(\mathbf{x};S/I)$. Then

$$
\begin{aligned}
\partial_3(e(w_4;\{1,3\})) &= \partial_3\big(\varepsilon(x_2x_4^2)\ e_1\wedge e_3\wedge e_6 - \varepsilon(x_2x_3x_4)\ e_1\wedge e_4\wedge e_6\big)\\
&= \varepsilon(x_1x_2x_4^2)\ e_3\wedge e_6 - \varepsilon(x_2x_3x_4^2)\ e_1\wedge e_6 + \varepsilon(x_2x_4^2x_6)\ e_1\wedge e_3\\
&- \ \varepsilon(x_1x_2x_3x_4)\ e_4\wedge e_6 + \varepsilon(x_2x_3x_4^2)\ e_1\wedge e_6 - \varepsilon(x_2x_3x_4x_6)\ e_1\wedge e_4\\
&= \ 0.
\end{aligned}
$$

In fact, the first, third, fourth and sixth terms vanish, as $\varepsilon(x_1x_2x_4^2) = \varepsilon(x_2x_4^2x_6) = \varepsilon(x_1x_2x_3x_4) = \varepsilon(x_2x_3x_4x_6) = 0$, and the second and fifth terms are opposite.

We illustrate how to obtain some of these elements.

Consider $w_3 = x_2x_3x_4x_6$. Then $\text{supp}_{\mathbf{t}}(w_3) = \text{supp}_{(1,0,2)}(x_2x_3x_4x_6) = \{2,4,5\}$ and $[\max(w_3)-1] \setminus \text{supp}_{\mathbf{t}}(w_3) = \{1,3\}$. Let $\vartheta = \{1,3\}$, then $\vartheta^{(w_3)} = \{2,4\}$. Moreover $\max(w_3) = 6 \notin \vartheta^{(w_3)}$. So, we can use equation (4.4) to compute the relevant Koszul cycles $e(w_3;\sigma)$. Of course, we may also use equation (4.2).

The monomial w_3 gives rise to the following Koszul cycles:

$$
\begin{array}{rrcl}
\sigma = \varnothing; & e(w_3;\varnothing) & = & \varepsilon(w_3/x_6)e_6 = \varepsilon(x_2x_3x_4)e_6,\\[2ex]
\sigma = \{1\}; & e(w_3;\{1\}) & = & -e(w_3;\varnothing)\wedge e_1 + e(x_1(w_3/x_2);\varnothing)\wedge e_2\\
 & & = & -\varepsilon(x_2x_3x_4)e_6\wedge e_1 + \varepsilon(x_1x_3x_4)e_6\wedge e_2\\
 & & = & \ \varepsilon(x_2x_3x_4)e_1\wedge e_6,\\[2ex]
\sigma = \{3\}; & e(w_3;\{3\}) & = & -e(w_3;\varnothing)\wedge e_3 + e(x_3(w_3/x_4);\varnothing)\wedge e_4\\
 & & = & -\varepsilon(x_2x_3x_4)e_6\wedge e_3 + \varepsilon(x_2x_3^2)e_6\wedge e_4\\
 & & = & \ \varepsilon(x_2x_3x_4)e_3\wedge e_6,\\[2ex]
\sigma = \{1,3\}; & e(w_3;\{1,3\}) & = & -e(w_3;\{1\})\wedge e_3 + e(x_3(w_3/x_4);\{1\})\wedge e_4\\
 & & = & -\varepsilon(x_2x_3x_4)e_1\wedge e_6\wedge e_3 + e(x_2x_3^2x_6;\{1\})\wedge e_4\\
 & & = & \ \varepsilon(x_2x_3x_4)e_1\wedge e_3\wedge e_6 + \varepsilon(x_2x_3^2)e_1\wedge e_6\wedge e_4\\
 & & = & \ \varepsilon(x_2x_3x_4)e_1\wedge e_3\wedge e_6.
\end{array}
$$

Our computations yield the Betti table of S/I,

	0	1	2	3
total:	1	4	5	2
0:	1	1	-	-
1:	-	-	-	-
2:	-	1	1	-
3:	-	2	4	2

Remark 4.2.10 The expression of our Koszul cycles is not so nice. Indeed, a basis element $e(u;\sigma)$ of $H_i(\mathbf{x};S/I)$, I a $\mathbf{t}$-spread strongly stable ideal, is a sum of 2^{i-1} wedge products! However, if $\mathbf{t}=(1,\dots,1,0,\dots,0)\in\mathbb{Z}_{\geq 0}^{d-1}$, $d\geq 2$, the element

$$z(u;\sigma)=\varepsilon(u/x_{\max(u)})e_\sigma\wedge e_{\max(u)}.$$

with $u\in G(I)$ and $\sigma\subseteq[\max(u)-1]\setminus\operatorname{supp}_{\mathbf{t}}(u)$ is easily seen to be a cycle. Indeed,

$$\begin{aligned}\partial_i(z(u;\sigma)) &= \sum_{j=1}^{i-1}(-1)^{j+1}\varepsilon(x_{k_j}(u/x_{\max(u)}))e_{\sigma\setminus\{k_j\}}\wedge e_{\max(u)}+(-1)^{i+1}\varepsilon(u)e_\sigma\\ &= 0,\end{aligned}$$

as $x_{k_j}(u/x_{\max(u)})\in I$ for all j and $u\in I$, since $\mathbf{t}=(1,\dots,1,0,\dots,0)$. It's easy to see that the homology classes $[z(u;\sigma)]$ are non zero and K-independent. Hence, they form a basis for $H_i(\mathbf{x})$, as the map $z: z(u;\sigma)\mapsto e(u;\sigma)$ is a bijection and the elements $e(u;\sigma)$ form a basis of $H_i(\mathbf{x})$ by Theorem 4.2.8. These Koszul cycles have been considered in the articles [10, 12]. But in general they are cycles only when the vector $\mathbf{t}$ has the form $\mathbf{t}=(1,\dots,1,0,\dots,0)$.

Example 4.2.11 Let $I=(x_1x_2\,,x_1x_3\,,x_1x_2^2\,,x_1x_2x_3\,,x_1x_2x_4\,,x_1x_3^2\,,x_1x_3x_4\,,x_1x_4^2)$ be a $(1,0)$-spread strongly stable ideal of $K[x_1,x_2,x_3,x_4]$. By Remark 4.2.10, since $\mathbf{t}=(1,0)$ and $G(I)=\{x_1x_2,x_1x_3,x_1x_4^2\}$, the relevant basis for the Koszul homologies of S/I are:

$H_1(\mathbf{x};S/I)$: $\varepsilon(x_1x_2/x_2)\ e_2$, $\varepsilon(x_1x_3/x_3)\ e_3$, $\varepsilon(x_1x_4^2/x_4)\ e_4$;
$H_2(\mathbf{x};S/I)$: $\varepsilon(x_1x_3/x_3)\ e_2\wedge e_3$, $\varepsilon(x_1x_4^2/x_4)\ e_2\wedge e_4$, $\varepsilon(x_1x_4^2/x_4)\ e_3\wedge e_4$;
$H_3(\mathbf{x};S/I)$: $\varepsilon(x_1x_4^2/x_4)\ e_2\wedge e_3\wedge e_4$;
$H_j(\mathbf{x};S/I)$: $\varnothing$, for all $j\geq 4$.

Therefore, using *Macaulay2* [82] the Betti table of I is

	0	1	2
total:	3	3	1
2:	2	1	-
3:	1	2	1

4.3 The minimal free resolution of vector-spread strongly stable ideals

In this section we construct the minimal free resolution of $\mathbf{t}$-spread strongly stable ideals of S. This resolution will generalize that of Eliahou and Kervaire [50], and also the squarefree lexsegment analogue in [12]. We will follow the construction given by Aramova and Herzog in [10].

Let $I \subset S = K[x_1, \dots, x_n]$ be a $\mathbf{t}$-spread strongly stable ideal. Note that, since $\mathrm{Tor}_i^S(K, S/I) \cong H_i(\mathbf{x}; S/I) = H_i(\mathbf{x})$, for all i, the minimal free resolution of S/I may be written as follows,

$$\mathbb{F} : \cdots \xrightarrow{d_3} S \otimes_K H_2(\mathbf{x}) \xrightarrow{d_2} S \otimes_K H_1(\mathbf{x}) \xrightarrow{d_1} S \otimes_K H_0(\mathbf{x}) \xrightarrow{d_0} S/I \to 0.$$

We set $F_i = S \otimes_K H_i(\mathbf{x})$, for all i, and note that $F_0 = S$. By Theorem 4.2.8 and also [10], for all $i \geq 1$ a basis of the graded free S-module F_i is given by the elements,

$$f(u;\sigma) = 1 \otimes (-1)^{(i-1)(i-2)/2}[e(u;\sigma)],$$

such that $u \in G(I)$, $\sigma \subseteq [\max(u) - 1] \setminus \mathrm{supp}_{\mathbf{t}}(u)$ and $|\sigma| = i - 1$. For later use, we shall make the following convention. If $\sigma \not\subseteq [\max(u) - 1] \setminus \mathrm{supp}_{\mathbf{t}}(u)$ we set $f(u;\sigma) = 0$.

Thus, it remains to describe the differentials d_i, for all $i \geq 0$. For this purpose, suppose the differentials $d_0, d_1, \dots, d_{i-1}$ have already been constructed such that

$$\mathbb{F}_{<i} : F_{i-1} \xrightarrow{d_{i-1}} F_{i-2} \xrightarrow{d_{i-2}} \cdots \xrightarrow{d_1} F_0 \xrightarrow{d_0} S/I \to 0$$

is exact. Fix a basis element $f(u;\sigma)$ of F_i. Let $\mathbb{K} = K_{\bullet}(\mathbf{x}; S/I)$ be the Koszul complex attached to $\mathbf{x}$ with respect to S/I whose ith module and differential are, respectively, K_i and $\partial_i : K_i \to K_{i-1}$. We consider the double complex $\mathbb{K} \otimes_S \mathbb{F}_{<i}$,

$$\begin{tikzcd}
 & \vdots \arrow[d, "\mathrm{id}\otimes d_2"'] & \vdots \arrow[d, "\mathrm{id}\otimes d_2"'] & & \vdots \arrow[d, "\mathrm{id}\otimes d_2"] & \\
0 \arrow[r] & K_n \otimes F_1 \arrow[r, "\partial_n\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes d_1"'] & K_{n-1} \otimes F_1 \arrow[r, "\partial_{n-1}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes d_1"'] & \cdots \arrow[r, "\partial_1\otimes\mathrm{id}"] & K_0 \otimes F_1 \arrow[r] \arrow[d, "\mathrm{id}\otimes d_1"] & 0 \\
0 \arrow[r] & K_n \otimes F_0 \arrow[r, "\partial_n\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes d_0"'] & K_{n-1} \otimes F_0 \arrow[r, "\partial_{n-1}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes d_0"'] & \cdots \arrow[r, "\partial_1\otimes\mathrm{id}"] & K_0 \otimes F_0 \arrow[r] \arrow[d, "\mathrm{id}\otimes d_0"] & 0 \\
0 \arrow[r] & K_n \otimes S/I \arrow[r, "\partial_n\otimes\mathrm{id}"'] \arrow[d] & K_{n-1} \otimes S/I \arrow[r, "\partial_{n-1}\otimes\mathrm{id}"'] \arrow[d] & \cdots \arrow[r, "\partial_1\otimes\mathrm{id}"'] & K_0 \otimes S/I \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & & 0 &
\end{tikzcd}$$

where "id" denotes each time a suitable identity function.

It is known by [10, Section 1] that to describe how the differential d_i acts on $f(u;\sigma)$ it suffices to determine elements $g_j \in K_{i-j} \otimes F_j$, $j = 0, \dots, i-1$, satisfying

$$(\mathrm{id}_{K_i} \otimes d_0)(g_0) = (-1)^{(i-1)(i-2)/2} 1 \otimes e(u;\sigma), \quad \text{and} \tag{4.11}$$
$$(\mathrm{id}_{K_{i-j-1}} \otimes d_{j+1})(g_{j+1}) = (\partial_{i-j} \otimes \mathrm{id}_{F_j})(g_j) \quad \text{for } j = 0, \dots, i-2. \tag{4.12}$$

To construct such a sequence is a difficult combinatorial task. Thus we restrict ourself to the case when $\mathbf{t} = (1, \dots, 1, 0, \dots, 0)$, (Remark 4.2.10). In this case we can replace the cycles $e(u;\sigma)$ by the cycles $z(u;\sigma)$. In order to construct the sequence of elements satisfying equations (4.11) and (4.12) we need the following notion. We recall that the *pure lexicographic order* is defined as follows: $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} >_{\mathrm{plex}} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ if and only if $a_1 = b_1,\ a_2 = b_2,\ \dots,\ a_{s-1} = b_{s-1}$ and $a_s > b_s$ for some $s \in \{1, \dots, n\}$.

Definition 4.3.1 Let $I \subset S$ be a $\mathbf{t}$-spread strongly stable ideal, $\mathbf{t} = (1, \dots, 1, 0, \dots, 0)$. Let $M(I)$ be the set of all monomials belonging to I. We define the map $g : M(I) \to G(I)$, as follows: for $w \in M(I)$, we set $g(w) = \max_{>_{\text{plex}}}\{u \in G(I) : u \text{ divides } w\}$. The map g is called the $\mathbf{t}$-*spread decomposition function* of I.

For $u \in G(I)$, $k \in [\max(u) - 1] \setminus \operatorname{supp}_{\mathbf{t}}(u)$, we set

$$u_k = g(x_k u) \quad \text{and} \quad v_k = (x_k u)/u_k.$$

We shall need also the following notations. For a subset σ of $[n]$ and for $k \in \sigma$ we define $\alpha(\sigma; k) = |\{s \in \sigma : s < k\}|$. For τ a subset of σ we let $\gamma(\tau) = \sum_{k \in \sigma \setminus \tau} \alpha(\sigma; k)$. In what follows, we denote $\sigma \setminus \{k\}$ by $\sigma \setminus k$, and $\sigma \cup \{k\}$ by $\sigma \cup k$, omitting the parentheses. To further simplify the notations, we set $\mathrm{id}_{K_{i-j-1}} \otimes d_{j+1} = d_{j+1}$ and $\partial_{i-j} \otimes \mathrm{id}_{F_j} = \partial_{i-j}$, for all $j = 0, \dots, i-2$.

The next theorem gives the desired differentials of the resolution $\mathbb{F}$ of S/I and generalize [10, Theorem 2.3]. To write the elements g_j more conveniently we switch the order in the tensor products, that is we think g_j as an element of $F_j \otimes K_{i-j}$.

Theorem 4.3.2 *Let* $g_0 = (-1)^{\frac{(i-1)(i-2)}{2}} 1 \otimes (u' e_\sigma \wedge e_{\max(u)})$, *and for* $j = 1, \dots, i-1$ *let*

$$g_j = (-1)^{i-j} \sum_{\substack{\tau \subset \sigma \\ |\tau| = j-1}} (-1)^{\gamma(\tau)} f(u; \tau) \otimes e_{\sigma \setminus \tau} + \sum_{\substack{\tau \subset \sigma \\ |\tau| = j}} (-1)^{\gamma(\tau)} s_\tau \otimes e_{\sigma \setminus \tau} \wedge e_{\max(u)},$$

where

$$s_\tau = \sum_{k \in \tau} (-1)^{\alpha(\tau; k)} \frac{v_k}{x_{\max(u)}} f(u_k; \tau \setminus k).$$

Then the elements $g_0, g_1, \dots, g_{i-1}$ *satisfy equations* (4.11) *and* (4.12). *Moreover, the* i*th differential of the minimal free resolution of* S/I *acting on* $f(u; \sigma)$ *is given by*

$$\begin{aligned} d_i(f(u; \sigma)) &= \partial_1(g_{i-1}) \\ &= \sum_{k \in \sigma} (-1)^{\alpha(\sigma; k)} (-x_k f(u; \sigma \setminus k) + v_k f(u_k; \sigma \setminus k)). \end{aligned} \tag{4.13}$$

Proof. Note that in the definition of s_τ, $x_{\max(u)}$ always divide v_k. Indeed, if we let $k^{(u)} = \min\{j \in \operatorname{supp}(u) : j > k\}$, then $w = x_k(u/x_{k^{(u)}}) \in I$ is again a $\mathbf{t}$-spread monomial. By Lemma 4.1.5, $w = w_1 w_2$ with $w_1 \in G(I)$ and $\max(w_1) \le \min(w_2)$. Consequently $\{y \in G(I) : y \text{ divides } x_k u\}$ is non empty and u_k exists. Proceeding as in the proof of Lemma 4.1.5 we see that $\max(u_k) \le \min(v_k)$. Finally, $v_k \neq 1$ otherwise $u_k = x_k u \in G(I)$, which is absurd. Hence $x_{\max(u)}$ divides v_k as wanted.

We proceed by induction on i. The case $i = 1$ is trivial. By induction, we can assume that the last formula for the differential d_ℓ holds for $\ell < i$. We need to verify the equations $\partial_{i-j}(g_j) = d_{j+1}(g_{j+1})$. For $j = 0$ this is trivial. Let $j > 0$.

Firstly, we calculate $\partial_{i-j}(g_j)$. Since $|\sigma \setminus \tau| = i - j - 1$, we have

$$\begin{aligned} \partial_{i-j}(g_j) = (-1)^{i-j} &\sum_{\substack{\tau \subset \sigma \\ |\tau| = j-1}} (-1)^{\gamma(\tau)} f(u; \tau) \otimes \Big(\sum_{k \in \sigma \setminus \tau} (-1)^{\alpha(\sigma \setminus \tau; k)} x_k e_{\sigma \setminus (\tau \cup k)}\Big) \\ + \sum_{\substack{\tau \subset \sigma \\ |\tau| = j}} (-1)^{\gamma(\tau)} s_\tau \otimes \Big(&\sum_{k \in \sigma \setminus \tau} (-1)^{\alpha(\sigma \setminus \tau; k)} x_k e_{\sigma \setminus (\tau \cup k)} \wedge e_{\max(u)} + (-1)^{i-j-1} x_{\max(u)} e_{\sigma \setminus \tau}\Big). \end{aligned}$$

We suitably rewrite both sums.

For the first sum, note that for $\tau \subseteq \sigma$, $|\tau| = j-1$ and $k \in \sigma \setminus \tau$, then setting $\rho = \tau \cup k$ we have that $|\rho| = j$, $\gamma(\tau) = \gamma(\rho \setminus k) = \sum_{s\in\sigma\setminus(\rho\cup k)} \alpha(\sigma; s) = \gamma(\rho) + \alpha(\sigma; k)$, $\alpha(\sigma; k) = \alpha(\sigma \setminus \tau; k) + \alpha(\tau; k)$ and also $\alpha(\tau; k) = \alpha(\rho; k)$ for it is $k \notin \tau$. Hence,

$$(-1)^{\gamma(\tau)}(-1)^{\alpha(\sigma\setminus\tau;k)} = (-1)^{\gamma(\rho)+\alpha(\sigma\setminus\tau;k)+\alpha(\tau;k)}(-1)^{\alpha(\sigma\setminus\tau;k)} = (-1)^{\gamma(\rho)}(-1)^{\alpha(\rho;k)}.$$

As $\tau \subseteq \sigma$, $|\tau| = j-1$ and $k \in \sigma \setminus \tau$ are arbitrary, $\rho = \tau \cup k \subseteq \sigma$ with $|\rho| = j$ is arbitrary too, thus the first sum of $\partial_{i-j}(g_j)$ can be rewritten as follows,

$$A = (-1)^{i-j-1} \sum_{\substack{\rho\subseteq\sigma \\ |\rho|=j}} (-1)^{\gamma(\rho)} \Big(\sum_{k\in\rho} (-1)^{\alpha(\rho;k)+1} x_k f(u;\rho\setminus k)\Big) \otimes e_{\sigma\setminus\rho}. \tag{4.14}$$

Analogously, the second sum can be written as $B + C$, where

$$B = \sum_{\substack{\vartheta\subseteq\sigma \\ |\vartheta|=j+1}} (-1)^{\gamma(\vartheta)} \Big(\sum_{k\in\vartheta} (-1)^{\alpha(\vartheta;k)} x_k s_{\vartheta\setminus k}\Big) \otimes e_{\sigma\setminus\vartheta} \wedge e_{\max(u)}, \tag{4.15}$$

$$C = (-1)^{i-j-1} \sum_{\substack{\rho\subseteq\sigma \\ |\rho|=j}} (-1)^{\gamma(\rho)} s_\rho \otimes x_{\max(u)} e_{\sigma\setminus\rho}. \tag{4.16}$$

Taking into account equations (4.14), (4.16), the inductive hypothesis and the definition of s_ρ we have that

$$\begin{aligned} A + C &= (-1)^{i-(j+1)} \sum_{\substack{\rho\subset\sigma \\ |\rho|=j}} (-1)^{\gamma(\rho)} \Big(\sum_{k\in\rho} (-1)^{\alpha(\rho;k)+1} x_k f(u;\rho\setminus k) + x_{\max(u)} s_\rho\Big) \otimes e_{\sigma\setminus\rho} \\ &= (-1)^{i-(j+1)} \sum_{\substack{\rho\subset\sigma \\ |\rho|=j}} (-1)^{\gamma(\rho)} d_{j+1}(f(u;\rho)) \otimes e_{\sigma\setminus\rho}. \end{aligned}$$

Thus, by the structure of g_{j+1}, to complete our proof we need to prove that

$$B = \sum_{\substack{\vartheta\subseteq\sigma \\ |\vartheta|=j+1}} (-1)^{\gamma(\vartheta)} d_{j+1}(s_\vartheta) \otimes e_{\sigma\setminus\vartheta} \wedge e_{\max(u)}.$$

That is, we have to prove

$$\begin{aligned} d_{j+1}(s_\vartheta) &= \sum_{k\in\vartheta} (-1)^{\alpha(\vartheta;k)} x_k s_{\vartheta\setminus k} \\ &= \sum_{k\in\vartheta} \sum_{r\in\vartheta\setminus k} (-1)^{\alpha(\vartheta;k)+\alpha(\vartheta\setminus k;r)} x_k \frac{v_r}{x_{\max(u)}} f(u_r; \vartheta\setminus\{k,r\}), \end{aligned} \tag{4.17}$$

for all $\vartheta \subseteq \sigma$, $|\vartheta| = j+1$.

Since $j < i-1$, then $j+1 < i$, and by inductive hypothesis,

$$\begin{aligned} d_{j+1}(s_\vartheta) = &\sum_{r\in\vartheta} (-1)^{\alpha(\vartheta;r)} \frac{v_r}{x_{\max(u)}} \Big(\sum_{k\in\vartheta\setminus r} (-1)^{\alpha(\vartheta\setminus r;k)} \times \\ &(-x_k f(u_r; \vartheta\setminus\{k,r\}) + \frac{x_k u_r}{g(x_k u_r)} f(g(x_k u_r); \vartheta\setminus\{k,r\}))\Big). \end{aligned}$$

For $u \in G(I)$ and $\tau \subseteq [n]$, we define

$$\Gamma(u;\tau) = \{r \in \tau : \tau \setminus r \subseteq [\max(u_r) - 1] \setminus \operatorname{supp}_{\mathbf{t}}(u_r)\},$$

where $u_r = g(x_r u)$. Note that for $r \in \tau \setminus \Gamma(u;\tau)$, $f(u_r;\tau \setminus r) = 0$.

Now, for the first sum of terms of $d_{j+1}(s_\vartheta)$, note that $\alpha(\vartheta;r) = \alpha(\vartheta \setminus k;r) + \alpha(k;r)$, $\alpha(\vartheta;k) = \alpha(\vartheta \setminus r;k) + \alpha(r;k)$ and $\alpha(k;r) - \alpha(r;k) = 1$ if $k < r$ or -1 if $k > r$. Thus,

$$(-1)^{\alpha(\vartheta;r)+\alpha(\vartheta\setminus r;k)+1} = (-1)^{\alpha(\vartheta\setminus k;r)+\alpha(k;r)+\alpha(\vartheta;k)-\alpha(r;k)+1} = (-1)^{\alpha(\vartheta;k)+\alpha(\vartheta\setminus k;r)}.$$

Taking into account this calculation and exchanging the indices k with r in the second sum of terms of $d_{j+1}(s_\vartheta)$, we can write $d_{j+1}(s_\vartheta)$ as $B_1 + B_2$, where

$$\begin{aligned} B_1 &= \sum_{r \in \Gamma(u;\vartheta)} \sum_{k \in \vartheta \setminus r} (-1)^{\alpha(\vartheta;k)+\alpha(\vartheta\setminus k;r)} x_k \frac{v_r}{x_{\max(u)}} f(u_r;\vartheta \setminus \{k,r\}),\\ B_2 &= \sum_{\substack{k \in \Gamma(u;\vartheta)\\ r \in \Gamma(u_k;\vartheta\setminus k)}} (-1)^{\alpha(\vartheta;k)+\alpha(\vartheta\setminus k;r)} \frac{x_r u_k v_k}{g(x_r u_k) x_{\max(u)}} f(g(x_r u_k);\vartheta \setminus \{k,r\}). \end{aligned}$$

In all terms of the right–hand side in equation (4.17), for $k, r \in \vartheta$, $k \neq r$, we have either $r \in \Gamma(u;\vartheta)$ or $r \notin \Gamma(u;\vartheta)$ and $r \in \Gamma(u;\vartheta \setminus k)$. Let B_3 be the sum of terms such that $r \in \Gamma(u;\vartheta)$, and let B_4 be the sum of terms such that $r \notin \Gamma(u;\vartheta)$ and $r \in \Gamma(u;\vartheta \setminus k)$. To finish the proof, it is enough to show that $B_1 = B_3$ and $B_2 = B_4$.

It is clear that $B_1 = B_3$.

Let us see that $B_2 = B_4$. The hypotheses $r \notin \Gamma(u;\vartheta)$ and $r \in \Gamma(u;\vartheta \setminus k)$ imply that $k \notin [\max(u_r) - 1] \setminus \operatorname{supp}_{\mathbf{t}}(u_r)$, where $u_r v_r = x_r u$ and $\max(u_r) \leq \min(v_r)$. But $k \in \vartheta \subseteq \sigma \subseteq [\max(u) - 1] \setminus \operatorname{supp}_{\mathbf{t}}(u)$. Thus, either $k \in [\max(u_r), \max(u) - 1]$ or $k \in \operatorname{supp}_{\mathbf{t}}(u_r) \setminus \operatorname{supp}_{\mathbf{t}}(u)$. We show in both cases that $g(x_r u_k) = u_r$.

If $k \in [\max(u_r), \max(u) - 1]$, then $k \geq \max(u_r) \geq r$, so $k > r$ since $k \neq r$. This implies that $r < k \leq \max(u_k)$ too. Hence, u_r divides $x_r u_k$. Finally, $g(x_r u_k) = u_r$.

If $k \in \operatorname{supp}_{\mathbf{t}}(u_r) \setminus \operatorname{supp}_{\mathbf{t}}(u)$, then $k > r$, and so $r < \max(u_k)$. Since $k \in \operatorname{supp}_{\mathbf{t}}(u_r)$ we have that $k < \max(u_r)$. Let us see that $\max(u_r) \leq \max(u_k)$. Suppose on the contrary that $\max(u_r) > \max(u_k)$. If $u = x_{j_1} x_{j_2} \cdots x_{j_d}$, then $u_k = x_k \cdot x_{j_1} \cdots x_{j_p}$ and $u_r = x_r \cdot x_{j_1} \cdots x_{j_q}$ are both $\mathbf{t}$-spread monomials of I with $p < q < d$. Then $x_r(u_k/x_k)$ is a $\mathbf{t}$-spread monomial of I that divides $x_r u$ and $x_r(u_k/x_k) >_{\text{plex}} u_r$, an absurd. Hence $\max(u_r) \leq \max(u_k)$, so u_r divides $x_r u_k$ and again $g(x_r u_k) = u_r$.

Thus, $g(x_r u_k) = u_r$ and

$$\begin{aligned} & \frac{x_r u_k v_k}{g(x_r u_k) x_{\max(u)}}\, f(g(x_r u_k);\vartheta \setminus \{k,r\}) && (\text{as } u_k v_k = x_k u),\\ = \;& \frac{x_r x_k u}{u_r x_{\max(u)}}\, f(u_r;\vartheta \setminus \{k,r\}) && (\text{as } x_r u = u_r v_r),\\ = \;& x_k \frac{v_r}{x_{\max(u)}}\, f(u_r;\vartheta \setminus \{k,r\}). \end{aligned}$$

This shows that $B_2 = B_4$ and completes our proof. □

We consider the ideal in Example 4.2.11 and construct the differentials of its minimal free resolution. Note that in this case $\mathbf{t} = (1, 0)$.

Example 4.3.3 Let $I \subset S = K[x_1, \dots, x_6]$ be the $(1,0)$-spread strongly stable ideal of Example 4.2.11 with minimal generating set

$$G(I) = \{w_1 = x_1x_2,\ w_2 = x_1x_3,\ w_3 = x_1x_4^2\}.$$

By Example 4.2.11, $\mathrm{pd}(S/I) = 3$. Let

$$\mathbb{F} : 0 \to F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 = S \xrightarrow{d_0} S/I \to 0$$

the minimal free resolution of S/I. We know that $d_0 = \varepsilon : S \to S/I$ is the canonical map. We shall describe the differentials d_1, d_2, d_3 by appropriate monomial matrices.

For $i = 1, 2, 3$, the basis of the free S-modules $F_i = S \otimes_K H_i(\mathbf{x})$ consists of

$$f(w_j; \sigma) = (-1)^{(i-1)(i-2)/2} 1 \otimes [z(w_j; \sigma)],$$

for $j = 1, \dots, 4$, $\sigma \subseteq [\max(w_j) - 1] \setminus \mathrm{supp}_{\mathbf{t}}(w_j)$ and $|\sigma| = i - 1$.

We introduce a natural order on the basis elements of F_i, as follows,

$$f(w_i; \sigma) \succ f(w_j; \vartheta) \iff i < j \quad \text{or} \quad i = j \quad \text{and} \quad e_\sigma > e_\vartheta,$$

where $e_\sigma > e_\vartheta$ with respect to the order on the wedge products given in Section 2.1.

For instance,

$$f(w_2; \{2\}) \succ f(w_3; \{2\}) \succ f(w_3; \{3\}). \tag{4.18}$$

Then, d_i, $i = 1, 2, 3$, may be represented by a matrix whose jth column is given by the components of $d_i(f_j)$ with respect to the ordered basis of F_{i-1}, where f_j is the jth basis element of F_i with respect to the order introduced.

By equation (4.13) we have that

$$\mathbb{F}: \quad 0 \to F_3 \xrightarrow{\begin{pmatrix} -x_4^2 \\ x_3 \\ -x_2 \end{pmatrix}} F_2 \xrightarrow{\begin{pmatrix} x_3^2 & x_4^2 & 0 \\ -x_2 & 0 & x_4^2 \\ 0 & -x_2 & -x_3 \end{pmatrix}}$$

$$F_1 \xrightarrow{\begin{pmatrix} x_1x_2 & x_1x_3 & x_2x_4^2 \end{pmatrix}} F_0 \xrightarrow{d_0} S/I \to 0.$$

For instance, taking into account the order given in (4.18), we have

$$\begin{aligned}
d_3(f(w_3; \{2,3\})) &= \begin{pmatrix} -x_4^2 \\ x_3 \\ -x_2 \end{pmatrix} (1) = -x_4^2 f(w_2; \{2\}) + x_3 f(w_3; \{2\}) - x_2 f(w_3; \{3\}), \\
d_2d_3(f(w_3; \{2,3\})) &= -x_4^2\big(-x_2 f(w_2; \varnothing) + x_3 f(w_1; \varnothing)\big) \\
&\quad + x_3\big(-x_2 f(w_3; \varnothing) + x_4^2 f(w_1; \varnothing)\big) \\
&\quad - x_2\big(-x_3 f(w_3; \varnothing) + x_4^2 f(w_2; \varnothing)\big) = 0.
\end{aligned}$$

4.4 Generalized algebraic shifting theory

In this final section, we extend algebraic shifting theory to vector-spread strongly stable ideals. From now on, K is a field of characteristic zero. We recall that by the symbol $\mathrm{Gin}(I)$ we mean the *generic initial ideal* of a monomial ideal $I \subset S$, with respect to the reverse lexicographic order, with $x_1 > x_2 > \cdots > x_n$ [89]. It is known that $\mathrm{Gin}(I)$ is a ($\mathbf{0}$-spread) strongly stable ideal.

Firstly, we need some notions.

Let $\mathbf{t},\mathbf{s} \in \mathbb{Z}_{\geq 0}^{d-1}$, $\mathbf{t} = (t_1,\dots,t_{d-1})$, $\mathbf{s} = (s_1,\dots,s_{d-1})$, with $d \geq 2$. We can transform any $\mathbf{t}$-spread monomial ideal into a $\mathbf{s}$-spread monomial ideal as follows: Let $\mathbf{0} \in \mathbb{Z}_{\geq 0}^{d-1}$ be the null vector with $d-1$ components. To denote the composition of functions

$$\mathrm{Mon}(T;\mathbf{t}) \xrightarrow{\sigma_{\mathbf{t},\mathbf{0}}} \mathrm{Mon}(T;\mathbf{0}) \xrightarrow{\sigma_{\mathbf{0},\mathbf{s}}} \mathrm{Mon}(T;\mathbf{s})$$

we use the symbol $\sigma_{\mathbf{t},\mathbf{s}}$, where $T = K[x_1,x_2,\dots,x_n,\dots]$. Note that $\sigma_{\mathbf{t},\mathbf{s}}(1) = 1$, $\sigma_{\mathbf{t},\mathbf{s}}(x_i) = x_i$, and for all monomials $u = x_{j_1}x_{j_2}\cdots x_{j_\ell} \in \mathrm{Mon}(T;\mathbf{t})$, $2 \leq \ell \leq d$,

$$\sigma_{\mathbf{t},\mathbf{s}}(x_{j_1}x_{j_2}\cdots x_{j_\ell}) = \prod_{k=1}^{\ell} x_{j_k - \sum_{r=1}^{k-1} t_r + \sum_{r=1}^{k-1} s_r}.$$

Finally, for I a $\mathbf{t}$-spread monomial ideal, we let $I^{\sigma_{\mathbf{t},\mathbf{s}}}$ the monomial ideal whose minimal generating set is $G(I^{\sigma_{\mathbf{t},\mathbf{s}}}) = \{\sigma_{\mathbf{t},\mathbf{s}}(u) : u \in G(I)\}$. Note that $I^{\sigma_{\mathbf{t},\mathbf{s}}} = (I^{\sigma_{\mathbf{t},\mathbf{0}}})^{\sigma_{\mathbf{0},\mathbf{s}}}$.

As mentioned in the introduction of this chapter, we define the $\mathbf{t}$-*spread algebraic shifting* as follows: for I a monomial ideal of T, we let $I^{s,\mathbf{t}}$ the following monomial ideal

$$I^{s,\mathbf{t}} = (\mathrm{Gin}(I))^{\sigma_{\mathbf{0},\mathbf{t}}}.$$

Note that for $\mathbf{t} = \mathbf{1} = (1,1,\dots,1)$, we obtain the classical algebraic shifting. Indeed, for $\mathbf{t} = \mathbf{1}$, $\sigma_{\mathbf{0},\mathbf{t}}$ is the *squarefree operator* defined in the introduction of the chapter.

We are going to verify the following four properties:

(Shift$_1$) $I^{s,\mathbf{t}}$ is a $\mathbf{t}$-spread strongly stable monomial ideal;

(Shift$_2$) $I^{s,\mathbf{t}} = I$ if I is a $\mathbf{t}$-spread strongly stable ideal;

(Shift$_3$) I and $I^{s,\mathbf{t}}$ have the same Hilbert function;

(Shift$_4$) If $I \subseteq J$, then $I^{s,\mathbf{t}} \subseteq J^{s,\mathbf{t}}$.

Proposition 4.4.1 *Let I be a monomial ideal. Then, I is a $\mathbf{t}$-spread strongly stable ideal if and only if $I^{\sigma_{\mathbf{t},\mathbf{s}}}$ is a $\mathbf{s}$-spread strongly stable ideal.*

Proof. Suppose that I is a $\mathbf{t}$-spread strongly stable ideal. Set $I' = I^{\sigma_{\mathbf{t},\mathbf{s}}}$. To show that I' is a $\mathbf{s}$-spread strongly stable ideal, it suffices to check condition (ii) of Corollary 4.1.6. So, let $u \in G(I)$, $u = x_{j_1}x_{j_2}\cdots x_{j_d}$, then

$$u_1 = \sigma_{\mathbf{t},\mathbf{s}}(u) = \prod_{k=1}^{d} x_{j_k - \sum_{r=1}^{k-1} t_r + \sum_{r=1}^{k-1} s_r} = x_{j_1'}x_{j_2'}\cdots x_{j_d'} \in G(I') = G(I^{\sigma_{\mathbf{t},\mathbf{s}}}).$$

Let $i \in \mathrm{supp}(u_1)$, $j < i$ such that $v_1 = x_j(u_1/x_i)$ is $\mathbf{s}$-spread, we prove that $v_1 \in I'$.

Now, $i = j_\ell' = j_\ell - \sum_{r=1}^{\ell-1} t_r + \sum_{r=1}^{\ell-1} s_r$, for some $\ell \in \{1,\dots,d\}$, and $j_{p-1}' + s_{p-1} \leq j \leq j_p' - 1$, for some $p \leq \ell$, in particular for $p = 1$, $j < j_1'$. Hence,

$$v_1 = x_j(u_1/x_i) = (\prod_{k=1}^{p-1} x_{j_k'})x_j(\prod_{k=p}^{\ell-1} x_{j_k'})(\prod_{k=\ell+1}^{d} x_{j_k'}).$$

Recall that $\sigma_{\mathbf{s},\mathbf{t}}$ is the inverse map of $\sigma_{\mathbf{t},\mathbf{s}}$. Set $v=\sigma_{\mathbf{s},\mathbf{t}}(v_1)$, then v is $\mathbf{t}$-spread, and

$$v=\sigma_{\mathbf{s},\mathbf{t}}(v_1)=(\prod_{k=1}^{p-1}x_{j_k})x_{j-\sum_{r=1}^{p-1}s_r+\sum_{r=1}^{p-1}t_r}(\prod_{k=p}^{\ell-1}x_{j_k-s_k+t_k})(\prod_{k=\ell+1}^{d}x_{j_k}).$$

Since $j_{k+1}-j_k\ge t_k$ for all k and $j_p'=j_p-\sum_{r=1}^{p-1}t_r+\sum_{r=1}^{p-1}s_r$, we have

$$\begin{aligned} j_k-s_k+t_k\le j_{k+1}-s_k\ &\le\ j_{k+1},\quad \text{for all } k=p,\dots,\ell-1,\ \text{and} &(4.19)\\ j-\textstyle\sum_{r=1}^{p-1}s_r+\sum_{r=1}^{p-1}t_r\ &<\ j_p'-\textstyle\sum_{r=1}^{p-1}s_r+\sum_{r=1}^{p-1}t_r=j_p. &(4.20)\end{aligned}$$

Setting

$$z_m\ =\ \begin{cases} x_{j-\sum_{r=1}^{p-1}s_r+\sum_{r=1}^{p-1}t_r}(u/x_{j_p}), & \text{for } m=1,\\ x_{j_{(p+m-2)}-s_{(p+m-2)}+t_{(p+m-2)}}(z_{m-1}/x_{j_{(p+m-1)}}), & \text{for } m=2,\dots,\ell+1-p,\end{cases}$$

we see that the monomials z_m are $\mathbf{t}$-spread. Moreover, as I is $\mathbf{t}$-spread strongly stable, $z_1\in I$ by (4.20), and inductively $z_m\in I$, by (4.19). So, $v=z_{\ell+1-p}\in I$, and by Lemma 4.1.5, $v=w_1w_2$ for unique monomials $w_1\in G(I)$, w_2 such that $\max(w_1)\le\min(w_2)$. Hence, $\sigma_{\mathbf{t},\mathbf{s}}(w_1)$ divides $v_1=\sigma_{\mathbf{t},\mathbf{s}}(v)$, with $\sigma_{\mathbf{t},\mathbf{s}}(w_1)\in G(I')=G(I^{\sigma_{\mathbf{t},\mathbf{s}}})$. Finally, $v_1\in I'=I^{\sigma_{\mathbf{t},\mathbf{s}}}$, as desired. The converse is trivially true as $I,\mathbf{t},\mathbf{s}$ are arbitrary. ☐

By virtue of this proposition, the property (Shift_1) is verified. Indeed, it is known that $\text{Gin}(I)$ is a $\mathbf{0}$-spread strongly stable ideal [89]. Consequently, $I^{s,\mathbf{t}}$ is a $\mathbf{t}$-spread strongly stable ideal, as desired.

The operators $\sigma_{\mathbf{t},\mathbf{s}}$ behave well, in fact they preserve the graded Betti numbers. We first note that Theorem 4.2.8 implies a formula for the graded Betti numbers. We remark that the next result holds whatever the characteristic of the field K is.

Corollary 4.4.2 *Let I be a $\mathbf{t}$-spread strongly stable ideal of S. Then,*

$$\beta_{i,i+j}(I)=\sum_{u\in G(I)_j}\binom{\max(u)-1-\sum_{\ell=1}^{j-1}t_\ell}{i},\quad \textit{for all}\ \ i,j\ge 0. \tag{4.21}$$

In particular, the graded Betti numbers of a vector-spread strongly stable ideal $I\subset S$ do not depend upon the characteristic of the field K.

Proof. By equation (2.2), we have $\beta_{i,i+j}(I)=\beta_{i+1,i+j}(S/I)=\dim_K H_{i+1}(\mathbf{x};S/I)_{i+j}$. By Theorem 4.2.8, the degree of a basis element $[e(u;\sigma)]$ of $H_{i+1}(\mathbf{x};S/I)_{i+j}$ is given by $|\sigma|+1+\deg(u)-1=i+j$. Thus $u\in G(I)_j$. For a fixed $u\in G(I)_j$, we have $\sigma\subseteq[\max(u)-1]\setminus\text{supp}_{\mathbf{t}}(u)$. Hence, there are

$$\binom{|[\max(u)-1]\setminus\text{supp}_{\mathbf{t}}(u)|}{i}=\binom{\max(u)-1-\sum_{\ell=1}^{j-1}t_\ell}{i}$$

possible choices for σ. Summing all these binomials over $u\in G(I)_j$, we obtain the formula in the statement. ☐

Suitable choices of $\mathbf{t}$ return several well known formulas for the graded Betti numbers. $\mathbf{t}=(0,0,\dots,0)$ returns the Eliahou–Kervaire formula for (strongly) stable ideals [50] (Theorem 2.4.5(c)); $\mathbf{t}=(1,1,\dots,1)$ gives the Aramova–Herzog–Hibi formula for squarefree (strongly) stable ideals [12] (Theorem 2.4.8(c)). In the uniform case, $\mathbf{t} = (t, t, \dots, t)$, we have the Ene–Herzog–Qureshi formula for uniform t-spread strongly stable ideals [53].

Let $P_I^S(y) = \sum_i \beta_i(I) y^i$ be the Poincaré series of I. Equation (4.21) implies

Corollary 4.4.3 *Let $I \subset S$ be a $\mathbf{t}$-spread strongly stable ideal. Then*

(a) $P_I^S(y) = \sum_{u \in G(I)} (1+y)^{\max(u)-1-\sum_{\ell=1}^{\deg(u)-1} t_\ell}$;

(b) $\operatorname{pd}(I) = \max\{\max(u) - 1 - \sum_{j=1}^{\deg(u)-1} t_j : u \in G(I)\}$;

(c) $\operatorname{reg}(I) = \max\{\deg(u) : u \in G(I)\}$.

Let us return now to our shifting operators. As announced, we have

Lemma 4.4.4 *Let I be a $\mathbf{t}$-spread strongly stable ideal. Then $I^{\sigma_{\mathbf{t},\mathbf{s}}}$ is a $\mathbf{s}$-spread strongly stable ideal, and for all $i, j \geq 0$,*

$$\beta_{i,i+j}(I) = \beta_{i,i+j}(I^{\sigma_{\mathbf{t},\mathbf{s}}}).$$

Proof. We have just proved that $I^{\sigma_{\mathbf{t},\mathbf{s}}}$ is a $\mathbf{s}$-spread strongly stable ideal with minimal generating set $G(I^{\sigma_{\mathbf{t},\mathbf{s}}}) = \{\sigma_{\mathbf{t},\mathbf{s}}(u) : u \in G(I)\}$. Moreover, for $u \in G(I)$, we have $\max(\sigma_{\mathbf{t},\mathbf{s}}(u)) = \max(u) - \sum_{\ell=1}^{\deg(u)-1} t_\ell + \sum_{\ell=1}^{\deg(u)-1} s_\ell$. Hence, Corollary 4.4.2 yields

$$\begin{aligned} \beta_{i,i+j}(I^{\sigma_{\mathbf{t},\mathbf{s}}}) &= \sum_{\sigma_{\mathbf{t},\mathbf{s}}(u) \in G(I^{\sigma_{\mathbf{t},\mathbf{s}}})_j} \binom{\max(\sigma_{\mathbf{t},\mathbf{s}}(u)) - 1 - \sum_{\ell=1}^{j-1} s_\ell}{i} \\ &= \sum_{u \in G(I)_j} \binom{\max(u) - \sum_{\ell=1}^{j-1} t_\ell + \sum_{\ell=1}^{j-1} s_\ell - 1 - \sum_{\ell=1}^{j-1} s_\ell}{i} \\ &= \sum_{u \in G(I)_j} \binom{\max(u) - 1 - \sum_{\ell=1}^{j-1} t_\ell}{i} = \beta_{i,i+j}(I). \end{aligned}$$

□

As a consequence, the property (Shift$_3$) is verified too. Indeed, it is known that I and $\operatorname{Gin}(I)$ have the same Hilbert function. Moreover, by Lemma 4.4.4, $\operatorname{Gin}(I)$ and $(\operatorname{Gin}(I))^{\sigma_{\mathbf{0},\mathbf{t}}}$ have the same graded Betti numbers and thus the same Hilbert function.

Note that condition (Shift$_4$) is trivially verified. Finally it remains to establish condition (Shift$_2$). This is accomplished in the next theorem.

Theorem 4.4.5 *Let K be a field of characteristic zero. Let $I \subset S$ be a $\mathbf{t}$-spread strongly stable ideal. Then*

$$I = (\operatorname{Gin}(I))^{\sigma_{\mathbf{0},\mathbf{t}}}.$$

Proof. We proceed by induction on the integer $\ell = \max\{\max(u) : u \in G(I)\} \geq 1$. If $\ell = 1$, then $G(I) = \{x_1^a\}$, $I = (x_1^a)$, and $\operatorname{Gin}(I) = I = (x_1^a)$, moreover $\sigma_{\mathbf{t},\mathbf{0}}(x_1^a) = x_1^a$, for some $a \geq 1$. So, the thesis holds for $\ell = 1$.

Let $\ell > 1$. By [89, Lemma 11.2.8] we can assume $\ell = n$. So, there exists a monomial $u \in G(I)$ with $\max(u) = n$. Let

$$p = \max\{p \ : \ x_n^p \text{ divides } w \text{ for some } w \in G(I)\},$$

our hypothesis implies that $p \geq 1$. We consider the following ideals:

$$I' = I : (x_n^p), \qquad I'' = (u \in G(I) : \max(u) < n).$$

Both are again $\mathbf{t}$-spread strongly stable ideals, and $I'' \subseteq I \subseteq I'$. By inductive hypothesis, $\mathrm{Gin}(I') = (I')^{\sigma_{\mathbf{t},\mathbf{0}}}$ and $\mathrm{Gin}(I'') = (I'')^{\sigma_{\mathbf{t},\mathbf{0}}}$. Equivalently,

$$I' = \mathrm{Gin}(I')^{\sigma_{\mathbf{0},\mathbf{t}}} \quad \text{and} \quad I'' = \mathrm{Gin}(I'')^{\sigma_{\mathbf{0},\mathbf{t}}}.$$

Therefore, $I'' \subseteq \mathrm{Gin}(I)^{\sigma_{\mathbf{0},\mathbf{t}}} \subseteq I'$.

Claim 2. It is

$$I \subseteq \mathrm{Gin}(I)^{\sigma_{\mathbf{0},\mathbf{t}}}. \tag{4.22}$$

To prove Claim 2, it is enough to show that each $u \in G(I)$ with $\max(u) = n$ belongs to $\mathrm{Gin}(I)^{\sigma_{\mathbf{0},\mathbf{t}}}$. Indeed, since $I'' \subseteq \mathrm{Gin}(I)^{\sigma_{\mathbf{0},\mathbf{t}}}$, all monomials $u \in G(I)$ with $\max(u) < n$ are in $\mathrm{Gin}(I)^{\sigma_{\mathbf{0},\mathbf{t}}}$.

Let $u \in G(I)$ with $\max(u) = n$. We set $a = n-1-\sum_{j=1}^{\deg(u)-1} t_j$ and $b = a+\deg(u)$. By Corollary 4.4.2 we have

$$\begin{aligned}
\beta_{a,b}(I) \;&=\; \sum_{\substack{v\in G(I)\\ \deg(v)=\deg(u)}} \binom{\max(v)-1-\sum_{j=1}^{\deg(u)-1} t_j}{n-1-\sum_{j=1}^{\deg(u)-1} t_j}\\
&=\; \big|\{v \in G(I) : \max(v) = n,\ \deg(v) = \deg(u)\}\big|.
\end{aligned}$$

Similarly, as $\mathrm{Gin}(I)^{\sigma_{\mathbf{0},\mathbf{t}}}$ is $\mathbf{t}$-spread strongly stable,

$$\begin{aligned}
\beta_{a,b}(\mathrm{Gin}(I)^{\sigma_{\mathbf{0},\mathbf{t}}}) \;&=\; \sum_{\substack{w\in G(\mathrm{Gin}(I)^{\sigma_{\mathbf{0},\mathbf{t}}})\\ \deg(w)=\deg(u)}} \binom{\max(w)-1-\sum_{j=1}^{\deg(u)-1} t_j}{n-1-\sum_{j=1}^{\deg(u)-1} t_j}\\
&=\; \big|\{w \in G(\mathrm{Gin}(I)^{\sigma_{\mathbf{0},\mathbf{t}}}) : \max(w) = n,\ \deg(w) = \deg(u)\}\big|.
\end{aligned}$$

Moreover, by [89, Corollary 3.3.3] and by Lemma 4.4.4, we have

$$\beta_{a,b}(I) \;\leq\; \beta_{a,b}(\mathrm{Gin}(I)) \;=\; \beta_{a,b}(\mathrm{Gin}(I)^{\sigma_{\mathbf{0},\mathbf{t}}}).$$

Hence

$$\begin{aligned}
&\big|\{w \in G(\mathrm{Gin}(I)^{\sigma_{\mathbf{0},\mathbf{t}}}) : \max(w) = n, \deg(w) = \deg(u)\}\big| \geq\\
&\qquad\big|\{v \in G(I) : \max(v) = n, \deg(v) = \deg(u)\}\big|.
\end{aligned} \tag{4.23}$$

Our aim is to prove that $u \in G(I)$ with $\max(u) = n$ belongs to $\mathrm{Gin}(I)^{\sigma_{\mathbf{0},\mathbf{t}}}$.

Let $w_1, \dots, w_s$ be the monomial generators in $G(\mathrm{Gin}(I)^{\sigma_{\mathbf{0},\mathbf{t}}})$ such that $\max(w_i) = n$ and $\deg(w_1) \leq \deg(w_2) \leq \dots \leq \deg(w_s)$. Since $\mathrm{Gin}(I)^{\sigma_{\mathbf{0},\mathbf{t}}} \subseteq I'$, we have $w_i x_n^p \in I$, for all $i = 1, \dots, s$. We prove that $w_i \in I$ for all i. Since $w_i x_n^p \in I$, there is a monomial $v_i \in G(I)$ such that v_i divides $w_i x_n^p$. We have $\deg(v_i) \leq \deg(w_i) + p$, for all $i = 1, \dots, s$.

If $\deg(v_1) < \deg(w_1)$, setting $u = v_1$ in (4.23), we would have an absurd. Hence $\deg(v_1) \geq \deg(w_1)$. By finite induction, $\deg(v_i) \geq \deg(w_i)$, for all $i = 1, \dots, s$.

Now, if $\deg(v_s) \geq \deg(w_s) + 1$, setting $u = v_s$ in (4.23), we would obtain an absurd. Hence, $\deg(v_s) \leq \deg(w_s)$, and since we have proved that $\deg(v_s) \geq \deg(w_s)$, we obtain $\deg(v_s) = \deg(w_s)$. Iterating this argument, $\deg(v_i) = \deg(w_i)$, for all $i = 1, \dots, s$.

If $v_i = (w_i x_n^p)/x_n^p = w_i$ we set $u_i = v_i$ and note that $u_i = w_i$ divides w_i. Otherwise, $v_i = (w_i x_n^p)/z_i$ for some monomial $z_i \neq x_n^p$, we note that v_i has bigger sorted indexes than w_i, thus since I is $\mathbf{t}$-spread strongly stable $w_i \in I$. Hence, there is a monomial $u_i \in G(I)$ that divides w_i. Finally, we have constructed monomials $u_1, \dots, u_s \in G(I)$ such that u_i divides w_i, for all $i = 1, \dots, s$. Repeating the same argument as before, using (4.23), we see that $\deg(u_i) \geq \deg(w_i)$, for all i, hence $u_i = w_i$, since u_i divides w_i, for all $i = 1, \dots, s$.

Thus, $w_i = u_i \in G(I)$, for all $i = 1, \dots, s$, and we get the inclusion

$$\{w \in G(\mathrm{Gin}(I)^{\sigma_{\mathbf{0},\mathbf{t}}}) : \max(w) = n\} \subseteq \{u \in G(I) : \max(u) = n\}.$$

This equation together with (4.23) yield

$$\{w \in G(\mathrm{Gin}(I)^{\sigma_{\mathbf{0},\mathbf{t}}}) : \max(w) = n\} = \{u \in G(I) : \max(u) = n\}.$$

Hence, Claim 2 is true.

Finally, I and $\mathrm{Gin}(I)$ have the same Hilbert function. Moreover, by Lemma 4.4.4, $\mathrm{Gin}(I)$ and $\mathrm{Gin}(I)^{\sigma_{\mathbf{0},\mathbf{t}}}$ have the same Hilbert function. Hence I and $\mathrm{Gin}(I)^{\sigma_{\mathbf{0},\mathbf{t}}}$ have the same Hilbert function. Formula (4.22) and this observation imply that $I = \mathrm{Gin}(I)^{\sigma_{\mathbf{0},\mathbf{t}}}$, or equivalently $\mathrm{Gin}(I) = I^{\sigma_{\mathbf{t},\mathbf{0}}}$, proving the theorem. □

We remark that the operator $\sigma_{\mathbf{t},\mathbf{s}}$ establishes a bijection between $\mathbf{t}$-spread strongly stable ideals and $\mathbf{s}$-spread strongly stable ideals.

Notes

Recently, the vector–spread algebraic shifting theory was used to generalize the classical Bigatti–Hulett theorem [89, Theorem 7.3.1] to the case of vector–spread strongly stable ideals [37]. In [34], we further investigated the algebraic properties of vector–spread strongly stable ideals. The mapping cone technique and the concept of ideals with linear quotients was lately considered in [38], for the case of exterior algebras, rather than in the symmetric algebra context. In such an article, the homological properties of vector–spread ideals in the exterior algebra were also investigated. Edge ideals of $\mathbf{t}$-spread d-partite hypergraphs were recently investigated in [130].

The reader may wonder why the Koszul cycles of a general $\mathbf{t}$-spread strongly stable ideal are so complicated, while in the classical cases ($\mathbf{t} = \mathbf{0}$ and $\mathbf{t} = \mathbf{1}$) they have a simple expression (Theorems 2.4.6(a) and 2.4.8(a)). Let $I \subset S$ be a monomial ideal. We recall that $z \in K_i(\mathbf{x}; S/I)$ is called a *monomial cycle* if

$$z = \varepsilon(u)e_\sigma, \qquad \text{for some monomial } u \in S, \quad \text{and} \quad \partial_i(z) = 0.$$

If we can find a basis of $H_i(\mathbf{x}; S/I)$ whose homology classes consisting of monomial cycles, we say that $H_i(\mathbf{x}; S/I)$ has a *monomial cycle basis*. Stable and squarefree stable ideals are examples of monomial ideals whose Koszul homology modules have monomial cycles bases. It is trivial to see that

Proposition. *$H_1(\mathbf{x}; S/I)$ has a monomial cycle basis, if I is a monomial ideal.*

Proof. Indeed, let $[z] \in H_1(\mathbf{x}; S/I)$, then we can write $z = \sum_i \varepsilon(f_i)e_{k_i}$, where each $f_i \in S$ is a monomial and k_i is an integer. Then

$$\partial_1(z) \;=\; \sum_i \varepsilon(x_{k_i} f_i) \;=\; \varepsilon(\sum_i x_{k_i} f_i) \;=\; 0.$$

Thus $\sum_i x_{k_i} f_i \in I$. We may assume that no two summands in this expression are opposite. Indeed, if for some j, h with $j \neq h$ we have $x_{k_j} f_j + x_{k_h} f_h = 0$, then $f_h = -x_{k_j} f_j / x_{k_h}$. Note that $w = \varepsilon(f_j/x_{k_h}) e_{k_h} \wedge e_{k_j} \in K_2(\mathbf{x}; S/I)$ and

$$\partial_2(w) = \varepsilon(f_j) e_{k_j} + \varepsilon(f_h) e_{k_h} \in \mathrm{Im}(\partial_2).$$

Thus $[z] = [z - w] = [\sum_{i \neq j,h} \varepsilon(f_i) e_{k_i}]$. This argument shows that we may assume from the very beginning that no two summands in $\sum_i x_{k_i} f_i$ are opposite. Since I is a monomial ideal, we have $x_{k_i} f_i \in I$ for all i. Thus $[\varepsilon(f_i) e_{k_i}] \in H_1(\mathbf{x}; S/I)$ for all i and $[z] = \sum_i [\varepsilon(f_i) e_{k_i}]$ is a sum of monomial Koszul cycles. Thus $H_1(\mathbf{x}; S/I)$ has a monomial cycle basis, as desired. $\square$

A less trivial but beautiful result of Dorin Popescu guarantees that $H_2(\mathbf{x}; S/I)$ also has a monomial cycle basis, if I is a monomial ideal [138, Theorem 1.5]. On the other hand, if $3 \leq i \leq n-1$, in general $H_i(\mathbf{x}; S/I)$ does not have a monomial cycle basis, as shown in [138].

Next, we provide an example of a $\mathbf{t}$-spread strongly stable ideal $I \subset S$ such that $H_3(\mathbf{x}; S/I)$ does not have a monomial cycle basis.

Example. Let $\mathbf{t} = (2,2)$. Then

$$I = (x_1x_3x_5, x_1x_3x_6, x_1x_3x_7, x_1x_3x_8, x_1x_4x_6, x_1x_4x_7)$$

is a $\mathbf{t}$-spread strongly stable ideal of $S = K[x_1, \dots, x_7]$.

Consider $u = x_1x_4x_7$ and let $\sigma = \{3, 6\}$. Then $\sigma = [\max(u) - 1] \setminus \mathrm{supp}_{\mathbf{t}}(u)$. By Theorem 4.2.8, $[e(u; \sigma)]$ is a basis element of $H_3(\mathbf{x}; S/I)$. Using the notation of Definition 4.2.3, $3^{(u)} = 4$, $6^{(u)} = 7$, $u' = x_1x_4$ and $\max(u) = 7$. Thus,

$$\begin{aligned} e(u;\sigma) &= \sum_{F \subseteq \{3,6\}} (-1)^{x_1x_4x_6(\sigma;F)} \varepsilon(\mathbf{x}_F(x_1x_4)/\mathbf{x}_{F^{(u)}}) e_{\{3,6\}\setminus F} \wedge e_{F^{(u)}} \wedge e_7 \\ &= \varepsilon(x_1x_4) e_3 \wedge e_6 \wedge e_7 + \varepsilon(x_1x_3) e_6 \wedge e_4 \wedge e_7. \end{aligned}$$

Then,

$$\begin{aligned} \partial_2(e(u;\sigma)) &= \varepsilon(x_1x_3x_4) e_6 \wedge e_7 - \varepsilon(x_1x_4x_6) e_3 \wedge e_7 + \varepsilon(x_1x_4x_7) e_3 \wedge e_6 \\ &\quad + \varepsilon(x_1x_3x_6) e_4 \wedge e_7 - \varepsilon(x_1x_3x_4) e_6 \wedge e_7 + \varepsilon(x_1x_3x_7) e_6 \wedge e_4 \\ &= 0, \end{aligned}$$

since the first and fifth summands are opposite and all other summands are zero, since $x_1x_4x_6, x_1x_4x_7, x_1x_3x_6, x_1x_3x_7 \in I$.

We claim that we can not replace $[e(u;\sigma)]$ with $[z]$, where z is a monomial cycle. This will show that $H_3(\mathbf{x}; S/I)$ does not have a monomial cycle basis. Indeed, if it was possible to find such z, then the multidegree of z would be the multidegree of the monomial $x_1x_3x_4x_6x_7$. Hence, z should be one of the following $\binom{5}{2} = 10$ elements:

$$\begin{array}{ll} \varepsilon(x_1x_3) e_4 \wedge e_6 \wedge e_7, & \varepsilon(x_1x_4) e_3 \wedge e_6 \wedge e_7, \\ \varepsilon(x_1x_6) e_3 \wedge e_4 \wedge e_7, & \varepsilon(x_1x_7) e_3 \wedge e_4 \wedge e_6, \\ \varepsilon(x_3x_4) e_1 \wedge e_6 \wedge e_7, & \varepsilon(x_3x_6) e_1 \wedge e_4 \wedge e_7, \\ \varepsilon(x_3x_7) e_1 \wedge e_4 \wedge e_6, & \varepsilon(x_4x_6) e_1 \wedge e_3 \wedge e_7, \\ \varepsilon(x_4x_7) e_1 \wedge e_3 \wedge e_6, & \varepsilon(x_6x_7) e_1 \wedge e_3 \wedge e_4. \end{array}$$

However, one can easily check that none of these elements is a cycle of $K_3(\mathbf{x}; S/I)$.

Hence, in general the Koszul homology modules of vector–spread strongly stable ideals do not have monomial cycle basis, and this justifies the construction of the cycles given in Section 4.2.

To complete the picture we prove the following.

Proposition. *$H_n(\mathbf{x};S/I)$ has a monomial cycle basis, if I is a monomial ideal.*

Proof. Let $\mathfrak{m}=(x_1,\dots,x_n)$ and $E=\bigoplus_{i=1}^n(S/\mathfrak{m})e_i$. Then
$$H_n(\mathbf{x};S/I)\ \cong\ (0:_{S/I}\mathfrak{m})\bigwedge^n E\ =\ ((I:\mathfrak{m})/I)\bigwedge^n E.$$
Denote by $\overline{f}=\varepsilon(f)$ the residue class of $f\in S$ modulo I. Since I is a monomial ideal, we can find a basis $\overline{f_1},\dots,\overline{f_\ell}$ of $(I:\mathfrak{m})/I$, where $f_1,\dots,f_\ell$ are the monomials of the set $G((I:\mathfrak{m}))\setminus I$. Under the above isomorphisms, we see that $H_n(\mathbf{x};S/I)$ has as a basis: $\varepsilon(f_1)e_1\wedge e_2\wedge\dots\wedge e_n$, $\varepsilon(f_2)e_1\wedge e_2\wedge\dots\wedge e_n$, $\dots$, $\varepsilon(f_\ell)e_1\wedge e_2\wedge\dots\wedge e_n$. The assertion follows. $\square$

Chapter 5

Multigraded syzygies of ideals with linear quotients

In this chapter, we investigate the *homological shift ideals* of monomial ideals. This concept has been recently introduced by Herzog, Moradi, Rahimbeigi and Zhu in [101]. For other developments in this theory, see also [15, 16, 17, 35, 36, 60, 59, 102].

Let $I \subset S = K[x_1, \dots, x_n]$ be a monomial ideal. Then, I is *multigraded*, that is, it is $\mathbb{Z}^n$-graded. For a monomial $u = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \in S$, the integral vector $\mathbf{a} = (a_1, a_2, \dots, a_n)$ is called the *multidegree* of u. We also write $u = \mathbf{x}^{\mathbf{a}}$. In particular, for $\mathbf{a} = \mathbf{0} = (0, 0, \dots, 0)$, $\mathbf{x}^{\mathbf{0}} = 1$. The minimal graded free S-resolution $\mathbb{F}$ of I is multigraded, that is:

$$\mathbb{F} : 0 \to \bigoplus_{\mathbf{a}} S(-\mathbf{a})^{\beta_{p,\mathbf{a}}(I)} \to \cdots \to \bigoplus_{\mathbf{a}} S(-\mathbf{a})^{\beta_{1,\mathbf{a}}(I)} \to \bigoplus_{\mathbf{a}} S(-\mathbf{a})^{\beta_{0,\mathbf{a}}(I)} \to I \to 0,$$

where $\beta_{i,\mathbf{a}}(I)$ is a *multigraded Betti number* of I.

The set $\{\mathbf{a}_{i,1}, \dots, \mathbf{a}_{i,k}\} = \{\mathbf{a} \in \mathbb{Z}^n_{\geq 0} : \beta_{i,\mathbf{a}}(I) \neq 0\}$ is called the set of the ith *multigraded shifts* of I.

We call

$$\mathrm{HS}_i(I) = (\mathbf{x}^{\mathbf{a}} \ : \ \beta_{i,\mathbf{a}}(I) \neq 0)$$

the ith *homological shift ideal* of I.

Note that $\mathrm{HS}_0(I) = I$ and $\mathrm{HS}_i(I) = 0$ if $i < 0$ or $i > \mathrm{pd}(I)$.

Example 5.0.1 Let $I = (x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2^2) \subset K[x_1, x_2, x_3]$. Then, the minimal multigraded free S-resolution of I is

$$\begin{aligned}\mathbb{F} : 0 &\to S\big(-(3,1,1)\big) \to S\big(-(3,1,0)\big) \oplus S\big(-(3,0,1)\big) \oplus S\big(-(2,2,0)\big) \oplus S\big(-(2,1,1)\big)\\ &\to S\big(-(3,0,0)\big) \oplus S\big(-(2,1,0)\big) \oplus S\big(-(2,0,1)\big) \oplus S\big(-(1,2,0)\big)\\ &\to I \to 0.\end{aligned}$$

We may identify the multigraded shifts by monomials. We denote $S(-\mathbf{a})$ by $[\mathbf{x}^{\mathbf{a}}]$. Hence, we can rewrite $\mathbb{F}$ as follows:

$$\begin{aligned}\mathbb{F} : 0 &\to [x_1^3x_2x_3] \to [x_1^3x_2] \oplus [x_1^3x_3] \oplus [x_1^2x_2^2] \oplus [x_1^2x_2x_3]\\ &\to [x_1^3] \oplus [x_1^2x_2] \oplus [x_1^2x_3] \oplus [x_1x_2^2]\\ &\to I \to 0.\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{HS}_0(I) &= I = (x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2^2),\\
\mathrm{HS}_1(I) &= (x_1^3x_2, x_1^3x_3, x_1^2x_2^2, x_1^2x_2x_3),\\
\mathrm{HS}_2(I) &= (x_1^3x_2x_3).
\end{aligned}$$

The main purpose of the theory of homological shift ideals is to understand what homological and combinatorial properties are enjoyed by all $\mathrm{HS}_i(I)$, $i = 0, \dots, \mathrm{pd}(I)$. We call any such property a *homological shift property* of I, or if the context is clear, simply *homological property*. Two important homological shift properties are the following ones: I has a *homological linear resolution* if $\mathrm{HS}_i(I)$ has a linear resolution for all i; I has *homological linear quotients* if $\mathrm{HS}_i(I)$ has linear quotients for all i.

5.1 The first homological shift of ideals with linear quotients

In this section, we show that the first homological shift ideal of an equigenerated monomial ideal with linear quotients has again linear quotients.

By Theorem 2.4.9, we have the following result.

Proposition 5.1.1 *Let $I \subset S$ be a monomial ideal with linear quotients and admissible order $u_1, u_2, \dots, u_m$ of $G(I)$. Then,*

$$\mathrm{HS}_k(I) = (\mathbf{x}_F u_i \ : \ i = 1, \dots, m,\ F \subseteq \mathrm{set}(u_i),\ |F| = k). \tag{5.1}$$

For the proof of our main result we need Corollary 5.1.3 of the following lemma.

Lemma 5.1.2 *Let $I \subset S$ be an equigenerated graded ideal with linear relations. Let $f_1, \dots, f_m$ be a minimal set of generators of I. Then, for any $1 \le i \le m$,*

$$(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_m) : f_i$$

is generated by linear forms.

Proof. To simplify the notation, we assume $i = m$, and we set $J = (f_1, \dots, f_{m-1}) : f_m$. Since the f_i are homogeneous elements, J is a graded ideal. Let $r_m \in J$ be an homogeneous element. Then, there exist $r_1, \dots, r_{m-1}$ such that $r_m f_m = -\sum_{i=1}^{m-1} r_i f_i$ with $\deg(r_i) = \deg(r_m)$ for $i = 1, \dots, m-1$. Therefore, $r = (r_1, \dots, r_m)$ is a homogeneous relation of I. By assumption, the relation module of I is generated by linear relations, say $\ell_i = (\ell_{i1}, \dots, \ell_{im})$ for $i = 1, \dots, t$. Therefore, there exist homogeneous elements $s_i \in S$ such that $r = \sum_{i=1}^{t} s_i \ell_i$. This implies that $r_m = \sum_{i=1}^{t} s_i \ell_{i,m}$. Since $\ell_{i,m} \in J$, the desired conclusion follows. $\square$

Corollary 5.1.3 *Let I be an equigenerated monomial ideal with linear quotients and let $u_1, \dots, u_m$ be its minimal monomial generators. Then, for any $1 \le i \le m$,*

$$(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_m) : u_i$$

is generated by variables.

Theorem 5.1.4 *Let $I \subset S$ be an equigenerated monomial ideal having linear quotients. Then* $\mathrm{HS}_1(I)$ *has linear quotients.*

Proof. We proceed by induction on $m \geq 1$. For $m = 1$ or $m = 2$ there is nothing to prove.

Let $m > 2$ and set $J = (u_1, \ldots, u_{m-1})$. Let $L = (x_i : i \in \mathrm{set}(u_m), x_i u_m \notin \mathrm{HS}_1(J))$. Then, by equation (5.1),

$$\mathrm{HS}_1(I) = \mathrm{HS}_1(J) + u_m L.$$

By inductive hypothesis, $\mathrm{HS}_1(J)$ has linear quotients. Let $v_1, \ldots, v_r$ be an admissible order of $\mathrm{HS}_1(J)$. If $L = (x_{j_1}, \ldots, x_{j_s})$, we claim that $v_1, \ldots, v_r, x_{j_1} u_m, \ldots, x_{j_s} u_m$ is an admissible order of $\mathrm{HS}_1(I)$. We only need to show that

$$(v_1, \ldots, v_r, x_{j_1} u_m, \ldots, x_{j_{t-1}} u_m) : x_{j_t} u_m \tag{5.2}$$

is generated by variables, for all $t = 1, \ldots, s$.

Note that each generator $x_{j_\ell} u_m : x_{j_t} u_m = x_{j_\ell}$, with $\ell < t$ is already a variable. Consider now a generator $v_\ell : x_{j_t} u_m$ for some $\ell = 1, \ldots, r$. Then $v_\ell = x_h u_j$ for some $j < m$ and $h \in \mathrm{set}(u_j)$. Moreover, we can write $x_{j_t} u_m = x_p u_k$ for some $k < m$.

If $j = k$, then

$$v_\ell : x_{j_t} u_m = x_h u_k : x_p u_k = x_h$$

is a variable and there is nothing to prove.

Suppose now $j \neq k$. Since $u_1, \ldots, u_{m-1}$ is an admissible order, by Corollary 5.1.3

$$Q = (u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_{m-1}) : u_k$$

is generated by variables. Since $j \neq k$ and $j < m$, $u_j : u_k$ belongs to Q. Hence, we can find $b < m$, $b \neq k$ such that $u_b : u_k = x_q$ and x_q divides $u_j : u_k$. Thus $x_q u_k \in \mathrm{HS}_1(J)$.

Note that x_q divides also $x_h u_j : x_p u_k$. Indeed x_q divides $u_j : u_k$. If x_q does not divide $x_h u_j : x_p u_k$, then necessarily $p = q$. But this would imply that $x_{j_t} u_m = x_q u_k \in \mathrm{HS}_1(J)$, against the fact that $x_{j_t} \in L$. Hence x_q divides $x_h u_j : x_p u_k$. But

$$x_q u_k : x_{j_t} u_m = x_q u_k : x_p u_k = x_q$$

belongs to the ideal (5.2). Hence $x_h u_j : x_p u_k$ is divided by a variable belonging to the ideal (5.2). This concludes our proof. $\square$

At present we are not able to generalize Theorem 5.1.4 for all monomial ideals with linear resolution. It is natural to ask the following question. Let $I \subset S$ be a monomial ideal having a linear resolution. Is it true that $\mathrm{HS}_1(I)$ has a linear resolution, too? In this case, one could expect even that $\mathrm{HS}_1(I)$ also has linear quotients, if I has a linear resolution.

Theorem 5.1.4 is no longer valid for monomial ideals with linear quotients generated in more than one degree, as next example of Bayati et all shows [17].

Example 5.1.5 ([17, Example 3.3]). Let $I = (x_1^2, x_1x_2, x_2^4, x_1x_3^4, x_1x_3^3x_4, x_1x_3^2x_4^2)$ be an ideal of $S = K[x_1, x_2, x_3, x_4]$. I is a (strongly) stable ideal whose Borel generators are $x_1x_2, x_2^4, x_1x_3^2x_4^2$. It is well–known that stable ideals have linear quotients. Thus I has linear quotients. Using *Macaulay2* [82] the package [59], we verified that

$$\begin{aligned}\mathrm{HS}_1(I) \;=\; &(x_1^2x_2,\, x_1x_2^4,\, x_1x_3^3x_4^2,\, x_1x_2x_3^2x_4^2,\, x_1^2x_3^2x_4^2,\, x_1x_3^4x_4,\\ &x_1x_2x_3^3x_4,\, x_1^2x_3^3x_4,\, x_1x_2x_3^4,\, x_1^2x_3^4)\end{aligned}$$

has the following Betti table

	0	1	2	3
3	1	.	.	.
4	.	.	.	.
5	1	1	.	.
6	8	15	8	1
7	.	.	.	.
8	.	3	5	2

We show that $\text{HS}_1(I)$ does not have linear quotients. Suppose by contradiction that $\text{HS}_1(I)$ has linear quotients. Then, since the Betti numbers of an ideal with linear quotients do not depend upon the characteristic of the underlying field K, we may assume that K has characteristic zero. Hence $\text{HS}_1(I)$ would be componentwise linear, see [89, Corollary 8.2.21]. However, this cannot be the case by virtue of [89, Theorems 8.2.22. and 8.2.23(a)]. Indeed $\beta_{1,1+8}(\text{HS}_1(I)) \neq 0$, while $\beta_{0,8}(\text{HS}_1(I)) = 0$.

5.2 The higher homological shifts of ideals with linear quotients

Let $I \subset S$ be a monomial ideal. In general, one can ask what properties of I are inherited by its homological shift ideals. However, even for special classes of monomial ideals, it is quite difficult to describe the homological shift ideals, see [101]. Furthermore, these ideals may depend upon the characteristic of the underlying field K. For this reason, one can focus on monomial ideals with linear quotients.

For our aim, a different description of the homological shift ideals of any equigenerated ideal with linear quotients given in Proposition 5.1.1 is required. Sometimes we will talk about the shifts of I, dropping the adjective multigraded.

For the next result, we use the Taylor resolution. Let I be a monomial ideal of S. Recall that $\mathbb{T}$ is a (multi)graded free S-resolution of I. Let $\beta_{i,\mathbf{a}}(I) = \dim_K \text{Tor}_i^S(K, I)_{\mathbf{a}}$ be a multigraded Betti number of I, and let $b_{i,\mathbf{a}} = \dim_K(T_i \otimes K)_{\mathbf{a}}$ the dimension of the $\mathbf{a}$th multigraded piece of the ith free module T_i of $\mathbb{T}$ tensorized by K.

Theorem 5.2.1 *Let $I \subset S$ be a monomial ideal and let $G(I) = \{u_1, u_2, \dots, u_m\}$ be its minimal monomial generating set. For all $j \geq 0$, the set of the jth multigraded shifts of I is a subset of*

$$\{\text{lcm}(u_i : i \in F) \ : \ F \subseteq [m], \ |F| = j + 1\}. \tag{5.3}$$

In particular,

$$\text{HS}_j(I) \ \subseteq \ (\text{lcm}(u_i : i \in F) \ : \ F \subseteq [m], \ |F| = j + 1).$$

Proof. To prove the statement, it is enough to show that for all i and $\mathbf{a}$, $\beta_{i,\mathbf{a}}(I) \ \leq \ b_{i,\mathbf{a}}$. Indeed, $b_{i,\mathbf{a}} \neq 0$ if and only if $\mathbf{x}^{\mathbf{a}} = \text{lcm}(u_j : j \in F)$ for some $F \subset [m]$ with $|F| = i+1$.

If $\mathbb{T}$ is the minimal free resolution of I, $\beta_{i,\mathbf{a}}(I) = b_{i,\mathbf{a}}$ for all i and $\mathbf{a}$. Otherwise we can find $\ell > 0$ and $\mathbf{e}_F \in T_\ell$ such that $f = \partial_\ell(\mathbf{e}_F) \in T_{\ell-1} \setminus \mathfrak{m}T_{\ell-1}$, where $\mathfrak{m} = (x_1, \dots, x_n)$. Let $\mathbb{U}$ be the subcomplex of $\mathbb{T}$ with $T_j = 0$ if $j \neq \ell, \ell - 1$, $T_\ell = S\mathbf{e}_F$ and $T_{\ell-1} = Sf$. Then $\mathbb{U}$ is an exact complex. Thus, the short exact sequence of complexes

$$0 \to \mathbb{U} \longrightarrow \mathbb{T} \longrightarrow \mathbb{T}/\mathbb{U} \to 0$$

induces the long exact sequence of homology modules

$$\cdots \to H_{i+1}(\mathbb{T}/\mathbb{U}) \to H_i(\mathbb{U}) \to H_i(\mathbb{T}) \to H_i(\mathbb{T}/\mathbb{U}) \to H_{i-1}(\mathbb{U}) \to \cdots .$$

Set $\mathbb{G} = \mathbb{T}/\mathbb{U}$. Since $H_i(\mathbb{T}) = H_i(\mathbb{U}) = 0$ for $i > 0$, $H_0(\mathbb{T}) = I$, and $H_0(\mathbb{U}) = 0$, from the above long exact sequence we see that $\mathbb{G}$ is an acyclic complex with 0th homology $H_0(\mathbb{G}) \cong H_0(\mathbb{T}) = I$. Furthermore, if we let $G_i = \bigoplus_{\mathbf{a}} S(-\mathbf{a})^{b'_{i,\mathbf{a}}}$ to be the ith free module of $\mathbb{G}$, then $G_j = T_j$ for all $j \neq \ell, \ell-1$, $G_\ell = T_\ell/S\mathbf{e}_F$, and $G_{\ell-1} = T_{\ell-1}/Sf$. So all G_j are multigraded free S-modules and $\mathbb{G}$ is a multigraded free resolution of I. Let $\mathbf{a}^*$ be the multidegree of $\mathbf{e}_F$. Since $b'_{\ell,\mathbf{a}^*} < b_{\ell,\mathbf{a}^*}$ and $b'_{\ell-1,\mathbf{a}^*} < b_{\ell-1,\mathbf{a}^*}$, by induction we may assume that $\beta_{i,\mathbf{a}}(I) \le b'_{i,\mathbf{a}}$ for all i and $\mathbf{a} \in \mathbb{Z}^n$. But we clearly have $b'_{i,\mathbf{a}} \le b_{i,\mathbf{a}}$ for all i and $\mathbf{a} \in \mathbb{Z}^n$. Hence our claim follows and the theorem is proved. $\square$

In view of this proposition, to determine $\mathrm{HS}_j(I)$ it suffices to determine the appropriate subset of the set given in (5.3).

As a nice corollary of Theorem 5.2.1 we can recover the following classical result due to Hochster. Recall that the integral vector $\mathbf{a}$ is called *squarefree* if $\mathbf{x}^{\mathbf{a}}$ is a *squarefree* monomial [89]. Thus $\mathbf{a}$ is squarefree if and only if its entries are 0 and 1.

Corollary 5.2.2 ([104], [89, Theorem 8.1.1(a)]). *Let $I \subset S$ be a squarefree monomial ideal. Then all the multigraded shifts of I are squarefree. In particular, all homological shift ideals* $\mathrm{HS}_j(I)$ *are squarefree monomial ideals, for all $j \ge 0$.*

Proof. In view of Theorem 5.2.1 it suffices to note that all monomials in $G(I)$ are squarefree and that the least common multiple (lcm) of squarefree monomials is a squarefree monomial too. $\square$

Our purpose is to describe precisely the subset of the set (5.3) when I is an equigenerated monomial ideal having linear quotients.

Given $u, v \in S$ two monomials of the same degree, we define the *distance* between u and v to be the integer:

$$d(u,v) \;=\; \frac{1}{2}\sum_{i=1}^{n} \big|\deg_{x_i}(u) - \deg_{x_i}(v)\big|.$$

This function satisfies the usual rules of a distance function.

The following lemma is pivotal for our aim.

Lemma 5.2.3 *Let u, v monomials of S of the same degree. Then, $u = x_k(v/x_\ell)$ for some $k \neq \ell$, if and only if $d(u,v) = 1$.*

Proof. Suppose $u = x_k(v/x_\ell)$. Write $v = x_1^{b_1}\cdots x_n^{b_n}$, then $u = (\prod_{i\neq k,\ell} x_i^{b_i})x_k^{b_k+1}x_\ell^{b_\ell-1}$. Note that

$$\big|\deg_{x_i}(u) - \deg_{x_i}(v)\big| \;=\; \begin{cases} |b_\ell - 1 - b_\ell| & \text{if } i = \ell, \\ |b_k + 1 - b_k| & \text{if } i = k, \\ \quad 0 & \text{otherwise.} \end{cases}$$

Thus $d(u,v) = \frac{1}{2}\sum_i |\deg_{x_i}(u) - \deg_{x_i}(v)| = \frac{1}{2}(|b_k+1-b_k| + |b_\ell-1-b_\ell|) = \frac{1}{2}(1+1) = 1$.

Conversely, assume $d(u,v) = 1$. From the definition of $d(u,v)$ it is clear that either $u = x_k^2 v$ or $u = x_k(v/x_\ell)$, for some $k \neq \ell$. But the first possibility does not occur, lest $\deg(u) > \deg(v)$. Therefore, the desired conclusion follows. $\square$

Lemma 5.2.3 allows us to describe the first homological shift ideal of any *equigenerated* monomial ideal with linear quotients, regardless of the admissible order of

I. The next result is a particular case of [101, Proposition 1.3], but for the sake of completeness a different proof for ideals with linear quotients is presented.

Proposition 5.2.4 *Let $I \subset S$ be an equigenerated monomial ideal with linear quotients. Then*

$$\text{HS}_1(I) \;=\; (\text{lcm}(u,v) \;:\; u,v \in G(I),\; d(u,v) = 1).$$

Proof. Suppose I has linear quotients with admissible order $u_1, u_2, \dots, u_m$ of $G(I)$, and let $\text{set}(u_r) = \{i : x_i \in (u_1, \dots, u_{r-1}) : u_r\}$. By equation (5.1) we have $\text{HS}_1(I) = (u_r x_\ell \;:\; r = 2, \dots, m,\; \ell \in \text{set}(u_r))$. Let $r \in \{2, \dots, m\}$ and $\ell \in \text{set}(u_r)$. Then, $u_r x_\ell \in (u_1, \dots, u_{r-1})$. Since I is equigenerated, $u_r x_\ell = u_s x_k$, for some $s < r$ and $k \neq \ell$. Whence, $u_r = x_k(u_s/x_\ell)$. By Lemma 5.2.3, we have $d(u_r, u_s) = 1$. Moreover,

$$\text{lcm}(u_r, u_s) = \text{lcm}(x_k(u_s/x_\ell), u_s) = x_k u_s = u_r x_\ell.$$

Thus, we have verified the inclusion $\text{HS}_1(I) \subseteq (\text{lcm}(u,v) : u,v \in G(I), d(u,v) = 1)$.

Conversely, let $\text{lcm}(u,v)$ with $u,v \in G(I)$, $d(u,v) = 1$. As $\deg(u) = \deg(v)$, Lemma 5.2.3 implies $u = x_k(v/x_\ell)$, hence $ux_\ell = vx_k$. Now, $u = u_r, v = u_s$, for some $r \neq s$. Assume $r > s$, then $x_\ell \in (u_1, \dots, u_s, \dots, u_{r-1}) : u_r$. Thus $\ell \in \text{set}(u_r)$ and $\text{lcm}(u_r, u_s) = u_r x_\ell \in G(\text{HS}_1(I))$, showing the other inclusion. $\square$

Before giving our main result, we consider some applications of Proposition 5.2.4.

Recall that a *monomial order* on S is a total order $\succ$ on the set of monomials of S such that $\mathbf{x^a} \succeq 1$ for all monomials $\mathbf{x^a} \in S$, and if $\mathbf{x^a} \succ \mathbf{x^b}$ then $\mathbf{x^c x^a} \succ \mathbf{x^c x^b}$, for all monomials $\mathbf{x^a}, \mathbf{x^b}, \mathbf{x^c} \in S$. In particular, $x_{\sigma(1)} \succ x_{\sigma(2)} \succ \cdots \succ x_{\sigma(n)}$ for some permutation σ of $[n]$. We may suppose $x_1 \succ x_2 \succ \cdots \succ x_n$, after an innocuous relabeling. In this case we say that $\succ$ *is induced by* $x_1 > x_2 > \cdots > x_n$. When we write that I has linear quotients with respect to the monomial order $\succ$ we mean that I has linear quotients with admissible order $u_1 \succ u_2 \succ \cdots \succ u_m$ of $G(I)$. A particular monomial order we are going to use is the *lexicographic order* $>_{\text{lex}}$ induced by $x_1 > x_2 > \cdots > x_n$. Let $\mathbf{x^a}, \mathbf{x^b}$ be monomials of S. Then $\mathbf{x^a} >_{\text{lex}} \mathbf{x^b}$ if

$$a_1 = b_1, \quad a_2 = b_2, \quad \dots, \quad a_{s-1} = b_{s-1}, \quad a_s > b_s,$$

for some $1 \leq s \leq n$ [89].

For a monomial $u \in S$, $u \neq 1$, we let $\text{supp}(u) = \{i \in [n] : \deg_{x_i}(u) > 0\}$ be its *support* and we let its *maximum* to be the integer $\max(u) = \max \text{supp}(u)$.

Lemma 5.2.5 *Let $I \subset S$ be an equigenerated monomial ideal with linear quotients with respect to a monomial order $\succ$ (e.g., $>_{\text{lex}}$) induced by $x_1 > x_2 > \cdots > x_n$. Then, for all $u \in G(I)$,*

$$\text{set}(u) \;\subseteq\; [\max(u) - 1].$$

Proof. Indeed, let $G(I)$ ordered as $u_1 \succ u_2 \succ \cdots \succ u_m$ and let $j \in \{1, \dots, m\}$. If $i \in \text{set}(u_j)$, then $x_i u_j \in (u_1, \dots, u_{j-1})$. Since $\deg(u_1) = \cdots = \deg(u_{j-1}) = \deg(u_j)$, there exists $s \in \text{supp}(u_j)$, $s \neq i$ such that $x_i(u_j/x_s) = u_p$ for some $p \leq j-1$. But $x_i(u_j/x_s) = u_p \succ u_j$. Since $\succ$ is a monomial order, this implies that $x_i u_j \succ x_s u_j$, *i.e.*, $x_i \succ x_s$. Thus $i < s$. But $s \leq \max(u_j)$ and so $i < \max(u_j)$. Summarizing our reasoning, we have shown that $\text{set}(u_j) \subseteq [\max(u_j) - 1]$, as desired. $\square$

In [101], the following general inclusion was shown.

Proposition 5.2.6 ([101, Proposition 1.4]). *Let $I \subset S$ be a monomial ideal with linear quotients. Then $\text{HS}_{j+1}(I) \subseteq \text{HS}_1(\text{HS}_j(I))$, for all j.*

In the same article, it was noted that for arbitrary equigenerated monomial ideals with linear quotients the equation $\text{HS}_{j+1}(I) = \text{HS}_1(\text{HS}_j(I))$ does not hold. Indeed, the equigenerated monomial ideal $I = (x_2x_4, x_1x_2, x_1x_3)$ has linear quotients with admissible order $x_2x_4 > x_1x_2 > x_1x_3$. We have $\text{HS}_1(I) = (x_1x_2x_3, x_1x_2x_4)$ and $\text{HS}_2(I) = (0)$, but $\text{HS}_1(\text{HS}_1(I)) = (x_1x_2x_3x_4) \neq (0) = \text{HS}_2(I)$.

We can *refine* Proposition 5.2.6 in the following special case. For an equigenerated monomial ideal $J \subset S$, by $J_{>\ell}$ we denote the monomial ideal whose minimal generating set is $G(J_{>\ell}) = \{u \in G(J) : |\text{supp}(u)| > \ell\}$. Although a part of the next corollary follows from Proposition 5.2.6 we include a short proof for completeness.

Corollary 5.2.7 *Let $I \subset S$ be an equigenerated monomial ideal with linear quotients with respect to a monomial order $\succ$ (e.g., $>_{\text{lex}}$) induced by $x_1 > x_2 > \cdots > x_n$. Then,*

$$\text{HS}_{j+1}(I) \ \subseteq \ (\text{HS}_1(\text{HS}_j(I)))_{>j+1}.$$

Proof. For $j = 0$ the thesis is immediate. Let $j > 0$. Firstly we show the inclusion $\text{HS}_{j+1}(I) \subseteq \text{HS}_1(\text{HS}_j(I))$. By Proposition 5.2.4,

$$\text{HS}_1(\text{HS}_j(I)) = (\text{lcm}(w_1, w_2) \ : \ w_1, w_2 \in G(\text{HS}_j(I)), \ d(w_1, w_2) = 1).$$

Take $w = \mathbf{x}_F u \in G(\text{HS}_{j+1}(I))$ with $u \in G(I)$, $F \subseteq \text{set}(u)$ and $|F| = j+1$. Since $j+1 \geq 2$ we can find $r, s \in F$, $r \neq s$. Then $w_1 = \mathbf{x}_{F\setminus\{r\}}u, w_2 = \mathbf{x}_{F\setminus\{s\}}u \in G(\text{HS}_j(I))$ and $d(w_1, w_2) = 1$. Thus $\text{lcm}(w_1, w_2) = w \in G(\text{HS}_1(\text{HS}_j(I)))$, as desired. It remains to prove that any $w = \mathbf{x}_F u \in G(\text{HS}_{j+1}(I))$ has $\text{supp}(w) > j+1$. By Lemma 5.2.5 we have $F \subseteq \text{set}(u) \subseteq [\max(u) - 1]$. Hence $\max(u) \notin F$ and $F \cup \{\max(u)\} \subseteq \text{supp}(w)$, and since $|F| = j+1$ we have that $\text{supp}(w) \geq |F| + 1 = j+2$, as desired. $\square$

Note that even for an ideal satisfying the hypothesis of Corollary 5.2.7, the inclusion $\text{HS}_{j+1}(I) \subseteq (\text{HS}_1(\text{HS}_j(I)))_{>j+1}$ could be strict. Indeed, consider again the equigenerated ideal $I = (x_2x_4, x_1x_2, x_1x_3)$. I has linear quotients with respect to the lexicographic order $>_{\text{lex}}$ induced by $x_2 > x_1 > x_3 > x_4$. We have $(\text{HS}_1(\text{HS}_1(I)))_{>2} = (x_1x_2x_3x_4)$ but $\text{HS}_2(I) = (0)$ and the inclusion $\text{HS}_2(I) \subseteq (\text{HS}_1(\text{HS}_1(I)))_{>2}$ is strict in such a case.

By Propositions 5.1.1 and 5.2.4 we obtain the desired description.

Theorem 5.2.8 *Let $I \subset S$ be an equigenerated monomial ideal with linear quotients with admissible order $u_1, u_2, \ldots, u_m$ of $G(I)$. Then, for all $j \geq 0$,*

$$\begin{aligned}\text{HS}_j(I) = (\text{lcm}(u_{i_1}, u_{i_2}, \ldots, u_{i_{j+1}}) = w \ &: i_1 < i_2 < \cdots < i_{j+1}, \ \deg(w) = \deg(u_{i_{j+1}}) + j, \\ &\textit{and} \ \ d(u_{i_\ell}, u_{i_{j+1}}) = 1, \ \textit{for all } \ell = 1, \ldots, j).\end{aligned}$$

Proof. We proceed by induction on $\mu(I) = |G(I)| = m \geq 1$. For $m = 1$ there is nothing to prove. Let $m = 2$, then by the Taylor complex $\text{pd}(I) \leq \mu(I) - 1 = 1$ and the claim follows from Proposition 5.2.4. Let $m > 2$ and set $J = (u_1, \ldots, u_{m-1})$. We note that $G(I)$ is the disjoint union of $G(J)$ and $\{u_m\}$. By Proposition 5.1.1,

$$\text{HS}_j(I) \ = \ \text{HS}_j(J) + u_m\big(\mathbf{x}_F \ : \ F \subseteq \text{set}(u_m), \ |F| = j\big).$$

By induction $\text{HS}_j(J)$ covers all required generators $w = \text{lcm}(u_{i_1}, u_{i_2}, \ldots, u_{i_{j+1}})$ with $i_1 < i_2 < \cdots < i_{j+1}$, $\deg(w) = \deg(u_{i_{j+1}}) + i$ and $d(u_{i_\ell}, u_{i_{j+1}}) = 1$, for all $\ell = 1, \ldots, j$ and with $i_{j+1} < mm$. To conclude we show that $u_m(\mathbf{x}_F : F \subseteq \text{set}(u_m), |F| = j)$ is generated by the same monomials as above but with $u_{i_{j+1}} = u_m$.

Indeed, let $F \subseteq \text{set}(u_m)$ with $|F| = j$ and consider $\mathbf{x}_F u_m$. For any $t \in F$, $x_t u_m \in J$ and since I is equigenerated, we have $x_t u_m = x_{r(t)} u_{i(t)}$ for some $1 \leq i(t) \leq m-1$ and some $r(t) \neq t$. By Lemma 5.2.3, $d(u_m, u_{i(t)}) = 1$ for all $t \in F$. Moreover $u_{i(t)} \neq u_{i(s)}$ for all $t, s \in F$, $t \neq s$, indeed $\deg_{x_t}(u_{i(t)}) > \deg_{x_t}(u_m) \geq \deg_{x_t}(u_{i(s)})$. Thus we have that $\mathbf{x}_F u_m = \text{lcm}(u_{i(t)}, u_m : i \in F)$ is as required.

Conversely, let $w = \text{lcm}(u_{i_1}, \dots, u_{i_j}, u_m)$ with $i_1 < \cdots < i_j < m$, $\deg(w) = \deg(u_m)+j$ and $d(u_{i_\ell}, u_m) = 1$, for all $\ell = 1, \dots, j$. Then for all ℓ, $u_{i_\ell} = x_{t(\ell)}(u_m/x_{s(\ell)})$ for some $t(\ell) \neq s(\ell)$. Thus,

$$\begin{aligned} w &= \text{lcm}(u_{i_\ell}, u_m : \ell = 1, \dots, j) = \text{lcm}(x_{t(\ell)}(u_m/x_{s(\ell)}), u_m : \ell = 1, \dots, j) \\ &= u_m \text{lcm}(x_{t(\ell)} : \ell = 1, \dots, j). \end{aligned}$$

By our hypothesis $\deg(w) = \deg(u_m) + j$, it follows that $x_{t(\ell)} \neq x_{t(p)}$ for all $\ell \neq p$. Moreover, $t(\ell) \in \text{set}(u_m)$ for all m. Setting $F = \{t(\ell) : \ell = 1, \dots, m\}$, we have $w = \mathbf{x}_F u_m$ and $w \in \text{HS}_j(I)$ by equation (5.1), concluding the proof. $\square$

The following two examples illustrate the previous result.

Example 5.2.9 Let $I = (x_2x_4, x_1x_2, x_1x_3)$ an ideal of $S = K[x_1, x_2, x_3]$. I has linear quotients. Note that $d(x_2x_4, x_1x_2) = d(x_1x_3, x_1x_2) = 1$, but $x_2x_4 > x_1x_3 > x_1x_2$ is not an admissible order of I. Hence by Theorem 5.2.8, $\text{lcm}(x_2x_4, x_1x_3, x_1x_2) = x_1x_2x_3x_4$ does not belong to $G(\text{HS}_2(I))$. Indeed $\text{pd}(I) = 1$ and $\text{HS}_2(I) = (0)$. So the condition $j_1 < j_2 < \cdots < j_{i+1}$ given in the previous theorem can not be removed.

Example 5.2.10 Let $I = (x_1^2x_3, x_1^2x_2, x_1x_2x_3)$ an ideal of $S = K[x_1, x_2, x_3]$. I has linear quotients. Note that $x_1^2x_3 > x_1^2x_2 > x_1x_2x_3$ is an admissible order, but $\deg(w) = 4 < 5 = \deg(x_1x_2x_3) + 2$. Hence $w = \text{lcm}(x_1^2x_3, x_1^2x_2, x_1x_2x_3) = x_1^2x_2x_3 \notin G(\text{HS}_2(I))$ by Theorem 5.2.8. Indeed $\text{pd}(I) = 1$ and $\text{HS}_2(I) = (0)$.

5.3 The highest homological shift of ideals with linear quotients

In this section we analyze the highest possible non zero homological shift ideal of an equigenerated monomial ideal with linear quotients.

Let $I \subset S$ be a monomial ideal. One can always suppose that $\text{supp}(I) = [n]$. Otherwise we replace I with $\widetilde{I} = I \cap R$, where $R = K[x_i : i \in \text{supp}(I)]$, without changing its projective dimension. By Hilbert syzygy theorem, $\text{pd}(S/I) \leq n$ and since $\text{pd}(I) = \text{pd}(S/I) - 1$ we have that $\text{pd}(I) \leq n - 1$. We say that I has *maximal projective dimension* if $\text{pd}(I) = |\text{supp}(I)| - 1 = n - 1$.

We are going to characterize equigenerated monomial ideals with linear quotients with respect to a monomial order $\succ$ having maximal projective dimension. We recall the concept of *socle* of a monomial ideal. Suppose that I has a d-linear resolution. Then the *socle ideal* of I, $\text{soc}(I)$, is the unique monomial ideal of S generated in degree $d-1$ [27, Proposition 1.4] such that

$$I : \mathfrak{m} = \text{soc}(I) + I.$$

In other words, $\text{soc}(I)$ is the unique monomial ideal with $G(\text{soc}(I)) = G(I : \mathfrak{m})_{d-1}$.

Lemma 5.3.1 *Let $I \subset S$ be a monomial ideal with a linear resolution and such that* $\text{supp}(I) = [n]$. *Then, the following conditions are equivalent.*

(a) *I has maximal projective dimension.*

(b) $\mathrm{soc}(I) \neq (0)$.

Proof. Assume I is generated in degree d, then I has a d-linear resolution.

(a) $\Rightarrow$ (b). Suppose I has maximal projective dimension, *i.e.*, $\mathrm{pd}(I) = n-1$. Then, since I has a d-linear resolution, the $(n-1)$th free module of the minimal free resolution of I is $S(-d-(n-1))^{\beta_{n-1}(I)} = S(-d-(n-1))^{\beta_{n-1,n-1+d}(I)}$. Hence

$$\begin{aligned} K(-d-(n-1))^{\beta_{n-1,n+d-1}(I)} &\cong \mathrm{Tor}^S_{n-1}(K,I)_{n+d-1} \\ &\cong \mathrm{Tor}^S_{n}(K,S/I)_{n+d-1} \\ &\cong H_n(x_1,\dots,x_n;S/I)_{n+d-1}. \end{aligned} \tag{5.4}$$

The natural basis of the K-vector space $H_n(x_1,\dots,x_n;S/I)_{n+d-1}$ consists of all the homology classes of the Koszul cycles $z = (w+I)e_1 \wedge e_2 \wedge \cdots \wedge e_n$ having degree $(d-1)+n$. Since $e_1 \wedge e_2 \wedge \cdots \wedge e_n$ has degree n it follows that $\deg(w) = d-1$. Therefore, since z is a cycle, we must have

$$\partial_n(z) = \sum_{k=1}^{n}(-1)^{k+1}(x_k w + I)e_1 \wedge \cdots \wedge e_{k-1} \wedge e_{k+1} \wedge \cdots \wedge e_n = 0.$$

Thus we have $x_k w + I = 0$ for all $k = 1,\dots,n$, that means $x_k w \in I$ for all $k = 1,\dots,n$. Hence $w \in G(\mathrm{soc}(I))$ and $\mathrm{soc}(I) \neq (0)$.

(b) $\Rightarrow$ (a). Conversely, any monomial $w \in G(\mathrm{soc}(I))$ gives rise to the following Koszul cycle $z = (w+I)e_1 \wedge e_2 \wedge \cdots \wedge e_n$ whose homology class is a basis element of $H_n(x_1,\dots,x_n;S/I)$. Since $\mathrm{soc}(I) \neq (0)$, we have $H_n(x_1,\dots,x_n;S/I) \neq 0$ and consequently I has maximal projective dimension.

□

Let $\mathbf{1}$ be the unit vector $(1,1,\dots,1)$ of $\mathbb{Z}^n$.

Corollary 5.3.2 *Let $I \subset S$ be a monomial ideal with a linear resolution and such that $\mathrm{supp}(I) = [n]$. Then,*

$$\mathrm{HS}_{n-1}(I) \;=\; x_1 x_2 \cdots x_n \cdot \mathrm{soc}(I).$$

Proof. Let F_{n-1} be the $(n-1)$th free S-module of the minimal free resolution of I. If I has not maximal projective dimension, then $F_{n-1} = 0$ and $\mathrm{HS}_{n-1}(I) = (0)$ too. By Lemma 5.3.1, $\mathrm{soc}(I) = (0)$ and the assertion follows in such a case.

Suppose now that I has maximal projective dimension. The isomorphism given in (5.4) is also a multigraded isomorphism. Thus $S(-\mathbf{a})$ is a direct summand of F_{n-1} if and only if $(\mathbf{x^a}/\mathbf{x}_{[n]}+I)e_1 \wedge e_2 \wedge \cdots \wedge e_n$ is a Koszul cycle. A comparison with the proof of Lemma 5.3.1 shows that the set of the $(n-1)$th multigraded shifts of I consists of all vectors $\mathbf{b}+\mathbf{1}$ such that $\mathbf{x^b} \in G(\mathrm{soc}(I))$. Hence $\mathrm{HS}_{n-1}(I) = x_1 x_2 \cdots x_n \mathrm{soc}(I)$. □

We recall that an equigenerated monomial ideal I with linear quotients has a linear resolution. So we can consider the socle ideal of I.

Theorem 5.3.3 *Let $I \subset S = K[x_1,\dots,x_n]$ be an equigenerated monomial ideal with linear quotients with respect to a monomial order $\succ$ induced by $x_1 > x_2 > \cdots > x_n$ and such that $\mathrm{supp}(I) = [n]$. Then, the following conditions are equivalent.*

(i) *I has maximal projective dimension.*

(ii) $\mathrm{soc}(I) \neq (0)$.

(iii) *There exists $u \in G(I)$ such that $x_i(u/x_n) \in G(I)$, for all $i = 1, \dots, n$.*

Furthermore,

$$G(\mathrm{soc}(I)) \;=\; \{u/x_n \;:\; x_i(u/x_n) \in G(I),\ i = 1, \dots, n\}. \tag{5.5}$$

Proof. (i) $\Leftrightarrow$ (ii) follows from Lemma 5.3.1. To prove (ii) $\Leftrightarrow$ (iii), it suffices to prove that equation (5.5) holds. From equation (5.1) and Corollary 5.3.2,

$$\begin{aligned} \mathrm{HS}_{n-1}(I) \;&=\; (\mathbf{x}_F u \;:\; u \in G(I),\ F \subseteq \mathrm{set}(u),\ |F| = n-1) \\ &=\; x_1 x_2 \cdots x_n \cdot \mathrm{soc}(I). \end{aligned} \tag{5.6}$$

On the other hand, by Lemma 5.2.5, if $u \in G(I)$ then $\mathrm{set}(u) \subseteq [\max(u) - 1]$ and so $|\mathrm{set}(u)| = n - 1$ if and only if $\max(u) = n$ and $\mathrm{set}(u) = [\max(u) - 1]$. Thus,

$$\mathrm{HS}_{n-1}(I) \;=\; (x_1 x_2 \cdots x_{n-1} u \;:\; u \in G(I),\ \max(u) = n,\ \mathrm{set}(u) = [n-1]). \tag{5.7}$$

A comparison with equation (5.6) shows that for any $u \in G(I)$ with $\mathrm{set}(u) = [n-1]$ we must have $\mathbf{x}_{\mathrm{set}(u)} u = x_1 x_2 \cdots x_{n-1} u \in x_1 x_2 \cdots x_n \mathrm{soc}(I)$. Thus $x_1 x_2 \cdots x_{n-1} u = x_1 x_2 \cdots x_n (u/x_n)$ with $u/x_n \in \mathrm{soc}(I)$. So equation (5.5) holds, as desired. □

Notes

Perhaps, the concept of homological shift ideal appeared for the first time in 2005, inside the book of Miller and Sturmfels [127]. In [127, Theorem 2.18], using a different language, the authors showed that the first homological shift ideal $\mathrm{HS}_1(I)$ of an equigenerated strongly stable ideal I has a linear resolution. This result was strengthened by Bayati, Jahani and Taghipour who showed that if I an equigenerated strongly stable ideal, then $\mathrm{HS}_1(I)$ even has linear quotients [17, Proposition 3.2].

One of the main motivations behind the study of homological shift ideals lies in the Bandari–Bayati–Herzog conjecture (Conjecture 6.1.1) which predicts that $\mathrm{HS}_i(I)$ is polymatroidal for all i, if I is polymatroidal. At present, we know that $\mathrm{HS}_1(I)$ is always polymatroidal if I is such (Theorem 6.1.2). Moreover, the conjecture was proved by Bayati for squarefree polymatroidal ideals (Corollary 6.1.5), by Herzog, Moradi, Rahimbeigi and Zhu for polymatroidal ideals that satisfy the strong exchange property (Corollary 6.1.6), and by Ficarra and Herzog for polymatroidal ideals generated in degree 2 (Theorem 7.3.3). This latter result was also recently recovered by Bayati in [15, Corollary 2.9] using a different method involving adjacency ideals and the so–called quasi–squarefree part.

The main general properties of homological shift ideals were investigated by Herzog, Moradi, Rahimbeigi and Zhu in [101]. An important result of Sbarra [141], which we are going to use later on, guarantees that taking homological shifts commutes with the polarization (Lemma 8.4.10). In the next four chapters, (Chapters 6–9), we study the homological shifts of polymatroidal ideals, edge ideals and cover ideals.

Chapter 6

Homological shifts of polymatroids

The interest in homological shift ideals has its origins in a meeting between Somayeh Bandari, Shamila Bayati and Jürgen Herzog that took place in Essen in 2012. Due to experimental evidence, the three authors conjectured that the property of being *polymatroidal* is an homological shift property. Recall that a *polymatroidal* ideal I is an equigenerated monomial ideal of S whose minimal generating set $G(I)$ corresponds to the base of a *discrete polymatroid*. Polymatroidal ideals constitute one of the most distinguished classes of monomial ideals. Indeed, the product of polymatroidal ideals is polymatroidal. Any polymatroidal ideal has linear quotients, linear resolution and so linear powers, [89, Corollary 12.6.4], a rare property among monomial ideals. In this chapter we study the homological shift ideals of polymatroidal ideals.

The chapter is organized as follows. In Section 6.1, we study the homological shift ideals of polymatroidal ideals. We are able to prove that $\text{HS}_1(I)$ is polymatroidal if I is polymatroidal (Theorem 6.1.2). Our proof is based on Proposition 5.2.4, whose main advantage consists in the fact that $\text{HS}_1(I)$ does not depend upon the admissible order of I. To study the higher homological shift ideals, firstly we note that for $j \geq 1$, $\text{HS}_{j+1}(I) \subseteq (\text{HS}_1(\text{HS}_j(I)))_{>j+1}$, (Corollary 5.2.7), where $J_{>j+1}$ is the monomial ideal with minimal generating set $G(J_{>j+1}) = \{u \in G(J) : |\text{supp}(u)| > j+1\}$. Unfortunately, equality in the above inclusion does not hold in general, (Example 6.1.7). Nonetheless, it holds for matroidal ideals, (Proposition 6.1.4). As a consequence Conjecture 6.1.1 holds for all matroidal ideals, (Corollary 6.1.5). It would be of interest to classify all polymatroidal ideals satisfying the equation $\text{HS}_{j+1}(I) = (\text{HS}_1(\text{HS}_j(I)))_{>j+1}$ for all $j < \text{pd}(I)$.

In Section 6.2, we discuss the highest possible homological shift ideal. By Corollary 5.3.2, $\text{HS}_{n-1}(I)$ is polymatroidal if and only if $\text{soc}(I)$ is polymatroidal. It is conjectured by Bandari and Herzog [13, page 760] that $\text{soc}(I)$ is a polymatroidal ideal if I is such (Conjecture 6.2.1). It turns out that if a polymatroidal ideal I has maximal projective dimension the same is true for I^k, $k \geq 1$, (Proposition 6.2.3). Hence if $\text{soc}(I) \neq (0)$, then $\text{soc}(I^k) \neq (0)$ too. In the last section we analyze four families of polymatroidal ideals, characterize when they have maximal projective dimension and prove that their socle ideals are polymatroidal. These four families are: matroidal ideals, *principal Borel ideals* [50], *PLP–polymatroidal ideals* [122] and *LP–polymatroidal ideals* [142]. It would be nice to establish Conjecture 6.1.1 for these families. Only for *transversal polymatroidal* ideals we are unable to establish Conjecture 6.2.1. In Question 6.2.18 we predict that the socle ideal of these ideals can be determined by the *spanning trees* of a certain *intersection graph* [96]. We answer positively to this question in a special case, (Proposition 6.2.22).

6.1 The first homological shift of polymatroidal ideals

Recall that an equigenerated monomial ideal $I \subset S = K[x_1, \dots, x_n]$ is a *polymatroidal* ideal if it satisfies the following *exchange property*,

($*$) for all $u, v \in G(I)$ and all i such that $\deg_{x_i}(u) > \deg_{x_i}(v)$, there exists j with $\deg_{x_j}(u) < \deg_{x_j}(v)$ and such that $x_j(u/x_i) \in G(I)$.

Such ideals are called *polymatroidal* because their minimal generating set $G(I)$ corresponds to the basis of a *discrete polymatroid*, see [89, Chapter 12].

For later use, we recall again the *dual exchange property*, (Lemma 3.4.1),

($**$) for all $u, v \in G(I)$ and all j such that $\deg_{x_j}(u) < \deg_{x_j}(v)$, there exists i with $\deg_{x_i}(u) > \deg_{x_i}(v)$ and such that $x_j(u/x_i) \in G(I)$.

It is expected that the following is true.

Conjecture 6.1.1 (Bandari–Bayati–Herzog [16, 101]). *Let $I \subset S$ be a polymatroidal ideal. Then all homological shift ideals $\mathrm{HS}_j(I)$ are again polymatroidal, for all $j \geq 0$.*

Using Theorem 3.4.2 and the description of $\mathrm{HS}_1(I)$ given in Proposition 5.2.4 we can prove the following result.

Theorem 6.1.2 *Let $I \subset S$ be a polymatroidal ideal. Then $\mathrm{HS}_1(I)$ is polymatroidal.*

Proof. It is convenient to note that by Proposition 5.2.4,

$$\begin{aligned}\mathrm{HS}_1(I) &= (x_iu \;:\; u \in G(I),\; i \in \mathrm{set}(u))\\ &= (\mathrm{lcm}(u,v) \;:\; u, v \in G(I),\; d(u,v) = 1).\end{aligned}$$

We must prove the following exchange property,

($*$) for all monomials $u, v \in G(I)$, all integers $k \in \mathrm{set}(u)$, $\ell \in \mathrm{set}(v)$ such that $u_1 = x_k u \neq x_\ell v = v_1$ and all i with $\deg_{x_i}(u_1) > \deg_{x_i}(v_1)$, there exists an integer j such that $\deg_{x_j}(u_1) < \deg_{x_j}(v_1)$ and $x_j(u_1/x_i) = x_j(x_k u)/x_i \in G(\mathrm{HS}_1(I))$.

We may assume that i is different both from k and ℓ. Indeed, if $k = i$, then as $k \in \mathrm{set}(u)$ we have $u_1 = x_p z$ for some $z \in G(I) \setminus \{u\}$, $p \neq k$, and we may use the element $x_p z$ with $p \neq k = i$. The same reasoning applies for ℓ. Thus, we can assume $i \neq k, \ell$. In particular, this assumption and our hypothesis $\deg_{x_i}(u_1) > \deg_{x_i}(v_1)$ imply that $\deg_{x_i}(u) > \deg_{x_i}(v)$ as well. Since I is a polymatroidal ideal, the following set

$$\Omega \;=\; \{h \in [n] \setminus \{i\} \;:\; \deg_{x_h}(u) < \deg_{x_h}(v) \;\; \textit{and} \;\; x_h(u/x_i) \in G(I)\}.$$

is non empty.

Let $h \in \Omega$ and set $w = x_h(u/x_i)$. We may distinguish two cases.

Case 1. Suppose that $k \in \Omega$. For $h = k \in \Omega$, we have $\deg_{x_k}(u) < \deg_{x_k}(v)$ and $w = x_k(u/x_i) \in G(I)$. We distinguish two more cases.

Subcase 1.1. Assume $w = v$. By hypothesis $v_1 = x_\ell v \in G(\mathrm{HS}_1(I))$. We show that the property ($*$) is verified for the integer $j = \ell$. Indeed, as $i \neq \ell$ and $w = v$,

$$\begin{aligned}\deg_{x_\ell}(u_1) = \deg_{x_\ell}(x_k u) = \deg_{x_\ell}(x_k u/x_i) = \deg_{x_\ell}(w)\\ = \deg_{x_\ell}(v) < \deg_{x_\ell}(x_\ell v),\end{aligned}$$

and $v_1 = x_\ell v = x_\ell w = x_\ell(x_k u)/x_i \in\ G(\mathrm{HS}_1(I))$, as desired.

SUBCASE 1.2. Assume $w \neq v$. Thus, for some r, $\deg_{x_r}(w) > \deg_{x_r}(v)$. Since I is polymatroidal, there exists an integer m with $\deg_{x_m}(w) < \deg_{x_m}(v)$ and such that $w_1 = x_m(w/x_r) \in G(I)$. Clearly $m \neq r$. Hence, Lemma 5.2.3 implies that $d(w, w_1) = 1$ and Proposition 5.2.4 implies that

$$\mathrm{lcm}(w, w_1) = x_m w = x_m(x_\ell u)/x_i \in G(\mathrm{HS}_1(I)). \tag{6.1}$$

It remains to prove that the integer m satisfies the first condition of property $(*)$, namely $\deg_{x_m}(u_1) < \deg_{x_m}(v_1)$. First note that $m \neq i$. Lest, if $i = m$, by hypothesis $\deg_{x_i}(w) < \deg_{x_i}(v)$ and then $\deg_{x_i}(u) = \deg_{x_i}(w) + 1 \leq \deg_{x_i}(v)$, against the fact that $\deg_{x_i}(u) > \deg_{x_i}(v)$. Thus, since $m \neq i$, we have $\deg_{x_m}(u) \leq \deg_{x_m}(w) < \deg_{x_m}(v)$. This inequality together with equation (6.1) show that the integer $j = m$ satisfies the property $(*)$ in such a case.

CASE 2. Suppose that $k \notin \Omega$. Nonetheless, for some $h \in \Omega$, with $h \neq k$, we have $\deg_{x_h}(u) < \deg_{x_h}(v)$ and $w = x_h(u/x_i) \in G(I)$. Since $k \in \mathrm{set}(u)$, there exist $z \in G(I)$, $z \neq u$ and $x_p \neq x_k$ such that $u_1 = x_k u = x_p z$.

SUBCASE 2.1. Suppose $d(w, z) = 1$. As $h \in \Omega$ but $k \notin \Omega$ we have $h \neq k$. Thus, as $w = x_h(u/x_i)$ and $z = x_k(u/x_p)$ it follows that $p = i$, lest $d(w, z) > 1$. Hence $p = i$ and Proposition 5.2.4 implies

$$\mathrm{lcm}(w, z) = \mathrm{lcm}(x_h(u/x_i), x_k(u/x_p)) = x_h(x_k u)/x_i \in G(\mathrm{HS}_1(I)).$$

Finally, we just need to check that $\deg_{x_h}(u_1) < \deg_{x_h}(v_1)$. Indeed, as $h \neq k$,

$$\deg_{x_h}(x_k u) = \deg_{x_h}(u) < \deg_{x_h}(v) \leq \deg_{x_h}(x_\ell v).$$

SUBCASE 2.2. Suppose $d(w, z) > 1$. Then $p \neq i$, lest $d(w, z) = 1$ by SUBCASE 2.1. Thus $d(w, z) = d(x_h(u/x_i), x_k(u/x_p)) = 2$, $i \neq h$, $h \neq k$, $k \neq p$, $p \neq i$ and

$$\begin{aligned} &\deg_{x_i}(w) < \deg_{x_i}(z), \qquad && \deg_{x_h}(w) > \deg_{x_h}(z),\\ &\deg_{x_k}(w) < \deg_{x_k}(z), && \deg_{x_p}(w) > \deg_{x_p}(z). \end{aligned}$$

Moreover, for all $q \neq i, h, k, p$ we have $\deg_{x_q}(w) = \deg_{x_q}(z)$. Since $w, z \in G(I)$ and $\deg_{x_i}(z) > \deg_{x_i}(w)$ we have $z_1 = x_h(z/x_i) \in G(I)$ or $z_2 = x_p(z/x_i) \in G(I)$. We distinguish two more cases.

SUBCASE 2.2.1. Suppose $z_1 = x_h(z/x_i) \in G(I)$. Note that

$$x_p(z_1/x_k) = x_p(x_h(z/x_i))/x_k = x_p x_h x_k((u/x_p)/x_i)/x_k = x_h(u/x_i) = w.$$

Since $k \neq p$, Lemma 5.2.3 implies that $d(z_1, w) = 1$. Thus, by Proposition 5.2.4

$$\begin{aligned} \mathrm{lcm}(z_1, w) &= \mathrm{lcm}(x_h(z/x_i), x_h(u/x_i))\\ &= \mathrm{lcm}(x_h(x_k(u/x_p)/x_i), x_h(u/x_i))\\ &= x_h(x_k u)/x_i \in G(\mathrm{HS}_1(I)), \end{aligned}$$

and the property $(*)$ is satisfied as $h \in \Omega$, that is $\deg_{x_h}(u) < \deg_{x_h}(v)$ and as $h \neq k$, we have $\deg_{x_h}(u_1) = \deg_{x_h}(x_k u) = \deg_{x_h}(u) < \deg_{x_h}(v) \leq \deg_{x_h}(x_\ell v) = \deg_{x_h}(v_1)$.

SUBCASE 2.2.2. Suppose $z_2 = x_p(z/x_i) \in G(I)$. Note that

$$z_2 = x_p(z/x_i) = x_p(x_k(u/x_p)/x_i) = x_k(u/x_i)$$

and $d(z_2, w) = 1$. Thus, Proposition 5.2.4 implies that

$$\mathrm{lcm}(z_2, w) = \mathrm{lcm}(x_k(u/x_i), x_h(u/x_i)) = x_h(x_k u)/x_i \in G(\mathrm{HS}_1(I)),$$

and as before $\deg_{x_h}(u_1) < \deg_{x_h}(v_1)$. The proof is complete. □

By Theorem 3.4.2, a polymatroidal ideal $I \subset S$ has linear quotients with respect to any lexicographic order $>_{\mathrm{lex}}$. Thus Corollary 5.2.7 implies that for all $j \geq 0$,

$$\mathrm{HS}_{j+1}(I) \ \subseteq \ (\mathrm{HS}_1(\mathrm{HS}_j(I)))_{>j+1}.$$

Next, we study when equality holds in the above equation. This is the case when I is actually matroidal, *i.e.*, it is squarefree polymatroidal. This fact was first noted in [16, Corollary 2.3]. However the proof in [16] makes use of *matroid theory* and *graph theory*. We provide here a totally algebraic proof.

Firstly, we note the following general fact.

Lemma 6.1.3 *Let $I \subset S$ be a squarefree monomial ideal. Then, for all $j \geq 0$,*

$$(\mathrm{HS}_1(\mathrm{HS}_j(I)))_{>j+1} \ = \ \mathrm{HS}_1(\mathrm{HS}_j(I)).$$

Proof. Firstly, by Corollary 5.2.2, all homological shift ideals $\mathrm{HS}_j(I)$ are squarefree. Secondly, we show that all monomials $w \in G(\mathrm{HS}_j(I))$ have $|\mathrm{supp}(w)| > j$. Let

$$\mathbb{F} \ : \ \cdots \xrightarrow{d_{j+1}} F_j \xrightarrow{d_j} F_{j-1} \xrightarrow{d_{j-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} I \to 0,$$

be the minimal free resolution of I. By induction, for all shifts $\mathbf{b}$ of F_{j-1} we have $|\mathrm{supp}(\mathbf{x}^{\mathbf{b}})| \geq j$. Since $\mathbb{F}$ is minimal, $\mathrm{Im}(d_j) \subseteq \mathfrak{m}F_{j-1}$, with $\mathfrak{m} = (x_1, \ldots, x_n)$. Since all shifts of F_j are squarefree, we see that for all shifts $\mathbf{a}$ of F_j we have $|\mathrm{supp}(\mathbf{x}^{\mathbf{a}})| \geq j+1$.

Finally, it is clear that $(\mathrm{HS}_1(\mathrm{HS}_j(I)))_{>j+1} \subseteq \mathrm{HS}_1(\mathrm{HS}_j(I))$. To conclude the proof it suffices to show the opposite inclusion. Let $y \in G(\mathrm{HS}_1(\mathrm{HS}_j(I)))$. We show that $|\mathrm{supp}(y)| > j+1$. By Proposition 5.2.4, $y = \mathrm{lcm}(w_1, w_2)$ with $w_1, w_2 \in G(\mathrm{HS}_j(I))$ such that $d(w_1, w_2) = 1$. By Lemma 5.2.3, $w_1 = x_k(w_2/x_\ell)$ for some $k \neq \ell$. Thus $y = \mathrm{lcm}(w_1, w_2) = x_\ell w_1$. We have shown that $w_1 \in G(\mathrm{HS}_j(I))$ has $|\mathrm{supp}(w_1)| \geq j+1$. Since $\ell \notin \mathrm{supp}(w_1)$, $|\mathrm{supp}(y)| = 1 + |\mathrm{supp}(w_1)| \geq j+2 > j+1$, as desired. □

Proposition 6.1.4 *Let $I \subset S$ be a matroidal ideal. Then $\mathrm{HS}_{j+1}(I) = \mathrm{HS}_1(\mathrm{HS}_j(I))$ for all $j < \mathrm{pd}(I)$.*

Proof. By Corollary 5.2.2 all homological shift ideals involved in the proof are squarefree. Fix the lexicographic order $>_{\mathrm{lex}}$ induced by $x_1 > x_2 > \cdots > x_n$. Then I has linear quotients with respect to $>_{\mathrm{lex}}$, [14, Theorem 2.4]. For $u \in G(I)$, we denote by $\mathrm{set}(u)$ the following set $\{i : x_i \in (v \in G(I) : v >_{\mathrm{lex}} u) : u\}$. By equation (5.1),

$$\mathrm{HS}_j(I) \ = \ (\mathbf{x}_F u \ : \ u \in G(I), \ F \subseteq \mathrm{set}(u), \ |F| = j). \tag{6.2}$$

For $j = 0$, there is nothing to prove. Let $1 \leq j < \mathrm{pd}(I)$. By Proposition 5.2.6 we have $\mathrm{HS}_{j+1}(I) \subseteq \mathrm{HS}_1(\mathrm{HS}_j(I))$. So we only need to prove that $\mathrm{HS}_1(\mathrm{HS}_j(I)) \subseteq \mathrm{HS}_{j+1}(I)$.

By Proposition 5.2.4, we have

$$\mathrm{HS}_1(\mathrm{HS}_j(I)) \;=\; (\mathrm{lcm}(w_1,w_2)\;:\;w_1,w_2\in G(\mathrm{HS}_j(I)),\ d(w_1,w_2)=1).$$

Thus, we must show that for all $w_1,w_2 \in G(\mathrm{HS}_j(I))$ with $d(w_1,w_2)=1$ we have $\mathrm{lcm}(w_1,w_2)\in G(\mathrm{HS}_{j+1}(I))$. By equation (6.2), $w_1=\mathbf{x}_Fu\neq w_2=\mathbf{x}_Gv$ with $u,v\in G(I)$, $F\subseteq \mathrm{set}(u)$, $G\subseteq\mathrm{set}(v)$ and $|F|=|G|=j$. Since $d(w_1,w_2)=1$, Lemma 5.2.3 gives $w_1=x_k(w_2/x_\ell)$ for some $k\neq\ell$. As observed before, w_1,w_2 are squarefree. Hence, $\mathrm{supp}(w_1)=\{k\}\cup(\mathrm{supp}(w_2)\setminus\{\ell\})$. Note that as $\ell\in\mathrm{supp}(w_2)$, we can find $z\in G(I)$ such that $\ell\in\mathrm{supp}(z)$. Indeed if $\ell\in\mathrm{supp}(v)$ then we can choose $z=v$. Otherwise, $\ell\in G\subseteq\mathrm{set}(v)$, and $x_\ell v=x_sz$ with $z\in G(I)\setminus\{v\}$, $s\neq\ell$ and so $\ell\in\mathrm{supp}(z)$.

Since $\ell\in\mathrm{supp}(z)\setminus\mathrm{supp}(u)$ it is $\deg_{x_\ell}(z)>\deg_{x_\ell}(u)$. By the dual exchange property $(**)$, the set of integers h such that $\deg_{x_h}(u)>\deg_{x_h}(z)$ and $x_\ell(u/x_h)\in G(I)$ is non empty. Hence, the following set is non empty too

$$\Omega\;=\;\{h\in[n]\setminus\{\ell\}\;:\;x_\ell(u/x_h)\in G(I)\}.$$

Case 1. Assume there exists $h\in\Omega$ with $h>\ell$. Then $x_\ell(u/x_h)>_{\mathrm{lex}}u$ and $\ell\in\mathrm{set}(u)$. Now $\mathrm{lcm}(w_1,w_2)=\mathrm{lcm}(w_1,x_\ell(w_1/x_k))=x_\ell w_1=x_\ell\mathbf{x}_Fu$. Since $\ell\notin\mathrm{supp}(w_1)$, we also have $\ell\notin F$. Since $\ell\in\mathrm{set}(u)$, we have that $F\cup\{\ell\}$ is a subset of $\mathrm{set}(u)$ having cardinality $j+1$ and $\mathrm{lcm}(w_1,w_2)=\mathbf{x}_{F\cup\{\ell\}}u\in G(\mathrm{HS}_{j+1}(I))$ by equation (5.1), as desired.

Case 2. Assume that for all $h\in\Omega$ we have $h<\ell$. Choose some $h\in\Omega$. Then, $u>_{\mathrm{lex}}x_\ell(u/x_h)\in G(I)$. Set $w=x_\ell(u/x_h)$. Hence, this time $h\in\mathrm{set}(w)$. Note that $h\in\mathrm{supp}(u)$ and since $w_1=\mathbf{x}_Fu$ is squarefree, $h\notin F$. We are going to show that $F\subseteq\mathrm{set}(w)$. Hence, we will have $F\cup\{h\}\subseteq\mathrm{set}(w)$ and then the desired conclusion:

$$\mathrm{lcm}(w_1,w_2)=x_\ell\mathbf{x}_Fu=x_h\mathbf{x}_F(x_\ell(u/x_h))=\mathbf{x}_{F\cup\{h\}}w\in G(\mathrm{HS}_{j+1}(I)).$$

Let $m\in F$, then for some $p\neq m$, $x_m(u/x_p)\in G(I)$ and $x_m(u/x_p)>_{\mathrm{lex}}u$. So $m<p$.

Subcase 2.1. Let $p=h$. Then $\ell>h=p>m$. Hence $x_m(u/x_p)=x_m(u/x_h)>_{\mathrm{lex}}x_\ell(u/x_h)=w$. Whence, $m\in\mathrm{set}(w)$ in this case, as desired.

Subcase 2.2. Let $p\neq h$. Then $d(x_m(u/x_p),w)=d(x_m(u/x_p),x_\ell(u/x_h))=2$, $h\neq m$, $h\neq\ell$, $\ell\neq m$, $p\neq h$, $p\neq m$, and

$$\begin{aligned}
&\deg_{x_h}(w)<\deg_{x_h}(x_m(u/x_p)), &&\deg_{x_\ell}(w)>\deg_{x_\ell}(x_m(u/x_p)),\\
&\deg_{x_m}(w)<\deg_{x_m}(x_m(u/x_p)), &&\deg_{x_p}(w)>\deg_{x_p}(x_m(u/x_p)).
\end{aligned}$$

Whereas, for all $q\neq h,\ell,m,p$ we have $\deg_{x_q}(w)=\deg_{x_q}(x_m(u/x_p))$. Since $x_m(u/x_p)$, $w\in G(I)$ and $\deg_{x_m}(x_m(u/x_p))>\deg_{x_m}(w)$, by the dual exchange property $(**)$ we have either $x_m(w/x_\ell)\in G(I)$ or $x_m(w/x_p)\in G(I)$.

Subcase 2.2.1. Assume $x_m(w/x_\ell)\in G(I)$. Note that

$$x_m(w/x_\ell)=x_m(x_\ell(u/x_h))/x_\ell=x_m(u/x_h)\in G(I).$$

Thus $m\in\Omega$ and by assumption $m<\ell$. Hence $x_m(w/x_\ell)>_{\mathrm{lex}}w$ and so $m\in\mathrm{set}(w)$.

Subcase 2.2.2. Assume $x_m(w/x_p)\in G(I)$. In this case, since $m<p$ we have $x_m(w/x_p)>_{\mathrm{lex}}w$, and again $m\in\mathrm{set}(w)$. The proof is complete $\square$

Proposition 6.1.4 and Theorem 6.1.2 yield another proof of Conjecture 6.1.1 for matroidal ideals, one was already obtained in [16, Theorem 2.2].

Corollary 6.1.5 (Bayati, 2019 [16, Theorem 2.2]). *Let $I \subset S$ be a matroidal ideal. Then $\mathrm{HS}_j(I)$ is again a matroidal ideal for all j.*

Unfortunately, the inclusion $\mathrm{HS}_{j+1}(I) \subseteq (\mathrm{HS}_1(\mathrm{HS}_j(I)))_{>j+1}$ could be strict for an arbitrary polymatroidal ideal I, as we show in next Example 6.1.7.

Recall that a polymatroidal ideal I satisfy the *strong exchange property* if

(†) for all $u, v \in G(I)$, all i such that $\deg_{x_i}(u) > \deg_{x_i}(v)$ and all j such that $\deg_{x_j}(u) < \deg_{x_j}(v)$ we have $x_j(u/x_i) \in G(I)$.

A monomial ideal I is called of *Veronese type* if there exist an integer d and an integral vector $\mathbf{b} = (b_1, \dots, b_n)$ with non negative entries such that

$$G(I) \;=\; \big\{\mathbf{x}^{\mathbf{c}} = x_1^{c_1}\cdots x_n^{c_n} \;:\; \sum_{i=1}^{n} c_i = d \text{ and } c_i \le b_i \text{ for } i \in [n]\big\}.$$

In this case, we denote the ideal I by $I_{\mathbf{b},n,d}$. It is known that a polymatroidal ideal I satisfies the strong exchange property (†) if and only if I is of Veronese type up to a multiplication by a monomial, [93, Theorem 1.1]. That means that $I = \mathbf{x}^{\mathbf{a}} I_{\mathbf{b},n,d}$ for some monomial $\mathbf{x}^{\mathbf{a}}$. It was proved in [101, Theorem 3.3.(a)] that

$$\mathrm{HS}_\ell(I_{\mathbf{b},n,d}) \;=\; (I_{\mathbf{b},n,d+\ell})_{>\ell}.$$

The ideal $(I_{\mathbf{b},n,d+\ell})_{>\ell}$ is polymatroidal by [101, Proposition 3.4]. Let I be a polymatroidal ideal satisfying the strong exchange property. Then $I = \mathbf{x}^{\mathbf{a}} I_{\mathbf{b},n,d}$. If

$$\mathbb{F} : \cdots \to \bigoplus_{j=1}^{\beta_i(I)} S(-\mathbf{a}_{i,j}) \to \bigoplus_{j=1}^{\beta_{i-1}(I)} S(-\mathbf{a}_{i-1,j}) \to \cdots \to I_{\mathbf{b},n,d} \to 0$$

is the minimal multigraded free resolution of $I_{\mathbf{b},n,d}$, then the minimal multigraded free resolution of $\mathbf{x}^{\mathbf{a}} I_{\mathbf{b},n,d} = I$ is

$$\mathbf{x}^{\mathbf{a}}\mathbb{F} \;:\; \cdots \to \bigoplus_{j=1}^{\beta_i(I)} S(-\mathbf{a}_{i,j} - \mathbf{a}) \to \bigoplus_{j=1}^{\beta_{i-1}(I)} S(-\mathbf{a}_{i-1,j} - \mathbf{a}) \to \cdots \to \mathbf{x}^{\mathbf{a}} I_{\mathbf{b},n,d} \to 0.$$

Thus $\mathrm{HS}_j(I) = \mathrm{HS}_j(\mathbf{x}^{\mathbf{a}} I_{\mathbf{b},n,d}) = \mathbf{x}^{\mathbf{a}}\mathrm{HS}_j(I_{\mathbf{b},n,d}) = (\mathbf{x}^{\mathbf{a}})(I_{\mathbf{b},n,d+\ell})_{>\ell}$ is a polymatroidal ideal for all $j \ge 0$, for it is a product of two polymatroidal ideals. By this discussion, we have the following consequence.

Corollary 6.1.6 [101, Corollary 3.6] *Let $I \subset S$ be a polymatroidal ideal satisfying the strong exchange property. Then $\mathrm{HS}_j(I)$ is again polymatroidal, for all j.*

Now, we are ready for our example.

Example 6.1.7 Let $I \subset S = K[x_1, \dots, x_5]$ be the polymatroidal ideal

$$\begin{aligned} I &= (x_1, x_2, x_3, x_4)(x_3, x_4, x_5) \\ &= (x_1x_3, x_1x_4, x_1x_5, x_2x_3, x_2x_4, x_2x_5, x_3^2, x_3x_4, x_3x_5, x_4^2, x_4x_5), \end{aligned}$$

where the minimal generators of I are sorted according to the *lexicographic order* $>_{\mathrm{lex}}$ induced by $x_1 > x_2 > x_3 > x_4 > x_5$. Using *Macaulay2*, [82], we have collected in the next tables the linear quotients of I,

	x_1x_3	x_1x_4	x_1x_5	x_2x_3	x_2x_4	x_2x_5	x_3^2
set	$\varnothing$	$\{3\}$	$\{3,4\}$	$\{1\}$	$\{1,3\}$	$\{1,3,4\}$	$\{1,2\}$

	x_3x_4	x_3x_5	x_4^2	x_4x_5
set	$\{1,2,3\}$	$\{1,2,3,4\}$	$\{1,2,3\}$	$\{1,2,3,4\}$

Thus, by equation (5.1) we have

$$\begin{aligned}
\mathrm{HS}_0(I) &= I = (x_1x_3,\ x_1x_4,\ x_1x_5,\ x_2x_3,\ x_2x_4,\ x_2x_5,\ x_3^2,\ x_3x_4,\ x_3x_5,\ x_4^2,\ x_4x_5),\\
\mathrm{HS}_1(I) &= (x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_3^2, x_1x_3x_4, x_1x_3x_5, x_1x_4^2, x_1x_4x_5, x_2x_3^2,\\
&\quad x_2x_3x_4, x_2x_3x_5, x_2x_4^2, x_2x_4x_5, x_3^2x_4, x_3^2x_5, x_3x_4^2, x_3x_4x_5, x_4^2x_5),\\
\mathrm{HS}_2(I) &= (x_1x_2x_3^2, x_1x_2x_3x_4, x_1x_2x_3x_5, x_1x_2x_4^2, x_1x_2x_4x_5, x_1x_3^2x_4,\\
&\quad x_1x_3^2x_5, x_1x_3x_4^2, x_1x_3x_4x_5, x_1x_4^2x_5, x_2x_3^2x_4, x_2x_3^2x_5,\\
&\quad x_2x_3x_4^2, x_2x_3x_4x_5, x_2x_4^2x_5, x_3^2x_4x_5, x_3x_4^2x_5),\\
\mathrm{HS}_3(I) &= (x_1x_2x_3^2x_4, x_1x_2x_3^2x_5, x_1x_2x_3x_4x_5, x_1x_2x_4^2x_5,\\
&\quad x_1x_3^2x_4x_5, x_1x_3x_4^2x_5, x_2x_3^2x_4x_5, x_2x_3x_4^2x_5),\\
\mathrm{HS}_4(I) &= (x_1x_2x_3^2x_4x_5, x_1x_2x_3x_4^2x_5),\\
\mathrm{HS}_j(I) &= (0),\quad j\ge 5.
\end{aligned}$$

Each of these ideals is polymatroidal, so they have linear quotients. Indeed, $\mathrm{HS}_0(I) = I$ is polymatroidal since I is and $\mathrm{HS}_1(I)$ is polymatroidal too by Theorem 6.1.2. For the other three ideals, note that $\mathrm{HS}_2(I) = (I_{(1,1,2,2,1),5,4})_{>2}$, $\mathrm{HS}_3(I) = (I_{(1,1,2,2,1),5,5})_{>3}$ and $\mathrm{HS}_4(I) = (I_{(1,1,2,2,1),5,6})_{>4}$ are all polymatroidal by [101, Proposition 3.4]. Note that for all $j = 1,2,3,4 = \mathrm{pd}(I)$ we have $\mathrm{HS}_{j+1}(I) = (\mathrm{HS}_1(\mathrm{HS}_j(I)))_{>j+1}$. However, for all $j = 1,2,3,4$ we have $\mathrm{HS}_{j+1}(I) \neq \mathrm{HS}_1(\mathrm{HS}_j(I))$. Indeed,

for $j = 1$, note that $x_3^2x_4, x_3x_4^2 \in G(\mathrm{HS}_1(I))$ and $d(x_3^2x_4, x_3x_4^2) = 1$. Proposition 5.2.4 implies that $\mathrm{lcm}(x_3^2x_4, x_3x_4^2) = x_3^2x_4^2 \in G(\mathrm{HS}_1(\mathrm{HS}_1(I)))$. However this monomial does not belong to $\mathrm{HS}_2(I)$. Thus $\mathrm{HS}_2(I) \neq \mathrm{HS}_1(\mathrm{HS}_1(I))$;

for $j = 2$, the monomials $x_3^2x_4x_5, x_3x_4^2x_5 \in G(\mathrm{HS}_2(I))$ have distance equal to 1, but $\mathrm{lcm}(x_3^2x_4x_5, x_3x_4^2x_5) = x_3^2x_4^2x_5 \in G(\mathrm{HS}_1(\mathrm{HS}_2(I))) \setminus G(\mathrm{HS}_3(I))$;

for $j = 3$, consider $x_2x_3^2x_4x_5, x_2x_3x_4^2x_5 \in G(\mathrm{HS}_3(I))$ whose distance is 1. But their $\mathrm{lcm}(x_2x_3^2x_4x_5, x_2x_3x_4^2x_5) = x_2x_3^2x_4^2x_5 \in G(\mathrm{HS}_1(\mathrm{HS}_3(I))) \setminus G(\mathrm{HS}_4(I))$;

for $j = 4$, we have $\mathrm{HS}_1(\mathrm{HS}_4(I)) = (x_1x_2x_3^2x_4^2x_5) \neq (0) = \mathrm{HS}_5(I)$.

Let $J = x_1x_2x_3x_4x_5I$. We verify that $\mathrm{HS}_{j+1}(J) \neq (\mathrm{HS}_1(\mathrm{HS}_j(J)))_{>j+1}$ for all $j = 1,2,3,4$. Note that

$$\mathrm{HS}_j(J) = x_1x_2x_3x_4x_5\mathrm{HS}_j(I) = \mathbf{x}_{[5]}\mathrm{HS}_j(I)$$

for all j. Furthermore, $(\mathrm{HS}_1(\mathrm{HS}_j(J)))_{>j+1} = \mathrm{HS}_1(\mathrm{HS}_j(J))$, for all j, because all monomials in J have support $[5]$. For $j = 1$, by what was shown above, we have that $\mathbf{x}_{[5]}x_3^2x_4^2 \in G(\mathrm{HS}_1(\mathrm{HS}_1(J))) \setminus G(\mathrm{HS}_2(J))$, hence $\mathrm{HS}_2(J) \neq (\mathrm{HS}_1(\mathrm{HS}_1(J)))_{>2}$. One can proceed similarly for $j = 2,3,4$.

We would like to point out that I does not satisfy the strong exchange property. Indeed, $u = x_1x_3, v = x_2x_4 \in G(I)$, $\deg_{x_3}(u) > \deg_{x_3}(v)$, $\deg_{x_2}(u) < \deg_{x_2}(v)$, but $x_2(u/x_3) = x_1x_2$ does not belong to $G(I)$.

6.2 Classes of polymatroidal ideals with maximal projective dimension

In view of Corollary 5.3.2, a positive answer to Conjecture 6.1.1 would imply a positive answer to the following conjecture mentioned for the first time in [13, page 760] and also studied in the article [27].

Conjecture 6.2.1 (Bandari–Herzog, Chu–Herzog–Lu), [13, 27]. *Let $I \subset S$ be a polymatroidal ideal. Then $\mathrm{soc}(I)$ is polymatroidal.*

Proposition 6.2.2 *Conjecture 6.2.1 holds if I is generated in degree two or $n \le 3$.*

Proof. If I is generated in degree two, then $\mathrm{soc}(I)$ is generated in degree at most one and it is polymatroidal. If $n = 1$ there is nothing to prove. If $n = 2$, $\mathrm{HS}_1(I) = x_1x_2 \cdot \mathrm{soc}(I)$ is polymatroidal, (Theorem 6.1.2). Finally, if $n = 3$, then I satisfy the strong exchange property [13, Proposition 2.7], and the statement follows from Corollary 6.1.6. □

If $\mathrm{pd}(I) < n-1$, then $\mathrm{soc}(I) = (0)$ and Conjecture 6.2.1 is trivially verified. So one only need to consider polymatroidal ideals with maximal projective dimension. To the best of our knowledge no classification of polymatroidal ideals with maximal projective dimension is known. However we have the following persistence property.

Proposition 6.2.3 *Let $I \subset S$ be a polymatroidal ideal with maximal projective dimension. Then I^k has maximal projective dimension, for all $k \ge 1$.*

Proof. The product of polymatroidal ideals is a polymatroidal ideal. Hence I^k is polymatroidal and it has linear quotients with respect to the lexicographic order $>_{\mathrm{lex}}$ induced by $x_1 > \cdots > x_n$, for all k. Theorem 5.3.3 guarantees that I^k has maximal projective dimension if and only if $G(\mathrm{soc}(I^k)) \neq \varnothing$. By hypothesis there exists $u \in G(I)$ with $u/x_n \in G(\mathrm{soc}(I))$. We claim that $u^k/x_n \in G(\mathrm{soc}(I^k))$. It suffices to show that $x_i(u^k/x_n) \in G(I^k)$, for all $i = 1,\dots,n$. But $x_i(u/x_n) \in G(I)$ for all i. Hence $x_i(u^k/x_n) = u^{k-1}(x_i(u/x_n)) \in I^{k-1}I = I^k$ is a minimal generator of $G(I^k)$. □

Next, we consider various classes of polymatroidal ideals and classify those with maximal projective dimension. In particular, we show that Conjecture 6.2.1 is true for the following classes of polymatroidal ideals: matroidal ideals, principal Borel ideals, PLP–polymatroidal ideals and a particular class of transversal polymatroidal ideals that coincides with the class of LP–polymatroidal ideals.

6.2.1 Matroidal ideals

For a matroidal ideal I, to decide when I has maximal projective dimension is rather trivial. Indeed, more generally we even have,

Proposition 6.2.4 *Let $I \subset S$ be an equigenerated squarefree monomial ideal with linear quotients with respect to a monomial order $\succ$ induced by $x_1 > x_2 > \cdots > x_n$ and such that $\mathrm{supp}(I) = [n]$. Then I has maximal projective dimension if and only if $I = \mathfrak{m} = (x_1,\dots,x_n)$. In such a case $\mathrm{HS}_{n-1}(\mathfrak{m}) = (x_1x_2\cdots x_n)$ and $\mathrm{soc}(\mathfrak{m}) = (1)$.*

Proof. By Theorem 5.3.3, I has maximal projective dimension if and only if there exists a monomial u such that $x_i(u/x_n) \in G(I)$ for all $i = 1,\dots,n$. Since I is squarefree, we must have $i \notin \mathrm{supp}(u)$ for $i = 1,\dots,n-1$. Hence $u = x_n \in G(I)$ and $x_i \in G(I)$ for $i = 1,\dots,n-1$ as well. Finally, $I = \mathfrak{m} = (x_1,\dots,x_n)$. □

Corollary 6.2.5 *Let $I \subset S$ be a matroidal ideal. Then I has maximal projective dimension if and only if $I = \mathfrak{m}$. Moreover, $\mathrm{soc}(I)$ is a matroidal ideal.*

6.2.2 Principal Borel ideals

Another interesting class of polymatroidal ideals is the class of *principal Borel ideals*. Recall that an ideal $I \subset S$ is *strongly stable* if for all $u \in G(I)$ and all $1 \leq j < i \leq n$ with $i \in \operatorname{supp}(u)$ it follows that $x_j(u/x_i) \in I$. If $u_1, \ldots, u_m$ are monomials of S, $B(u_1, \ldots, u_m)$ denotes the unique smallest strongly stable monomial ideal of S containing $u_1, \ldots, u_m$. It is clear that for any strongly stable monomial ideal I there exist unique monomials $u_1, \ldots, u_m$ such that $I = B(u_1, \ldots, u_m)$. These monomials are called the *Borel generators* of I. If $m = 1$, then $I = B(u)$ is called a *principal Borel ideal*. Note that if $u = x_{i_1}x_{i_2}\cdots x_{i_d}$, $1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq n$, then $v = x_{j_1}x_{j_2}\cdots x_{j_d} \in B(u)$ if and only if $i_1 \leq j_1,\ i_2 \leq j_2,\ \ldots,\ i_d \leq j_d$. It is known that any principal Borel ideal is a polymatroidal ideal.

Thanks to the *Eliahou–Kervaire formula* for a (strongly) stable ideal $I \subset S$, [50],

$$\beta_{i,i+j}(I) \;=\; \sum_{u \in G(I)_j} \binom{\max(u) - 1}{j},$$

we obtain the following corollary.

Corollary 6.2.6 *Let $I \subset S$ be a strongly stable ideal. Then I has maximal projective dimension if and only if there exists a monomial $u \in G(I)$ with $\max(u) = n$.*

Note that an equigenerated strongly stable ideal has a linear resolution. So the notion of socle ideal for such an ideal makes sense.

Proposition 6.2.7 *Let $I = B(u_1, \ldots, u_m) \subset S = K[x_1, \ldots, x_n]$ be an equigenerated strongly stable ideal. Then, $\operatorname{soc}(I) = B(u_j/x_n : \max(u_j) = n, j = 1, \ldots, m)$, and*

$$\mathrm{HS}_{n-1}(I) \;=\; x_1x_2\cdots x_n \cdot B(u_j/x_n : \max(u_j) = n, j = 1, \ldots, m).$$

Proof. Theorem 5.3.3 and Corollary 6.2.6 imply $\operatorname{soc}(I) = (v/x_n : v \in G(I), \max(v) = n)$. We prove that $\operatorname{soc}(I)$ is a strongly stable ideal whose Borel generators are the monomials u_j such that $\max(u_j) = n$, $j = 1, \ldots, m$. Indeed let $w = v/x_n \in \operatorname{soc}(I)$, and let $1 \leq j \leq i \leq \max(w) < n$ such that $i \in \operatorname{supp}(w)$. Then $v \in G(I)$, $i \in \operatorname{supp}(v)$ and $i < n$. Since I is strongly stable and equigenerated, $\widetilde{v} = x_j(v/x_i) \in G(I)$. Moreover, $\max(\widetilde{v}) = n$ since $i < n$. Hence, $\widetilde{v}/x_n = x_j(v/x_i)/x_n = x_j(w/x_i) \in \operatorname{soc}(I)$, as desired. Finally, it is clear that the monomials u_j/x_n such that $\max(u_j) = n$ are the Borel generators of $\operatorname{soc}(I)$, and all our statements follow. □

Any strongly stable ideal has linear quotients with respect to the lexicographic order induced by $x_1 > x_2 > \cdots > x_n$. So the previous proposition implies immediately

Corollary 6.2.8 *Let $I \subset S$ be an equigenerated strongly stable ideal having maximal projective dimension. Then $\mathrm{HS}_{n-1}(I)$ has linear quotients.*

Corollary 6.2.9 *Let $I = B(u) \subset S$ be a principal Borel ideal having maximal projective dimension. Then $\mathrm{HS}_{n-1}(I)$ is again a principal Borel ideal and in particular it is polymatroidal.*

Now we discuss the socle of powers of principal Borel ideals.

Proposition 6.2.10 *Let $I = B(u)$ be a principal Borel ideal. Then, for all $k \geq 1$, $I^k = (B(u))^k = B(u^k)$. Moreover, if I has maximal projective dimension, for all $k \geq 1$,*

$$\operatorname{soc}((B(u))^k) = B(u^k/x_n).$$

Proof. The inclusion $I^k \subset B(u^k)$ is clear. Conversely, let $u = x_{i_1}x_{i_2}\cdots x_{i_d}$ with $1 \le i_1 \le i_2 \le \cdots \le i_d \le n$. Let $w = x_{j_1}x_{j_2}\cdots x_{j_{kd}} \in B(u^k)$ with $1 \le j_1 \le j_2 \le \cdots \le j_{kd} \le n$. By definition of $B(u^k)$ it follows that

$$\begin{aligned} j_1,\ j_2,\ \dots,\ j_d &\le i_1, \\ j_{d+1},\ j_{d+2},\ \dots,\ j_{2d} &\le i_2, \\ &\vdots \\ j_{(k-1)d+1},\ j_{(k-1)d+2},\ \dots,\ j_{kd} &\le i_d. \end{aligned}$$

Define $w_\ell = x_{j_\ell}x_{j_{\ell+d}}\cdots x_{j_{\ell+(k-1)d}}$, for $\ell = 1,\dots,d$. Then $w_\ell \in I = B(u)$ for all ℓ, and $w = w_1w_2\cdots w_d \in I^k = (B(u))^k$ proving the inclusion $B(u^k) \subseteq (B(u))^k$. $\square$

6.2.3 Pruned path lattice polymatroidal ideals

In this Subsection we consider a special class of polymatroidal ideals introduced by Schweig in [142] and by Lu in [122].

Hereafter, given integral vectors $\mathbf{a} = (a_1,\dots,a_n)$ and $\mathbf{b} = (b_1,\dots,b_n)$ of non negative integers, when we write $\mathbf{a} \le \mathbf{b}$ we mean that $a_i \le b_i$ for all $i = 1,\dots,n$. We say that a monomial $u = \mathbf{x^c}$ is $\mathbf{b}$-*bounded* if $\mathbf{c} \le \mathbf{b}$. We denote the null vector $(0,0,\dots,0)$ by $\mathbf{0}$ and the unit vector $(1,1,\dots,1)$ by $\mathbf{1}$.

Let $\mathbf{a}$, $\mathbf{b}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ be four integral vectors of non negative integers such that $\mathbf{a} \le \mathbf{b}$, $\boldsymbol{\alpha} \le \boldsymbol{\beta}$, $\alpha_1 \le \cdots \le \alpha_n = d$, $\beta_1 \le \cdots \le \beta_n = d$ with $d \ge 1$. The *pruned path lattice polymatroidal ideal* or simply the *PLP–polymatroidal ideal* of type $(\mathbf{a},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})$ is the following polymatroidal ideal of $S = K[x_1,\dots,x_n]$, [122, 142],

$$I_{(\mathbf{a},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})} \ = \ (\mathbf{x^c} \ : \ \mathbf{a} \le \mathbf{c} \le \mathbf{b},\ \alpha_i \le c_1 + \cdots + c_i \le \beta_i,\ i \in [n]).$$

If $\mathbf{a} = \mathbf{0}$, $I_{(\mathbf{0},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})}$ is called a *basic* PLP–polymatroidal ideal. Any PLP–polymatroidal is the product of a basic PLP–polymatroidal and a monomial, [122, Remark 4.1],

$$I_{(\mathbf{a},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})} = \mathbf{x^a} I_{(\mathbf{0},\mathbf{b}-\mathbf{a}|\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)}.$$

Thus we restrict our attention on basic PLP–polymatroidal ideals of type $(\mathbf{0},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})$.

Examples 6.2.11 (a) If $\mathbf{a} = \mathbf{0}$ and $\mathbf{b} \ge d\mathbf{1} = (d,d,\dots,d)$, then $I = I_{(\mathbf{a},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})}$ is a *lattice path polymatroidal* or simply a *LP–polymatroidal ideal* [122].

(b) If $\boldsymbol{\alpha} = \mathbf{0}$ and $\boldsymbol{\beta} = d\mathbf{1} = (d,d,\dots,d)$, then $I_{(\mathbf{a},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})} = \mathbf{x^a} I_{\mathbf{b}-\mathbf{a},n,d}$ is an ideal that satisfy the strong exchange property (†).

(c) The ideal $I = (x_1,x_2,x_3,x_4)(x_3,x_4,x_5) \subset K[x_1,\dots,x_5]$ of Example 6.1.7 is a PLP–polymatroidal ideal of type $\big(\mathbf{0},(1,1,2,2,1)\big|(0,0,0,1,2),(1,1,2,2,2)\big)$. As noted in Example 6.1.7, I does not satisfy the strong exchange property. Hence not all PLP–polymatroidal ideals satisfy the strong exchange property.

The following result is essentially due to Lu.

Lemma 6.2.12 (Lu, 2017 [122, Lemma 4.2(b)]). *Let $I_{(\mathbf{0},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})} \subset S$ be a PLP–polymatroidal ideal and suppose that $G(I_{(\mathbf{0},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})})$ is ordered with respect to the lexicographic order $>_{\text{lex}}$ induced by $x_1 > x_2 > \cdots > x_n$. Then, for $u \in G(I_{(\mathbf{0},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})})$,*

$$\operatorname{set}(u) \;=\; \big\{i \in [\max(u)-1] : \deg_{x_i}(u) < b_i,\; \sum_{k=1}^{i} \deg_{x_k}(u) < \beta_i\big\}.$$

Proof. Let $u \in G(I_{(\mathbf{0},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})})$ and $i \in \operatorname{set}(u)$. By [122, Lemma 4.2(b)] it follows that $i \in \operatorname{set}(u)$ if and only if $\deg_{x_i}(u) < b_i$ and $\sum_{k=1}^{i} \deg_{x_k}(u) < \beta_i$. Whereas, by Lemma 5.2.5 it follows that $\operatorname{set}(u) \subseteq [\max(u)-1]$. $\square$

As a consequence we recover [27, Proposition 3.3] due to Chu, Herzog and Lu. Let $\boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\varepsilon}_n$ be the standard basis vectors of $\mathbb{Q}^n$, *i.e.*, $\boldsymbol{\varepsilon}_i$ has all component zeros except for the ith component which is equal to 1. Note that $\mathbf{1} = \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2 + \cdots + \boldsymbol{\varepsilon}_n$.

Corollary 6.2.13 *Let $I_{(\mathbf{0},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})} \subset S$ be a PLP–polymatroidal ideal. Then*

$$\operatorname{soc}(I_{(\mathbf{0},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})}) \;=\; I_{(\mathbf{0},\mathbf{b}-\mathbf{1}|\boldsymbol{\alpha}-\boldsymbol{\varepsilon}_n,\boldsymbol{\beta}-\mathbf{1})}.$$

Proof. By Theorem 5.3.3, $G(\operatorname{soc}(I)) = \{u/x_n : x_i(u/x_n) \in G(I), i = 1, \dots, n\}$. Let $u \in G(I)$ such that $u/x_n \in G(\operatorname{soc}(I))$. Since $x_i(u/x_n) \in G(I)$ for all i, we see that $\operatorname{set}(u) = [n-1]$, where we consider that I has linear quotients with respect to $>_{\text{lex}}$ induced by $x_1 > x_2 > \cdots > x_n$. By Lemma 6.2.12, we have for all $i < n$, $\deg_{x_i}(u/x_n) = \deg_{x_i}(u) < b_i$ and $\sum_{k=1}^{i} \deg_{x_k}(u/x_n) = \sum_{k=1}^{i} \deg_{x_k}(u) < \beta_i$. Since $\deg_{x_n}(u/x_n) < \deg_{x_n}(u) \le b_n$ too, we see that u is $\mathbf{b}-\mathbf{1}$-bounded. Finally, for $i = n$, $\sum_{k=1}^{i} \deg_{x_k}(u/x_n) = \deg(u) - 1 = \alpha_n - 1 = \beta_n - 1 = d-1$. Therefore

$$\operatorname{soc}(I_{(\mathbf{0},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})}) \;\subseteq\; I_{(\mathbf{0},\mathbf{b}-\mathbf{1}|\boldsymbol{\alpha}-\boldsymbol{\varepsilon}_n,\boldsymbol{\beta}-\mathbf{1})}.$$

The opposite inclusion is trivial and the proof is complete. $\square$

Corollary 6.2.14 *Let $I_{(\mathbf{0},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})} \subset S$ be a PLP–polymatroidal ideal. Then the last homological shift ideal $\mathrm{HS}_{n-1}(I_{(\mathbf{0},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})})$ is again a PLP–polymatroidal ideal.*

Proof. Indeed, $\mathrm{HS}_{n-1}(I_{(\mathbf{0},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})}) = \mathbf{x}_{[n]} \cdot \operatorname{soc}(I_{(\mathbf{0},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})}) = \mathbf{x}_{[n]} \cdot I_{(\mathbf{0},\mathbf{b}-\mathbf{1}|\boldsymbol{\alpha}-\boldsymbol{\varepsilon}_n,\boldsymbol{\beta}-\mathbf{1})}$ is the product of a monomial and a basic PLP–polymatroidal ideal. $\square$

We can classify all PLP–polymatroidal ideals with maximal projective dimension.

Proposition 6.2.15 *Let $I_{(\mathbf{0},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})} \subset S$ be a PLP–polymatroidal ideal. Then $I_{(\mathbf{0},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})}$ has maximal projective dimension if and only if*

$$\begin{aligned} \beta_i + b_{i+1} + \cdots + b_j &\ \ge\ \alpha_j + (j+1-i), && \textit{for all } 1 \le i \le j < n, \textit{ and} \\ \beta_i + b_{i+1} + \cdots + b_n &\ \ge\ \alpha_n + (n-i), && \textit{for all } 1 \le i \le n. \end{aligned}$$

Proof. Let $\mathbf{b}^* = \mathbf{b} - \mathbf{1}$, $\boldsymbol{\alpha}^* = \boldsymbol{\alpha} - \boldsymbol{\varepsilon}_n$ and $\boldsymbol{\beta}^* = \boldsymbol{\beta} - \mathbf{1}$. I has maximal projective dimension if and only if $\operatorname{soc}(I_{(\mathbf{0},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})}) = I_{(\mathbf{0},\mathbf{b}^*|\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)}$ is non zero. By [27, Lemma 3.1] due to Chu, Herzog and Lu, $I_{(\mathbf{0},\mathbf{b}^*|\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)}$ is non zero if and only if

$$(\boldsymbol{\beta}^*)[i] + \sum_{k=i+1}^{j} (\mathbf{b}^*)[k] \;\ge\; (\boldsymbol{\alpha}^*)[j], \quad \text{for all } 1 \le i \le j \le n,$$

where $(\mathbf{c})[i]$ stands for c_i the ith component of the vector $\mathbf{c} = (c_1, c_2, \dots, c_n) \in \mathbb{Z}^n$. By expanding these inequalities we obtain the inequalities in the statement. □

For instance, the basic PLP–polymatroidal of Example 6.2.11(c) has maximal projective dimension. Indeed, the inequalities in the previous proposition are all satisfied. Next, we turn to powers.

Corollary 6.2.16 *Let $I_{(\mathbf{0},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})} \subset S$ be a PLP–polymatroidal ideal having maximal projective dimension. Then, for all $k \geq 1$,*

$$\operatorname{soc}((I_{(\mathbf{0},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})})^k) \;=\; I_{(\mathbf{0},k\mathbf{b}-\mathbf{1}|k\boldsymbol{\alpha}-\boldsymbol{\varepsilon}_n,k\boldsymbol{\beta}-\mathbf{1})}.$$

Proof. By [122, Proposition 2.10], $(I_{(\mathbf{0},\mathbf{b}|\boldsymbol{\alpha},\boldsymbol{\beta})})^k = I_{(\mathbf{0},k\mathbf{b}|k\boldsymbol{\alpha},k\boldsymbol{\beta})}$. So the statement follows at once from Corollary 6.2.13. □

6.2.4 Transversal polymatroidal ideals

Let $A \subseteq [n]$ be non empty. It is clear that $\mathfrak{p}_A = (x_i : i \in A)$ is polymatroidal. A polymatroidal ideal $I \subset S$ is called a *transversal polymatroidal ideal* if $I = \mathfrak{p}_{A_1} \cdots \mathfrak{p}_{A_t}$ for some non empty subsets $A_1, \dots, A_t$ of $[n]$. The ideal of Example 6.1.7 is a transversal PLP–polymatroidal ideal. However, the ideal $(x_1, x_3)(x_2, x_4)$ is a transversal but non PLP–polymatroidal ideal [122, Example 2.5].

Note that for a transversal polymatroidal ideal the condition $\operatorname{supp}(I) = [n]$ is fulfilled if and only if $\bigcup_{i=1}^t A_i = [n]$.

Theorem 6.2.17 *Let $I = \prod_{i=1}^t \mathfrak{p}_{A_i} \subset S$ be a transversal polymatroidal ideal such that $\bigcup_{i=1}^t A_i = [n]$. Then, the following conditions are equivalent.*

(i) *I has maximal projective dimension.*

(ii) *for all pair of integers $1 \leq i < j \leq t$ there exist integers $i = \ell_1, \ell_2, \dots, \ell_{r-1}, \ell_r = j$ such that $A_{\ell_s} \cap A_{\ell_{s+1}} \neq \varnothing$, for all $s = 1, \dots, r-1$.*

Proof. Following [96], we attach to I the following graph, which is called the *intersection graph* of I. Let $G_I = (V, E)$ be the simple graph whose vertex set V is $\{A_i : i \in [t]\}$ and whose edge set E consists of all the unordered pairs $\{k, \ell\}$ such that $A_k \cap A_\ell \neq \varnothing$. By [96, Theorem 4.3], $\mathfrak{m} \in \operatorname{Ass}(I)$ if and only if G_I is connected. Now, $\operatorname{pd}(I) = n - 1$ if and only if $\operatorname{depth}(S/I) = 0$. This is the case, if and only if $\mathfrak{m} \in \operatorname{Ass}(I)$, concluding the proof. □

We recall some concepts of graph theory. It is known that any connected graph G has a *spanning tree* $\mathcal{T}$. That means a collection $\mathcal{T}$ of edges of G such that $\bigcup_{e \in \mathcal{T}} e = V$, here V is the vertex set of G, and such that the graph $T = (V, \mathcal{T})$ is a tree.

The proof of [96, Theorem 4.3] shows that for all spanning trees $\mathcal{T} = \{e_k\}_{k=1,\dots,t-1}$ of G_I, any monomial $x_{\ell_1} x_{\ell_2} \cdots x_{\ell_{t-1}}$ with $\ell_k \in A_{i_k} \cap A_{j_k}$ is such that $I : w = \mathfrak{m}$, *i.e.*, w is in $G(\operatorname{soc}(I))$. Thus, one may wonder if all monomial generators of $G(\operatorname{soc}(I))$ arise in this way. We are led to ask the following question.

Question 6.2.18 *Let $I = \prod_{i=1}^t \mathfrak{p}_{A_i} \subset S$ be a transversal polymatroidal ideal such that I has maximal projective dimension. Is it true that*

$$\begin{aligned}\operatorname{soc}(I) \;=\; (x_{\ell_1} x_{\ell_2} \cdots x_{\ell_{t-1}} \;:\; & \ell_k \in A_{i_k} \cap A_{j_k},\; e_k = \{i_k, j_k\} \in E(G_I),\; k = 1, \dots, t-1, \\ & \textit{and } \mathcal{T} = \{e_k\}_{k=1,\dots,t-1} \textit{ is a spanning tree of } G_I)\ ?\end{aligned}$$

At present we do not know if Question 6.2.18 has a positive answer in general. Therefore, we restrict to a particular subclass of transversal polymatroidal ideals. Given integers $p \leq q$, we denote by $\mathfrak{p}_{[p,q]}$ the monomial prime ideal $(x_i : i \in [p,q])$. Let $\boldsymbol{\alpha} : \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_t$ and $\boldsymbol{\beta} : \beta_1 \leq \beta_2 \leq \cdots \leq \beta_t$ be two non–decreasing sequences of positive integers such that $\alpha_i \leq \beta_i$ for all $i = 1, \dots, t$. We define the transversal polymatroidal ideal

$$I_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \;=\; \mathfrak{p}_{[\alpha_1,\beta_1]} \cdot \mathfrak{p}_{[\alpha_2,\beta_2]} \cdots \mathfrak{p}_{[\alpha_t,\beta_t]}.$$

The class of all such ideals is the class of *LP–polymatroidal ideals*, [122, Lemma 2.1].

Note that if $\operatorname{supp}(I_{(\boldsymbol{\alpha},\boldsymbol{\beta})}) = [n]$ then $\alpha_1 = 1$ and $\beta_t = n$. So, from now on we assume that $\alpha_1 = 1$ and $\beta_t = n$.

Proposition 6.2.19 *Let $I_{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \prod_{i=1}^{t} \mathfrak{p}_{[\alpha_i,\beta_i]} \subset S$ be a LP–polymatroidal ideal with $\alpha_1 = 1$, $\beta_t = n$. Then $I_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ has maximal projective dimension if and only if*

$$\alpha_{i+1} \leq \beta_i, \quad \textit{for all} \;\; i = 1, \dots, t-1.$$

Proof. Let $A_i = [\alpha_i, \beta_i]$, $i = 1, \dots, t$. If $\alpha_{i+1} \leq \beta_i$ for $i = 1, \dots, t-1$, then $A_1 \cap A_2 \neq \varnothing$, $A_2 \cap A_3 \neq \varnothing$, $\dots$, $A_{t-1} \cap A_t \neq \varnothing$. So by Theorem 6.2.17(ii) it follows that $I_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ has maximal projective dimension in such a case, for $G_{I_{(\boldsymbol{\alpha},\boldsymbol{\beta})}}$ is connected.

Conversely, we must have $\alpha_{\ell+1} \leq \beta_\ell$ for $\ell = 1, \dots, t-1$. Indeed, suppose on the contrary that for some $1 \leq \ell \leq t-1$ we have $\alpha_{\ell+1} > \beta_\ell$, then

$$\Big(\bigcup_{k=1}^{\ell} A_k\Big) \cap \Big(\bigcup_{k=\ell+1}^{t} A_k\Big) \;=\; [1, \beta_\ell] \cap [\alpha_{\ell+1}, n] \;=\; \varnothing.$$

Hence the graph $G_{I_{(\boldsymbol{\alpha},\boldsymbol{\beta})}}$ would have two connected components and it would be not connected. This violates condition (ii) of Theorem 6.2.17, a contradiction. $\square$

The proof of the previous proposition points out that if the ideal $I_{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \prod_{i=1}^{t} \mathfrak{p}_{[\alpha_i,\beta_i]}$ has maximal projective dimension, setting $A_i = [\alpha_i, \beta_i]$, then the graph $G_{I_{(\boldsymbol{\alpha},\boldsymbol{\beta})}}$ has edge set $\{\{1,2\}, \{2,3\}, \dots, \{t-1,t\}\}$. Hence, we have the following result.

Corollary 6.2.20 *Let $I_{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \prod_{i=1}^{t} \mathfrak{p}_{[\alpha_i,\beta_i]} \subset S$ be a LP–polymatroidal ideal having maximal projective dimension. Then $G_{I_{(\boldsymbol{\alpha},\boldsymbol{\beta})}}$ is a path graph.*

Let $I \subset S$ be a monomial ideal. We define the *bounding multidegree* of I to be the integral vector $\mathbf{deg}(I) = (\deg_{x_1}(I), \deg_{x_2}(I), \dots, \deg_{x_n}(I)) \in \mathbb{Z}^n$ defined by

$$\deg_{x_i}(I) \;=\; \max_{u \in G(I)} \deg_{x_i}(u).$$

As an immediate consequence of Theorem 5.2.1 we have that

Corollary 6.2.21 *Let $I \subset S$ be a monomial ideal and let $\mathbf{deg}(I)$ be its bounding multidegree. For all multigraded shifts $\mathbf{a}$ of I, we have $\mathbf{a} \leq \mathbf{deg}(I)$.*

Proposition 6.2.22 *Let $I_{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \prod_{i=1}^{t} \mathfrak{p}_{[\alpha_i,\beta_i]} \subset S$ be a LP–polymatroidal ideal with $t \geq 2$, $\alpha_1 = 1$, $\beta_t = n$ and having maximal projective dimension. Then,*

$$\operatorname{soc}(I_{(\boldsymbol{\alpha},\boldsymbol{\beta})}) \;=\; \prod_{i=1}^{t-1} \mathfrak{p}_{[\alpha_{i+1},\beta_i]}.$$

In particular, Question 6.2.18 is true in this case, and $\mathrm{HS}_{n-1}(I_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ *is again a transversal polymatroidal ideal.*

Proof. Let $A_i = [\alpha_i, \beta_i]$, for $i \in [t]$. By Corollary 6.2.20, $G_{I_{(\boldsymbol{\alpha},\boldsymbol{\beta})}}$ is a path graph, with edge set $\{\{1,2\},\{2,3\},\dots,\{t-1,t\}\}$. Hence $G_{I_{(\boldsymbol{\alpha},\boldsymbol{\beta})}}$ has only one spanning tree, namely itself. As said before, the proof of [96, Theorem 4.3] shows that all the monomials $w = x_{\ell_1}x_{\ell_2}\cdots x_{\ell_{t-1}}$ with $\ell_i \in A_i \cap A_{i+1} = [\alpha_i,\beta_i]\cap[\alpha_{i+1},\beta_{i+1}] = [\alpha_{i+1},\beta_i]$, $i = 1,\dots,t-1$, are minimal generators of $\mathrm{soc}(I_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$. Hence, we have proved the inclusion

$$\prod_{i=1}^{t-1} \mathfrak{p}_{[\alpha_{i+1},\beta_i]} \subseteq \mathrm{soc}(I_{(\boldsymbol{\alpha},\boldsymbol{\beta})}).$$

Now we apply Corollary 6.2.21 to show the opposite inclusion. For this purpose, let us compute the bounding multidegree of $I_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$. An easy calculation shows that

$$\begin{aligned}(\mathbf{deg}(I_{(\boldsymbol{\alpha},\boldsymbol{\beta})}))[j] &= \deg_{x_j}(I_{(\boldsymbol{\alpha},\boldsymbol{\beta})}) = \big|\{i \in [t] : j \in A_i\}\big| \\ &= \begin{cases} 1 & \text{if } j \in [n] \setminus (\bigcup_{i=1}^{t-1}[\alpha_{i+1},\beta_i]), \\ 1 + \big|\{\ell \in [t] : \alpha_{\ell+1} \le j \le \beta_\ell\}\big| & \text{if } j \in \bigcup_{i=1}^{t-1}[\alpha_{i+1},\beta_i]. \end{cases}\end{aligned}$$

By Corollary 5.3.2, $\mathrm{HS}_{n-1}(I_{(\boldsymbol{\alpha},\boldsymbol{\beta})}) = x_1x_2\cdots x_n\cdot\mathrm{soc}(I_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$, so each minimal generator w of $\mathrm{soc}(I_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ has multidegree bounded by $\mathbf{deg}(I_{(\boldsymbol{\alpha},\boldsymbol{\beta})}) - \mathbf{1}$. Thus w is a monomial of degree $t-1$ whose support is a subset of $\bigcup_{i=1}^{t-1}[\alpha_{i+1},\beta_i]$. For $t = 2$, we furthermore have that $w = x_s$ with $s \in [\alpha_2,\beta_1]$. Thus in this case $\mathrm{soc}(I_{(\boldsymbol{\alpha},\boldsymbol{\beta})}) = \mathfrak{p}_{[\alpha_2,\beta_1]}$.

Now, let $t > 2$, and consider

$$\boldsymbol{\alpha}^* : \alpha_1 \le \alpha_2 \le \cdots \le \alpha_{t-1} \quad\text{and}\quad \boldsymbol{\beta}^* : \beta_1 \le \beta_2 \le \cdots \le \beta_{t-1}.$$

We have $I_{(\boldsymbol{\alpha},\boldsymbol{\beta})} = I_{(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)}\mathfrak{p}_{[\alpha_t,\beta_t]}$. Let $u \in G(I_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ with $u/x_n \in \mathrm{soc}(I_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$. Then $\max(u/x_n) \in [\alpha_t,\beta_{t-1}]$. It is clear that $\max(u/x_n) \le \beta_{t-1}$. Suppose by contradiction that $\max(u/x_n) < \alpha_t$, then if $\ell < \alpha_t$ we would have $\mathrm{supp}(x_\ell(u/x_n)) \subseteq [1,\alpha_t-1]$ and by the structure of $I_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$, $x_\ell(u/x_n) \notin G(I_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$, an absurd. So $s = \max(u/x_n) \ge \alpha_t$. We claim that $w = u/(x_sx_n) \in \mathrm{soc}(I_{(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)})$. Let $j \in \{1,\dots,\beta_{t-1}\}$, then $x_j(u/x_n) \in G(I_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$. If $j < \alpha_t$, we see that $x_j \cdot u/(x_sx_n) \in G(I_{(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)})$, as desired. If $j = s$, then $x_ju/(x_sx_n) = u/x_n \in G(I_{(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)})$. Finally, if $j \in [\alpha_t,\beta_{t-1}] \setminus \{s\}$, then since $u/(x_sx_n) \in \prod_{i=1}^{t-2}\mathfrak{p}_{[\alpha_i,\beta_i]}$, we see that $x_j \cdot u/(x_sx_n) \in \prod_{i=1}^{t-1}\mathfrak{p}_{[\alpha_i,\beta_i]} = G(I_{(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)})$. Summarizing, we have shown that $\mathrm{soc}(I_{(\boldsymbol{\alpha},\boldsymbol{\beta})}) \subseteq \mathfrak{p}_{[\alpha_t,\beta_{t-1}]}\cdot\mathrm{soc}(I_{(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)})$. Since by inductive hypothesis, $\mathrm{soc}(I_{(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)}) = \prod_{i=1}^{t-2}\mathfrak{p}_{[\alpha_{i+1},\beta_i]}$, the conclusion follows. □

Example 6.2.23 Let $I = \mathfrak{p}_{[1,4]}\mathfrak{p}_{[3,5]} \subset S = K[x_1,\dots,x_5]$ be the LP–polymatroidal ideal considered in Example 6.1.7. In such a case $\mathbf{deg}(I) = (1,1,2,2,1)$, and I has maximal projective dimension ($\mathrm{pd}(I) = 4$) since $[1,4]\cap[3,5] = [3,4] \neq \varnothing$. By Proposition 6.2.22 we have $\mathrm{soc}(I) = \mathfrak{p}_{[1,4]\cap[3,5]} = \mathfrak{p}_{[3,4]} = (x_3,x_4)$ and by Corollary 5.3.2 we can compute the highest homological shift ideal of I as

$$\mathrm{HS}_4(I) = \mathbf{x}_{[5]}\cdot\mathrm{soc}(I) = (x_1x_2x_3^3x_4x_5, x_1x_2x_3x_4^2x_5).$$

The many characterizations of ideals with maximal projective dimension we have obtained, for the various classes of polymatroidal ideals we have considered, seem to be very different in nature. Thus, it seems unlikely that a nice characterization of all polymatroidal ideals with maximal projective dimension is possible.

Chapter 7

Homological shifts of edge ideals

The classification of all Cohen–Macaulay edge ideals and the classification of all edge ideals with linear resolution are fundamental problems. While the first problem is widely open and considered to be intractable in general, for the second problem we have a complete answer as follows by Dirac and Fröberg (Theorems 3.3.4). In this chapter we discuss the algebraic consequences of Dirac's theorem on chordal graphs related to the theory of homological shift ideals of edge ideals.

The chapter is structured as follows.

Sections 7.1 and 7.2 are devoted to homological shifts of edge ideals with linear resolution. Let G be a graph. Unfortunately, even if $I(G)$ has linear resolution, it may not have homological linear resolution (Example 7.1.1). At present we do not have a complete classification of all edge ideals with homological linear quotients or homological linear resolution. Thus, we determine many classes of cochordal graphs whose edge ideals have homological linear resolution. In particular, for *proper interval graphs* and *forests*, we prove that the edge ideals of their complementary graphs have homological linear quotients, (Theorems 7.1.2 and 7.2.1). To prove the first result we introduce the class of *reversible* chordal graphs, and show that any proper interval graph is a reversible graph, (Lemma 7.1.3). For the second result, we consider two operations on chordal graphs that preserve the homological linear quotients property. Namely, adding whiskers to a chordal graph and taking unions of disjoint chordal graphs, (Propositions 7.2.2 and 7.2.4). Using these results, it is easy to see that $I(G)$ has homological linear quotients, if G^c is a forest. Indeed, any forest is the union of pairwise disjoint trees, and any tree can be constructed by iteratively adding whiskers to a previously constructed tree on a smaller vertex set.

In the last section, we consider polymatroidal ideals. An equigenerated monomial ideal I is called *polymatroidal* if its minimal set of monomial generators $G(I)$ corresponds to the set of bases of a *discrete polymatroid*, see [89, Chapter 12]. Polymatroidal ideals are characterized by the fact that they have linear quotients with respect to the lexicographic order induced by any ordering of the variables (Theorem 3.4.2). It is conjectured by Bandari, Bayati and Herzog that all homological shift ideals of a polymatroidal ideal are polymatroidal (Conjecture 6.1.1). At present this conjecture is widely open. On the other hand, Bayati proved that the conjecture holds for any squarefree polymatroidal ideal [16]. Herzog, Moradi, Rahimbeigi and Zhu proved that it holds for polymatroidal ideals that satisfy the strong exchange property [101, Corollary 3.6]; in [60] it was proved that $\text{HS}_1(I)$ is again polymatroidal if I is such, pointing towards the validity of the conjecture in general.

We settle Conjecture 6.1.1 for polymatroidal ideals generated in degree two (Theorem 7.3.3). In the squarefree case, I may be seen as the edge ideal of a cochordal graph and we apply our criterion on reversibility of perfect elimination orders. Unfortunately our methods are very special and they can not be applied for a general polymatroidal ideal.

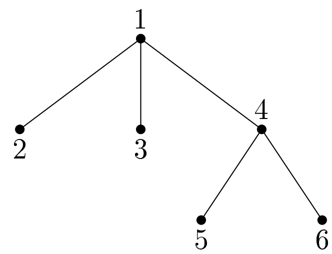
7.1 Homological shifts of proper interval graphs

Let G be a finite simple graph with vertex set $V(G) = [n]$ and edge set $E(G)$. Let K be a field. The *edge ideal* of G is the squarefree monomial ideal $I(G)$ of $S = K[x_1, \dots, x_n]$ generated by the monomials $x_i x_j$ such that $\{i, j\} \in E(G)$.

By Theorem 3.3.6 $I(G)$ has linear resolution if and only if it has linear quotients. Thus, the theorems of Dirac and Fröberg classify all edge ideals with linear quotients. It is known that if $x_1 > x_2 > \cdots > x_n$ is a perfect elimination order of G^c, then $I(G)$ has linear quotients with respect to the lexicographic order $>_{\text{lex}}$ induced by $x_1 > x_2 > \cdots > x_n$.

Now we turn to the homological shifts of edge ideals with linear quotients. Unfortunately, in general an edge ideal with linear quotients does not even has homological linear resolution as next example shows.

Example 7.1.1 Let G be the following cochordal graph on six vertices.



Let $I = I(G) \subset S = K[x_1, \dots, x_6]$. Using the package [59] we verified that $\text{HS}_0(I)$ and $\text{HS}_1(I)$ have linear quotients. However the last homological shift ideal $\text{HS}_2(I) = (x_1x_2x_3x_4,\, x_1x_4x_5x_6)$ has the following non-linear resolution

$$0 \to S(-6) \to S(-4)^2 \to (x_1x_2x_3x_4, x_1x_4x_5x_6) \to 0.$$

In graph theory, one distinguished class of chordal graphs is the family of *proper interval graphs*. A graph G is called an *interval graph*, if one can label its vertices with some intervals on the real line so that two vertices are adjacent in G, when the intersection of their corresponding intervals is non-empty. A *proper interval graph* is an interval graph such that no interval properly contains another.

Now we are ready to state our main result in the section.

Theorem 7.1.2 *Let G be a cochordal graph on $[n]$ whose complementary graph G^c is a proper interval graph. Then, $I(G)$ has homological linear quotients.*

In order to prove the theorem we introduce a more general class of graphs.

We call a perfect elimination order $x_1 > x_2 > \cdots > x_n$ of a chordal graph G *reversible* if $x_n > x_{n-1} > \cdots > x_1$ is also a perfect elimination order of G. We call a chordal graph G *reversible* if G admits a reversible perfect elimination order. Moreover, a cochordal graph G is called reversible if and only if G^c is reversible.

Lemma 7.1.3 *Let G be a proper interval graph. Then G is reversible.*

Proof. By [121, Theorem 1 and Lemma 1], up to a relabeling of the vertex set of G, the following property is satisfied:

(∗) for all $i < j$, $\{i, j\} \in E(G)$ implies that the induced subgraph of G on $\{i, i + 1 \dots, j\}$ is a *clique*, *i.e.*, a complete subgraph.

With such a labeling, both $x_1 > x_2 > \cdots > x_n$ and $x_n > x_{n-1} > \cdots > x_1$ are perfect elimination orders of G. By symmetry, it is enough to show that $x_1 > x_2 > \cdots > x_n$ is a perfect elimination order. Let $i \in [n]$, $j, k \in N_G(i)$ with $j, k > i$. We prove that $\{j, k\} \in E(G)$. Suppose $j > k$. By (∗), the induced subgraph of G on $\{i, i + 1 \dots, j\}$ is a clique. Since $j > k > i$, we obtain that $\{j, k\} \in E(G)$, as wanted. □

With this lemma at hand, Theorem 7.1.2 follows from the following more general result.

Theorem 7.1.4 *Let G be a cochordal graph on $[n]$, and let $x_1 > \cdots > x_n$ be a reversible perfect elimination order of G^c. Then, $\mathrm{HS}_k(I(G))$ has linear quotients with respect to the lexicographic order $>_{\text{lex}}$ induced by $x_1 > \cdots > x_n$, for all $k \geq 0$.*

For the proof of this theorem, we need a description of the homological shift ideals.

Lemma 7.1.5 *Let G be a cochordal graph on $[n]$, and let $x_1 > x_2 > \cdots > x_n$ be a perfect elimination order of G^c. Then, for all $\{i, j\} \in E(G)$, with $i < j$,*

$$\mathrm{set}(x_i x_j) = \{1, \dots, i - 1\} \cup (\{i + 1, \dots, j - 1\} \cap N_G(i)). \tag{7.1}$$

In particular,

$$\mathrm{HS}_k(I(G)) = \big(\mathbf{x}_A \mathbf{x}_B \ : A, B \subseteq [n], A, B \neq \varnothing, \max(A) < \min(B), |A \cup B| = k + 2, \\ \{\max(A), b\} \in E(G), \ \textit{for all } b \in B\big).$$

Proof. As remarked before $I(G)$ has linear quotients with respect to the lexicographic order $>_{\text{lex}}$ induced by $x_1 > x_2 > \cdots > x_n$. Let $\{i, j\} \in E(G)$ with $i < j$. Let us determine $\mathrm{set}(x_i x_j)$. If $k \in \mathrm{set}(x_i x_j)$, then $x_k(x_i x_j)/x_\ell \in I(G)$ and $x_k(x_i x_j)/x_\ell >_{\text{lex}} x_i x_j$ for some $\ell \in \{i, j\}$. Note that $k < j$, indeed for $k > j$, both $x_i x_k, x_j x_k$ are smaller than $x_i x_j$ in the lexicographic order. Thus either $k < i$ or $i < k < j$. We distinguish the two possible cases.

Case 1. Suppose $k < i$. Assume that none of $x_k x_i, x_k x_j$ is in $I(G)$. Then $\{k, i\}, \{k, j\} \in E(G^c)$. Since $x_1 > x_2 > \cdots > x_n$ is a perfect elimination order, the induced graph of G_i^c on the vertex set $N_{G_k^c}(k)$ is complete. But $i, j > k$ and $i, j \in N_{G_k^c}(k)$. Thus we would have $\{i, j\} \in E(G^c)$, that is, $x_i x_j \notin I(G)$, absurd.

Case 2. Suppose $i < k < j$. Since $k > i$, $x_k x_j <_{\text{lex}} x_i x_j$. Thus $k \in \mathrm{set}(x_i x_j)$ if and only if $x_i x_k \in E(G)$, that is $k \in N_G(i)$.

The two cases above show that equation (7.1) holds. The formula for $\mathrm{HS}_k(I(G))$ follows immediately by applying equations (5.1) and (7.1). □

Proof of Theorem 7.1.4. We proceed by induction on $n \geq 1$. Let G' be the induced subgraph of G on the vertex set $\{2, 3, \dots, n\}$. Then $x_2 > x_3 > \cdots > x_n$ is again a reversible perfect elimination order of $(G')^c$ and G' is a reversible cochordal graph.

Let $J = (x_i : x_1 x_i \in I(G))$. Then, $I(G) = x_1 J + I(G')$ is a Betti splitting, because $G(I(G))$ is the disjoint union of $G(x_1 J)$ and $G(I(G'))$, and $x_1 J$, $I(G')$ have linear resolutions, see [74, Corollary 2.4]. Since $I(G') \cap x_1 J = x_1 I(G')$, [35, Proposition 1.7] gives

$$\mathrm{HS}_k(I(G)) = x_1(\mathrm{HS}_{k-1}(I(G')) + \mathrm{HS}_k(J)) + \mathrm{HS}_k(I(G')).$$

We claim that $\mathrm{HS}_k(I(G))$ has linear quotients with respect to the lexicographic order $>_{\mathrm{lex}}$ induced by $x_1 > x_2 > \cdots > x_n$. For $k = 0$ this is true. Let $k > 0$.

Let $u = x_{i_1}x_{j_1}\mathbf{x}_{F_1}, v = x_{i_2}x_{j_2}\mathbf{x}_{F_2} \in G(\mathrm{HS}_k(I(G)))$, with $u >_{\mathrm{lex}} v$, $i_1 < j_1, i_2 < j_2$, $x_{i_1}x_{j_1},\ x_{i_2}x_{j_2} \in I(G)$, $F_1 \subseteq \mathrm{set}(u)$, $F_2 \subseteq \mathrm{set}(v)$. We are going to prove that there exists $w \in G(\mathrm{HS}_k(I(G)))$ such that $w >_{\mathrm{lex}} v$, $w : v = x_p$ and x_p divides $u : v$.

We can write

$$u = x_{p_1}x_{p_2}\cdots x_{p_{k+2}}, \qquad v = x_{q_1}x_{q_2}\cdots x_{q_{k+2}},$$

with $p_1 < p_2 < \cdots < p_{k+2}$, $q_1 < q_2 < \cdots < q_{k+2}$. Since $u >_{\mathrm{lex}} v$ then $p_1 = q_1$, $p_2 = q_2$, $\ldots$, $p_{s-1} = q_{s-1}$, $p_s < q_s$ for some $s \in \{1, \ldots, k+2\}$. If $s = k+2$, then $u : v = x_{p_{k+2}} = x_{j_1}$ and there is nothing to prove. Therefore, we may assume $s < k+2$. Thus $p_s < q_s < q_{k+2} = j_2$. Set $p = p_s$ and $q = q_s$, then x_p divides $u : v$.

Suppose for the moment that x_1 divides v. Then by definition of $>_{\mathrm{lex}}$, $p_1 = q_1 = 1$ and x_1 divides u, too. There are four cases to consider.

Case 1. Suppose $i_1 = i_2 = 1$. Setting $u' = u/x_1$ and $v' = v/x_1$, we have $u', v' \in G(\mathrm{HS}_k(J))$ and $u' >_{\mathrm{lex}} v'$. Since J is an ideal generated by variables, it has homological linear quotients with respect to $>_{\mathrm{lex}}$. Hence, there exists $w' \in G(\mathrm{HS}_k(J))$ with $w' >_{\mathrm{lex}} v'$ such that $w' : v' = x_\ell$ and x_ℓ divides $u' : v'$. Setting $w = x_1w'$, we have that $w >_{\mathrm{lex}} v$ and $w \in G(x_1\mathrm{HS}_k(J)) \subseteq G(\mathrm{HS}_k(I(G)))$. Hence $w : v = w' : v' = x_\ell$ and x_ℓ divides $u : v = u' : v'$.

Case 2. Suppose $i_1 > 1$ and $i_2 > 1$. Setting $u' = u/x_1$ and $v' = v/x_1$, we have $u', v' \in G(\mathrm{HS}_{k-1}(I(G')))$ and $u' >_{\mathrm{lex}} v'$. By inductive hypothesis, $I(G')$ has homological linear quotients with respect to $>'_{\mathrm{lex}}$ induced by $x_2 > x_3 > \cdots > x_n$. Hence, there exists $w' \in G(\mathrm{HS}_{k-1}(I(G')))$ with $w' >'_{\mathrm{lex}} v'$ such that $w' : v' = x_\ell$ and x_ℓ divides $u' : v'$. Setting $w = x_1w'$, we have that $w >_{\mathrm{lex}} v$ and $w \in G(x_1\mathrm{HS}_{k-1}(I(G'))) \subseteq G(\mathrm{HS}_k(I(G)))$. Hence $w : v = w' : v' = x_\ell$ and x_ℓ divides $u : v = u' : v'$.

Case 3. Suppose $i_1 > 1$ and $i_2 = 1$. Then $1 = i_2 < p < j_2$.

Subcase 3.1. Assume $x_1x_p \in I(G)$, then $p \in \mathrm{set}(x_{i_2}x_{j_2})$. Setting $w = x_p(v/x_q)$, by equation (5.1) $w \in G(\mathrm{HS}_k(I(G)))$, and $w >_{\mathrm{lex}} v$, because $p < q$. Moreover $w : v = x_p$ and x_p divides $u : v$.

Subcase 3.2. Assume that $x_1x_p \notin I(G)$. By hypothesis, $x_n > x_{n-1} > \cdots > x_1$ is also a perfect elimination order of G^c. Thus, by Lemma 7.1.5, we can write $u = \mathbf{x}_A\mathbf{x}_B$ with $A = \{p_{k+2}, p_{k+1}, \ldots, p_r\}$, $B = \{p_{r-1}, \ldots, p_2, p_1\}$ for some $r > 1$ and with $\{p_r, p_\ell\} \in E(G)$ for all $\ell = r-1, \ldots, 2, 1$. Since $\{1, p\} = \{p_1, p_s\} \notin E(G)$, by the above presentation of u, $s > r$. Using again Lemma 7.1.5, but considering the reversed perfect elimination order $x_n > x_{n-1} > \cdots > x_1$, we see that

$$\begin{aligned} w &= x_{q_{s+1}}x_{q_{s+2}}\cdots x_{q_{k+2}}u/(x_{p_{s+1}}x_{p_{s+2}}\cdots x_{p_{k+2}}) \\ &= \mathbf{x}_{(A\setminus\{p_{s+1},p_{s+2},\ldots,p_{k+2}\})\cup\{q_{s+1},q_{s+2},\ldots,q_{k+2}\}}\mathbf{x}_B \in G(\mathrm{HS}_k(I(G))). \end{aligned}$$

Moreover, $w >_{\mathrm{lex}} v$, $w : v = x_p$ and x_p divides $u : v$, as desired.

Case 4. Suppose $i_1 = 1$ and $i_2 > 1$. Recall that $p < j_2$. Moreover $p \neq i_2$, because x_p divides $u : v$ but x_{i_2} divides v. Thus there are two cases to consider.

Subcase 4.1. Assume $p < i_2$. By Lemma 7.1.5, $p \in \mathrm{set}(x_{i_2}x_{j_2})$. If $q \neq i_2$, then $q < j_2$ and by equation (5.1) $w = x_p(v/x_q)$ is a minimal generator of $\mathrm{HS}_k(I(G))$. Moreover $w >_{\mathrm{lex}} v$ and $w : v = x_p$ divides $u : v$, as wanted. Suppose now that $q = i_2$. If there exists ℓ such that x_ℓ divides v and $i_2 < \ell < j_2$, then $\ell > p$ and $w = x_p(v/x_\ell)$

is a minimal generator of $\mathrm{HS}_k(I(G))$ such that $w >_{\text{lex}} v$ and with $w : v = x_p$ dividing $u : v$, as wanted. Otherwise, suppose no such integer ℓ exists. Then, $s = k+1$, $q_{k+1} = i_2$ and $q_{k+2} = j_2$. Since $p \in \mathrm{set}(x_{i_2}x_{j_2})$, then $x_p x_\ell \in I(G)$, where $\ell \in \{i_2, j_2\}$. Then $p < \ell$ and by Lemma 7.1.5 we see that $w = x_p(v/x_\ell)$ is a minimal generator of $\mathrm{HS}_k(I(G))$ such that $w >_{\text{lex}} v$ and with $w : v = x_p$ dividing $u : v$.

Subcase 4.2. Assume now $i_2 < p < j_2$. If $x_{i_2}x_p \in I(G)$, by Lemma 7.1.5, $p \in \mathrm{set}(x_{i_2}x_{j_2})$. Setting $w = x_p(v/x_q)$, we have $w \in G(\mathrm{HS}_k(I(G)))$, $w >_{\text{lex}} v$ and $w : v = x_p$ divides $u : v$. Suppose now that $x_{i_2}x_p \notin I(G)$. By hypothesis, $x_n > x_{n-1} > \cdots > x_1$ is also a perfect elimination order of G^c. Thus, by Lemma 7.1.5, we can write $u = \mathbf{x}_A\mathbf{x}_B$ with $A = \{p_{k+2}, p_{k+1}, \dots, p_r\}$, $B = \{p_{r-1}, \dots, p_2, p_1\}$ for some $r > 1$ and with $\{p_r, p_\ell\} \in E(G)$ for all $\ell = r-1, \dots, 2, 1$. Note that $i_2 < p$, so x_{i_2} divides u. Since $\{i_2, p\} = \{i_2, p_s\} \notin E(G)$, by the above presentation of u, $s > r$. Using again Lemma 7.1.5, but considering the reversed perfect elimination order $x_n > x_{n-1} > \cdots > x_1$, we see that

$$\begin{aligned} w &= x_{q_{s+1}}x_{q_{s+2}}\cdots x_{q_{k+2}}u/(x_{p_{s+1}}x_{p_{s+2}}\cdots x_{p_{k+2}}) \\ &= \mathbf{x}_A\mathbf{x}_{(B\setminus\{p_{s+1},p_{s+2},\dots,p_{k+2}\})\cup\{q_{s+1},q_{s+2},\dots,q_{k+2}\}} \in G(\mathrm{HS}_k(I(G))). \end{aligned}$$

Moreover, $w >_{\text{lex}} v$, $w : v = x_p$ and x_p divides $u : v$, as desired.

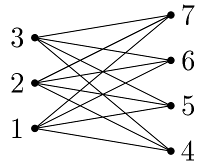
Suppose now that x_1 does not divide v. Then $v \in G(\mathrm{HS}_k(I(G')))$. If x_1 does not divide u, then $u \in G(\mathrm{HS}_k(I(G')))$, too. Let $>'_{\text{lex}}$ be the lexicographic order induced by $x_2 > x_3 > \cdots > x_n$. Since by induction $I(G')$ has homological linear quotients with respect to $>'_{\text{lex}}$ and also $u >'_{\text{lex}} v$, there exists $w \in G(\mathrm{HS}_k(I(G')))$, with $w >'_{\text{lex}} v$, $w : v = x_\ell$ and x_ℓ divides $u : v$. But also we have $w \in G(\mathrm{HS}_k(I(G)))$ and $w >_{\text{lex}} v$. Otherwise if x_1 divides u, then x_1 divides $u : v$. Since $\mathrm{HS}_k(I(G')) \subseteq \mathrm{HS}_{k-1}(I(G'))$ and $k > 0$, we can write $v = x_t w'$ with $w' \in G(\mathrm{HS}_{k-1}(I(G')))$. Let $w = x_1 w'$. Then $w >_{\text{lex}} v$ and $w : v = x_1$ divides $u : v$.

Hence, the inductive proof is complete and the theorem is proved. □

Remark 7.1.6 Let $x_1 > x_2 > \cdots > x_n$ be a reversible perfect elimination order of G^c. By symmetry, Theorem 7.1.4 shows also that $\mathrm{HS}_k(I(G))$ has linear quotients with respect to the lexicographic order induced by $x_n > x_{n-1} > \cdots > x_1$.

Example 7.1.7 Let n, m be two positive integers.

(a) Let $G = K_{n,m}$ be the *complete bipartite graph*. That is, $V(G) = [n+m]$ and $E(G) = \big\{\{i,j\} : i \in [n], j \in \{n+1, \dots, n+m\}\big\}$. For example, for $n = 3$ and $m = 4$



It is easy to see that G^c is the disjoint union of two complete graphs Γ_1 and Γ_2 on vertex sets $[n]$ and $\{n+1, \dots, n+m\}$ respectively. Furthermore, any ordering of the vertices is a perfect elimination order of G^c. Applying the previous theorem,

$$I(G) = (x_1, \dots, x_n)(x_{n+1}, \dots, x_m)$$

has homological linear quotients with respect to the lexicographic order induced by any ordering of the variables.

(b) Let G be the graph with vertex set $V(G) = [n+m]$ and edge set

$$E(G) = \big\{\{i,j\} : i \in [n+m], n+1 \leq j \leq n+m, i < j\big\}.$$

We claim that G is a reversible cochordal graph. Indeed G^c is the disjoint union of the complete graph K_n on the vertex set $[n]$ together with the set of isolated vertices $\{n+1, \dots, n+m\}$. It easily seen that any ordering of the vertices is a perfect elimination order of G^c. Applying Theorem 7.1.4

$$I(G) = (x_1, \dots, x_n)(x_{n+1}, \dots, x_m) + (x_i x_j : n+1 \leq i < j \leq n+m)$$

has homological linear quotients with respect to the lexicographic order induced by any ordering of the variables.

7.2 Homological shifts of trees

In this section we construct several classes of edge ideals with homological linear quotients, by considering various operations on cochordal graphs that preserve the homological linear quotients property. As a main application of all these results we will prove the following theorem.

Theorem 7.2.1 *Let G be a graph such that G^c is a forest. Then $I(G)$ has homological linear quotients.*

For the next proof we recall that the squarefree Veronese ideal $I_{n,d}$ has homological linear quotients, (see for instance [101, Corollary 3.2]).

The first operation we consider consists in adding *whiskers*. Let Γ' be a graph on vertex set $[n-1]$. Let $i \in [n-1]$ and let Γ be the graph with vertex set $[n]$ and edge set $V(\Gamma) = V(\Gamma') \cup \{\{i,n\}\}$. Γ is called the *whisker graph* of Γ' obtained by adding the whisker $\{i,n\}$ to Γ'.

Proposition 7.2.2 *Let Γ' be a graph on vertex set $[n-1]$ and Γ be the graph on vertex set $[n]$ and edge set $V(\Gamma) = V(\Gamma') \cup \{\{i,n\}\}$ for some $i \in [n-1]$. Set $G = \Gamma^c$. Suppose $I((\Gamma')^c)$ has homological linear quotients. Then $I(G)$ has homological linear quotients, too.*

Proof. Since Γ' is chordal, obviously Γ is chordal, too. Set $J = I((\Gamma')^c)$, $I = I(G)$ and $L = (x_j : j \in [n-1] \setminus \{i\})$. Since $N_{G^c}(n) = \{i\}$, we have the Betti splitting:

$$I = x_n L + J. \tag{7.2}$$

Since G is cochordal, $\mathrm{HS}_0(I)$ and $\mathrm{HS}_1(I)$ have linear quotients. So we only have to show that $\mathrm{HS}_k(I)$ has linear quotients for $k \geq 2$. By equation (7.2), for all $k \geq 2$,

$$\mathrm{HS}_k(I) = x_n \mathrm{HS}_k(L) + x_n \mathrm{HS}_{k-1}(J) + \mathrm{HS}_k(J).$$

Note that $\mathrm{HS}_k(L)$ is the squarefree Veronese ideal of degree $k+1$ in the polynomial ring $K[x_j : j \in [n-1] \setminus \{i\}]$. Thus $\mathrm{HS}_k(L)$ has admissible order, say, $u_1, \dots, u_m$. Let $v_1, \dots, v_r$ and $w_1, \dots, w_s$ be admissible orders of $\mathrm{HS}_{k-1}(J)$ and $\mathrm{HS}_k(J)$, respectively.

Let $v_{j_1},\dots,v_{j_p}$, with $j_1<\dots<j_p$, the monomials in $G(\mathrm{HS}_{k-1}(J))\setminus G(\mathrm{HS}_k(L))$. We claim that the following is an admissible order of $\mathrm{HS}_k(J)$:

$$x_nu_1,\dots,x_nu_m,\ x_nv_{j_1},\dots,x_nv_{j_p},\ w_1,\dots,w_s \tag{7.3}$$

Note that $(x_nu_1,\dots,x_nu_{\ell-1}):x_nu_\ell=(u_1,\dots,u_{\ell-1}):u_\ell$ is generated by variables. Let $\ell\in[p]$. We show that

$$\begin{aligned}Q&=(x_nu_1,\dots,x_nu_m,x_nv_{j_1},\dots,x_nv_{j_{\ell-1}}):x_nv_{j_\ell}\\&=(u_1,\dots,u_m,v_{j_1},\dots,v_{j_{\ell-1}}):v_{j_\ell}\end{aligned}$$

is generated by variables. Consider $v_{j_q}:v_{j_\ell}$, then we can find $d<j_\ell$ such that $v_d:v_{j_\ell}$ is a variable that divides $v_{j_q}:v_{j_\ell}$. Either $d=j_b$, for some $b<\ell$, or $v_d\in\mathrm{HS}_k(L)$. In any case, $v_d\in(u_1,\dots,u_m,v_{j_1},\dots,v_{j_{\ell-1}})$ and $v_d:v_{j_\ell}\in Q$ divides $v_{j_q}:v_{j_\ell}$.
Consider now $u_q:v_{j_\ell}$, $1\le q\le m$. Hence x_i divides v_{j_ℓ}, lest $v_{j_\ell}\in G(\mathrm{HS}_k(L))$. But then $v_{j_\ell}/x_i\in\mathrm{HS}_{k-1}(L)$. Let x_t dividing $u_q:v_{j_\ell}$. Then $u=x_tv_{j_\ell}/x_i\in\mathrm{HS}_k(L)$ and $u:v_{j_\ell}=x_t\in Q$ divides $u_q:v_{j_\ell}$.

Finally, let $\ell\in[s]$. We show that

$$\begin{aligned}Q&=(x_nu_1,\dots,x_nu_m,x_nv_{j_1},\dots,x_nv_{j_p},w_1,\dots,w_{\ell-1}):w_\ell\\&=(x_n\mathrm{HS}_k(L)+x_n\mathrm{HS}_{k-1}(J)):w_\ell+(w_1,\dots,w_{\ell-1}):w_\ell\end{aligned}$$

is generated by variables. Since $w_1,\dots,w_s$ is an admissible order, $(w_1,\dots,w_{\ell-1}):w_\ell$ is generated by variables. Consider now a generator $x_nz:w_\ell$ with $z\in\mathrm{HS}_k(L)$ or $z\in\mathrm{HS}_{k-1}(J)$. Then x_n divides $x_nz:w_\ell$. On the other hand $w_\ell/x_t\in\mathrm{HS}_{k-1}(J)$ for some t. But then $x_nw_\ell/x_t:w_\ell=x_n\in Q$ divides our generator.

The three cases above show that (7.3) is an admissible order, as desired. □

Since any tree can be constructed iteratively by adding a whisker to a tree on a smaller vertex set at each step, the previous proposition implies immediately

Corollary 7.2.3 *Let G be a graph such that G^c is a tree. Then $I(G)$ has homological linear quotients.*

The second operation we consider consists in joining disjoint graphs. Two graphs Γ_1 and Γ_2 are called *disjoint* if $V(\Gamma_1)\cap V(\Gamma_2)=\varnothing$. The *join* of Γ_1 and Γ_2 is the graph Γ with vertex set $V(\Gamma)=V(\Gamma_1)\cup V(\Gamma_2)$ and edge set $E(\Gamma)=E(\Gamma_1)\cup E(\Gamma_2)$.

Proposition 7.2.4 *Let Γ_1 and Γ_2 be disjoint chordal graphs such that $I(\Gamma_1^c),I(\Gamma_2^c)$ have homological linear quotients. Let Γ be the join of Γ_1 and Γ_2 and set $G=\Gamma^c$. Then $I(G)$ has homological linear quotients, too.*

Proof. Obviously Γ is chordal, too. Let $G_1=\Gamma_1^c$, $G_2=\Gamma_2^c$, $V(G_1)=[n]$ and $V(G_2)=[n+1,n+m]$. Set $L=(x_1,\dots,x_n)(x_{n+1},\dots,x_m)$. Then,

$$I(G)=I(G_1)+I(G_2)+L$$

Suppose $x_1>\dots>x_n$ and $x_{n+1}>\dots>x_{n+m}$ are perfect elimination orders of Γ_1 and Γ_2. Then $G=\Gamma^c$ is cochordal. Indeed, $x_1>\dots>x_n>x_{n+1}>\dots>x_{n+m}$ is a perfect elimination order of Γ. Let $>_{\mathrm{lex}}$ be the lexicographic order induced by such an ordering of the variables. Set, $I=I(G)$, $I_1=I(G_1)$ and $I_2=I(G_2)$. Then, I,I_1,I_2 and J have linear quotients with respect to $>_{\mathrm{lex}}$.

Let $k\ge0$ and $u\in G(\mathrm{HS}_k(I))$ such that x_ix_j divides u for some integers $i\in[n]$, $n+1\le j\le n+m$. We claim that $u\in G(\mathrm{HS}_k(L))$. Let $i_0=\max\{i\in[n]:x_i \text{ divides } u\}$

and $j_0 = \max\{j \in [n+1, n+m] : x_j \text{ divides } u\}$. Let $u/(x_{i_0}x_{j_0}) = \mathbf{x}_F$. Then $F \subseteq [i_0 - 1] \cup [n+1, j_0 - 1] = \text{set}_I(x_{i_0}x_{j_0})$ and $x_{i_0}x_{j_0} \in L$. Thus, by equation (5.1), $u = x_{i_0}x_{j_0}\mathbf{x}_F \in \text{HS}_k(L)$, as desired. This argument shows that any squarefree monomial $w \in K[x_1, \dots, x_{n+m}]$ of degree $k+2$, containing as a factor any monomial x_ix_j with $i \in [n]$ and $n+1 \le j \le n+m$, is a generator of $\text{HS}_k(L)$.

From this remark, for all $k \ge 0$, it follows that

$$\text{HS}_k(I) = \text{HS}_k(L) + \text{HS}_k(I_1) + \text{HS}_k(I_2).$$

Note that L is the edge ideal of a complete bipartite graph. By Examples 7.1.7(a), L has homological linear quotients. Let $u_1, \dots, u_m$ be an admissible order of $\text{HS}_k(L)$. Moreover, let $v_1, \dots, v_r$ and $w_1, \dots, w_s$ be admissible orders of $\text{HS}_k(I_1)$ and $\text{HS}_k(I_2)$, respectively. Note that the monomials u_i, v_j, w_t are all different, because all monomials u_i contain a factor $x_{i_0}x_{j_0}$ with $i_0 \in [n]$ and $j_0 \in [n+1, n+m]$. Whereas, the v_j are monomials in $K[x_1, \dots, x_n]$ and the w_t are monomials in $K[x_{n+1}, \dots, x_{n+m}]$.

We claim that the following is an admissible order of $\text{HS}_k(I)$:

$$u_1, \dots, u_m,\ v_1, \dots, v_r,\ w_1, \dots, w_s \tag{7.4}$$

Let $\ell \in [m]$. Then $(u_1, \dots, u_{\ell-1}) : u_\ell$ is generated by variables.

Let $\ell \in [r]$. We show that

$$Q = (u_1, \dots, u_m, v_1, \dots, v_{\ell-1}) : v_\ell$$

is generated by variables. Clearly $(v_1, \dots, v_{\ell-1}) : v_\ell$ is generated by variables. Consider now $u_q : v_\ell$, $1 \le q \le m$. Recall that v_ℓ is a monomial in $K[x_1, \dots, x_n]$. Thus x_j divides $u_q : v_\ell$ for some $j \in \{n+1, \dots, n+m\}$. Consider v_ℓ/x_t for some t. Then $u = x_j(v_\ell/x_t) \in \text{HS}_k(L)$ and $u : v_\ell = x_j \in Q$, as desired.

Finally, let $\ell \in [s]$. We show that $Q = (u_1, \dots, u_m, v_1, \dots, v_r, w_1, \dots, w_{\ell-1}) : w_\ell$ is generated by variables.

Since $w_1, \dots, w_s$ is an admissible order, $(w_1, \dots, w_{\ell-1}) : w_\ell$ is generated by variables. Consider now a generator $z : w_\ell$ with $z = u_q$ or $z = v_q$, for some q. Since w_ℓ is a monomial in $K[x_{n+1}, \dots, x_{n+m}]$, $z : w_\ell$ is divided by a variable x_i, where $i \in [n]$. Consider w_ℓ/x_t for some t. Then $u = x_i(w_\ell/x_t) \in \text{HS}_k(L)$ and $u : w_\ell = x_i \in Q$, as desired.

The three cases above show that (7.4) is an admissible order, as desired. □

Proof of Theorem 7.2.1. Let $\Gamma = G^c$ be a forest and let c be the number of connected components of Γ. If $c = 1$, then Γ is a tree, and by Corollary 7.2.3, $I(G)$ has homological linear quotients. Suppose $c > 1$ and write $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are disjoint forests. The numbers of connected components of Γ_1 and Γ_2 are smaller than c. Thus, by induction $I(\Gamma_1^c)$ and $I(\Gamma_2^c)$ have homological linear quotients. Applying Proposition 7.2.4, it follows that $I(G)$ has homological linear quotients, too. □

Let G be a complete multipartite graph, then G^c is the disjoint union of some complete graphs. Repeated applications of Proposition 7.2.4 yield

Corollary 7.2.5 *Let G be a complete multipartite graph. Then $I(G)$ has homological linear quotients.*

7.3 Polymatroidal ideals generated in degree two

As we said before, it is expected by Bandari, Bayati and Herzog that the homological shift ideals $\text{HS}_k(I)$ of a polymatroidal ideal I are all polymatroidal, see [16, 101]. In this section, we provide an affirmative answer to this conjecture for all polymatroidal ideals generated in degree two.

Firstly, we deal with the squarefree case.

Lemma 7.3.1 *Let $I \subset S$ be a matroidal ideal generated in degree two, and let G be the simple graph on $[n]$ such that $I = I(G)$. Then, any ordering of the variables is a perfect elimination order of G^c.*

Proof. Up to relabeling, we can consider the ordering $x_1 > x_2 > \cdots > x_n$. Let $j, k \in N_{G^c}(i)$ with $j, k > i$. We must prove that $\{j, k\} \in E(G^c)$. By our assumption, $\{i, j\}, \{i, k\} \notin E(G)$, that is $x_i x_j, x_i x_k \notin I(G) = I$. Suppose by contradiction that $\{j, k\} \notin E(G^c)$, then $\{j, k\} \in E(G)$, that is, $x_j x_k \in I(G)$. Pick any monomial $x_i x_s \in I(G)$. Then $\deg_{x_i}(x_i x_s) > \deg_{x_i}(x_j x_k)$. By Lemma 3.4.1, we can find ℓ with $\deg_{x_\ell}(x_i x_s) < \deg_{x_\ell}(x_j x_k)$ and $x_i(x_j x_k)/x_\ell \in I(G)$. Thus, either $x_i x_j \in I(G)$ or $x_i x_k \in I(G)$. This is a contradiction. Hence $\{j, k\} \in E(G^c)$, as desired. $\square$

Corollary 7.3.2 *Let $I \subset S$ be a matroidal ideal generated in degree two. Then $\text{HS}_k(I)$ is a matroidal ideal, for all $k \geq 0$.*

Proof. Let G be the simple graph on $[n]$ such that $I = I(G)$. By Lemma 7.3.1 and Theorem 3.3.5, G^c is a reversible chordal graph and any ordering of the variables is a reversible perfect elimination order of G^c. By Theorem 7.1.4, for all $k \geq 0$, $\text{HS}_k(I)$ has linear quotients with respect to the lexicographic order induced by any ordering of the variables. Thus, by Theorem 3.4.2, $\text{HS}_k(I)$ is matroidal, for all $k \geq 0$. $\square$

Now, we turn to the non squarefree case.

Theorem 7.3.3 *Let $I \subset S$ be a polymatroidal ideal generated in degree two. Then, $\text{HS}_k(I)$ is polymatroidal, for all $k \geq 0$.*

Proof. If I is squarefree, the thesis follows from Corollary 7.3.2. Suppose I is not squarefree. Up to a suitable relabeling, we can write $I = (J, x_1^2, x_2^2, \dots, x_t^2)$, where J is the squarefree part of I, *i.e.*, $G(J) = \{u \in G(I) : u \text{ is squarefree}\}$ and $1 \leq t \leq n$. Then J is a matroidal ideal. Let G be the simple graph on $[n]$ with $J = I(G)$, then G^c is cochordal. Let $u_1, \dots, u_m$ be an admissible order of J. We claim that

$$u_1, \dots, u_m, x_1^2, x_2^2, \dots, x_t^2$$

is an admissible order of I. We only need to prove that

$$Q = (u_1, \dots, u_m, x_1^2, \dots, x_{\ell-1}^2) : x_\ell^2 = (J, x_1^2, \dots, x_{\ell-1}^2) : x_\ell^2$$

is generated by variables. Indeed, let $x_i x_j : x_\ell^2 \in Q$ be a generator with $i \leq j$. If $x_i x_j : x_\ell^2$ is a variable, there is nothing to prove. Otherwise $x_i x_j : x_\ell^2 = x_i x_j$, and $\ell \neq i, j$. Since $\deg_{x_\ell}(x_\ell^2) > \deg_{x_\ell}(x_i x_j)$, by the exchange property, $w = x_k(x_\ell^2)/x_\ell = x_k x_\ell \in I$, with $k = i$ or $k = j$. Then $k \neq \ell$, $w = x_k x_\ell \in J$ and $w : x_\ell^2 = x_k \in Q$ is a variable that divides $x_i x_j : x_\ell^2$, as desired.

We claim that $\text{set}(x_\ell^2) = [n] \setminus \{\ell\}$, for all $\ell = 1, \dots, t$. Let $i \in [n] \setminus \{\ell\}$. Then $x_i x_j \in G(I)$ for some j. If $j = \ell$, then $x_i x_\ell \in I$. Suppose $j \neq \ell$, then $\deg_{x_j}(x_i x_j) > \deg_{x_j}(x_\ell^2)$. By the exchange property, $x_i x_\ell \in I$, as desired.

By equation (5.1), for all $k>0$,

$$\mathrm{HS}_k(I) \;=\; \mathrm{HS}_k(J)+\sum_{\ell=1}^{t} x_\ell^2\cdot \mathrm{HS}_{k-1}((x_i : i\in[n]\setminus\{\ell\})).$$

We set $J_\ell=(x_i : i\in[n]\setminus\{\ell\})$, $\ell=1,\dots,t$. Since J is matroidal, $\mathrm{HS}_k(J)$ is matroidal by Corollary 7.3.2. Moreover, each ideal J_ℓ is generated by variables, and so it is matroidal. Hence all ideals $x_\ell^2\cdot\mathrm{HS}_{k-1}(J_\ell)$ are polymatroidal.

To verify that $\mathrm{HS}_k(I)$ is polymatroidal, we check the exchange property. Let $u,v\in G(\mathrm{HS}_k(I))$ and i such that $\deg_{x_i}(u)>\deg_{x_i}(v)$.

To achieve our goal, we note the following fact. Let $w\in S$ be any squarefree monomial of degree $k+1$ and let $\ell\in[t]$. Then $x_\ell w\in\mathrm{HS}_k(I)$. Indeed, if x_ℓ divides w, then $x_\ell w\in x_\ell^2\cdot\mathrm{HS}_{k-1}(J_\ell)\subset\mathrm{HS}_k(I)$. Suppose x_ℓ does not divide w. For all i such that x_i divides w, $x_ix_\ell\in J$ because $i\neq\ell$. Fix a lexicographic order $\succ$ such that $x_\ell>x_i$ for all $i\in[n]\setminus\{\ell\}$. Up to relabeling, we can assume $\ell=1$ and that $\succ$ is induced by $x_1>x_2>\dots>x_n$. Writing $x_\ell w=x_\ell x_{j_2}\cdots x_{j_{k+2}}$ with $\ell=1<j_2<\dots<j_{k+2}\le n$, then $x_\ell x_{j_{k+2}}\in J$, $x_\ell x_{j_i}\in J$ and $x_\ell x_{j_i}\succ x_\ell x_{j_{k+2}}$, for $i=2,\dots,k+1$. Hence

$$\{j_2,\dots,j_{k+1}\}\subseteq\{j \mid x_j\in(u\in G(J): u\succ x_\ell x_{j_{k+2}}) : x_\ell x_{j_{k+2}}\}.$$

This shows that $x_\ell w\in\mathrm{HS}_k(J)\subset\mathrm{HS}_k(I)$, because by Theorem 3.4.2, J has linear quotients with respect to $\succ$.

If $u,v\in\mathrm{HS}_k(J)$ or $u,v\in x_\ell^2\cdot\mathrm{HS}_{k-1}(J_\ell)$, we can find j with $\deg_{x_j}(u)<\deg_{x_j}(v)$ such that $x_j(u/x_i)\in\mathrm{HS}_k(I)$, because both $\mathrm{HS}_k(J), x_\ell^2\cdot\mathrm{HS}_{k-1}(J_\ell)$ are polymatroidal.

Suppose now $u\in\mathrm{HS}_k(J)$ and $v\in x_\ell^2\cdot\mathrm{HS}_{k-1}(J_\ell)$. Then $\deg_{x_\ell}(u)<\deg_{x_\ell}(v)$ and $x_\ell(u/x_i)\in\mathrm{HS}_k(I)$, because u/x_i is a squarefree monomial of degree $k+1$.

Suppose $u\in x_\ell^2\cdot\mathrm{HS}_{k-1}(J_\ell)$ and $v\in\mathrm{HS}_k(J)$. Let j such that $\deg_{x_j}(u)<\deg_{x_j}(v)$. Then $\deg_{x_j}(u)=0$. If $i=\ell$, then $x_j(u/x_\ell)\in\mathrm{HS}_k(I)$ because it is the product of x_ℓ times a squarefree monomial of degree $k+1$. If $i\neq\ell$, then $x_j(u/x_i)$ can also be written as such a product. In any case $x_j(u/x_i)\in\mathrm{HS}_k(I)$.

Finally, suppose $u\in x_\ell\cdot\mathrm{HS}_{k-1}(J_\ell)$ and $v\in x_h^2\cdot\mathrm{HS}_{k-1}(J_h)$ with $\ell\neq h$. Suppose $i=\ell$ and let j such that $\deg_{x_j}(u)<\deg_{x_j}(v)$. Then $u'=x_j(u/x_i)$ is either x_ℓ times a squarefree monomial of degree $k+1$, or is equal to x_h times a squarefree monomial of degree $k+1$. In both cases, $u'\in\mathrm{HS}_k(I)$. Suppose now $i\neq\ell$. If there exist more than one j with $\deg_{x_j}(u)<\deg_{x_j}(v)$, we can choose $j\neq h$. Then $\deg_{x_j}(v)=1$ and so x_j does not divide u. Consequently $x_j(u/x_i)$ is equal to x_ℓ times a squarefree monomial of degree $k+1$, and so $x_j(u/x_i)\in\mathrm{HS}_k(I)$. If there is only one j such that $\deg_{x_j}(u)<\deg_{x_j}(v)$, then $j=h$. We claim that x_h does not divide u, then $x_h(u/x_i)$ is equal to x_ℓ times a squarefree monomial of degree $k+1$, and so $x_h(u/x_i)\in\mathrm{HS}_k(I)$, as wanted. Writing $v=x_h^2x_{j_1}\cdots x_{j_k}$, with $j_p\in[n]\setminus\{h\}$, $p=1,\dots,k$, then $\deg_{x_{j_p}}(v)=1\le\deg_{x_{j_p}}(u)$, for all $p=1,\dots,k$. Then $x_{j_1}\cdots x_{j_k}$ divides $u/(x_ix_\ell)$ because $\deg_{x_\ell}(u)>1\ge\deg_{x_\ell}(v)$ and $\deg_{x_i}(u)=1>\deg_{x_i}(v)$. This implies that $u=x_ix_\ell\cdot x_{j_1}\cdots x_{j_k}$. From this presentation it follows that x_h does not divide u, because $i,\ell\neq h$ and $j_p\neq h$ for $p=1,\dots,k$, as well.

The cases above show that the exchange property holds for all monomials of $G(\mathrm{HS}_k(I))$. Hence $\mathrm{HS}_k(I)$ is polymatroidal and the proof is complete. $\square$

Chapter 8

Homological shifts of very well–covered graphs

Since the foundations of *Algebraic Graph Theory*, one of the fundamental problem has been to classify all Cohen–Macaulay graphs, that is all Cohen–Macaulay edge ideals [149]. This is an hopeless task. However, the dual problem, namely to classify all Cohen–Macaulay cover ideals of a graph, is completely solved. Indeed the cover ideal $J(G)$ of a graph G is the *Alexander dual* of the edge ideal of G, *i.e.*, $J(G) = I(G)^\vee$, and the fundamental *Eagon–Reiner Criterion* says that $I(G)^\vee$ is Cohen–Macaulay if and only if $I(G)$ has a linear resolution [46]. All graphs G such that $I(G)$ has a linear resolution have been characterized by Ralph Fröberg in [75]. Furthermore, Herzog, Hibi and Zheng showed that for any graph G such that $I(G)$ has a linear resolution, then $I(G)$ has *linear powers*, *i.e.*, for all $k \geq 1$, $I(G)^k$ has a linear resolution [94].

Inspired by the problems above, in this chapter we consider the dual problem of classifying the graphs G for which $J(G)$ has a *homological linear resolution* and those with *homological linear quotients*. Of course, first one needs to know all graphs G such that $J(G)$ has a linear resolution. One can observe that for $J(G)$ to have a linear resolution, it is necessary that it is *equigenerated*, which is equivalent to requiring that G is *unmixed* in the sense that all the minimal vertex covers of G have the same cardinality. On the other hand, if G is unmixed and without isolated vertices, then $2\,\text{height}(I(G)) \geq |V(G)|$ [76]. Therefore, to study the question above it is natural to restrict ourself to the "best possible class" of graphs: *all unmixed graphs G without isolated vertices and such that* $2\,\text{height}(I(G)) = |V(G)|$. In such a case, G is called a *very well–covered graph*. Such a class of graphs has been studied by many authors (see, for instance, [40, 58, 115, 116, 117, 119, 124]).

In Section 8.2, we investigate Betti splittings of the cover ideals of Cohen–Macaulay very well–covered graphs (Proposition 8.2.3). The Betti splitting technique introduced in [74] by Francisco, Ha and Van Tuyl is a very useful tool throughout.

In Section 8.3, if G is a Cohen–Macaulay very well–covered graph, we use the Betti splitting, discussed in Section 8.2, to construct explicitly a minimal free resolution of $J(G)$. A minimal free resolution of $J(G)$ has also been constructed in [115] by different tools. The main advantage of the Betti splitting technique is that one does not need to verify if the defined complex is exact or minimal, but only to determine the suitable *comparison maps* [74, Proposition 2.1]. The features of our resolution are required in Section 8.4 to achieve our main goal. In the last part of Section 8.3, we draw some consequences of Theorem 8.3.5. Firstly, we get a formula for the Betti numbers of $J(G)$ which is independent on the characteristic of the base field K (Corollary 8.3.6). Then, some formulas for the projective dimension of the cover ideal of a Cohen–Macaulay very well–covered graph G are determined. In particular, in Corollary 8.3.7, we recover a result due to Mahmoudi *et all* [124, Lemma 3.4] and, furthermore, we provide a new proof of it by Betti splitting (Remark 8.3.8). Finally, we show that $J(G)$ has the *alternating sum property* [150, Definition 4.1] and also we determine a formula for the multiplicity of $S/J(G)$ (Proposition 8.3.10).

Section 8.4 contains our main result which states that the cover ideal of a Cohen–Macaulay very well–covered graph has homological linear quotients (Theorem 8.4.1). Then, we infer that a very well–covered graph G is Cohen–Macaulay if and only if $J(G)$ has homological linear resolution which is equivalent to $J(G)$ having homological linear quotients (Theorem 8.4.2). An interesting consequence about the cover ideal of the *whisker graph* G^* of a graph G is drawn in Corollary 8.4.3.

Our experiments and the results obtained have lead us to conjecture that given a Cohen–Macaulay very well–covered graph G, all the powers of $J(G)$ have homological linear quotients (Conjecture 8.4.4). At present we are able to prove our conjecture only for the subclass of Cohen–Macaulay bipartite graphs. For this goal, we carefully study the powers of the *Hibi ideals*. In 1987, Hibi introduced a fundamental class of ideals associated to a finite poset [103]. More precisely, let $(P, \succeq)$ be a finite poset, a *poset ideal* $\mathcal{I}$ of P is a subset of P such that for any $\alpha \in \mathcal{I}$ and any $\beta \in P$ with $\beta \preceq \alpha$, then $\beta \in \mathcal{I}$. If $\mathcal{J}(P)$ is the set of all poset ideals of P, ordered by inclusion, then $\mathcal{J}(P)$ is a distributive lattice. Indeed, Birkhoff's fundamental theorem [143, Theorem 3.4.1] shows that any distributive lattice arises in such a way. To any $\mathcal{I} \in \mathcal{J}(P)$ one can associate the squarefree monomial $u_{\mathcal{I}} = (\prod_{p\in\mathcal{I}} x_p)(\prod_{p\in P\setminus\mathcal{I}} y_p)$ in the polynomial ring $K[\{x_p, y_p\}_{p\in P}]$. Then the Hibi ideal H_P associated to $(P, \succeq)$ is the squarefree monomial ideal generated by all $u_{\mathcal{I}}$, $\mathcal{I} \in \mathcal{J}(P)$. The importance of such a class lies in the fact that the class of Hibi ideals coincides with the class of the cover ideals of Cohen–Macaulay bipartite graphs [88] (see, also, [89, Lemma 9.1.9 and Theorem 9.1.13]). Therefore, one can focus on the homological shifts of powers of Hibi ideals. In Construction 8.4.6, we associate to any poset $(P, \succeq)$ and any integer $\ell \geq 1$ a new poset $(P(\ell), \succeq_\ell)$ and then we show that the ℓth power of H_P is equal to $H_{P(\ell)}$ up to *polarization* (Theorem 8.4.9).

Finally, observing that polarization commutes with homological shifts (Lemma 8.4.10) and preserves also the linear quotients property (Lemma 3.1.2), we state that all the powers of an Hibi ideal have homological linear quotients (Corollary 8.4.11).

All the examples in this chapter have been verified using *Macaulay2* [82] and the package `HomologicalShiftIdeals` [59].

8.1 A glimpse to very well–covered graphs.

In this section, we analyze the class of *very well–covered graphs* (see, for instance, [116] and the references therein). Let G be an unmixed graph without isolated vertices and let $I(G)$ be its edge ideal in S. Gitler and Valencia [76, Corollary 3.4] showed that

$$2\,\mathrm{height}(I(G)) \geq |V(G)|.$$

Definition 8.1.1 A graph G is called *very well–covered* if it is unmixed without isolated vertices and with $2\,\mathrm{height}(I(G)) = |V(G)|$.

Hereafter, with abuse of notation and to simplify the notation, we denote a vertex $i \in V(G)$ by x_i, and an edge $\{i, j\} \in E(G)$ by $x_i x_j$.

By [58, Theorem 1.2], very well–covered graphs have always perfect matchings. Hence, for a very well–covered graph G with $2n$ vertices, we may assume

$(*)$ $V(G) = X \cup Y$, $X \cap Y = \varnothing$, with $X = \{x_1, \dots, x_n\}$ a minimal vertex cover of G and $Y = \{y_1, \dots, y_n\}$ a maximal independent set of G such that $\{x_1y_1, \dots, x_ny_n\} \subseteq E(G)$.

It is important to point out that when we assume the condition $(*)$ for a very well–covered graph, we do not force any restriction on the graph. Indeed, it is only a relabeling of the vertices.

For the remainder of this chapter, we set $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with K a field. For a positive integer n, we denote the set $\{1, 2, \ldots, n\}$ by $[n]$.

Theorem 8.1.2 (Crupi–Rinaldo–Terai, 2011 [40, Theorem 3.6]). *Let G be a graph with $2n$ vertices, which are not isolated, with $\operatorname{height}(I(G)) = n$. We assume condition $(*)$ and also we assume that if $x_iy_j \in E(G)$ then $i \leq j$. Then, the following conditions are equivalent:*

(a) *G is a Cohen–Macaulay very well–covered graph.*

(b) *The following conditions hold:*

  (i) *if $x_iy_j \in E(G)$ then $x_ix_j \notin E(G)$,*

  (ii) *if $z_ix_j, y_jx_k \in E(G)$ then $z_ix_k \in E(G)$ for any distinct i, j, k and $z_i \in \{x_i, y_i\}$.*

For our convenience, we reformulate Theorem 8.1.2, as follows.

Characterization 8.1.3 ([40], [124, Lemma 3.1]). *Let G be a very well–covered graph with $2n$ vertices. Then, the following conditions are equivalent.*

(a) *G is Cohen–Macaulay.*

(b) *There exists a relabeling of $V(G) = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ such that*

  (i) *$X = \{x_1, \ldots, x_n\}$ is a minimal vertex cover of G and $Y = \{y_1, \ldots, y_n\}$ is a maximal independent set of G,*

  (ii) *$x_iy_i \in E(G)$ for all $i \in [n]$,*

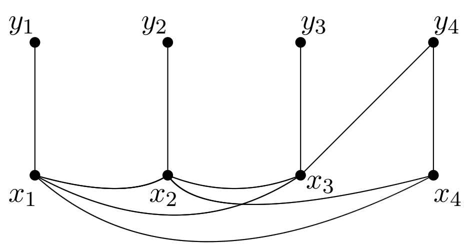
  (iii) *if $x_iy_j \in E(G)$ then $i \leq j$,*

  (iv) *if $x_iy_j \in E(G)$ then $x_ix_j \notin E(G)$,*

  (v) *if $z_ix_j, y_jx_k \in E(G)$ then $z_ix_k \in E(G)$ for any distinct i, j, k and $z_i \in \{x_i, y_i\}$.*

The next example illustrates the previous characterization.

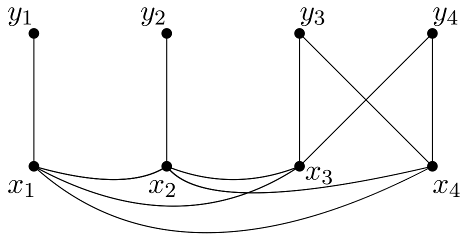
Example 8.1.4 The following graph G is an example of a Cohen–Macaulay very well–covered graph with 8 vertices.



G

Indeed, for $S = K[x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4]$, $\dim S/I(G) = 4 = \operatorname{depth} S/I(G)$ and moreover, its minimal vertex covers are: $\{x_1, x_2, x_3, x_4\}$, $\{x_1, y_2, x_3, x_4\}$, $\{y_1, x_2, x_3, x_4\}$, $\{x_1, x_2, x_3, y_4\}$, $\{x_1, x_2, y_3, y_4\}$.

The next graph G is an example of a not Cohen–Macaulay very well–covered graph.



G

Indeed, G is very well–covered. Its minimal vertex covers are the sets $\{x_1, x_2, x_3, x_4\}$, $\{x_1, y_2, x_3, x_4\}$, $\{y_1, x_2, x_3, x_4\}$, $\{x_1, x_2, y_3, y_4\}$. But G is not Cohen–Macaulay. Note that $\dim S/I(G) = 4 \neq \operatorname{depth} S/I(G) = 3$, with $S = K[x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4]$.

8.2 Betti splittings of cover ideals of very well–covered graphs

In this section we analyze the Betti splittings of the cover ideals of the class of Cohen–Macaulay very well–covered graphs.

Our first result shows that if we remove some pairs of vertices of a Cohen–Macaulay very well–covered graph in a suitable way, then we obtain "smaller graphs" which are again Cohen–Macaulay very well–covered graphs.

Proposition 8.2.1 *Let G be a Cohen–Macaulay very well–covered graph with $2n$ vertices and assume the condition $(*)$. Then $G \setminus \{x_i, y_i : i \in A\}$ is a Cohen–Macaulay very well–covered graph, for any subset $A \subseteq [n]$.*

Proof. Firstly, note that since G is a Cohen–Macaulay very well–covered graph with $2n$ vertices and condition $(*)$ holds, then $\text{height}(I(G)) = n$. Moreover, from [40, Lemma 3.5], we may assume that if $x_iy_j \in E(G)$ then $i \leq j$.

If $A = \varnothing$, there is nothing to prove. Now, let $A \neq \varnothing$ and set $A = \{i_1, i_2, \dots, i_t\}$. We show that

$$G_0 = G \setminus \{x_i, y_i : i \in A\} = G \setminus \{x_{i_1}, y_{i_1}, x_{i_2}, y_{i_2}, \dots, x_{i_t}, y_{i_t}\}$$

is a Cohen–Macaulay very well–covered graph. Note that G_0 has $2(n-t)$ vertices that are not isolated. Moreover, G_0 satisfies condition $(*)$ for $X_0 = X \setminus \{x_{i_1}, \dots, x_{i_t}\}$ and $Y_0 = Y \setminus \{y_{i_1}, \dots, y_{i_t}\}$, and $I(G_0)$ has height $n-t$, since X_0 is a minimal vertex cover of G_0. It is clear that G_0 satisfies the conditions (i)–(ii) of Theorem 8.1.2(b), because G satisfies such conditions. By Theorem 8.1.2, we get that G_0 is also a Cohen–Macaulay very well–covered graph. □

Let $F \subseteq [n]$ be a non empty set, we set $\mathbf{x}_F = \prod_{i \in F} x_i$, $\mathbf{y}_F = \prod_{i \in F} y_i$. Otherwise, we set $\mathbf{x}_\varnothing = \mathbf{y}_\varnothing = 1$. For a monomial $u \in S = K[x_1, \dots, x_n, y_1, \dots, y_n]$, we define *support* of u the set

$$\text{supp}(u) = \{x_i : x_i \text{ divides } u\} \cup \{y_j : y_j \text{ divides } u\}.$$

From now, when we tell about a Cohen–Macaulay very well–covered graph G with $2n$ vertices, we tacitly assume that there exists a relabeling of the set of vertices $V(G) = \{x_1, \dots, x_n, y_1, \dots, y_n\}$ which satisfy the conditions (i)–(v) of Characterization 8.1.3.

Lemma 8.2.2 *Let G be a Cohen–Macaulay very well–covered graph with $2n$ vertices. For each $u \in G(J(G))$ there exists a unique subset F of $[n]$ such that $u = \mathbf{x}_F\mathbf{y}_{[n]\setminus F}$.*

Proof. Let $u \in G(J(G))$. By definition, u is a squarefree monomial whose support $C = \operatorname{supp}(u)$ is a minimal vertex cover of G. Since G is very well–covered, u has degree $|V(G)|/2 = n$. By Characterization 8.1.3(b)(ii), $x_ny_n \in E(G)$. Hence $z_n \in C$ with $z_n \in \{y_n, x_n\}$. Note that $C_1 = C \setminus \{z_n\}$ is a vertex cover of $G_1 = G \setminus \{y_n, x_n\}$. By Proposition 8.2.1, G_1 is again Cohen–Macaulay very well–covered. Since $|C_1| = |C| - 1 = |V(G)|/2 - 1 = |V(G_1)|/2$, then C_1 is a minimal vertex cover of G_1. Thus the monomial u_1 whose support is C_1 is a minimal generator of $J(G_1)$. By induction, $u_1 = \mathbf{x}_{F_1}\mathbf{y}_{[n-1]\setminus F_1}$ for a unique subset F_1 of $[n-1]$. If $z_n = x_n$, let $F = F_1 \cup \{n\}$. Otherwise, if $z_n = y_n$, let $F = F_1$. In both cases, $u = z_nu_1 = \mathbf{x}_F\mathbf{y}_{[n]\setminus F}$. $\square$

If G is a Cohen–Macaulay very well–covered graph with $2n$ vertices, then

$$N[x_n] = \{x_{i_1}, x_{i_2}, \dots, x_{i_t}, x_n, y_n\}, \text{ with } i_r < n,\ r \in [t],$$

$$N[y_n] = \{x_{j_1}, x_{j_2}, \dots, x_{j_p}, x_n, y_n\}, \text{ with } j_q < n,\ q \in [p].$$

Moreover, from Characterization 8.1.3, $i_r \neq j_q$, for all $r \in [t]$ and $q \in [p]$. We will consider again such sets in the next section (see, Setup 8.3.1 and Lemma 8.3.2).

Proposition 8.2.3 *Let G be a Cohen–Macaulay very well–covered graph with $2n$ vertices. Let $N[x_n] = \{x_{i_1}, x_{i_2}, \dots, x_{i_t}, x_n, y_n\}$, $N[y_n] = \{x_{j_1}, x_{j_2}, \dots, x_{j_p}, x_n, y_n\}$ and*

$$\begin{aligned} G_1 &= G \setminus \{x_{i_1}, y_{i_1}, x_{i_2}, y_{i_2}, \dots, x_{i_t}, y_{i_t}, x_n, y_n\}, \\ G_2 &= G \setminus \{x_{j_1}, y_{j_1}, x_{j_2}, y_{j_2}, \dots, x_{j_p}, y_{j_p}, x_n, y_n\}. \end{aligned}$$

Then

$$J(G) = x_{i_1}x_{i_2}\cdots x_{i_t}\cdot y_nJ(G_1) + x_{j_1}x_{j_2}\cdots x_{j_p}\cdot x_nJ(G_2)$$

is a Betti splitting.

Proof. Let $J_1 = x_{i_1}x_{i_2}\cdots x_{i_t}\cdot y_nJ(G_1)$ and $J_2 = x_{j_1}x_{j_2}\cdots x_{j_t}\cdot x_nJ(G_2)$. Note that y_n does not divide any minimal monomial generator of J_2. Thus $G(J_1)\cap G(J_2) = \emptyset$. We claim that J_1 and J_2 have n–linear resolutions. Indeed, by Proposition 8.2.1, G_1 and G_2 are again Cohen–Macaulay very well–covered graphs and our claim follows from Criterion 3.2.4. By virtue of Proposition 2.4.13, to prove that $J(G) = J_1 + J_2$ is a Betti splitting, it is enough to show that $J(G) = J_1 + J_2$ is a y_n-partition of $J(G)$. Indeed, let u be a minimal generator of $J(G)$. Then $C = \operatorname{supp}(u)$ is a minimal vertex cover of G. By Lemma 8.2.2, $u = \mathbf{x}_F\mathbf{y}_{[n]\setminus F}$ for some $F \subseteq [n]$. Thus, either $y_n \in C$ or $x_n \in C$. We distinguish two cases.

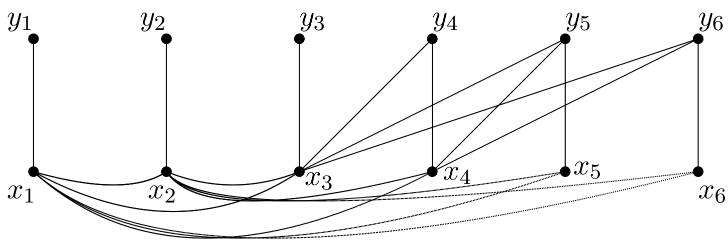
Case 1. Suppose $y_n \in C$. Since $x_n \notin C$, but $x_{i_1}x_n, \dots, x_{i_t}x_n \in E(G)$ and C is a vertex cover, then we obtain that $x_{i_1}, \dots, x_{i_t} \in C$. Hence $x_{i_1}x_{i_2}\cdots x_{i_t}\cdot y_n$ divides u. Note that the support of $v = u/(x_{i_1}x_{i_2}\cdots x_{i_t}\cdot y_n)$ is a vertex cover of G_1. Furthermore, $\operatorname{supp}(v)$ is a minimal vertex cover, since $|\operatorname{supp}(v)| = n - (t+1) = |V(G_1)|/2$ and G_1 is a very well–covered graph (Proposition 8.2.1). Hence $v \in J(G_1)$ and $u \in J_1$.

Case 2. Suppose $x_n \in C$. Since $y_n \notin C$, but $x_{j_1}y_n, \dots, x_{j_p}y_n \in E(G)$ and C is a vertex cover, then $x_{j_1}, \dots, x_{j_p} \in C$. Hence $x_{j_1}x_{j_2}\cdots x_{j_p}\cdot x_n$ divides u. Note that the support of $w = u/(x_{j_1}x_{j_2}\cdots x_{j_p}\cdot x_n)$ is a vertex cover of G_2. Furthermore, $\operatorname{supp}(w)$ is a minimal vertex cover, since $|\operatorname{supp}(w)| = n - (p+1) = |V(G_2)|/2$ and G_2 is a very well–covered graph (Proposition 8.2.1). Hence, $w \in J(G_2)$ and $u \in J_2$.

Hence, we have shown that $G(J(G))$ is contained in $G(J_1) \cup G(J_2)$.

For the opposite inclusion, let $v \in G(J(G_1))$, then $\text{supp}(v)$ is a minimal vertex cover of G_1. We claim that $C = \text{supp}(x_{i_1}\cdots x_{i_t}\cdot y_n v)$ is a minimal vertex cover of G. Indeed, all edges of G incident with a vertex belonging to $\{x_{i_1},\dots,x_{i_t},y_n\}$ are incident with a vertex of C. Since $N[x_n]\setminus\{x_n,y_n\} = \{x_{i_1},\dots,x_{i_t}\}\subset C$, then each edge which is incident with x_n is also incident with a vertex of C. Finally, let e be an edge incident with y_{i_k} for some $k\in[t]$. Then $e = x_jy_{i_k}\in E(G)$. Since $x_jy_{i_k}$ and $x_{i_k}x_n$ are both edges of G, by Characterization 8.1.3(v), $x_jx_n\in E(G)$, too. It follows that $x_j\in N[x_n]\setminus\{x_n,y_n\}$ and $\{x_j,y_{i_k}\}\cap C\neq\varnothing$. Finally, C is a minimal vertex cover of G. This shows that $G(J_1)\subseteq G(J(G))$. Similarly, one proves that $G(J_2)\subseteq G(J(G))$ by exploiting again condition (v) of Characterization 8.1.3. The result follows. $\square$

Example 8.2.4 Let $S = K[x_1,\dots,x_6,y_1,\dots,y_6]$. Consider the following graph G.



G

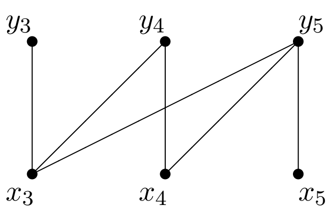
By Characterization 8.1.3, one verifies that G is a Cohen–Macaulay very well–covered graph with 12 vertices. We have

$$\begin{aligned} I(G) &= (x_1y_1,x_2y_2,x_3y_3,x_4y_4,x_5y_5,x_6y_6,x_1x_2,x_1x_3,x_1x_4,x_1x_5,\\ &\quad\; x_1x_6,x_2x_3,x_2x_4,x_2x_5,x_2x_6,x_3y_4,x_3y_5,x_3y_6,x_4y_5,x_4y_6),\\ J(G) &= (x_1x_2x_3x_4x_5x_6,y_1x_2x_3x_4x_5x_6,x_1y_2x_3x_4x_5x_6,x_1x_2x_3x_4y_5x_6,\\ &\quad\; x_1x_2x_3x_4x_5y_6,x_1x_2x_3x_4y_5y_6,x_1x_2x_3y_4y_5y_6,x_1x_2y_3y_4y_5y_6). \end{aligned}$$

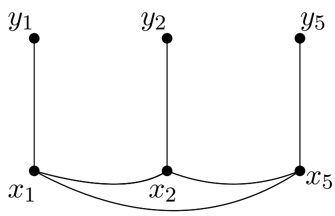
Furthermore, $N[x_6] = \{x_1,x_2,x_6,y_6\}$, $N[y_6] = \{x_3,x_4,x_6,y_6\}$ and consequently

$$G_1 = G\setminus\{x_1,y_1,x_2,y_2,x_6,y_6\},\ \ G_2 = G\setminus\{x_3,y_3,x_4,y_4,x_6,y_6\},$$

i.e.,



G_1



G_2

It follows that

$$\begin{aligned} J(G_1) &= (x_3x_4x_5,x_3x_4y_5,x_3y_4y_5,y_3y_4y_5),\\ J(G_2) &= (x_1x_2x_5,y_1x_2x_5,x_1y_2x_5,x_1x_2y_5). \end{aligned}$$

Finally,

$$J(G) = x_1x_2y_6J(G_1) + x_3x_4x_6J(G_2)$$

is a Betti splitting.

8.3 The resolution of cover ideals of very well–covered graphs

In this section we construct the minimal free resolution of the cover ideal $J(G)$ of any Cohen–Macaulay very well–covered graph G. Our method uses induction on half of the number of vertices of G and the mapping cone construction. Indeed, the mapping cone applied to the Betti splitting of $J(G)$ (Proposition 8.2.3) yields a minimal free resolution of $J(G)$, if we know the minimal free resolutions of the cover ideals of three suitable subgraphs which we can associate to the given graph G.

Firstly, let G_1 and G_2 be two graphs, we define *intersection graph* of G_1 ad G_2 the graph G with $V(G) = V(G_1) \cap V(G_2)$ and $E(G) = \{e : e \in E(G_1) \cap E(G_2)\}$. We denote G by $G_1 \cap G_2$.

Setup 8.3.1 Let G be a Cohen–Macaulay very well–covered graph with $2n$ vertices. Let $N[x_n] = \{x_{i_1}, x_{i_2}, \ldots, x_{i_t}, x_n, y_n\}$, $N[y_n] = \{x_{j_1}, x_{j_2}, \ldots, x_{j_p}, x_n, y_n\}$ and define

$$\begin{aligned} G_1 &= G \setminus \{x_{i_1}, y_{i_1}, x_{i_2}, y_{i_2}, \ldots, x_{i_t}, y_{i_t}, x_n, y_n\}, \\ G_2 &= G \setminus \{x_{j_1}, y_{j_1}, x_{j_2}, y_{j_2}, \ldots, x_{j_p}, y_{j_p}, x_n, y_n\}. \end{aligned}$$

By Proposition 8.2.3, both G_1 and G_2 are Cohen–Macaulay very well–covered graphs. Furthermore, $|V(G_1)| = 2(n-1-t)$ and $|V(G_2)| = 2(n-1-p)$.
Set $J = J(G)$, $J_1 = x_{i_1}x_{i_2}\cdots x_{i_t} \cdot y_n J(G_1)$ and $J_2 = x_{j_1}x_{j_2}\cdots x_{j_p} \cdot x_n J(G_2)$. Then, Proposition 8.2.3 implies that $J = J_1 + J_2$ is a Betti splitting of J.
Finally, let us consider the subgraph $G_3 = G_1 \cap G_2$ of G. Since the structure of G_1 and G_2, it is clear that G_3 is Cohen–Macaulay very well–covered, too.

For instance, in Example 8.2.4, $G_3 = G \setminus \{x_i, y_i : i \neq 5\}$.

The following lemmas will be crucial in the sequel.

Lemma 8.3.2 *Assume Setup 8.3.1. Then*

$$N[x_n] \cap N[y_n] = \{x_n, y_n\}.$$

Proof. For all $s \in [p]$, $x_{j_s}y_n \in E(G)$. Thus Characterization 8.1.3(iv) implies that $x_{j_s}x_n \notin E(G)$, and so $x_{j_s} \notin N[x_n]$, for all $s \in [p]$. Finally $N[x_n] \cap N[y_n] = \{x_n, y_n\}$, as desired. □

Lemma 8.3.3 *Assume Setup 8.3.1. Then*

$$J_1 \cap J_2 = x_{i_1}x_{i_2}\cdots x_{i_t} \cdot x_{j_1}x_{j_2}\cdots x_{j_p} \cdot x_n y_n J(G_3).$$

Proof. Set $J_{1,2} = J_1 \cap J_2$. Since $J = J_1 + J_2$ is a Betti splitting and J has an n–linear resolution, then $J_{1,2}$ has an $(n+1)$–linear resolution (Proposition 2.4.14). A generating set for $J_{1,2}$ is the set $\{\mathrm{lcm}(u_1, u_2) : u_1 \in G(J_1), u_2 \in G(J_2)\}$. Since $J_{1,2}$ is equigenerated in degree $n+1$, then

$$G(J_{1,2}) = \{\mathrm{lcm}(u_1, u_2) : u_1 \in G(J_1),\ u_2 \in G(J_2),\ \deg(\mathrm{lcm}(u_1, u_2)) = n+1\}.$$

By Lemma 8.3.2, $N[x_n] \cap N[y_n] = \{x_n, y_n\}$. Thus, by the presentation of J_1 and J_2, we get that each monomial $w \in G(J_{1,2})$ is divided by $x_{i_1}\cdots x_{i_t} \cdot x_{j_1}\cdots x_{j_p} \cdot x_n y_n$. Note that $(X \cup Y) \setminus \{x_{i_1}, y_{i_1}, \ldots, x_{i_t}, y_{i_t}, x_{j_1}, y_{j_1}, \ldots, x_{j_p}, y_{j_p}, x_n, y_n\}$ is the vertex set of the graph G_3. Moreover the support of $v_1 = u_1/(x_{i_1}x_{i_2}\cdots x_{i_t} \cdot x_{j_1}x_{j_2}\cdots x_{j_p} \cdot y_n)$ is a minimal vertex cover of the very well–covered graph G_3. Since $\mathrm{lcm}(u_1, u_2) = x_n u_1$, we get that $\mathrm{lcm}(u_1, u_2) = x_{i_1} \cdots x_{i_t} \cdot x_{j_1} \cdots x_{j_p} \cdot x_n y_n v_1 \in G(x_{i_1} \cdots x_{i_t} \cdot x_{j_1} \cdots x_{j_p} \cdot x_n y_n J(G_3))$.

Conversely, let $w = x_{i_1} \cdots x_{i_t} \cdot x_{j_1} \cdots x_{j_p} \cdot x_n y_n v \in G(x_{i_1} \cdots x_{i_t} \cdot x_{j_1} \cdots x_{j_p} \cdot x_n y_n J(G_3))$, with $v \in J(G_3)$. Then $\mathrm{supp}(x_{j_1} x_{j_2} \cdots x_{j_p} \cdot v)$ is a minimal vertex cover of G_1 and $\mathrm{supp}(x_{i_1} x_{i_2} \cdots x_{i_t} \cdot v)$ is a minimal vertex cover of G_2. Thus $u_1 = w/x_n \in G(J_1)$ and $u_2 = w/y_n \in G(J_2)$. Since $w = \mathrm{lcm}(u_1, u_2)$ and $\deg(w) = n + 1$, we have that $w \in G(J_{1,2})$, showing the other inclusion. □

We now turn to the construction of the minimal free resolution. For a subset C of the set of variables $X \cup Y = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, we set

$$\mathbf{z}_C = \mathbf{x}_{C_x} \mathbf{y}_{C_y}, \tag{8.1}$$

with $C_x = \{i : x_i \in C\}$ and $C_y = \{j : y_j \in C\}$.

If G is graph such that $V(G) = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ and $C \in \mathcal{C}(G)$, we define the set $\mathcal{C}(G; C) = \{x_s : y_s \in C \text{ and } (C \setminus y_s) \cup x_s \in \mathcal{C}(G)\}$. Recall that $\mathcal{C}(G)$ is the set of all minimal vertex cover of G.

For instance, in Example 8.1.4, for $C = \{x_1, x_2, y_3, y_4\}$, $\mathcal{C}(G; C) = \{x_3\}$. Indeed, $\{x_1, x_2, y_4\} \cup x_3 = \{x_1, x_2, x_3, y_4\}$ is a minimal vertex cover of the given graph G. One can observe that $x_4 \notin \mathcal{C}(G; C)$, since $\{x_1, x_2, y_3\} \cup x_4$ is not a vertex cover of G. Moreover, for $C = \{x_1, x_2, x_3, x_4\}$, $\mathcal{C}(G; C) = \varnothing$.

In what follows, we denote by $\binom{\mathcal{C}(G;C)}{i}$ the set of all subsets of size i of $\mathcal{C}(G; C)$, $0 \le i \le |\mathcal{C}(G; C)|$. With abuse of notation, for $\sigma \in \binom{\mathcal{C}(G;C)}{i}$, we set $\mathbf{x}_\sigma = \prod_{x_s \in \sigma} x_s$. In particular, $\mathbf{x}_\varnothing = 1$.

Construction 8.3.4 Assume Setup 8.3.1. Let

$$\mathbb{F} \ : \quad \cdots \to F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} J \to 0$$

be the complex

- whose ith free module F_i has as a basis the symbols $\mathbf{f}(C; \sigma)$ having multidegree $\mathbf{z}_C \mathbf{x}_\sigma$, where $C \in \mathcal{C}(G)$ and $\sigma \in \binom{\mathcal{C}(G;C)}{i}$, *i.e.*, $\sigma \subseteq \mathcal{C}(G; C)$ is a subset of size i;
- and whose ith differential is given by $d_0(\mathbf{f}(C; \varnothing)) = \mathbf{z}_C$ for $i = 0$ and for $i > 0$ is defined as follows:

$$d_i(\mathbf{f}(C; \sigma)) \ = \ \sum_{x_s \in \sigma} (-1)^{\alpha(\sigma; x_s)} \big[y_s \mathbf{f}((C \setminus y_s) \cup x_s; \sigma \setminus x_s) - x_s \mathbf{f}(C; \sigma \setminus x_s) \big],$$

where $\alpha(\sigma; x_s) = |\{x_j \in \sigma : j > s\}|$.

From now on, we set $J_{1,2} = J_1 \cap J_2$ and denote by $(\mathbb{F}_{J_{1,2}}, d^{J_{1,2}})$, $(\mathbb{F}_{J_1}, d^{J_1})$, $(\mathbb{F}_{J_2}, d^{J_2})$ the minimal free resolutions of $J_{1,2}$, J_1, J_2, respectively.

Theorem 8.3.5 *The complex $\mathbb{F}$ given in Construction 8.3.4 is the minimal free resolution of $J = J(G)$.*

Proof. We proceed by strong induction on $|V(G)|/2 = n$. For $n = 1$, $I(G) = (x_1 y_1)$ and $J = J(G) = (x_1, y_1)$. In this case one readily verifies that the complex $\mathbb{F}$ of Construction 8.3.4 is the minimal free resolution of $J = J(G)$. So, let $n > 1$. The graphs G_1, G_2, G_3 have vertex sets whose cardinality is less than $|V(G)|$, thus by induction we can assume that the minimal free resolutions of $J(G_1), J(G_2), J(G_3)$ are as given in Construction 8.3.4. As a consequence, we know explicitly the resolutions $\mathbb{F}_{J_{1,2}}, \mathbb{F}_{J_1}, \mathbb{F}_{J_2}$, since each of these resolutions is equal to one of the three previously mentioned resolutions up to multiplication by a suitable monomial.

Let $(\mathbb{F}_J, d^J)$ be the resolution obtained by the mapping cone applied to the Betti splitting $J = J_1 + J_2$. Then $\mathbb{F}_J$ is the minimal free resolution of J. We show that $\mathbb{F}_J$ can be identified with $\mathbb{F}$. We achieve this goal in three steps.

Step 1. Let us show that the free modules of $\mathbb{F}_J$ have the basis described in Construction 8.3.4. By the mapping cone, we have $F_i^J = F_{i-1}^{J_{1,2}} \oplus F_i^{J_1} \oplus F_i^{J_2}$.

The bases of $F_i^{J_1}, F_i^{J_2}$ and $F_i^{J_{1,2}}$ are the following ones, respectively:

$$\mathcal{B}_1 = \Big\{x_{i_1}x_{i_2}\cdots x_{i_t}\cdot y_n \cdot \mathbf{f}_{J_1}(C_1;\sigma_1)\ :\ C_1 \in \mathcal{C}(G_1),\ \sigma_1 \in \binom{\mathcal{C}(G_1;C_1)}{i}\Big\},$$

$$\mathcal{B}_2 = \Big\{x_{j_1}x_{j_2}\cdots x_{j_p}\cdot x_n \cdot \mathbf{f}_{J_2}(C_2;\sigma_2)\ :\ C_2 \in \mathcal{C}(G_2),\ \sigma_2 \in \binom{\mathcal{C}(G_2;C_2)}{i}\Big\},$$

$$\mathcal{B}_3 = \Big\{x_{i_1}\cdots x_{i_t}\cdot x_{j_1}\cdots x_{j_p}\cdot x_n y_n \cdot \mathbf{f}_{J_{1,2}}(C_3;\sigma_3)\ :\ C_3 \in \mathcal{C}(G_3),\ \sigma_3 \in \binom{\mathcal{C}(G_3;C_3)}{i}\Big\}.$$

Let $\mathbf{f}(C;\sigma)$ be a basis element of F_i as in Construction 8.3.4, *i.e.*, with multidegree $\mathbf{z}_C\mathbf{x}_\sigma$ and with $C \in \mathcal{C}(G)$, $\sigma \in \binom{\mathcal{C}(G;C)}{i}$. We distinguish three possible cases.

Case 1.1. Let $x_n \in C$. Then $N[y_n] \setminus N[x_n] = \{x_{j_1}, \dots, x_{j_p}\} \subset C$. Indeed, C is a vertex cover of G, $y_n \notin C$ (Lemma 8.2.2) but $x_{j_r}y_n \in E(G)$ for $r \in [p]$. Hence, $C_2 = C \setminus \{x_{j_1}, \dots, x_{j_p}, x_n\}$ is a minimal vertex cover of G_2. Furthermore, for all $x_s \in \sigma$, $(C_2 \setminus y_s) \cup x_s$ is a minimal vertex cover of G_2 because $(C \setminus y_s) \cup x_s$ is a minimal vertex cover of G. Since $\mathbf{b} = x_{j_1}x_{j_2}\cdots x_{j_p}\cdot x_n \cdot \mathbf{f}_{J_2}(C_2;\sigma) \in \mathcal{B}_2$ has the same multidegree of $\mathbf{f}(C;\sigma)$, we can identify $\mathbf{f}(C;\sigma)$ with $\mathbf{b}$.

Case 1.2. Let $y_n \in C$ and $x_n \notin \sigma$. Then $C_1 = C\setminus\{x_1, \dots, x_{i_t}, y_n\}$ is a minimal vertex cover of G_1. As before, we can identify $\mathbf{f}(C;\sigma)$ with $x_{i_1}\dots x_{i_t}\cdot y_n \cdot \mathbf{f}_{J_1}(C_1;\sigma) \in \mathcal{B}_1$.

Case 1.3. Let $y_n \in C$ and $x_n \in \sigma$. Then both C and $(C\setminus y_n)\cup x_n$ are minimal vertex covers of G. Hence, $(N[x_n] \cup N[y_n]) \setminus \{x_n, y_n\} = \{x_{i_1}, \dots, x_{i_t}, x_{j_1}, \dots, x_{j_p}\} \subset C$. Setting $C_3 = C \setminus \{x_{i_1}, \dots, x_{i_t}, x_{j_1}, \dots, x_{j_p}, y_n\}$ and $\sigma_3 = \sigma \setminus x_n$, we obtain that C_3 and $(C_3 \setminus y_s) \cup x_s$ are both minimal vertex covers of G_3 for all $x_s \in \sigma_3$. Hence, in this case we can identify $\mathbf{f}(C;\sigma)$ with $x_{i_1}\cdots x_{i_t}\cdot x_{j_1}\cdots x_{j_p}\cdot x_n y_n \cdot \mathbf{f}_{J_{1,2}}(C_3;\sigma_3) \in \mathcal{B}_3$.

Conversely, any basis element $\mathbf{b} \in \mathcal{B}_i$ $(i = 1,2,3)$ can be identified with a basis element $\mathbf{f}(C;\sigma)$, as given in Construction 8.3.4. Thus, we realize that the modules F_i^J have the required bases as described in Construction 8.3.4.

Step 2. Let $\psi_{-1} : u \in J_{1,2} \mapsto (u, -u) \in J_1 \oplus J_2$. In order to apply the mapping cone, we need to construct the comparison maps ψ_i making the following diagram

$$\begin{tikzcd}
\mathbb{F}_{J_{1,2}} : \cdots \arrow[r] & F_2^{J_{1,2}} \arrow[r, "d_2^{J_{1,2}}"] \arrow[d, "\psi_2"'] & F_1^{J_{1,2}} \arrow[r, "d_1^{J_{1,2}}"] \arrow[d, "\psi_1"'] & F_0^{J_{1,2}} \arrow[r, "d_0^{J_{1,2}}"] \arrow[d, "\psi_0"'] & J_{1,2} \arrow[r] \arrow[d, "\psi_{-1}"] & 0 \\
\mathbb{F}_{J_1} \oplus \mathbb{F}_{J_2} : \cdots \arrow[r] & F_2^{J_1} \oplus F_2^{J_2} \arrow[r, "d_2^{J_1}\oplus d_2^{J_2}"'] & F_1^{J_1} \oplus F_1^{J_2} \arrow[r, "d_1^{J_1}\oplus d_1^{J_2}"'] & F_0^{J_1} \oplus F_0^{J_2} \arrow[r, "d_0^{J_1}\oplus d_0^{J_2}"'] & J_1 \oplus J_2 \arrow[r] & 0
\end{tikzcd}$$

commutative. Using the notation in (8.1), let $\mathbf{b} = x_{i_1}\cdots x_{i_t}\cdot x_{j_1}\cdots x_{j_p}\cdot x_n y_n \cdot$

$\mathbf{f}_{J_{1,2}}(C_3;\sigma_3) = \mathbf{z}_{N[x_n]\cup N[y_n]}\cdot\mathbf{f}_{J_{1,2}}(C_3;\sigma_3)\in\mathcal{B}_3$ be a basis element of $F_i^{J_{1,2}}$. We define

$$\psi_i(\mathbf{b}) \;=\; \big(\mathbf{z}_{N[x_n]}\cdot\mathbf{f}_{J_1}(C_3\cup\{x_{j_1},\dots,x_{j_p}\};\sigma_3),\ -\mathbf{z}_{N[y_n]}\cdot\mathbf{f}_{J_2}(C_3\cup\{x_{i_1},\dots,x_{i_t}\};\sigma_3)\big).$$

To simplify the notation, we set $C_1 = C_3\cup\{x_{j_1},\dots,x_{j_p}\}$ and $C_2 = C_3\cup\{x_{i_1},\dots,x_{i_t}\}$. One can note that ψ_i is well defined. Indeed, $C_1\in\mathcal{C}(G_1)$, $C_2\in\mathcal{C}(G_2)$, $\sigma_3\subseteq\mathcal{C}(G_1;C_1)$, and $\sigma_3\subseteq\mathcal{C}(G_2;C_2)$.

We need to verify that for all $i\ge 0$ and all $\mathbf{b}\in\mathcal{B}_3$, it is

$$\psi_{i-1}\circ d_i^{J_{1,2}}(\mathbf{b}) \;=\; (d_i^{J_1}\oplus d_i^{J_2})\circ\psi_i(\mathbf{b}).$$

For $i=0$, we have $\sigma_3=\varnothing$, thus $\mathbf{x}_{\sigma_3}=1$ and both sides of the equation are equal to

$$(\mathbf{z}_{N[x_n]\cup N[y_n]}\mathbf{z}_{C_3},-\mathbf{z}_{N[x_n]\cup N[y_n]}\mathbf{z}_{C_3}).$$

Let $i>0$. To further simplify the notation, let us write the generic basis element $(\mathbf{z}_{N[x_n]}\mathbf{f}_{J_1}(C_1;\sigma_1),\mathbf{z}_{N[y_n]}\mathbf{f}_{J_2}(C_2;\sigma_2))\in F_i^{J_1}\oplus F_i^{J_2}$ as $\mathbf{z}_{N[x_n]}\mathbf{f}_{J_1}(C_1;\sigma_1)+\mathbf{z}_{N[y_n]}\mathbf{f}_{J_2}(C_2;\sigma_2)$. By induction we know explicitly $d_i^{J_{1,2}}, d_i^{J_1}, d_i^{J_2}$. Let us compute $\psi_{i-1}\circ d_i^{J_{1,2}}(\mathbf{b})$. We have

$$d_i^{J_{1,2}}(\mathbf{b}) = \mathbf{z}_{N[x_n]\cup N[y_n]}\sum_{x_s\in\sigma_3}(-1)^{\alpha(\sigma_3;x_s)}\big[y_s\mathbf{f}_{J_{1,2}}((C_3\setminus y_s)\cup x_s;\sigma_3\setminus x_s)-x_s\mathbf{f}_{J_{1,2}}(C_3;\sigma_3\setminus x_s)\big].$$

Hence,

$$\begin{aligned}\psi_{i-1}\circ d_i^{J_{1,2}}(\mathbf{b}) \;=\; \sum_{x_s\in\sigma_3}(-1)^{\alpha(\sigma_3;x_s)}\big[\,& y_s\mathbf{z}_{N[x_n]}\cdot\mathbf{f}_{J_1}((C_1\setminus y_s)\cup x_s;\sigma_3\setminus x_s)\\ &-y_s\mathbf{z}_{N[y_n]}\cdot\mathbf{f}_{J_2}((C_2\setminus y_s)\cup x_s;\sigma_3\setminus x_s)\\ &-x_s\mathbf{z}_{N[x_n]}\cdot\mathbf{f}_{J_1}(C_1;\sigma_3\setminus x_s)\\ &+x_s\mathbf{z}_{N[y_n]}\cdot\mathbf{f}_{J_2}(C_2;\sigma_3\setminus x_s)\big].\end{aligned}\tag{8.2}$$

Now, let us compute $(d_i^{J_1}\oplus d_i^{J_2})\circ\psi_i(\mathbf{b})$. Firstly, we have

$$\psi_i(\mathbf{b}) \;=\; \mathbf{z}_{N[x_n]}\cdot\mathbf{f}_{J_1}(C_1;\sigma_3)-\mathbf{z}_{N[y_n]}\cdot\mathbf{f}_{J_2}(C_2;\sigma_3),$$

then

$$\begin{aligned}(d_i^{J_1}\oplus d_i^{J_2})\circ\psi_i(\mathbf{b}) &= \mathbf{z}_{N[x_n]}\cdot d_i^{J_1}(\mathbf{f}_{J_1}(C_1;\sigma_3))-\mathbf{z}_{N[y_n]}\cdot d_i^{J_2}(\mathbf{f}_{J_2}(C_2;\sigma_3))\\ &= \mathbf{z}_{N[x_n]}\sum_{x_s\in\sigma_3}(-1)^{\alpha(\sigma_3;x_s)}\big[y_s\mathbf{f}_{J_1}((C_1\setminus y_s)\cup x_s;\sigma_3\setminus x_s)-x_s\mathbf{f}_{J_1}(C_1;\sigma_3\setminus x_s)\big]\\ &\quad-\mathbf{z}_{N[y_n]}\sum_{x_s\in\sigma_3}(-1)^{\alpha(\sigma_3;x_s)}\big[y_s\mathbf{f}_{J_2}((C_2\setminus y_s)\cup x_s;\sigma_3\setminus x_s)-x_s\mathbf{f}_{J_2}(C_2;\sigma_3\setminus x_s)\big].\end{aligned}$$

A comparison with equation (8.2) shows that $\psi_{i-1}\circ d_i^{J_{1,2}}(\mathbf{b}) \;=\; (d_i^{J_1}\oplus d_i^{J_2})\circ\psi_i(\mathbf{b})$.

Step 3. It remains to prove that the differentials d_i^J act as described in Construction 8.3.4. So, let $\mathbf{f}(C;\sigma)$, $C\in\mathcal{C}(G)$ and $\sigma\in\binom{\mathcal{C}(G;C)}{i}$ be a basis element. Let us compute $d_i^J(\mathbf{f}(C;\sigma))$. For $i=0$, we easily see that $d_0^J(\mathbf{f}(C;\varnothing))=\mathbf{z}_C$, as desired.

Let $i>0$. We distinguish three cases.

Case 3.1. Let $x_n\in C$. Set $C_2 = C\setminus\{x_{j_1},\dots,x_{j_p},x_n\}$. By the mapping cone, we have:

$$
\begin{aligned}
d_i^J(\mathbf{f}(C;\sigma)) &= d_i^{J_2}(x_{j_1}\cdots x_{j_p}\cdot x_n\cdot \mathbf{f}_{J_2}(C_2;\sigma))\\
&= x_{j_1}\cdots x_{j_p}\cdot x_n \sum_{x_s\in\sigma}(-1)^{\alpha(\sigma;x_s)}\big[y_s\mathbf{f}_{J_2}((C_2\setminus y_s)\cup x_s;\sigma\setminus x_s)-x_s\mathbf{f}_{J_2}(C_2;\sigma\setminus x_s)\big]\\
&= \sum_{x_s\in\sigma}(-1)^{\alpha(\sigma;x_s)}\big[y_s\mathbf{f}((C\setminus y_s)\cup x_s;\sigma\setminus x_s)-x_s\mathbf{f}(C;\sigma\setminus x_s)\big],
\end{aligned}
$$

as required.

CASE 3.2. Let $y_n\in C$ and $x_n\notin\sigma$. Setting $C_1=C\setminus\{x_{i_1},\ldots,x_{i_t},x_n\}$, by the mapping cone we see that $d_i^J(\mathbf{f}(C;\sigma))=d_i^{J_1}(x_{i_1}x_{i_2}\cdots x_{i_t}\cdot y_n\cdot \mathbf{f}_{J_1}(C_1;\sigma))$ has the required expression.

CASE 3.3. Let $y_n\in C$ and $x_n\in\sigma$. Setting $C_3=C\setminus\{x_{i_1},\ldots,x_{i_t},x_{j_1},\ldots,x_{j_p},y_n\}$, $\sigma_3=\sigma\setminus x_n$ and $\mathbf{b}=x_{i_1}\cdots x_{i_t}\cdot x_{j_1}\cdots x_{j_p}\cdot x_ny_n\cdot \mathbf{f}_{J_{1,2}}(C_3;\sigma_3)$, by the mapping cone, we have that

$$d_i^J(\mathbf{f}(C;\sigma)) \;=\; -d_{i-1}^{J_{1,2}}(\mathbf{b})+\psi_{i-1}(\mathbf{b}).$$

Let us compute $-d_{i-1}^{J_{1,2}}(\mathbf{b})$. Since $\mathbf{z}_{N[x_n]\cup N[y_n]}=x_{i_1}\cdots x_{i_t}\cdot x_{j_1}\cdots x_{j_p}\cdot x_ny_n$, we have

$$
\begin{aligned}
-d_{i-1}^{J_{1,2}}(\mathbf{b}) &= -\mathbf{z}_{N[x_n]\cup N[y_n]}\sum_{x_s\in\sigma_3}(-1)^{\alpha(\sigma_3;x_s)}\big[y_s\mathbf{f}_{J_{1,2}}((C_3\setminus y_s)\cup x_s;\sigma_3\setminus x_s)-x_s\mathbf{f}_{J_{1,2}}(C_3;\sigma_3\setminus x_s)\big]\\
&= -\sum_{x_s\in\sigma_3}(-1)^{\alpha(\sigma\setminus x_n;x_s)}\big[y_s\mathbf{f}((C\setminus y_s)\cup x_s;\sigma\setminus x_s)-x_s\mathbf{f}(C;\sigma\setminus x_s)\big]\\
&= \sum_{x_s\in\sigma\setminus x_n}(-1)^{\alpha(\sigma;x_s)}\big[y_s\mathbf{f}((C\setminus y_s)\cup x_s;\sigma\setminus x_s)-x_s\mathbf{f}(C;\sigma\setminus x_s)\big],
\end{aligned}
$$

where the last equation follows from the fact that $\alpha(\sigma;x_s)=\alpha(\sigma\setminus x_n;x_s)+|\{x_n\}|$ for all $x_s\in\sigma\setminus x_n$.

Whereas, for the term $\psi_{i-1}(\mathbf{b})$, by the definition of the comparison maps, we have

$$
\begin{aligned}
\psi_{i-1}(\mathbf{b}) &= \mathbf{z}_{N[x_n]}\cdot\mathbf{f}_{J_1}(C_3\cup\{x_{j_1},\ldots,x_{j_p}\};\sigma_3)-\mathbf{z}_{N[y_n]}\cdot\mathbf{f}_{J_2}(C_3\cup\{x_{i_1},\ldots,x_{i_t}\};\sigma_3)\\
&= y_n\mathbf{f}((C\setminus y_n)\cup x_n;\sigma\setminus x_n)-x_n\mathbf{f}(C;\sigma\setminus x_n)\\
&= (-1)^{\alpha(\sigma;x_n)}\big[y_n\mathbf{f}((C\setminus y_n)\cup x_n;\sigma\setminus x_n)-x_n\mathbf{f}(C;\sigma\setminus x_n)\big],
\end{aligned}
$$

where the last equation follows because $\alpha(\sigma;x_n)=|\varnothing|=0$. Hence, we see that

$$d_i^J(\mathbf{f}(C;\sigma)) \;=\; \sum_{x_s\in\sigma}(-1)^{\alpha(\sigma;x_s)}\big[y_s\mathbf{f}((C\setminus y_s)\cup x_s;\sigma\setminus x_s)-x_s\mathbf{f}(C;\sigma\setminus x_s)\big],$$

as desired. The induction is complete and the result follows. □

Corollary 8.3.6 *Let G be a Cohen–Macaulay very well–covered graph. Then*

$$
\begin{aligned}
\beta_i(J(G)) &= \sum_{C\in\mathcal{C}(G)}\binom{|\mathcal{C}(G;C)|}{i}, \qquad i\geq 0,\\
\mathrm{pd}(J(G)) &= \max\{\big|\mathcal{C}(G;C)\big| \;:\; C\in\mathcal{C}(G)\}.
\end{aligned}
$$

In particular, the graded Betti numbers of $J(G)$ do not depend upon the characteristic of the underlying field K.

Let G be a graph and let $z_iz_j, z_kz_\ell\in E(G)$ be a pair of edges. We say that z_iz_j and z_kz_ℓ are 3-*disjoint* if the induced subgraph of G on the vertex set $\{z_i,z_j,z_k,z_\ell\}$

consists of two disjoint edges. The maximum size of a set of pairwise 3-disjoint edges in G is denoted by $a(G)$ [124].

The next result holds.

Corollary 8.3.7 *Let G be a Cohen–Macaulay very well–covered graph. Then*

$$a(G) = \operatorname{reg}(S/I(G)) = \max\{|\mathcal{C}(G;C)| \ : \ C \in \mathcal{C}(G)\} = \operatorname{pd}(J(G)).$$

Proof. By a result of Terai, [89, Proposition 8.1.10], we have $\operatorname{pd}(J(G)) = \operatorname{reg}(S/I(G))$. Thus the assertion follows from [124, Lemma 3.4] and Corollary 8.3.6. ☐

Remark 8.3.8 The equality $a(G) = \operatorname{reg}(S/I(G))$ has been firstly proved in [124, Lemma 3.4]. Here we give a new proof by Betti splittings. By [114, Lemma 2.2], one always have $\operatorname{reg}(S/I(G)) \geq a(G)$. It remains to prove that $\operatorname{pd}(J(G)) \leq a(G)$. Assume Setup 8.3.1.

By [74, Corollary 2.2] applied to the Betti splitting provided in Proposition 8.2.3, and by Lemma 8.3.3, we have

$$\operatorname{pd}(J(G)) \ = \ \max\{\operatorname{pd}(J(G_1)),\ \operatorname{pd}(J(G_2)),\ \operatorname{pd}(J(G_3)) + 1\}.$$

By induction on $n = |V(G)|/2$, we may assume that $\operatorname{pd}(J(G_i)) \leq a(G_i)$ for $i = 1, 2, 3$. Note that $a(G_i) \leq a(G)$, for $i = 1, 2, 3$, clearly. Thus it remains to prove the inequality $a(G_3) + 1 \leq a(G)$. Let D be a set of pairwise 3-disjoint edges of G_3 with $|D| = a(G_3)$. Then $D \cup x_n y_n$ is again a set of pairwise 3-disjoint edges of G. Indeed, if $e = z_k z_\ell \in D$, then $\{z_k, z_\ell\} \subseteq V(G_3) = V(G) \setminus \{x_i, y_i : x_i \in N[x_n] \cup N[y_n]\}$. Thus there can not be any edge connecting z_k or z_ℓ with either x_n or y_n. This shows that the induced subgraph with vertex set $\{z_k, z_\ell, x_n, y_n\}$ has only two edges. Hence $a(G) \geq |D| + 1 = a(G_3) + 1$.

Let M be a finitely generated S-module of dimension d, and let P_M be the Hilbert polynomial of M [23, 89]. Then, $P_M(t) = \sum_{i=0}^{d-1}(-1)^{d-1-i} e_{d-1-i}\binom{t+i}{i}$, $e_{d-1-i} \in \mathbb{Q}$, for all i. We define the *multiplicity* of M as

$$e(M) \ = \ \begin{cases} e_0 & \text{if } d > 0, \\ \operatorname{length}(M) & \text{if } d = 0. \end{cases}$$

Now, we verify that $J(G)$ has the *alternating sum property* and thanks to Corollary 8.3.6 we get a formula for the *multiplicity* of $S/J(G)$, $e(S/J(G))$.

We quote next definition from [150, Definition 4.1].

Definition 8.3.9 Let I be a monomial ideal in S with $G(I) = \{u_1, \ldots, u_m\}$ and let $d = \min\{\deg(u_i) : i = 1, \ldots, m\}$. We say that I has the *alternating sum property*, if

$$\sum_{i \geq 1}(-1)^i \beta_{i,i+j}(S/I) = \begin{cases} -1, & \text{for } j = d, \\ 0, & \text{for } j > d. \end{cases}$$

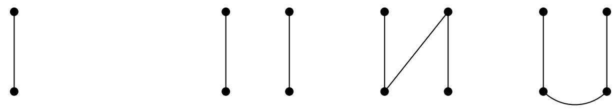
Proposition 8.3.10 *Let G be a Cohen–Macaulay very well–covered graph. Then*

(a) $\sum_{i \geq 1}(-1)^i \beta_i(S/J(G)) = -1$.

(b) $e(S/J(G)) = |E(G)| = \frac{1}{2}\sum_{C \in \mathcal{C}(G)} \sum_{i=0}^{|\mathcal{C}(G;C)|}(-1)^{i+1}\binom{|\mathcal{C}(G;C)|}{i}(n+i)^2$.

Proof. Let $|V(G)| = 2n$.

(a). We proceed by induction on $n \geq 1$. For $n = 1, 2$ the only Cohen–Macaulay very well–covered graphs are the following ones.



For each of these graphs, (a) holds. Let $n > 2$. Using the same notation as in Setup 8.3.1, from Proposition 8.2.3, Lemma 8.3.3 and by the inductive hypothesis on the graphs G_1, G_2, G_3, we have that

$$\begin{aligned}
&\sum_{i\geq 1}(-1)^i\beta_i(S/J(G)) = \sum_{i\geq 0}(-1)^i\left[\beta_i(S/J(G_1)) + \beta_i(S/J(G_2)) + \beta_{i-1}(S/J(G_3))\right] \\
&= \sum_{i\geq 0}(-1)^i\beta_i(S/J(G_1)) + \sum_{i\geq 0}(-1)^i\beta_i(S/J(G_2)) - \sum_{i\geq 0}(-1)^i\beta_i(S/J(G_3)) \ = \\
&= \ -1-1+1 \ = \ -1.
\end{aligned}$$

(b). Since $J(G) = \bigcap_{z_iz_j\in E(G)}(z_i, z_j)$ we see that $J(G)$ has height two and, furthermore, $e(S/J(G)) = |E(G)|$ by [89, Corollary 6.2.3]. The first equality holds true.

On the other hand, by a formula of Peskine and Szpiro [136] (see also [89, Corollary 6.1.7]), since $\beta_i(S/J(G)) = \beta_{i-1}(J(G))$, one has that

$$\begin{aligned}
e(S/J(G)) &= \frac{(-1)^h}{h!}\sum_{i=1}^{\mathrm{pd}(S/J(G))}(-1)^{i+1}\sum_{j=1}^{\beta_i(S/J(G))} a_{ij}^h = \\
&= \frac{(-1)^h}{h!}\sum_{i=0}^{\mathrm{pd}(J(G))}(-1)^{i+1}\sum_{j=1}^{\beta_i(J(G))} a_{ij}^h,
\end{aligned}$$

where $h = \mathrm{height}(J(G))$ and a_{ij} are the shifts of the ith free module of the minimal free resolution $\mathbb{F}$ of $S/J(G)$, *i.e.*, $\bigoplus_{j=1}^{\beta_i(S/J(G))} S(-a_{ij})$. Since $J(G)$ has a n–linear resolution and $\mathrm{height}(J(G)) = h = 2$, from Corollary 8.3.6, we have that

$$\begin{aligned}
e(S/J(G)) &= \frac{1}{2}\sum_{i\geq 0}(-1)^{i+1}\sum_{C\in\mathcal{C}(G)}\binom{|\mathcal{C}(G;C)|}{i}(n+i)^2 \ = \\
&= \frac{1}{2}\sum_{C\in\mathcal{C}(G)}\sum_{i=0}^{|\mathcal{C}(G;C)|}(-1)^{i+1}\binom{|\mathcal{C}(G;C)|}{i}(n+i)^2.
\end{aligned}$$

□

Example 8.3.11 Let us consider the Cohen–Macaulay well–covered graph G with 12 vertices in Example 8.2.4. Set $C_1 = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $C_2 = \{y_1, x_2, x_3, x_4, x_5, x_6\}$, $C_3 = \{x_1, y_2, x_3, x_4, x_5, x_6\}$, $C_4 = \{x_1, x_2, x_3, x_4, y_5, x_6\}$, $C_5 = \{x_1, x_2, x_3, x_4, x_5, y_6\}$, $C_6 = \{x_1, x_2, x_3, x_4, y_5, y_6\}$, $C_7 = \{x_1, x_2, x_3, y_4, y_5, y_6\}$ and $C_8 = \{x_1, x_2, y_3, y_4, y_5, y_6\}$.

Then

$\mathcal{C}(G)$	$\mathcal{C}(G;C)$	$\lvert\mathcal{C}(G;C)\rvert$
C_1	$\varnothing$	0
C_2	$\{x_1\}$	1
C_3	$\{x_2\}$	1
C_4	$\{x_5\}$	1
C_5	$\{x_6\}$	1
C_6	$\{x_5, x_6\}$	2
C_7	$\{x_4\}$	1
C_8	$\{x_3\}$	1

Using [82], one can verify that $\beta_1(S/J(G)) = 8$, $\beta_2(S/J(G)) = 8$, $\beta_3(S/J(G)) = 1$. Hence, $\sum_{i\geq 1}(-1)^i\beta_i(S/J(G)) = -1$ and the alternating sum property holds. Moreover, $a(G) = \operatorname{reg}(S/I(G)) = 2 = \max\{\lvert\mathcal{C}(G;C)\rvert \;:\; C \in \mathcal{C}(G)\} = \operatorname{pd}(J(G))$.

Finally, for the multiplicity of $S/J(G)$, we have $e(S/J(G)) = |E(G)| = 20$.

8.4 Homological shifts of vertex cover ideals of very well–covered graphs and applications to Hibi ideals

In this section, we investigate the homological shift ideals of powers of the vertex cover ideal of Cohen–Macaulay very well–covered graphs. Here is our main result.

In what follows we refer to the notation in Setup 8.3.1.

Theorem 8.4.1 *Let G be a Cohen–Macaulay very well–covered graph with $2n$ vertices. Then $\mathrm{HS}_k(J(G))$ has linear quotients with respect to the lexicographic order $>_{\mathrm{lex}}$ induced by $x_n > y_n > x_{n-1} > y_{n-1} > \cdots > x_1 > y_1$, for all $k \geq 0$.*

Proof. We proceed by strong induction on $n = |V(G)|/2 \geq 1$. For $n = 1$, $I(G) = (x_1y_1)$ and $J = J(G) = (x_1, y_1)$. Hence, $\mathrm{HS}_0(J) = J$, $\mathrm{HS}_1(J) = (x_1y_1)$ have linear quotients with respect to $>_{\mathrm{lex}}$.

Let $n > 1$. By Proposition 8.2.3,

$$J(G) \;=\; x_{j_1}x_{j_2}\cdots x_{j_p}\cdot x_nJ(G_2) + x_{i_1}x_{i_2}\cdots x_{i_t}\cdot y_nJ(G_1) \qquad (8.3)$$

is a Betti splitting.

Set $f = x_{j_1}\cdots x_{j_p}\cdot x_n$ and $g = x_{i_1}\cdots x_{i_t}\cdot y_n$. Since $\mathrm{HS}_k(wI) = w\cdot\mathrm{HS}_k(I)$ for all monomial ideals I in S and all non zero monomials $w \in S$, from (8.3), by Lemma 8.3.3, we have that

$$\mathrm{HS}_k(J(G)) \;= fg\cdot\mathrm{HS}_{k-1}(J(G_3)) + f\cdot\mathrm{HS}_k(J(G_2)) + g\cdot\mathrm{HS}_k(J(G_1)).$$

Note that, since $|V(G_j)| < |V(G)|$ $(j = 1, 2, 3)$, by the inductive hypothesis, $\mathrm{HS}_k(J(G_j))$ have linear quotients with respect to $>_{\mathrm{lex}}$.

Let

$$G(\mathrm{HS}_{k-1}(J(G_3))) \;=\; \{w_1 >_{\mathrm{lex}} w_2 >_{\mathrm{lex}} \cdots >_{\mathrm{lex}} w_q\}, \qquad (8.4)$$
$$G(\mathrm{HS}_k(J(G_2))) \;=\; \{v_1 >_{\mathrm{lex}} v_2 >_{\mathrm{lex}} \cdots >_{\mathrm{lex}} v_r\}, \qquad (8.5)$$
$$G(\mathrm{HS}_k(J(G_1))) \;=\; \{u_1 >_{\mathrm{lex}} u_2 >_{\mathrm{lex}} \cdots >_{\mathrm{lex}} u_s\}, \qquad (8.6)$$

thus $G(\mathrm{HS}_k(J(G)))$ is ordered as follows:

$$fgw_1 >_{\text{lex}} \cdots >_{\text{lex}} fgw_q \quad >_{\text{lex}} \quad fv_1 >_{\text{lex}} \cdots >_{\text{lex}} fv_r \quad >_{\text{lex}} \quad gu_1 >_{\text{lex}} \cdots >_{\text{lex}} gu_s.$$

We prove that such an order is an admissible order of $\mathrm{HS}_k(J(G))$. For our purpose, we need to show that all the following three colon ideals

$$(fgw_1, \dots, fgw_{\ell-1}) : fgw_\ell, \quad \ell \in \{2, \dots, q\}, \tag{8.7}$$
$$(fg\mathrm{HS}_{k-1}(J(G_3)), fv_1, \dots, fv_{\ell-1}) : fv_\ell, \quad \ell \in \{1, \dots, r\}, \tag{8.8}$$
$$(fg\mathrm{HS}_{k-1}(J(G_3)), f\mathrm{HS}_k(J(G_2)), gu_1, \dots, gu_{\ell-1}) : gu_\ell, \quad \ell \in \{1, \dots, s\}, \tag{8.9}$$

are generated by variables.

First Colon Ideal. Let us consider the colon ideal in (8.7). Since,

$$(fgw_1, \dots, fgw_{\ell-1}) : fgw_\ell = (w_1, \dots, w_{\ell-1}) : w_\ell,$$

the assertion follows from the fact that $\mathrm{HS}_{k-1}(J(G_3))$ has linear quotients with respect to the order in (8.4).

Second Colon Ideal. Let us consider the colon ideal in (8.8).
Set $P = (fg\mathrm{HS}_{k-1}(J(G_3)), fv_1, \dots, fv_{\ell-1}) : fv_\ell$. One can observe that

$$P = (fg\mathrm{HS}_{k-1}(J(G_3))) : fv_\ell + (v_1, \dots, v_{\ell-1}) : v_\ell,$$

with $(v_1, \dots, v_{\ell-1}) : v_\ell$ generated by variables. Indeed, $\mathrm{HS}_k(J(G_2))$ has linear quotients with respect to the order in (8.5).
Thus, if we show that each generator of $(fg\mathrm{HS}_{k-1}(J(G_3))) : fv_\ell$ is divided by a variable of P, we conclude that P is generated by variables, as wanted. The colon ideal $(fg\mathrm{HS}_{k-1}(J(G_3))) : fv_\ell$ is generated by the monomials

$$\frac{\mathrm{lcm}(fgw_j, fv_\ell)}{fv_\ell}, \quad j = 1, \dots, q.$$

Fix $j \in \{1, \dots, q\}$. By Construction 8.3.4, we have that

$$\begin{aligned} fgw_j &= w\mathbf{x}_\sigma, \quad w \in G(J(G)),\ y_n \in \mathrm{supp}(w),\ \sigma \subseteq \mathcal{C}(G; \mathrm{supp}(w)),\ x_n \in \sigma,\ |\sigma| = k, \\ fv_\ell &= v\mathbf{x}_\tau, \quad v \in G(J(G)),\ x_n \in \mathrm{supp}(v),\ \tau \subseteq \mathcal{C}(G; \mathrm{supp}(v)),\ x_n \notin \tau,\ |\tau| = k. \end{aligned}$$

Let $h = \mathrm{lcm}(w\mathbf{x}_\sigma, v\mathbf{x}_\tau)/(v\mathbf{x}_\tau)$. If $\deg(h) = 1$, h is a variable and there is nothing to prove. Suppose $\deg(h) > 1$. Let us consider the following integer

$$i \ = \ \min\big\{i \ : \ z_i \text{ divides } h = \frac{\mathrm{lcm}(w\mathbf{x}_\sigma, v\mathbf{x}_\tau)}{v\mathbf{x}_\tau},\ z_i \in \{x_i, y_i\}\big\}. \tag{8.10}$$

Since $\deg(h) > 1$ and y_n divides h, we have that $i < n$. From Lemma 8.2.2, we have that y_i divides at least one of the monomials $w\mathbf{x}_\sigma$, $v\mathbf{x}_\tau$. Indeed, if y_i does not divide any of these monomials, then x_i will divide both these monomials, and consequentially, $z_i \in \{x_i, y_i\}$ does not divide h. Against our assumption.
Now, let $N[y_i] = \{x_{k_1}, \dots, x_{k_b}, x_i, y_i\}$. We claim that $x_{k_1} \cdots x_{k_b}$ divides both monomials w, v. Indeed, x_i divides at least one of the monomials w, v. For instance, say x_i divides $w\mathbf{x}_\sigma$, then either x_i divides w or $x_i(w/y_i) \in G(J(G))$. In both cases, $x_{k_1} \cdots x_{k_b}$ must divide w, since w or $x_i(w/y_i)$ are minimal vertex covers. The same reasoning works if x_i divides $v\mathbf{x}_\tau$. Thus $x_{k_1} \cdots x_{k_b}$ must divide w or v. Without loss

of generality, suppose that $x_{k_1}\cdots x_{k_b}$ divides w. Then, by Construction 8.3.4, none of the variables $y_{k_1},\dots,y_{k_b}$ divides w. By Characterization 8.1.3(iii), $k_1<\dots<k_b<i$. Thus, by the meaning of i, $z_d\in\{x_d,y_d\}$ does not divide h for all $d=k_1,\dots,k_b$. Hence, we see that $x_{k_1}\cdots x_{k_b}$ divides both the monomials w, v. Now, we distinguish two cases.

Case 1. Suppose x_iy_i divides one between the monomials $w\mathbf{x}_\sigma$ and $v\mathbf{x}_\tau$, and that y_i divides the other one. Suppose, for instance, that x_iy_i divides $w\mathbf{x}_\sigma$ and that y_i divides $v\mathbf{x}_\tau$. Then, $x_i(v/y_i)\mathbf{x}_\tau>_{\text{lex}}v\mathbf{x}_\tau$. Moreover, $x_i(v/y_i)\in G(J(G))$, because $x_{k_1}\cdots x_{k_b}$ divides v and so $\operatorname{supp}(x_i(v/y_i))$ is again a minimal vertex cover of G.
We claim that $x_i(v/y_i)\mathbf{x}_\tau\in\operatorname{HS}_k(J(G))$. Indeed for all $c\in\tau$, $N[y_c]\setminus\{x_c,y_c\}\subseteq\operatorname{supp}(v)$. Since $N[y_c]\setminus\{x_c,y_c\}$ is a subset of $X=\{x_1,\dots,x_n\}$ and $\operatorname{supp}(x_i(v/y_i))=(\operatorname{supp}(v)\setminus y_i)\cup x_i$, we see that $N[y_c]\setminus\{x_c,y_c\}$ is again a subset of $\operatorname{supp}(x_i(v/y_i))$. Thus, for all $c\in\tau$, $x_cx_i(v/y_i)/y_c\in G(J(G))$, *i.e.*, $\tau\subseteq\mathcal{C}(G;\operatorname{supp}(x_i(v/y_i)))$. Finally, $x_i(v/y_i)\mathbf{x}_\tau\in\operatorname{HS}_k(J(G))$. Moreover,

$$\frac{\operatorname{lcm}(x_i(v/y_i)\mathbf{x}_\tau,v\mathbf{x}_\tau)}{v\mathbf{x}_\tau}=x_i\ \text{ and }\ x_i\ \text{ divides }\ h.$$

It follows that h is divided by the variable x_i belonging to P, as desired.

Otherwise, suppose x_iy_i divides $v\mathbf{x}_\tau$ and y_i divides $w\mathbf{x}_\sigma$. Then, $x_i(w/y_i)\mathbf{x}_\sigma>_{\text{lex}}w\mathbf{x}_\sigma$ and, as before, $x_i(w/y_i)\mathbf{x}_\sigma\in G(\operatorname{HS}_k(J(G)))$. Moreover,

$$h'=\frac{\operatorname{lcm}(x_i(w/y_i)\mathbf{x}_\sigma,v\mathbf{x}_\tau)}{v\mathbf{x}_\tau}=h/x_i\ \text{ divides }\ h$$

and so $h'\in P$ and $\deg(h')<\deg(h)$. Hence, we can iterate the previous reasoning by considering the integer i' arising from h', as in formula (8.10). In such a situation it is $i'>i$.

Case 2. Suppose x_i divides one of the monomials $w\mathbf{x}_\sigma$, $v\mathbf{x}_\tau$ and y_i divides the other one. Suppose x_i divides $w\mathbf{x}_\sigma$ and y_i divides $v\mathbf{x}_\tau$. Then $x_i(v/y_i)\mathbf{x}_\tau>_{\text{lex}}v\mathbf{x}_\tau$, and arguing as before, one gets $x_i(v/y_i)\mathbf{x}_\tau\in\operatorname{HS}_k(J(G))$. Moreover,

$$\frac{\operatorname{lcm}(x_i(v/y_i)\mathbf{x}_\tau,v\mathbf{x}_\tau)}{v\mathbf{x}_\tau}=x_i\ \text{ and }\ x_i\ \text{ divides }\ h.$$

Therefore, h is divided by the variable x_i belonging to P, as desired.

Otherwise, suppose x_i divides $v\mathbf{x}_\tau$ and y_i divides $w\mathbf{x}_\sigma$. It follows that $x_i(w/y_i)\mathbf{x}_\sigma\in G(\operatorname{HS}_k(J(G)))$ and $x_i(w/y_i)\mathbf{x}_\sigma>_{\text{lex}}w\mathbf{x}_\sigma$. Moreover,

$$h'=\frac{\operatorname{lcm}(x_i(w/y_i)\mathbf{x}_\sigma,v\mathbf{x}_\tau)}{v\mathbf{x}_\tau}=h/x_i\ \text{ divides }\ h.$$

Thus $h'\in P$ and $\deg(h')<\deg(h)$. In such a case, we iterate the reasoning made above by considering the integer i' arising from h' as in (8.10), and this time $i'>i$. It is clear that iterating the reasoning above we get that P is generated by variables.

Third Colon Ideal. Set $P=(fg\operatorname{HS}_{k-1}(J(G_3)),f\operatorname{HS}_k(J(G_2)),gu_1,\dots,gu_{\ell-1}):gu_\ell$. One can observe that P is generated by the monomials

$$\frac{\operatorname{lcm}(fgw_i,gu_\ell)}{gu_\ell},\quad\frac{\operatorname{lcm}(fv_j,gu_\ell)}{gu_\ell},\quad\frac{\operatorname{lcm}(gu_h,gu_\ell)}{gu_\ell},$$

with $i\in[q]$, $j\in[r]$ and $h\in[\ell-1]$. To prove that P is generated by variables it is enough to show that any of the monomials above is divided by a variable $z_b\in P$.

Note that $\mathrm{HS}_k(J(G_1))$ has linear quotients with respect to the order in (8.6), thus the colon ideal $(gu_1,\dots,gu_{\ell-1}):gu_\ell=(u_1,\dots,u_{\ell-1}):u_\ell$ is generated by variables. Thus, $\mathrm{lcm}(gu_h,gu_\ell)/(gu_\ell)\in(u_1,\dots,u_{\ell-1}):u_\ell$ is divided by a variable of P. For the other two type of monomial generators it suffices to repeat the same argument as in the SECOND COLON IDEAL case. $\square$

As an immediate consequence we have the following classification.

Theorem 8.4.2 *Let G be a very well–covered graph with $2n$ vertices. Then, the following conditions are equivalent:*

(i) *G is Cohen–Macaulay.*

(ii) *$J(G)$ has homological linear quotients.*

(iii) *$J(G)$ has homological linear resolution.*

Let G be a finite simple graph with vertex set $V(G)=X=\{x_1,\dots,x_n\}$. The *whisker graph* G^* of G is the graph with vertex set $V(G^*)=\{x_1,\dots,x_n\}\cup\{y_1,\dots,y_n\}$ and edge set $E(G^*)=E(G)\cup\{x_iy_i:i=1,\dots,n\}$. In other words, the whisker graph of G is obtained by adding to each vertex x_i a *whisker*, that means that for each vertex $x_i\in V(G)$ we add a new vertex y_i and an edge x_iy_i connecting these two vertices.

Corollary 8.4.3 *Let G be any simple graph with n vertices. Then the vertex cover ideal of the whisker graph G^* of G has homological linear quotients.*

Proof. It is easy to see that G^* satisfies the conditions (i)–(v) of Characterization 8.1.3. Thus G^* is a Cohen–Macaulay very well–covered graph and the result follows from the previous theorem. $\square$

Our experiments and the results above, suggest the following conjecture.

Conjecture 8.4.4 *Let G be a Cohen–Macaulay very well–covered graph with 2n vertices. Then $\mathrm{HS}_k((J(G))^\ell)$ has linear quotients with respect to the lexicographic order induced by $x_n>y_n>x_{n-1}>y_{n-1}>\dots>x_1>y_1$, for all $k\geq 0$, and all $\ell\geq 1$.*

Note that our conjecture would also imply that $J(G)$ has linear powers.

At present it seems too difficult to prove our conjecture in full generality. Therefore we concentrate our attention on the subclass of Cohen–Macaulay *bipartite* graphs. For this purpose, we need to recall what an *Hibi ideal* is [103].

Let $(P,\succeq)$ be a finite partially ordered set (a *poset*, for short) and set $P=\{p_1,\dots,p_n\}$. A *poset ideal* of P is a subset $\mathcal{I}$ of P such that if $p_i\in P$, $p_j\in\mathcal{I}$ and $p_i\preceq p_j$, then $p_i\in\mathcal{I}$ [88]. To any poset ideal $\mathcal{I}$ of P, we associate the squarefree monomial $u_{\mathcal{I}}=(\prod_{p_i\in\mathcal{I}}x_i)(\prod_{p_i\in P\setminus\mathcal{I}}y_i)$ in the polynomial ring $S=K[x_1,\dots,x_n,y_1,\dots,y_n]$. Then the *Hibi ideal* (associated to P) is the monomial ideal of S defined as follows:

$$H_P\ =\ (u_{\mathcal{I}}\ :\ \mathcal{I}\text{ is a poset ideal of }P).$$

As an immediate consequence of Theorem 8.4.1 we have the next result.

Corollary 8.4.5 *Let $(P,\succeq)$ be a finite poset. Then H_P has homological linear quotients.*

Proof. By [89, Lemma 9.1.11], the Alexander dual $H_P^\vee$ may be seen as the edge ideal of a Cohen–Macaulay bipartite graph G_P. In particular G_P is a Cohen–Macaulay very well–covered graph. Hence, seeing H_P as the cover ideal of G_P, the result follows immediately from Theorem 8.4.1. $\square$

Now, we turn to the powers of an Hibi ideal. We denote by $\mathcal{J}(P)$ the distributive lattice consisting of all poset ideals of $(P, \succeq)$ ordered by inclusion.

From now on, with abuse of language but with the aim of simplifying the notation, we identify each $p_i \in P$ with the variable x_i, $i \in [n]$.

Construction 8.4.6 Let $(P, \succeq)$ be a finite poset. For any integer $\ell \geq 1$, we construct a new poset $(P(\ell), \succeq_\ell)$ defined as follows:

- $P(\ell) \;=\; \{x_{i,1}, x_{i,2}, \dots, x_{i,\ell} \;:\; i = 1, \dots, n\}$,
- and $x_{i,r} \succeq_\ell x_{j,s}$ if and only if $x_i \succeq x_j$ and $r \geq s$.

By [103, pag 99] we have the following useful property.

Lemma 8.4.7 *Let $(P, \succeq)$ be a finite poset. Each minimal monomial generator of H_P^ℓ posses a unique expression of the form*

$$u_{\mathcal{I}_1} u_{\mathcal{I}_2} \cdots u_{\mathcal{I}_\ell}, \quad \textit{with} \quad \mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \cdots \subseteq \mathcal{I}_\ell, \quad \mathcal{I}_i \in \mathcal{J}(P), \quad i = 1, \dots, \ell.$$

Now, we need the technique of polarization. Let $(P, \succeq)$ be a finite poset and $\ell \geq 1$ be a positive integer. Note that X and $\varnothing$ are both poset ideals of P, thus both $u_X = x_1 x_2 \cdots x_n$ and $u_\varnothing = y_1 y_2 \cdots y_n$ belong to H_P. Thus, we have that $u_X^\ell = x_1^\ell x_2^\ell \cdots x_n^\ell$ and $u_\varnothing^\ell = y_1^\ell y_2^\ell \cdots y_n^\ell$ belong to H_P^ℓ. Hence, the polynomial ring in which H_P^ℓ lives is

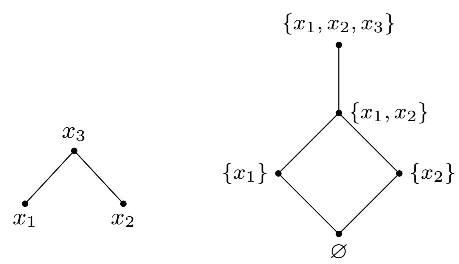
$$R = K[x_{i,j}, y_{i,j} \;:\; i = 1, \dots, n, \; j = 1, \dots, \ell].$$

In order to preserve the structure of Hibi ideals, we innocuously *modify* polarization. More precisely, let $1 \leq k \leq \ell$ and $i \in [n]$, then we set

$$\begin{aligned} (x_i^k)^\wp &= x_{i,1} x_{i,2} \cdots x_{i,\ell}, \\ (y_i^k)^\wp &= y_{i,\ell} y_{i,\ell-1} \cdots y_{i,\ell+1-k}, \end{aligned}$$

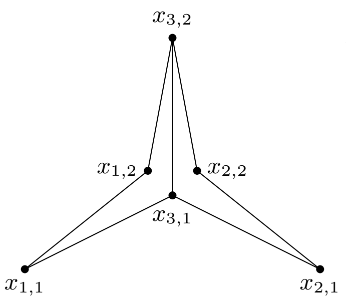
and extend the polarization of an arbitrary monomial in the obvious way. In other words, with respect to the usual polarization, we are just applying the relabeling of the variables $y_{i,j} \mapsto y_{i,\ell+1-j}$ for $i = 1, \dots, n$ and $j = 1, \dots, \ell$.

Example 8.4.8 Consider the poset $(P, \succeq)$ with $P = \{x_1, x_2, x_3\}$, $x_3 \succ x_1$ and $x_3 \succ x_2$. The poset $(P, \succeq)$ and the distributive lattice $\mathcal{J}(P)$ are depicted below:

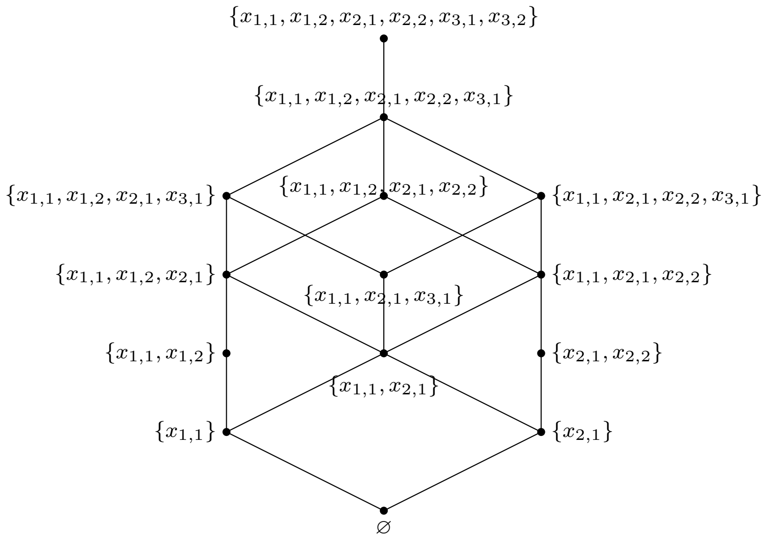


$(P, \succeq)$ $\mathcal{J}(P)$

The poset $(P(2),\succeq_2)$ and the distributive lattice $\mathcal{J}(P(2))$ are the following ones:



$(P(2),\succeq_2)$



$\mathcal{J}(P(2))$

We have that

$$\begin{aligned}
H_P &= (x_1x_2x_3,\ x_1x_2y_3,\ x_1y_2y_3,\ y_1x_2y_3,\ y_1y_2y_3),\\
H_{P(2)} &= (x_{1,1}x_{1,2}x_{2,1}x_{2,2}x_{3,1}x_{3,2},\ x_{1,1}x_{1,2}x_{2,1}x_{2,2}x_{3,1}y_{3,2},\ x_{1,1}x_{1,2}x_{2,1}x_{2,2}y_{3,1}y_{3,2},\\
&\quad x_{1,1}x_{1,2}x_{2,1}y_{2,2}x_{3,1}y_{3,2},\ x_{1,1}y_{1,2}x_{2,1}y_{2,2}x_{3,1}y_{3,2},\ x_{1,1}y_{1,2}x_{2,1}x_{2,2}x_{3,1}y_{3,2},\\
&\quad x_{1,1}x_{1,2}x_{2,1}y_{2,2}y_{3,1}y_{3,2},\ x_{1,1}y_{1,2}x_{2,1}y_{2,2}y_{3,1}y_{3,2},\ x_{1,1}y_{1,2}x_{2,1}x_{2,2}y_{3,1}y_{3,2},\\
&\quad x_{1,1}x_{1,2}y_{2,1}y_{2,2}y_{3,1}y_{3,2},\ y_{1,1}y_{1,2}x_{2,1}x_{2,2}y_{3,1}y_{3,2},\ x_{1,1}y_{1,2}y_{2,1}y_{2,2}y_{3,1}y_{3,2},\\
&\quad y_{1,1}y_{1,2}x_{2,1}y_{2,2}y_{3,1}y_{3,2},\ y_{1,1}y_{1,2}y_{2,1}y_{2,2}y_{3,1}y_{3,2}).
\end{aligned}$$

One can easily verify that $(H_P^2)^{\wp}=H_{P(2)}$ with respect to our modified polarization. For instance, consider $(x_1y_2y_3)(y_1x_2y_3)\in G(H_P^2)$. Then

$$\big((x_1y_2y_3)(y_1x_2y_3)\big)^{\wp}\ =\ (x_1y_1x_2y_2y_3^2)^{\wp}\ =\ x_{1,1}y_{1,2}x_{2,1}y_{2,2}y_{3,1}y_{3,2}\ =\ u_{\mathcal{I}}\in G(H_{P(2)}),$$

where $\mathcal{I}=\{x_{1,1},x_{2,1}\}\in\mathcal{J}(P(2))$.

Theorem 8.4.9 *Let $(P,\succeq)$ be a finite poset. Then, for any $\ell\geq 1$,*

$$(H_P^\ell)^{\wp}\ =\ H_{P(\ell)}.$$

Proof. By Lemma 8.4.7, H_P^ℓ is minimally generated by the monomials

$$u_{\mathcal{I}_1}u_{\mathcal{I}_2}\cdots u_{\mathcal{I}_\ell},\quad \text{with}\quad \mathcal{I}_1\subseteq\mathcal{I}_2\subseteq\cdots\subseteq\mathcal{I}_\ell,\quad \mathcal{I}_j\in\mathcal{J}(P),\quad j=1,\dots,\ell,$$

while $H_{P(\ell)}\subset K\big[x_{i,j},y_{i,j}:i\in[n],j\in[\ell]\big]$ is generated by the squarefree monomials

$$u_{\mathcal{I}}\ =\ \Big(\prod_{x_{i,j}\in\mathcal{I}}x_{i,j}\Big)\Big(\prod_{x_{i,j}\in P(\ell)\setminus\mathcal{I}}y_{i,j}\Big),\qquad \mathcal{I}\in\mathcal{J}(P(\ell)).$$

Hence, to get the assertion, we must show that for all $u_{\mathcal{I}_1}u_{\mathcal{I}_2}\cdots u_{\mathcal{I}_\ell}\in G(H_P^\ell)$, the monomial $(u_{\mathcal{I}_1}u_{\mathcal{I}_2}\cdots u_{\mathcal{I}_\ell})^{\wp}$ is equal to $u_{\mathcal{I}}$, for some poset ideal $\mathcal{I}\in\mathcal{J}(P(\ell))$, and conversely, given any $u_{\mathcal{I}} \in G(H_{P(\ell)})$, with $\mathcal{I} \in \mathcal{J}(P(\ell))$, then there exist $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \cdots \subseteq \mathcal{I}_\ell$, $\mathcal{I}_i \in \mathcal{J}(P)$, $i = 1, \ldots, \ell$, such that $u_{\mathcal{I}} = (u_{\mathcal{I}_1} u_{\mathcal{I}_2} \cdots u_{\mathcal{I}_\ell})^{\wp}$.

Let $u_{\mathcal{I}_1} u_{\mathcal{I}_2} \cdots u_{\mathcal{I}_\ell} \in G(H_P^\ell)$, with $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \cdots \subseteq \mathcal{I}_\ell$. Set $k_i = |\{r \ : \ x_i \in \mathcal{I}_r\}|$ and $m_i = |\{s \ : \ x_i \notin \mathcal{I}_s\}|$. Therefore, $k_i + m_i = \ell$ and

$$(u_{\mathcal{I}_1} u_{\mathcal{I}_2} \cdots u_{\mathcal{I}_\ell})^{\wp} \ = \ (\prod_{x_i \in P} \prod_{r=1}^{k_i} x_{i,r})(\prod_{x_i \in P} \prod_{s=k_i+1}^{\ell} y_{i,s}).$$

We claim that $\mathcal{I} = \{x_{i,r} \ : \ r = 1, \ldots, k_i\}$ is a poset ideal of $P(\ell)$. From this, it will follow that $(u_{\mathcal{I}_1} u_{\mathcal{I}_2} \cdots u_{\mathcal{I}_\ell})^{\wp} = u_{\mathcal{I}} \in G(H_{P(\ell)})$, as wanted. Indeed, let $x_{i,r} \in \mathcal{I}$ and $x_{j,s} \in P(\ell)$ such that $x_{j,s} \preceq_\ell x_{i,r}$. We must prove that $x_{j,s} \in \mathcal{I}$. By definition of the order $\succeq_\ell$ we have $x_j \preceq x_i$ and $r \geq s$. If $j = i$, then also $x_{i,s} \in \mathcal{I}$, by the polarization technique. Let $j \neq i$. We note that $k_j \geq k_i$. Indeed, for any c such that $x_i \in \mathcal{I}_c$, one has that $x_j \in \mathcal{I}_c$, since $x_j \preceq x_i$. Thus, if $x_{i,r}$ divides $(u_{\mathcal{I}_1} u_{\mathcal{I}_2} \cdots u_{\mathcal{I}_\ell})^{\wp}$, then $x_{j,r}$ divides $(u_{\mathcal{I}_1} u_{\mathcal{I}_2} \cdots u_{\mathcal{I}_\ell})^{\wp}$. For any $d \leq r$, $x_{j,d}$ divides $(u_{\mathcal{I}_1} u_{\mathcal{I}_2} \cdots u_{\mathcal{I}_\ell})^{\wp}$, by the polarization technique. Since $s \leq r$, then $x_{j,s}$ divides $(u_{\mathcal{I}_1} u_{\mathcal{I}_2} \cdots u_{\mathcal{I}_\ell})^{\wp}$, as wanted.

Conversely, let $\mathcal{I} \in \mathcal{J}(P(\ell))$ be a poset ideal and $u_{\mathcal{I}} \in G(H_{P(\ell)})$. Set

$$\mathcal{I}_k = \{x_i \ : \ x_{i,\ell+1-k} \in \mathcal{I}\}, \quad \text{for} \quad k = 1, \ldots, \ell.$$

We claim that the sets $\mathcal{I}_j$ are poset ideals of P and that $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \cdots \subseteq \mathcal{I}_\ell$. From this, it will follow that $u_{\mathcal{I}} = (u_{\mathcal{I}_1} u_{\mathcal{I}_2} \cdots u_{\mathcal{I}_\ell})^{\wp} \in G((H_P^\ell)^{\wp})$, as wanted.

Fix $k \in [\ell]$ and let $x_i \in \mathcal{I}_k$ and $x_j \preceq x_i$. We must prove that $x_j \in \mathcal{I}_k$, too. Since $x_j \preceq x_i$, then $x_{j,\ell+1-k} \preceq_\ell x_{i,\ell+1-k}$ by definition of the order $\succeq_\ell$. Thus $x_{j,\ell+1-k} \in \mathcal{I}$ and consequently $x_j \in \mathcal{I}_k$, as wanted.

Now, let $k \in [\ell]$ with $k < \ell$. We prove that $\mathcal{I}_k \subseteq \mathcal{I}_{k+1}$. As a consequence, it follows that $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \cdots \subseteq \mathcal{I}_\ell$, as desired. Let $x_i \in \mathcal{I}_k$, then $x_{i,\ell+1-k} \in \mathcal{I}$. Since $\ell+1-k > \ell+1-(k+1)$, we get that $x_{i,\ell+1-k} \succeq_\ell x_{i,\ell+1-(k+1)}$ and so $x_{i,\ell+1-(k+1)} \in \mathcal{I}$, too. Thus $x_i \in \mathcal{I}_{k+1}$. The assertion follows. ☐

Lemma 3.1.2 obviously also holds with respect to our modified polarization.

The polarization commutes with the homological shift ideals as follows from the next result of Sbarra [141].

Lemma 8.4.10 (Sbarra, 2001 [141, Corollary 1.8], [101, Proposition 1.14]). *Let $I \subset S$ be a monomial ideal. Then $\mathrm{HS}_i(I^{\wp}) = \mathrm{HS}_i(I)^{\wp}$ for all $i \geq 0$.*

Finally, we obtain the next result which gives a positive answer to Conjecture 8.4.4 for the class of bipartite graphs.

Corollary 8.4.11 *Let G be a Cohen–Macaulay bipartite graph with $2n$ vertices. Then, for all $k \geq 0$ and all $\ell \geq 1$, $\mathrm{HS}_k((J(G))^\ell)$ has linear quotients with respect to the lexicographic order induced by $x_n > y_n > x_{n-1} > y_{n-1} > \cdots > x_1 > y_1$.*

Proof. By [89, Theorem 9.1.13], there exists a poset $(P, \succeq)$ such that $J(G) = H_P$. Let $\ell \geq 1$. Then by Theorem 8.4.9, $(H_P^\ell)^{\wp} = H_{P(\ell)}$. By Theorem 8.4.1, for all $k \geq 0$, $\mathrm{HS}_k(H_{P(\ell)})$ has linear quotients with respect to the lexicographic order induced by

$$x_{n,\ell} > y_{n,\ell} > x_{n,\ell-1} > y_{n,\ell-1} > \cdots > x_{n,1} > y_{n,1} > x_{n-1,\ell} > y_{n-1,\ell} > \cdots > x_{1,1} > y_{1,1}.$$

By Lemma 8.4.10, $\mathrm{HS}_k(H_{P(\ell)}) = \mathrm{HS}_k((H_P^\ell)^{\wp}) = \mathrm{HS}_k(H_P^\ell)^{\wp}$. Finally, applying Lemmas 3.1.2 and 9.3.3, we obtain that $\mathrm{HS}_k(H_P^\ell) = \mathrm{HS}_k((J(G))^\ell)$ has linear quotients with respect to the lexicographic order induced by $x_n > y_n > x_{n-1} > y_{n-1} > \cdots > x_1 > y_1$, as wanted. ☐

Chapter 9

Powers of very well–covered graphs

Very well–covered graphs have been studied from view points of both Commutative Algebra and Combinatorics. See, for instance, [40, 58, 76, 115, 116, 117, 124, 56]. In [35] very well–covered graphs were studied by means of *Betti splittings* [74]. Recently, many authors have managed the Betti splitting technique for studying algebraic and combinatorial properties of classes of monomial ideals (see, for instance, [33, 39, 66] and references therein).

In the present chapter, we continue the algebraic study of Cohen–Macaulay very well–covered graphs started in [35]. If G is a graph in such a class, our main tool will be the Rees algebra of the cover ideal $J(G)$. We state that if G is a Cohen–Macaulay very well–covered graph, then the Rees algebra of $J(G)$ is a normal Cohen–Macaulay domain and as a consequence we obtain some relevant properties on the behavior of the powers of $J(G)$, when G is a *whisker graph*. Adding a *whisker* to a graph G at a vertex v means adding a new vertex w and an edge vw to the set $E(G)$. If a whisker is added to every vertex of G, then the resulting graph, denoted by G^*, is called the *whisker graph* or *suspension* of G. It is important to point out that the whisker graph G^* of a graph G with n vertices is a very well-covered Cohen–Macaulay graph with $2n$ vertices (see, for instance, [116] and references therein). In [84, Question 6.6] it is asked in which way attaching whiskers to a graph H gives rise to a graph G such that $J(G)$ has linear powers. In Corollary 9.3.6 we partially answer this question. See also [123, Corollary 4.5] and [129, Theorem 2.3].

Here are the outline of the chapter. In Section 9.1 we discuss a normality criterion for squarefree monomial ideals (Criterion 9.1.2). This result is borrowed from [133]. Section 9.2 deeply investigates the Rees algebra $\mathcal{R}(J(G))$ of $J(G)$, with G a Cohen–Macaulay very well–covered graph. Our main result states that $\mathcal{R}(J(G))$ is a normal Cohen–Macaulay domain (Theorem 9.2.1). To obtain this result we use Criterion 9.1.2 as well as the structure theorem of Cohen–Macaulay very well–covered graphs (Characterization 8.1.3) stated in [40]. In Section 9.3, if G is a whisker graph with $2n$ vertices, we prove that $J(G)$ satisfies the ℓ-exchange property (Theorem 9.3.5). As a consequence, we state that $J(G)^k$ has linear quotients, for all $k \geq 1$, and then that $J(G)$ has linear powers (Corollary 9.3.6). Furthermore, if G is a whisker graph, we show that each power of $J(G)$ has homological linear quotients (Theorem 9.3.8). This result supports a conjecture stated in [35, Conjecture 4.4]. Moreover, as applications of the previous results, we compute the limit depth, the depth stability and the analytic spread of $J(G)$. Finally, if G is a whisker graph with $2n$ vertices, we get a partial result on the structure of the reduced Gröbner basis of the presentation ideal of $\mathcal{R}(J(G))$ (Corollary 9.3.11). At present we do not know the reduced Gröbner basis of the presentation ideal of $\mathcal{R}(J(G))$. However, our experiments in *Macaulay2* [82] suggest that for a suitable monomial order, the reduced Gröbner basis is quadratic and hence that $\mathcal{R}(J(G))$ is Koszul (Conjecture 9.3.13), for any Cohen–Macaulay very well–covered graph with $2n$ vertices.

9.1 A normality criterion for monomial ideals

Let I be an ideal of a domain R. An element $f \in R$ is *integral over* I if it satisfies an equation of the type

$$f^k + a_1 f^{k-1} + \cdots + a_{k-1} f + a_k = 0, \quad a_i \in I^i.$$

The set of all these elements, denoted by $\overline{I}$, is an ideal containing I and called the *integral closure* of I. We say that I is *integrally closed* if $\overline{I} = I$, and we say that I is *normal* if all its powers I^k, $k \geq 1$, are integrally closed.

Let I be an ideal of a commutative ring R generated by $u_1, \ldots, u_m$. The *Rees algebra* of I, denoted by $\mathcal{R}(I)$ or $R[It]$, is the subring of $R[t]$, defined as follows

$$\mathcal{R}(I) = R[It] = R[u_1 t, \ldots, u_m t] = \bigoplus_{k \geq 0} I^k t^k \subset \; R[t],$$

where t is a new variable.

We quote the next fundamental result from [97] (see, also, [149, Theorem 4.3.17]).

Theorem 9.1.1 *Let I be an ideal of a normal domain R. Then the following are equivalent:*

(a) *I is a normal ideal;*

(b) *the Rees algebra $\mathcal{R}(I)$ is normal.*

Now, let $R = K[x_1, \ldots, x_n]$ be the standard graded polynomial ring with coefficients in a field K and let I be a monomial ideal of R. As usual we denote by $G(I) = \{u_1, \ldots, u_m\}$ the unique minimal set of monomial generators of I. Then the Rees algebra of I is the following K-algebra

$$\mathcal{R}(I) \;=\; K[x_1, \ldots, x_n, u_1 t, \ldots, u_m t] \;\subset\; R[t].$$

The next criterion quickly follows from [133, Theorem 3.1] (or [131, Theorem 3.1]).

Criterion 9.1.2 *Let $I_1, I_2 \subset K[x_2, \ldots, x_n]$ be two squarefree monomial ideals. If $I_1 \subseteq I_2$ are normal ideals, then $I = I_1 + x_1 I_2 \subset R$ is a normal ideal, too.*

Indeed, by [133, Theorem 3.1], it is enough to check that $I_1 + I_2$ is normal and that $\gcd(x_1, u) = 1$ for all $u \in G(I_1) \cup G(I_2)$. Since $I_1 \subseteq I_2$, the first assertion follows because $I_1 + I_2 = I_2$ is normal by hypothesis. The second assertion follows because the generators of I_1 and I_2 are monomials of $K[x_2, \ldots, x_n]$.

9.2 The Rees algebra

In this section we study the Rees algebra of the vertex cover ideal of a Cohen–Macaulay very well–covered graph.

The main result in this section is the following.

Theorem 9.2.1 *Let G be a Cohen–Macaulay very well–covered graph. Then the Rees algebra $\mathcal{R}(J(G))$ is a normal Cohen–Macaulay domain.*

As in the previous chapter, if G is a Cohen–Macaulay very well–covered graph with $2n$ vertices, we assume that its set of vertices $V(G) = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ satisfies the conditions (i)–(v) of Characterization 8.1.3, without having to relabel it.

Hereafter, denote by S the polynomial ring $K[x_1,\ldots,x_n,y_1,\ldots,y_n]$ in the $2n$ variables $x_1,\ldots,x_n,y_1,\ldots,y_n$ with coefficients in the field K.

Lemma 9.2.2 *Let G be a Cohen–Macaulay very well–covered graph with $2n$ vertices. Then*

$$J(G) \;=\; \mathbf{z}_{N(x_1)}J(G_1)+x_1J(G\setminus\{x_1,y_1\}), \tag{9.1}$$

where $G_1 = G\setminus\{x_i,y_i : i\in N(x_1)_x\cup N(x_1)_y\}$.

Proof. The proof is similar to that of [35, Proposition 2.3]. We include it for completeness. Let $u\in G(J(G))$. By Lemma 8.2.2, either x_1 divides u or y_1 divides u.

Case 1. Suppose x_1 divides u. Note that $N(y_1)=\{x_1\}$. Indeed, by Characterization 8.1.3(i), $N(y_1)$ is a subset of X, since Y is a maximal independent set. Moreover, by (iii) if $x_iy_1\in E(G)$ then $i\le 1$. Hence, $N(y_1)=\{x_1\}$. Consequently $\mathbf{z}_{N(y_1)}=x_1$ and the support C' of $u/\mathbf{z}_{N(y_1)}=u/x_1$ is a vertex cover of $G\setminus\{x_1,y_1\}$. But C' is a minimal vertex cover, for u/x_1 has degree $n-1$ and $G\setminus\{x_1,y_1\}$ is a Cohen–Macaulay very well–covered graph with $2(n-1)$ vertices (Proposition 8.2.1). Thus $u/x_1\in G(J(G\setminus\{x_1,y_1\}))$ and so $u\in G(x_1J(G\setminus\{x_1,y_1\}))$.

Case 2. Suppose y_1 divides u. Since the support C of u is a minimal vertex cover of G and $x_1\notin C$, then $z_i\in C$ for all $z_i\in N(x_1)$. Consequently, the support C_1 of $u/\mathbf{z}_{N(x_1)}$ is a vertex cover of G_1. But C_1 is a minimal vertex cover of G_1, for $|C_1|=n-|N(x_1)|$ and G_1 is a Cohen–Macaulay very well–covered graph with $2(n-|N(x_1)|)$ vertices (Proposition 8.2.1). Hence $u\in G(\mathbf{z}_{N(x_1)}J(G_1))$.

These two cases show the inclusion "$\subseteq$" in equation (9.1). The other inclusion is acquired as in the last part of the proof of [35, Proposition 2.3]. $\square$

Corollary 9.2.3 *Let G be a Cohen–Macaulay very well–covered graph with $2n$ vertices. Then $J(G)$ is a normal ideal.*

Proof. By Lemma 9.2.2, equation (9.1) holds. Set $J=J(G)$, $J_1=\mathbf{z}_{N(x_1)\setminus y_1}J(G_1)$ and $J_2=J(G\setminus\{x_1,y_1\})$. Thus

$$J \;=\; y_1J_1+x_1J_2.$$

Since $y_1J_1, J_2\subset K[x_2,\ldots,x_n,y_1,y_2,\ldots,y_n]$, it is enough to show that $J_1\subseteq J_2$. Then $y_1J_1\subset J_2$ and the result follows from Criterion 9.1.2 and induction on n.

Let $u\in G(J_1)$. We must prove that $u\in G(J_2)$, too. That is, we must show that $C=\operatorname{supp}(u)$ is a minimal vertex cover of $G\setminus\{x_1,y_1\}$. It is enough to prove C is a vertex cover of $G\setminus\{x_1,y_1\}$. Minimality follows because $|C|=n-1$. Hence, we must prove that $e\cap C\neq\varnothing$ for all edges $e\in E(G\setminus\{x_1,y_1\})$. Let $e\in E(G\setminus\{x_1,y_1\})$. Since $y_1u\in G(J)$, it follows that $C\cup y_1$ is a minimal vertex cover of G. Hence $e\cap(C\cup y_1)\neq\varnothing$. Therefore $e\cap C\neq\varnothing$ because $y_1\notin e$. Our assertion follows. $\square$

Finally, we are in the position to prove the main result in the section.

Proof of Theorem 9.2.1. By Corollary 9.2.3, $J(G)$ is a normal ideal. Hence, the Rees algebra $\mathcal{R}(J(G))$ is normal (Theorem 9.1.1). Next, by a theorem of Hochster [105], since $\mathcal{R}(J(G))$ is a normal affine semigroup ring, it follows that $\mathcal{R}(J(G))$ is Cohen–Macaulay. $\square$

The *toric ring* of $G(J(G))$ is the K-algebra $K[J(G)]=K[u : u\in G(J(G))]\subset S$.

Corollary 9.2.4 *Let G be a Cohen–Macaulay very well–covered graph. Then the toric ring $K[J(G)]$ is a normal Cohen–Macaulay domain.*

Proof. Since $J(G)$ is generated in one degree, the statement follows from Theorem 9.2.1 together with [149, Proposition 4.3.42]. □

Now let I be an ideal of a noetherian ring R. As usual, denote by $V(I)$ the set of prime ideals containing I and by $\mathrm{Ass}(I)$ the set of associated prime ideals of R/I. For all $P \in \mathrm{Spec}(R)$, we denote by $\mathfrak{m}_P$ the maximal ideal of the local ring R_P. Recall that I satisfies the *persistence property* (with respect to associated ideals) if

$$\mathrm{Ass}(I) \subseteq \mathrm{Ass}(I^2) \subseteq \mathrm{Ass}(I^3) \subseteq \cdots .$$

In [95], Herzog and Qureshi introduced the notion of *strong persistence property*. More in detail, let $P \in V(I)$. We say that I satisfies the *strong persistence property with respect to* P if for all k and all $f \in (I_P^k : \mathfrak{m}_P) \setminus I_P^k$ there exists $g \in I_P$ such that $fg \notin I_P^{k+1}$. The ideal I is said to satisfy the *strong persistence property* if it satisfies the strong persistence property for all $P \in V(I)$. The strong persistence property implies the persistence property [95] (see, also, [133, Proposition 2.1]).

Theorem 9.2.1 yields the next result.

Corollary 9.2.5 *Let G be a Cohen–Macaulay very well–covered graph. Then $J(G)$ satisfies the strong persistence property, and in particular, the persistence property.*

Proof. The assertion follows from Theorem 9.2.1 and [95, Corollary 1.6]. □

9.3 Whisker graphs

In this section we study some algebraic properties of the powers of the cover ideals of a special class of Cohen–Macaulay very well–covered graphs. Our main tool is the so called ℓ-exchange property introduced in [93].

Let $I \subset R = K[x_1, \ldots, x_n]$ be a monomial ideal generated in one degree, and let $K[I] = K[u : u \in G(I)]$ be the toric ring of $G(I)$. Then $K[I]$ has the presentation

$$\psi : T = K[t_u : u \in G(I)] \to K[I]$$

defined by $\psi(t_u) = u$ for all $u \in G(I)$. The kernel $\mathrm{Ker}(\psi) = J$ is called the *toric ideal* of $K[I]$.

Fix a monomial order $>$ on T. We say that the monomial $t_{u_1} \cdots t_{u_N} \in T$ is *standard with respect to* $>$, if $t_{u_1} \cdots t_{u_N}$ does not belong to the initial ideal, $\mathrm{in}_<(J)$, of the toric ideal J of $K[I]$.

Definition 9.3.1 ([43, Definition 3.3]). The equigenerated monomial ideal $I \subset R$ satisfies the *ℓ-exchange property with respect to* $>$, if the following condition is satisfied: for all standard monomials $t_{u_1} \cdots t_{u_N}, t_{v_1} \cdots t_{v_N} \in T$ of degree N such that

(i) $\deg_{x_i}(u_1 \cdots u_N) = \deg_{x_i}(v_1 \cdots v_N)$, for all $1 \le i \le j-1$ with $j \le n-1$,

(ii) $\deg_{x_j}(u_1 \cdots u_N) < \deg_{x_j}(v_1 \cdots v_N)$,

there exist h and k with $j < h \le n$ and $1 \le k \le N$, such that $x_j(u_k/x_h) \in G(I)$.

The following lemmata will be needed later.

Lemma 9.3.2 *Let G be a Cohen–Macaulay very well–covered graph with $2n$ vertices. Let $C \in \mathcal{C}(G)$ such that $C_y \neq \varnothing$ and let $i = \min C_y$. Then $(C \setminus y_i) \cup x_i \in \mathcal{C}(G)$.*

Proof. Firstly we prove that $C' = (C \setminus y_i) \cup x_i$ is a vertex cover of G. Let $e \in E(G)$, we must show that $e \cap C'$ is non empty. Since C is a vertex cover of G, then $e \cap C \neq \varnothing$. If $\{x_i, y_i\} \cap e = \varnothing$, then $e \cap C' \neq \varnothing$, too. If $x_i \in e$ then $e \cap C'$ contains x_i and therefore the intersection is non empty. Finally, suppose $y_i \in e$ but $x_i \notin e$. Since Y is a maximal independent set, it follows that $N(y_i) \subseteq X$. Hence, $e = x_j y_i$ for some j. By Characterization 8.1.3(iii) we have $j \leq i$. Thus $j < i$, because $x_i \notin e$. Since $i = \min C_y$ and $j < i$, it follows from Lemma 8.2.2 that $x_j \in C$. Hence $x_j \in e \cap C'$ and again the intersection is non empty.

The fact that C' is a minimal vertex cover of G follows because $|C'| = n$. □

Lemma 9.3.3 *Let G be a Cohen–Macaulay very well–covered graph with $2n$ vertices. Then, for all $k \geq 1$, all $i \in [n]$ and all $u \in G(J(G)^k)$ we have*

$$\deg_{x_i}(u) + \deg_{y_i}(u) = k.$$

Proof. By Lemma 8.2.2, for all $u \in G(J(G))$, we have

$$\deg_{x_i}(u) + \deg_{y_i}(u) = 1 \quad \text{for all} \quad 1 \leq i \leq n.$$

Since $J(G)$ is generated in a single degree, the minimal generators of $J(G)^k$ are the products $u = u_1 \cdots u_k$ of k arbitrary monomials of $G(J(G))$. Hence, for all $1 \leq i \leq n$, we have $\deg_{x_i}(u) + \deg_{y_i}(u) = \sum_{j=1}^{k} [\deg_{x_i}(u_j) + \deg_{y_i}(u_j)] = k$. □

Now, we consider a wide class of Cohen–Macaulay very well–covered graphs.

Let H be a graph on the vertex set $X = \{x_1, \dots, x_n\}$ and take a new set of variables $Y = \{y_1, \dots, y_n\}$. Then, the *whisker graph* $G = H^*$ of H is the graph obtained from H by attaching to each vertex x_i a new vertex y_i and the edge $x_i y_i$. The edge $x_i y_i$ is called a *whisker*. More in detail, the *whisker graph* $G = H^*$ of H is the graph on the vertex set $X \cup Y = \{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}$ and the edge set $E(G) \cup \{x_1 y_1, x_2 y_2, \dots, x_n y_n\}$.

Lemma 9.3.4 *Let G be a whisker graph with vertex set $X \cup Y$. Then, for all $C \in \mathcal{C}(G)$ and all $y_i \in C$ we have that $(C \setminus y_i) \cup x_i \in \mathcal{C}(G)$.*

Proof. By assumption $G = H^*$, for some graph H on the vertex set $X = \{x_1, \dots, x_n\}$, thus a Cohen–Macaulay very well–covered graph with the $2n$ vertices $x_1, \dots, x_n$, $y_1, \dots, y_n$. Since for all i, the only vertex adjacent to y_i is x_i, then for any labeling of $X \cup Y$, the conditions (i)–(v) of Characterization 8.1.3 are satisfied. Hence, if $C \in \mathcal{C}(G)$ and $y_i \in C$, we can choose a labeling such that $\min_y C = i$. The assertion follows by applying Lemma 9.3.2. □

From now on, if G is a whisker graph with $2n$ vertices, we implicitly assume that

- G is the whisker graph associated to a given graph whose vertex set is the set $X = \{x_1, \dots, x_n\}$, and with whiskers $x_i y_i$, $i = 1, \dots, n$, that is, $V(G) = X \cup Y$, with $Y = \{y_1, \dots, y_n\}$.

- G is a very well–covered Cohen–Macaulay graph whose vertex set $X \cup Y$ satisfies the conditions (i)–(v) of Characterization 8.1.3.

Theorem 9.3.5 *Let G be a whisker graph with $2n$ vertices. Then $J(G)$ satisfies the ℓ-exchange property with respect to the lexicographic order $>_{\mathrm{lex}}$ induced by $x_1 > y_1 > x_2 > y_2 > \cdots > x_n > y_n$.*

Proof. Set $z_{2p-1} = x_p$ and $z_{2p} = y_p$, for $p = 1, \dots, n$. Then

$$z_1 > z_2 > z_3 > z_4 > \cdots > z_{2n-1} > z_{2n}.$$

We prove the following slightly more general statement.

($*$) For all monomials $t_{u_1} \cdots t_{u_N}$, $t_{v_1} \cdots t_{v_N}$ of $K[t_u : u \in G(J(G))]$ such that

(i) $\deg_{z_i}(u_1 \cdots u_N) = \deg_{z_i}(v_1 \cdots v_N)$, for all $1 \le i \le j-1$ with $j \le 2n-1$,

(ii) $\deg_{z_j}(u_1 \cdots u_N) < \deg_{z_j}(v_1 \cdots v_N)$,

there exist h and k, with $j < h \le 2n$ and $1 \le k \le N$, such that $z_j(u_k/z_h) \in G(J(G))$.

Let $t_{u_1} \cdots t_{u_N}$, $t_{v_1} \cdots t_{v_N}$ monomials of $K[t_u : u \in G(J(G))]$ satisfying the conditions (i) and (ii).

We claim that the integer j is odd. Suppose for a contradiction that j is even, then $j = 2p$ for some $p \in [n]$. Thus $z_j = y_p$ and

$$\deg_{y_p}(u_1 \cdots u_N) < \deg_{y_p}(v_1 \cdots v_N). \tag{9.2}$$

On the other hand, since $J(G)$ is generated in a single degree, $u_1 \cdots u_N$ and $v_1 \cdots v_N$ belong to $G(J(G)^N)$. Thus, Lemma 9.3.3 gives

$$\deg_{x_p}(u_1 \cdots u_N) + \deg_{y_p}(u_1 \cdots u_N) = \deg_{x_p}(v_1 \cdots v_N) + \deg_{y_p}(v_1 \cdots v_N) = N. \tag{9.3}$$

Equations (9.2) and (9.3) yield $\deg_{x_p}(u_1 \cdots u_N) > \deg_{x_p}(v_1 \cdots v_N)$, but this contradicts condition (i), since $x_p = z_{j-1}$. Hence j is odd, and so $z_j = x_p$ for some $p \in [n]$.

Since $\deg_{x_p}(u_1 \cdots u_N) < \deg_{x_p}(v_1 \cdots v_N) \le N$, by Lemma 9.3.3 it follows that $\deg_{y_p}(u_1 \cdots u_N) > 0$. Hence, there exists k with $1 \le k \le N$ such that y_p divides u_k. By Lemma 9.3.4, it follows that $x_p(u_k/y_p) \in G(J(G))$. Since $y_p = z_{j+1}$, and $z_j > z_{j+1}$, the claim ($*$) is proved. □

We recall that an ideal I of a polynomial ring $R = K[x_1, \dots, x_n]$ has *linear powers* if I^k has linear resolution, for all $k \ge 1$. Moreover, a monomial ideal I of R, has *linear quotients* if for some order $u_1, \dots, u_m$ of its minimal generating set $G(I)$, all colon ideals $(u_1, \dots, u_{\ell-1}) : u_\ell$, $\ell = 2, \dots, m$, are generated by a subset of the set of variables $\{x_1, \dots, x_n\}$.

As a first consequence of Theorem 9.3.5 we prove that the cover ideal of a whisker graph G has linear powers. Such a result has been recently obtained in [123, Corollary 4.5] (see, also, [129, Theorem 2.3]) by showing that the ordinary powers of $J(G)$ are weakly polymatroidal, which implies having linear powers. It also follows from a more general result proved in [90, Theorem 3.3].

Corollary 9.3.6 *Let G be a whisker graph with $2n$ vertices. Then,*

(a) *for all $k \ge 1$, $J(G)^k$ has linear quotients with respect to the lexicographic order $>_{\mathrm{lex}}$ induced by $x_1 > y_1 > x_2 > y_2 > \cdots > x_n > y_n$.*

(b) *$J(G)$ has linear powers. In particular, the depth function $\mathrm{depth} S/J(G)^k$ is a non-increasing function of k, that is, $\mathrm{depth} S/J(G)^k \ge \mathrm{depth} S/J(G)^{k+1}$ for all $k \ge 1$.*

Proof. (a) Since $J(G)$ is generated in a single degree, each minimal monomial generator of $J(G)^N$ is a product $u_1 \cdots u_N$ of N arbitrary, non necessarily distinct, monomials $u_i \in G(J(G))$. Let $u = u_1 \cdots u_N \in G(J(G)^N)$, where each $u_i \in G(J(G))$. Setting $P = (v_1 \cdots v_N : v_i \in G(J(G)), v_1 \cdots v_N >_{\text{lex}} u_1 \cdots u_N)$, we must prove that the ideal $P : u$ is generated by variables.

Let $v = v_1 \cdots v_N \in G(P)$. Using the labeling z_i on the variables, given in the proof of Theorem 9.3.5, by the definition of $>_{\text{lex}}$, for some i and j we have

(i) $\deg_{z_i}(v_1 \cdots v_N) = \deg_{z_i}(u_1 \cdots u_N)$, for all $1 \leq i \leq j-1$ with $j \leq 2n-1$,

(ii) $\deg_{z_j}(v_1 \cdots v_N) > \deg_{z_j}(u_1 \cdots u_N)$.

Hence, by the property $(*)$ proved in Theorem 9.3.5, there exist integers k and h such that $z_j = x_h$ and $x_h(u_k/y_h) \in G(J(G))$. Since $x_h > y_h$, we have $x_h(u_k/y_h) >_{\text{lex}} u_k$. Consequently,

$$u' = x_h(u/y_h) = u_1 \cdots u_{k-1} \cdot x_h(u_k/y_h) \cdot u_{k+1} \cdots u_N \in P$$

and $x_h = u'/\gcd(u', u) \in P : u$ divides the monomial $v/\gcd(v, u) \in P$. Indeed, the set $\{v/\gcd(v, u) : v \in G(P)\}$ generates $P : u$ ([89, Proposition 1.2.2]). Hence, we see that $P : u$ is generated by variables, as desired.
(b) That $J(G)$ has linear powers follows from (a) and the fact that all powers $J(G)^k$ are monomial ideals generated in a single degree. The claim about the non-increasingness of the function $\operatorname{depth} S/J(G)^k$ follows from [89, Proposition 10.3.4]. $\square$

A weaker form of Conjecture 8.4.4 is the following one.

Conjecture 9.3.7 ([35, Conjecture 4.4]). *Let G be a Cohen–Macaulay very well–covered graph with $2n$ vertices. Then $\text{HS}_k(J(G)^\ell)$ has linear quotients, for all $k \geq 0$, and all $\ell \geq 1$.*

In [35], we gave a positive answer to this conjecture for $\ell = 1$ (Theorem 8.4.1) and for all Cohen–Macaulay bipartite graphs (Corollary 8.4.11). Now we prove that the powers of cover ideals of whisker graphs have homological linear quotients, partially answering Conjecture 9.3.7.

Theorem 9.3.8 *Let G be a whisker graph with $2n$ vertices. Then, for all $\ell \geq 1$ and all $k \geq 0$, $\text{HS}_k(J(G)^\ell)$ has linear quotients with respect to the lexicographic order $>_{\text{lex}}$ induced by $x_1 > x_2 > \cdots > x_n > y_1 > y_2 > \cdots > y_n$.*

Proof. Let $>$ be the lexicographic order induced by $x_1 > y_1 > x_2 > y_2 > \cdots > x_n > y_n$. Then, by Corollary 9.3.6(a), $J(G)^\ell$ has linear quotients with respect to $>$ for all $\ell \geq 1$. Let $u \in G(J(G)^\ell)$, we define

$$\text{set}(u) \;=\; \{i \;:\; z_i \in \{x_i, y_i\}, z_i \in (v \in G(J(G)^\ell) : v > u) : (u)\}.$$

The definition of the order $>$ and Lemma 9.3.3 imply that the set of variables generating the ideal $(v \in G(J(G)^\ell) : v > u) : (u)$ is a subset of $X = \{x_1, \ldots, x_n\}$.

Thus, by [60, Proposition 1.2] we have

$$\text{HS}_k(J(G)^\ell) \;=\; (\mathbf{x}_F u \;:\; u \in G(J(G)^\ell),\ F \subseteq \text{set}(u),\ |F| = k).$$

Let $\mathbf{x}_D v \in G(\text{HS}_k(J(G)^\ell))$, $D \subseteq \text{set}(v)$, $v \in G(J(G)^\ell)$ and consider the colon ideal

$$P \;=\; (\mathbf{x}_F u \in G(\text{HS}_k(J(G)^\ell)) : \mathbf{x}_F u >_{\text{lex}} \mathbf{x}_D v) : (\mathbf{x}_D v).$$

We must prove that P is generated by variables.

Let $\mathbf{x}_F u \in G(\mathrm{HS}_k(J(G)^\ell))$, $F \subseteq \mathrm{set}(u)$, $u \in G(J(G)^\ell)$, such that $\mathbf{x}_F u >_{\mathrm{lex}} \mathbf{x}_D v$. Let $h = \mathrm{lcm}(\mathbf{x}_F u, \mathbf{x}_D v)/(\mathbf{x}_D v)$. If $\deg(h) = 1$, h is a variable. Assume $\deg(h) > 1$. Let z_i be the labeling on the variables such that $z_1 = x_1$, $z_2 = x_2$, $\ldots$, $z_n = x_n$, $z_{n+1} = y_1$, $z_{n+2} = y_2$, $\ldots$, $z_{2n} = y_n$. Then, by definition of $>_{\mathrm{lex}}$, there exists p such that $\deg_{z_j}(\mathbf{x}_F u) = \deg_{z_j}(\mathbf{x}_D v)$ for all $j < p$ and

$$\deg_{z_p}(\mathbf{x}_F u) > \deg_{z_p}(\mathbf{x}_D v). \tag{9.4}$$

Now, we distinguish two cases.

CASE 1. Suppose $z_p = x_i$ for some i. We claim that

$$\deg_{x_i}(\mathbf{x}_F u) \leq \ell. \tag{9.5}$$

Indeed, by Lemma 9.3.3 and the structure of $\mathbf{x}_F u$, it follows that $\deg_{x_i}(\mathbf{x}_F u) \leq \ell + 1$. Suppose by contradiction that $\deg_{x_i}(\mathbf{x}_F u) = \ell + 1$, then $i \in \mathrm{set}(u)$. Necessarily y_i must divide u. But this would imply that $\deg_{x_i}(\mathbf{x}_F u) + \deg_{y_i}(\mathbf{x}_F u)$ exceeds $\ell + 1$, which is impossible. Hence, equation (9.5) follows.

By Lemma 9.3.3 and equations (9.4) and (9.5), $\ell \geq \deg_{x_i}(\mathbf{x}_F u) > \deg_{x_i}(\mathbf{x}_D v)$ and $\deg_{y_i}(\mathbf{x}_D v) > 0$. Writing $v = v_1 v_2 \cdots v_\ell$, with each $v_q \in G(J(G))$, we have that y_i divides v_q for some q. Then $i \in \mathrm{set}(v)$. Indeed, $x_i(v_q/y_i) \in G(J(G))$ and $v' = x_i(v/y_i) = v_1 \cdots v_{q-1}(x_i(v_q/y_i))v_{q+1} \cdots v_\ell \in G(J(G)^\ell)$. We distinguish two cases.

SUBCASE 1.1. Let $i \notin D$. Then $\mathbf{x}_D v' \in G(\mathrm{HS}_k(J(G)^\ell))$ and $\mathbf{x}_D v' >_{\mathrm{lex}} \mathbf{x}_D v$. Moreover, $\mathrm{lcm}(\mathbf{x}_D v', \mathbf{x}_D v)/(\mathbf{x}_D v) = x_i \in P$ divides h.

SUBCASE 1.2. Let $i \in D$. Then $\deg_{x_i}(\mathbf{x}_D v) + \deg_{y_i}(\mathbf{x}_D v) = \ell + 1$ (Lemma 9.3.3). Since by (9.4) and (9.5) $\deg_{x_i}(\mathbf{x}_D v) < \ell$, it follows that $\deg_{y_i}(v) \geq 2$. Hence, there exist $h_1 \neq h_2$ such that y_i divides v_{h_1} and v_{h_2}. Set $v' = v_1 \cdots (x_i(v_{h_1})/y_i) \cdots v_{h_2} \cdots v_\ell$. Then, it follows that $v' \in G(J(G)^\ell)$ and $D \subseteq \mathrm{set}(v')$. Thus $\mathbf{x}_D v' >_{\mathrm{lex}} \mathbf{x}_D v$ and moreover $\mathrm{lcm}(\mathbf{x}_D v', \mathbf{x}_D v)/(\mathbf{x}_D v) = x_i \in P$ divides h.

CASE 2. Suppose $z_p = y_i$ for some i. For all j such that $\deg_{y_j}(\mathbf{x}_F u) > \deg_{y_j}(\mathbf{x}_D v)$, since $\deg_{x_j}(\mathbf{x}_F u) = \deg_{x_j}(\mathbf{x}_D v)$, Lemma 9.3.3 gives

$$\ell + 1 = \deg_{x_j}(\mathbf{x}_F u) + \deg_{y_j}(\mathbf{x}_F u) > \deg_{x_j}(\mathbf{x}_D v) + \deg_{y_j}(\mathbf{x}_D v) = \ell.$$

Hence $j \notin D$ and $\deg_{y_j}(\mathbf{x}_F u) - \deg_{y_j}(\mathbf{x}_D v) = 1$. Let $j_1, \ldots, j_t$ be the integers such that $\deg_{y_{j_s}}(\mathbf{x}_F u) > \deg_{y_{j_s}}(\mathbf{x}_D v)$, $s = 1, \ldots, t$. Then, the above argument shows that $h = y_{j_1} y_{j_2} \cdots y_{j_t}$ and $j_s \notin D$ for all $s = 1, \ldots, t$. Since $\deg(h) > 1$ we have $t \geq 2$. As before $v' = x_{j_2} \cdots x_{j_t} v/(y_{j_2} \cdots y_{j_t}) \in G(J(G)^\ell)$, $D \subseteq \mathrm{set}(v')$ and $\mathbf{x}_D v' >_{\mathrm{lex}} \mathbf{x}_D v$. Finally $\mathrm{lcm}(\mathbf{x}_D v', \mathbf{x}_D v)/(\mathbf{x}_D v) = y_{j_1} \in P$ divides h .

The above CASES 1 and 2 show that P is generated by variables, as wanted. □

Another relevant consequence of Theorem 9.3.5 concerns the *limit depth* of $J(G)$. The role of the Rees algebra of the cover ideal $J(G)$ will be crucial to calculating it.

Let I be a graded ideal of a polynomial ring R with n variables. By a theorem of Brodmann [22], $\mathrm{depth} R/I^k$ is constant for k large enough. This eventually constant value is called the *limit depth* of I, and it is denoted by $\lim_{k\to\infty} \mathrm{depth} R/I^k$.

The *depth stability* of I, denoted by $\mathrm{dstab}(I)$, is the least integer k_0 such that $\mathrm{depth} R/I^k = \mathrm{depth} R/I^{k_0}$ for all $k \geq k_0$. Brodmann proved that

$$\lim_{k\to\infty} \mathrm{depth} R/I^k \leq n - \ell(I),$$

where $\ell(I)$ is the *analytic spread* of I, that is, the Krull dimension of the fiber ring $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$, where $\mathfrak{m}$ is the maximal graded ideal of R.

If the Rees algebra of I is Cohen–Macaulay, then by [89, Proposition 10.3.2] (see, also, [49] combined with [107, Proposition 1.1]), we have

$$\lim_{k\to\infty} \operatorname{depth} R/I^k = n - \ell(I). \tag{9.6}$$

Hence, we have

Theorem 9.3.9 *Let G be a whisker graph with $2n$ vertices. Then*

$$\lim_{k\to\infty} \operatorname{depth} S/J(G)^k \;=\; n-1.$$

Moreover, $\operatorname{dstab}(J(G)) \leq n$ *and* $\ell(J(G)) = n+1$.

For the proof of this result, we need the next more general lemma. Let $u \in S$ be a monomial. Using the notation in Section 9.2, we have $\operatorname{supp}(u)_x = \{i : x_i \textit{ divides } u\}$ and $\operatorname{supp}(u)_y = \{i : y_i \textit{ divides } u\}$.

Lemma 9.3.10 *Let G be a Cohen–Macaulay very well–covered graph with $2n$ vertices. Then, for all $i \in [n]$, there exists $u \in G(J(G))$ such that $\min\operatorname{supp}(u)_y = i$.*

Proof. We proceed by induction on $n \geq 1$. For the base case, $J(G) = (x_1, y_1)$ and the statement holds. Now, let $n > 1$. Then, by Lemma 9.2.2 we have $J(G) = \mathbf{z}_{N(x_1)}J(G_1) + x_1J(G \setminus \{x_1, y_1\})$. If $u \in G(\mathbf{z}_{N(x_1)}J(G_1))$, then $\min\operatorname{supp}(u)_y = 1$. Let $i \in [n]$ with $i \neq 1$. Since $G \setminus \{x_1, y_1\}$ is a Cohen–Macaulay very well–covered graph (Proposition 8.2.1), by induction there exists $v \in G(J(G \setminus \{x_1, y_1\}))$ with $\min\operatorname{supp}(v)_y = i$. Setting $u = x_1v$, we have $u \in G(J(G))$ and $\min\operatorname{supp}(u)_y = \min\operatorname{supp}(v)_y = i$. The proof is complete. □

Now, we are in the position to prove Theorem 9.3.9.

Proof of Theorem 9.3.9. For any $N \geq 1$ and any $u_1 \cdots u_N \in G(J(G)^N)$, with $u_i \in G(J(G))$, let $P = (v_1 \cdots v_N : v_i \in G(J(G)), v_1 \cdots v_N >_{\text{lex}} u_1 \cdots u_N)$ and denote by $s(u_1, \ldots, u_N)$ the number of variables generating the colon ideal $P : u_1 \cdots u_N$. Since $J(G)^N$ has linear quotients with respect to $>_{\text{lex}}$ (Corollary 9.3.6(a)) we have that $\operatorname{depth} S/J(G)^N = 2n - \max\{s(u_1, \ldots, u_N) + 1 : u_1, \ldots, u_N \in G(J(G))\}$. This follows from [89, Corollary 8.2.2] and the Auslander–Buchsbaum formula. The definition of the order $>_{\text{lex}}$ and Lemma 9.3.3 imply that the set of variables generating the ideal $P : u_1 \cdots u_N$ is a subset of $X = \{x_1, \ldots, x_n\}$. Pick n monomials $u_i = \mathbf{x}_{F_i}\mathbf{y}_{[n]\setminus F_i}$, where $\min([n] \setminus F_i) = i$, for $i = 1, \ldots, n$. The existence of these monomials follows from Lemma 9.3.10. By Lemma 9.3.2, $x_i(u_i/y_i) \in G(J(G))$ for $i = 1, \ldots, n$. Hence, $s(u_1, \ldots, u_n) = n$. This shows that $\operatorname{depth} S/J(G)^n = n - 1$. We claim that $\operatorname{depth} S/J(G)^N = n - 1$ for all $N \geq n$. It is enough to consider $u_1, \ldots, u_n$ and $N - n$ arbitrary monomials $v_{n+1}, \ldots, v_N \in G(J(G))$. Then $s(u_1, \ldots, u_n, v_{n+1}, \ldots, v_n) = n$ and $\operatorname{depth} S/J(G)^N = n - 1$. Hence, $\operatorname{dstab}(J(G)) \leq n$. Moreover, from Theorem 9.2.1 and equation (9.6), since S is a polynomial ring in $2n$ variables, $\ell(J(G)) = 2n - \lim_{k\to\infty} \operatorname{depth} S/J(G)^k = n + 1$. □

We close the section with some remarks on the reduced Gröbner basis of the presentation ideal J of $\mathcal{R}(J(G))$. Hereafter, we follow closely [51, Section 6.4.1].

Let G be a Cohen–Macaulay very well–covered graph with $2n$ vertices $X \cup Y = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. Let $G(J(G)) = \{u_1, \ldots, u_m\}$ and let $\mathcal{R}(J(G))$ be the Rees algebra of $J(G)$. Let $\mathbf{x} = x_1, \ldots, x_n$, $\mathbf{y} = y_1, \ldots, y_n$ and $\mathbf{t} = t_{u_1}, \ldots, t_{u_m}$.

Then the Rees algebra $\mathcal{R}(J(G))$ has the presentation

$$\varphi : S' = K[\mathbf{x}, \mathbf{y}, \mathbf{t}] \to \mathcal{R}(J(G))$$

defined by setting

$$\varphi(x_i) = x_i, \quad \varphi(y_i) = y_i, \quad \text{for} \quad 1 \leq i \leq n,$$

and

$$\varphi(t_{u_j}) = u_j t \quad \text{for} \quad 1 \leq j \leq m.$$

The ideal $J = \text{Ker}(\varphi)$ is called the *presentation ideal* of $\mathcal{R}(J(G))$.

Analogously, the toric ring $K[J(G)] = K[u_1, \ldots, u_m]$ has the presentation

$$\psi : T = K[\mathbf{t}] \to K[u_1, \ldots, u_m]$$

defined by setting $\psi(t_{u_j}) = u_j$ for $1 \leq j \leq m$. The ideal $L = \text{Ker}(\psi)$ is called the *toric ideal* of $K[u_1, \ldots, u_m]$. Let $\mathfrak{m} = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ be the graded maximal ideal of S. Since $J(G)$ is generated in a single degree, the fiber ring $\mathcal{R}(J(G))/\mathfrak{m}\mathcal{R}(J(G))$ is isomorphic to the toric ring $K[J(G)]$.

Let $>'$ be an arbitrary monomial order on T and let $>_{\text{lex}}$ be the lexicographic order on S induced by $x_1 > y_1 > x_2 > y_2 > \cdots > x_n > y_n$. We define the monomial order $>'_{\text{lex}}$ as follows: for two monomials $w_1 t_{u_1}^{a_1} \cdots t_{u_m}^{a_m}$ and $w_2 t_{u_1}^{b_1} \cdots t_{u_m}^{b_m}$ in S', with $w_1, w_2 \in S$, we set $w_1 t_{u_1}^{a_1} \cdots t_{u_m}^{a_m} >'_{\text{lex}} w_2 t_{u_1}^{b_1} \cdots t_{u_m}^{b_m}$ if and only if $w_1 >_{\text{lex}} w_2$ or $w_1 = w_2$ and $t_{u_1}^{a_1} \cdots t_{u_m}^{a_m} >' t_{u_1}^{b_1} \cdots t_{u_m}^{b_m}$. According to [51, Section 2] the order $>'_{\text{lex}}$ is the product order of $>'$ and $>_{\text{lex}}$.

From Theorem 9.3.5 and [51, Theorem 6.24], we get the next result.

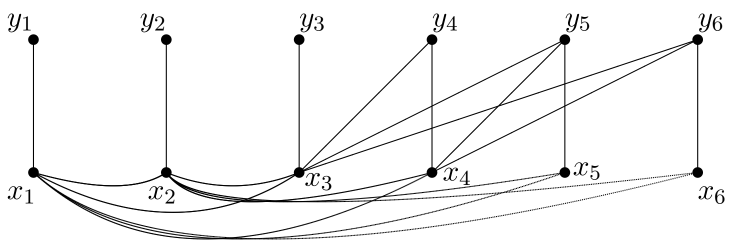
Corollary 9.3.11 *Let G be a whisker graph with $2n$ vertices. Then the reduced Gröbner basis of the presentation ideal J of $\mathcal{R}(J(G))$ with respect to $>'_{\text{lex}}$ consists of all binomials belonging to the reduced Gröbner basis of L with respect to $>'$ together with the binomials*

$$x_i t_u - y_i t_{x_i(u/y_i)},$$

where $u, x_i(u/y_i) \in G(J(G))$.

The statement of Corollary 9.3.11 seems to be true for all Cohen–Macaulay very well–covered graphs.

Example 9.3.12 Consider the graph G with 12 vertices depicted below



G

By Characterization 8.1.3, G is a Cohen–Macaulay very well–covered graph with 12 vertices. We have

$$\begin{aligned} I(G) &= (x_1y_1, x_2y_2, x_3y_3, x_4y_4, x_5y_5, x_6y_6, x_1x_2, x_1x_3, x_1x_4, x_1x_5,\\ &\quad x_1x_6, x_2x_3, x_2x_4, x_2x_5, x_2x_6, x_3y_4, x_3y_5, x_3y_6, x_4y_5, x_4y_6),\\ J(G) &= (x_1x_2x_3x_4x_5x_6,\ x_1x_2x_3x_4x_5y_6,\ x_1x_2x_3x_4y_5x_6,\ x_1x_2x_3x_4y_5y_6,\\ &\quad x_1x_2x_3y_4y_5y_6,\ x_1x_2y_3y_4y_5y_6,\ x_1y_2x_3x_4x_5x_6,\ y_1x_2x_3x_4x_5x_6). \end{aligned}$$

We order the monomials $u_1, \dots, u_8$ of $G(J(G))$ with respect to the lexicographic order induced by $x_1 > y_1 > \cdots > x_6 > y_6$. Thus, for instance $u_1 = x_1x_2x_3x_4x_5x_6$, $u_2 = x_1x_2x_3x_4x_5y_6$ and so on.

Now, let

$$\varphi : S' = K[\mathbf{x}, \mathbf{y}, \mathbf{t}] \to \mathcal{R}(J(G))$$

be the map defined by setting $\varphi(x_i) = x_i$, $\varphi(y_i) = y_i$, for $1 \le i \le 6$, and $\varphi(t_{u_j}) = u_jt$ for $1 \le j \le 8$. Furthermore, let $T = K[\mathbf{t}]$.

Let $>'_{\text{lex}}$ be the product order of the lexicographic order $>_{\text{lex}}$ on S induced by $x_1 > y_1 > \cdots > x_n > y_n$, and the lexicographic order $>'$ on T such that $t_{u_i} >' t_{u_j}$ if and only if $u_i >_{\text{lex}} u_j$. By using *Macaulay2* [82], we have that the reduced Gröbner basis of $\ker(\varphi)$ with respect to the order $>'_{\text{lex}}$ is the following one:

$$\begin{aligned} \mathcal{G} \;=\; \{&x_6t_{u_2} - y_6t_{u_1},\ x_5t_{u_3} - y_5t_{u_1},\ x_6t_{u_4} - y_6t_{u_3},\ x_5t_{u_3} - y_5t_{u_1},\ x_4t_{u_5} - y_4t_{u_4},\\ &x_3t_{u_6} - y_3t_{u_5},\ x_2t_{u_7} - y_2t_{u_1},\ x_1t_{u_8} - y_1t_{u_1},\ t_{u_1}t_{u_4} - t_{u_2}t_{u_3}\}. \end{aligned}$$

Our experiments using *Macaulay2* [82] suggest the next conjecture.

Conjecture 9.3.13 *Let G be a Cohen–Macaulay very well–covered graph with $2n$ vertices. Then the presentation ideal of the Rees algebra of $J(G)$ has a quadratic reduced Gröbner basis with respect to the product order $>'_{\text{lex}}$ of the lexicographic order $>_{\text{lex}}$ on S induced by $x_1 > y_1 > \cdots > x_n > y_n$, and the lexicographic order $>'$ on T such that $t_{u_i} >' t_{u_j}$ if and only if $u_i >_{\text{lex}} u_j$. In particular, $\mathcal{R}(J(G))$ is a Koszul algebra.*

Notes

To determine when a homogeneous ideal has linear powers, or more generally when all powers of I are componentwise linear is a classical question in Commutative Algebra. One of the strongest results in this direction is the theorem of Herzog, Hibi and Zheng about edge ideals with linear powers (Theorem 3.3.6). It says that $I(G)$ has linear powers, that is, $\text{reg}(I(G)) = 2k$ for all $k \ge 1$, if and only if G is a cochordal graph.

More generally, the study of the regularity of powers of a homogeneous ideal $I \subset S$ is motivated by the result of Cutkosky, Herzog and Trung [41], as well as that of Kodiyalam [118], which says that $\text{reg}(I^k)$ is asymptotically a linear function $a(I)k + b(I)$, with $a(I)$ a positive integer less or equal to the highest degree of a minimal generator of I, and $b(I) \ge 0$. For edge ideals, a key problem is to determine the constants $a(I(G))$ and $b(I(G))$ appearing in $\text{reg}(I(G)^k) = a(I(G))k + b(I(G))$ ($k \gg 0$) in terms of the combinatorics of G. However, this problem is doomed to be a dream in general. Indeed, while $a(I(G)) = 2$, Minh and Vu exhibited an example of an edge ideal whose Castelnuovo–Mumford regularity of each of its powers depends on the characteristic of the underlying field [128, Remark 5.5]. Thus the above problem can be solved only if we confine ourself to particular classes of graphs.

Our Conjecture 8.4.4 is inspired by the Herzog–Hibi–Ohsugi conjecture [91, Conjecture 2.5] which predicts that $J(G)^k$ is componentwise linear for all $k \geq 1$, if G is a chordal graph. In such paper [91], componentwise linearity is characterized as follows.

Theorem. (Herzog–Hibi–Ohsugi, 2011 [91, Corollary 1.5]). *Let K be an infinite field, $I \subset S = K[x_1, \dots, x_n]$ be a graded ideal and $\mathbf{z}$ a generic K-basis of S_1. Then, the following conditions are equivalent.*

(a) *All powers of I are componentwise linear.*

(b) *$\mathbf{z}$ is a d-sequence with respect to the Rees algebra $\mathcal{R}(I)$ of I.*

The concept of d-sequence was introduced by Huneke in [108] as a generalization of the concept of regular sequence. There is a characterization for d-sequences, similar to the case of regular sequences (Theorem 2.2.1), in terms of the acyclicity of the so–called *approximation complexes*, see [98] and [99, Theorem 4.1].

The above theorem, while being a very powerful algebraic characterization, in practice can be rarely applied to concrete examples. Indeed, most of the partial results that support Herzog–Hibi–Ohsugi conjecture use either Rees algebras methods involving Gröbner basis and the so–called x-condition [94], or linear quotients techniques, like checking the weakly polymatroidal property. For recent progress on the conjecture, see the survey [84] of Ha and Van Tuyl.

In a similar direction, Fakhari considered the problem of determining all graphs G whose cover ideal $J(G)$ has linear powers. The following beautiful result holds.

Theorem. (Fakhari, 2021 [57, Theorem 3.4]). *Let G be a simple graph. Then, the following conditions are equivalent:*

(a) *$J(G)$ has linear powers.*

(b) *$J(G)^{(k)}$ has linear resolution, for some $k \geq 2$.*

(c) *$J(G)^{(k)}$ has a linear presentation, for all $k \geq 1$.*

(d) *$J(G)^{(k)}$ has a linear presentation, for some $k \geq 2$.*

(e) *G is a Cohen–Macaulay very well–covered graph.*

Symbolic powers are in general different from the ordinary powers. For an edge ideal it is known that $I(G)^k = I(G)^{(k)}$ for all $k \geq 1$, if and only if, G is a bipartite graph. In the bipartite case, we also have $J(G)^k = J(G)^{(k)}$ for all $k \geq 1$. On the other hand, for a Cohen–Macaulay very well–covered graph G which is not bipartite, in general we have $J(G)^k \neq J(G)^{(k)}$.

In a private communication with Crupi and Fakhari, we learned that the initial idea of Fakhari [57] was to prove that all ordinary powers of the cover ideal $J(G)$ of a Cohen–Macaulay very well–covered graph G have a linear resolution. On the other hand, our Conjecture 8.4.4 is much stronger, because it would imply not just that $J(G)$ has linear powers, but that also all homological shifts of $J(G)^k$ have linear resolution (indeed even linear quotients), for all k.

Chapter 10

Asymptotic behaviour of the v-number of homogeneous ideals

In 1921, Emmy Noether revolutionized Commutative Algebra by establishing the primary decomposition Theorem for Noetherian rings [134]. It says that any ideal I of a Noetherian ring R can be decomposed as the irredundant intersection of finitely many primary ideals $I = Q_1 \cap \cdots \cap Q_t$ and $\mathrm{Ass}(I) = \{\sqrt{Q_1}, \ldots, \sqrt{Q_t}\}$, the *set of associated primes of* I, is uniquely determined. This fundamental result is a landmark in Commutative Algebra, and always inspires new exciting research trends. A basic question in the seventies was the following. What is the asymptotic behaviour of the set $\mathrm{Ass}(I^k)$ for $k \gg 0$ large enough? In 1976, it was predicted by Ratliff [139], and later proved by Brodmann in 1979 [21], that $\mathrm{Ass}(I^k)$ stabilizes. That is, there exists $k_0 > 0$ such that $\mathrm{Ass}(I^{k+1}) = \mathrm{Ass}(I^k)$ for all $k \geq k_0$. Another remarkable result of Brodmann says that $\mathrm{depth}(R/I^k)$ is constant for $k \gg 0$ [22]. Suppose furthermore that I is a graded ideal of a standard graded polynomial ring $S = K[x_1, \ldots, x_n]$ with coefficients in a field K. In 1999, Kodiyalam [118], and, independently, Cutkosky, Herzog and Trung [41], showed that the Castelnuovo–Mumford regularity of S/I^k is a linear function in k for $k \gg 0$. The legacy of Brodmann theorem has opened up the most flourished research topic in Commutative Algebra: the asymptotic behaviour of the homological invariants of (ordinary) powers of graded ideals, see [25].

Now, let $S = K[x_1, \ldots, x_n]$ be the standard graded polynomial ring with coefficients in a field K, $I \subset S$ be a graded ideal and $\mathfrak{m} = (x_1, \ldots, x_n)$ be the maximal ideal. Note that S is Noetherian. The graded version of the primary decomposition theorem says that for any prime $\mathfrak{p} \in \mathrm{Ass}(I)$, there exists a homogeneous element $f \in S$ such that $(I : f) = \mathfrak{p}$. It is natural to define the following invariants. Denote by S_d the dth graded component of S. The v-*number of* I *at* $\mathfrak{p}$ is defined as

$$\mathrm{v}_{\mathfrak{p}}(I) \ = \ \min\{d \ : \ \textit{there exists } f \in S_d \textit{ such that } (I : f) = \mathfrak{p}\}.$$

Whereas, the v-*number of* I is defined as

$$\mathrm{v}(I) \ = \ \min\{d \ : \ \textit{there exists } f \in S_d \textit{ such that } (I : f) \in \mathrm{Ass}(I)\}.$$

The concept of v-number was introduced by Cooper *et all* in [31], and further studied in [6, 29, 62, 83, 112, 111, 140, 137].

This invariant plays an important role in Algebraic Geometry and in the theory of (*projective*) *Reed–Muller–type codes* [31, 45, 77, 81, 80, 79, 78]. Let $\mathbb{X}$ be a finite set of points of the projective plane $\mathbb{P}^{s-1}$, and let $\delta_{\mathbb{X}}(d)$ be the minimum distance function of the projective Reed–Muller–type code $C_{\mathbb{X}}(d)$. Then $\delta_{\mathbb{X}}(d) = 1$ if and only if $\mathrm{v}(I(\mathbb{X})) \leq d$ [31, Corollary 5.6]. In such article, for a radical complete intersection ideal I, the famous Eisenbud-Green-Harris conjecture [48] is shown to be equivalent to [31, Conjecture 6.2] (see [31, Proposition 6.8]). This latter conjecture is related to the v-number. Indeed, for such an ideal I we have $\mathrm{v}(I) = \mathrm{reg}(S/I)$. For a nice summary, see [137, Section 12].

The v-number of edge ideals was studied in [111]. A graph G belongs to the class W_2 if and only if G is well–covered without isolated vertices, and $G \setminus v$ is well–covered for all vertices $v \in V(G)$. Let $I(G) \subset S = K[x_v : v \in V(G)]$ be the edge ideal of G. Then G is in W_2 if and only if $\mathrm{v}(I(G)) = \dim(S/I(G))$ [111, Theorem 4.5]. The v-number of binomial edge ideals was recently considered in [6] and [112].

In this chapter, we investigate the eventual behaviour of the function $\mathrm{v}(I^k)$ for $k \gg 0$, where $I \subset S$ is a graded ideal. Such a function, for large k measures the "asymptotic homogeneous growth" of the primary decomposition of I^k.

The chapter is structured as follows. In Section 10.1, we recall how to compute the v-number of a graded ideal $I \subset S$ (Theorem 10.1.1) as shown by Grisalde, Reyes and Villarreal [83]. Hereafter, for a finitely generated graded S-module $M = \bigoplus_d M_d \neq 0$, we set $\alpha(M) = \min\{d : M_d \neq 0\}$ and $\omega(M) = \max\{d : (M/\mathfrak{m}M)_d \neq 0\}$. Firstly, we determine the "local" numbers $\mathrm{v}_\mathfrak{p}(I)$, for all $\mathfrak{p} \in \mathrm{Ass}(I)$. After computing a basis of the S-module $(I : \mathfrak{p})/I$, we select a generator $\overline{f} \in (I : \mathfrak{p})/I$ of least degree d such that $(I : f) = \mathfrak{p}$. This latter condition is automatically satisfied if $\mathfrak{p} \in \mathrm{Max}(I)$. Then $\mathrm{v}_\mathfrak{p}(I) = d$ and $\mathrm{v}(I) = \min\{\mathrm{v}_\mathfrak{p}(I) : \mathfrak{p} \in \mathrm{Ass}(I)\}$.

Let $I \subset S$ be a graded ideal, we call $\mathrm{v}(I^k)$ the v-*function* of I. In Section 10.2, we investigate the asymptotic behaviour of $\mathrm{v}(I^k)$ for $k \gg 0$. By Theorem 10.1.1(b), we have $\mathrm{v}(I^k) = \min\{\mathrm{v}_\mathfrak{p}(I^k) : \mathfrak{p} \in \mathrm{Ass}(I^k)\}$. By Brodmann [21], $\mathrm{Ass}(I^k) = \mathrm{Ass}(I^{k+1})$ for all $k \gg 0$. We denote this common set by $\mathrm{Ass}^\infty(I)$, and call each prime $\mathfrak{p} \in \mathrm{Ass}^\infty(I)$ a *stable prime ideal* of I. Let $\mathrm{Max}^\infty(I)$ be the set of stable prime ideals of I maximal with respect to the inclusion. Thus, $\mathrm{v}(I^k) = \min\{\mathrm{v}_\mathfrak{p}(I^k) : \mathfrak{p} \in \mathrm{Ass}^\infty(I)\}$. To understand the asymptotic behaviour of the v-function, we consider the $\mathrm{v}_\mathfrak{p}$-*functions* $\mathrm{v}_\mathfrak{p}(I^k)$ for each $\mathfrak{p} \in \mathrm{Ass}^\infty(I)$. In the classical case, to prove the asymptotic linearity of the Castelnuovo–Mumford regularity of the powers of I, $\mathrm{reg}(I^k)$, one introduces the Rees ring of I, $\mathcal{R}(I) = \bigoplus_{k \geq 0} I^k$, and shows that this is a bigraded finitely generated module over a suitable polynomial ring [41].

Let $\mathfrak{p} \in \mathrm{Ass}^\infty(I)$. For the $\mathrm{v}_\mathfrak{p}$-function $\mathrm{v}_\mathfrak{p}(I^k)$, we consider a similar approach as above. It should be noted, however, that the S-module $(I^k : \mathfrak{p})/I^k$ has a more subtle module structure than the ordinary power I^k. We introduce the module

$$\mathrm{Soc}_\mathfrak{p}(I) \;=\; \bigoplus_{k\geq 0}(I^k : \mathfrak{p})/I^k.$$

over the ring $\mathcal{F}_\mathfrak{p}(I) = \bigoplus_{k \geq 0}(I^k/\mathfrak{p}I^k)$.

A priori, it is not clear that $\mathrm{Soc}_\mathfrak{p}(I)$ is a finitely generated bigraded $\mathcal{F}_\mathfrak{p}(I)$-module. This is shown in Theorem 10.2.2 by carefully analyzing the module structure of $\mathrm{Soc}_\mathfrak{p}(I)$. We prove that $\mathrm{Soc}_\mathfrak{p}(I)$ is equal to a truncation of the ideal $(0 :_{\mathrm{gr}_I(S)} \mathfrak{p})$, and this ideal of $\mathrm{gr}_I(S)$ is finitely generated as a $\mathcal{F}_\mathfrak{p}(I)$-module. Here $\mathrm{gr}_I(S) = \bigoplus_{k \geq 0}(I^k/I^{k+1})$ denotes the associated graded ring of I. The proof relies essentially on a property showed by Ratliff [139, Corollary 4.2], namely that $(I^{k+1} : I) = I^k$ for all $k \gg 0$, and on the fact that $\mathrm{gr}_I(S)$ is Noetherian ring [126, Proposition (10.D)].

The first main result in Section 10.2 is Theorem 10.2.1, which states that

$$\alpha((I^k : \mathfrak{p})/I^k) \leq \mathrm{v}_\mathfrak{p}(I^k) \leq \omega((I^k : \mathfrak{p})/I^k)$$

for all $k \gg 0$ and all $\mathfrak{p} \in \mathrm{Ass}^\infty(I)$, and that the functions $\alpha((I^k : \mathfrak{p})/I^k)$, $\omega((I^k : \mathfrak{p})/I^k)$ are linear in k for $k \gg 0$. In particular, Theorem 10.2.1(b) follows by a careful analysis of the bigraded structure of $\mathrm{Soc}_{\mathfrak{p}}(I)$ and in the end boils down to a linear programming argument (Proposition 10.2.5).

The second main result in the section is Theorem 10.2.6, which states, under reasonable assumptions that the functions $\mathrm{v}(I^k)$ and $\mathrm{v}_{\mathfrak{p}}(I^k)$ ($\mathfrak{p} \in \mathrm{Ass}^{\infty}(I)$) are linear in k for $k \gg 0$. This result is valid for several classes of graded ideals. To name a few, ideals of maximal minors, binomial edge ideals of closed graphs, and normally torsionfree squarefree monomial ideals (Example 10.2.7). We believe that the functions $\mathrm{v}_{\mathfrak{p}}(I^k)$, $\mathfrak{p} \in \mathrm{Ass}^{\infty}(I)$, and $\mathrm{v}(I^k)$ are always linear in k for $k \gg 0$. We conclude Section 10.2 with an estimate on the growth of $\mathrm{v}_{\mathfrak{p}}(I^k)$, $\mathfrak{p} \in \mathrm{Ass}^{\infty}(I)$. We prove that $\mathrm{v}_{\mathfrak{p}}(I^{k+1}) \leq \mathrm{v}_{\mathfrak{p}}(I^k) + \omega(I)$, for all $k \gg 0$. Thus, we have $\mathrm{v}(I^{k+1}) \leq \mathrm{v}(I^k) + \omega(I)$ for $k \gg 0$.

Section 10.3 concerns monomial ideals $I \subset S = K[x, y]$ in two variables. Denote by $G(I) = \{u_1, \ldots, u_m\}$ the unique minimal monomial generating set of I, with $u_i = x^{a_i} y^{b_i}$ for all i. Then I determines the sequences $\mathbf{a} : a_1 > a_2 > \cdots > a_m$ and $\mathbf{b} : b_1 < b_2 < \cdots < b_m$, and we set $I = I_{\mathbf{a},\mathbf{b}}$. Conversely, given any two such sequences $\mathbf{a}$ and $\mathbf{b}$, $\{x^{a_1} y^{b_1}, \ldots, x^{a_m} y^{b_m}\}$ is the minimal generating set of a monomial ideal of S. In terms of $\mathbf{a}$ and $\mathbf{b}$ we determine $\mathrm{Ass}^{\infty}(I_{\mathbf{a},\mathbf{b}})$ (Corollary 10.3.4) and the v-number $\mathrm{v}(I_{\mathbf{a},\mathbf{b}})$ (Theorem 10.3.7). We prove that $\mathrm{v}(I^k_{\mathbf{a},\mathbf{b}})$ is a linear function $f(k) = ak + b$ for $k \gg 0$ (Theorem 10.3.1). Our experiments in *Macaulay2* [82] suggest that $b \geq -1$. On the other hand, for any integers $a \geq 1$ and $b \geq -1$, we construct a monomial ideal of S such that $\mathrm{v}(I^k) = ak + b$ for all $k \geq 1$ (Theorem 10.3.8).

In the last section, we study the v-function of ideals with linear powers. We expect that if I is a graded ideal with linear powers, then $\mathrm{v}(I^k) = \alpha(I)k - 1$, for all $k \geq 1$ (Conjecture 10.4.1). We settle this conjecture for edge ideals with linear resolution, polymatroidal ideals and Hibi ideals. To prove these results, we use an inductive argument based on a bound proved by Saha and Sengupta (Proposition 10.4.2). On the other hand, if I does not have linear powers, then the conclusion of Conjecture 10.4.1 is no longer valid, as we show with an example due to Terai [30, Remark 3]. For such an ideal I, we have $\mathrm{v}(I) = \alpha(I) = 3$ and $\mathrm{v}(I^k) = \alpha(I)k - 1$ for all $k \geq 2$. Nonetheless, the v-function of I is linear.

10.1 How to compute the v-number of a graded ideal?

Let I be an ideal of a Noetherian domain R. We denote the set of associated primes of I by $\mathrm{Ass}(I)$, and by $\mathrm{Max}(I)$ the set of associated primes of I that are maximal with respect to the inclusion. It is clear that I has no embedded primes if and only if $\mathrm{Ass}(I) = \mathrm{Max}(I)$.

Let $S = K[x_1, \ldots, x_n] = \bigoplus_d S_d$ be the standard graded polynomial ring with n variables and coefficients in a field K, and let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the graded maximal ideal. The concept of v-number was introduced by Cooper *et all* in [31]. Let $I \subset S$ be a graded ideal and let $\mathfrak{p} \in \mathrm{Ass}(I)$. Then, the v-*number of I at* $\mathfrak{p}$ is defined as

$$\mathrm{v}_{\mathfrak{p}}(I) \;=\; \min\{d \;:\; \textit{there exists } f \in S_d \textit{ such that } (I : f) = \mathfrak{p}\}.$$

Whereas, the v-*number of I* is defined as

$$\mathrm{v}(I) \;=\; \min\{d \;:\; \textit{there exists } f \in S_d \textit{ such that } (I : f) \in \mathrm{Ass}(I)\}.$$

Note that if $I = \mathfrak{m} = (x_1, \ldots, x_n)$, then $\mathrm{v}_{\mathfrak{m}}(I) = \mathrm{v}(I) = 0$.

The following result due to Grisalde, Reyes and Villarreal [83, Theorem 3.2] shows how to compute the v-number of a graded ideal. For a finitely generated graded S-module $M = \bigoplus_d M_d \neq 0$, we call $\alpha(M) = \min\{d : M_d \neq 0\}$ the *initial degree* of M. In the next theorem, the bar $^-$ denotes the residue class modulo I.

Theorem 10.1.1 *Let $I \subset S$ be a graded ideal and let $\mathfrak{p} \in \mathrm{Ass}(I)$. The following hold.*

(a) *If $\mathcal{G} = \{\overline{g_1}, \dots, \overline{g_r}\}$ is a homogeneous minimal generating set of $(I : \mathfrak{p})/I$, then*

$$\mathrm{v}_{\mathfrak{p}}(I) = \min\{\deg(g_i) \ : \ 1 \leq i \leq r \text{ and } (I : g_i) = \mathfrak{p}\}.$$

(b) $\mathrm{v}(I) = \min\{\mathrm{v}_{\mathfrak{p}}(I) : \mathfrak{p} \in \mathrm{Ass}(I)\}$.

(c) $\mathrm{v}_{\mathfrak{p}}(I) \geq \alpha((I : \mathfrak{p})/I)$, *with equality if* $\mathfrak{p} \in \mathrm{Max}(I)$.

(d) *If I has no embedded primes, then* $\mathrm{v}(I) = \min\{\alpha((I : \mathfrak{p})/I) : \mathfrak{p} \in \mathrm{Ass}(I)\}$.

10.2 Asymptotic behaviour of the v-number

Let R be a commutative Noetherian domain and $I \subset R$ an ideal. It is known by Brodmann [21] that $\mathrm{Ass}(I^k)$ stabilizes for large k. That is, $\mathrm{Ass}(I^{k+1}) = \mathrm{Ass}(I^k)$ for all $k \gg 0$. A prime ideal $\mathfrak{p} \subset R$ such that $\mathfrak{p} \in \mathrm{Ass}(I^k)$ for all $k \gg 0$, is called a *stable prime of I*.

The set of the stable primes of I is denoted by $\mathrm{Ass}^\infty(I)$. Likewise, $\mathrm{Max}^\infty(I)$ denotes the set of stable primes of I, maximal with respect to the inclusion. The least integer k_0 such that $\mathrm{Ass}(I^k) = \mathrm{Ass}(I^{k_0})$ for all $k \geq k_0$ is denoted by $\mathrm{astab}(I)$.

Now, let $S = K[x_1, \dots, x_n]$ be the standard graded polynomial ring, with K a field, and unique graded maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. Let $I \subset S$ be a graded ideal and let $\mathfrak{p} \in \mathrm{Ass}^\infty(I)$. In light of Theorem 10.1.1 and Brodmann result, to understand the asymptotic behaviour of the function $\mathrm{v}_{\mathfrak{p}}(I^k)$, one has to understand the asymptotic behaviour of the modules $(I^k : \mathfrak{p})/I^k$ for $k \gg 0$.

Let $M \neq 0$ be a finitely generated graded S-module. Let $\omega(M)$ be the highest degree of a homogeneous element of the K-vector space $M/\mathfrak{m}M$. Equivalently, the highest degree j such that the graded Betti number $\beta_{0,j}(M)$ is non-zero. Thus

$$\omega(M) = \max\{d : \beta_{0,d}(M) \neq 0\} = \max\{d : \mathrm{Tor}_0^S(S/\mathfrak{m}, M)_d \neq 0\}.$$

Similarly, one has that $\alpha(M) = \min\{d : \mathrm{Tor}_0^S(S/\mathfrak{m}, M)_d \neq 0\}$.

The following theorem provides natural asymptotic upper and lower bounds for the v-function $\mathrm{v}(I^k)$ which are linear functions in k for $k \gg 0$.

Theorem 10.2.1 *Let $I \subset S = K[x_1, \dots, x_n]$ be a graded ideal, and let $\mathfrak{p} \in \mathrm{Ass}^\infty(I)$. Then, the following holds.*

(a) *For all $k \geq 1$, we have*

$$\alpha((I^k : \mathfrak{p})/I^k) \ \leq \ \mathrm{v}_{\mathfrak{p}}(I^k) \ \leq \ \omega((I^k : \mathfrak{p})/I^k).$$

(b) *The functions $\alpha((I^k : \mathfrak{p})/I^k)$, $\omega((I^k : \mathfrak{p})/I^k)$ are linear in k for $k \gg 0$.*

(c) *There exist eventually linear functions $f(k)$ and $g(k)$ such that*

$$f(k) \leq \mathrm{v}(I^k) \leq g(k), \quad \textit{for all} \ \ k \gg 0.$$

Statement (a) follows immediately from Theorem 10.1.1(a). Assume for a moment that statement (b) holds, then (c) can be proved as follows. By Brodmann, we have $\mathrm{v}(I^k) = \min\{\mathrm{v}_{\mathfrak{p}}(I^k) : \mathfrak{p} \in \mathrm{Ass}^\infty(I)\}$ for all $k \gg 0$. Thus, by (a), for all $k \gg 0$

$$\min_{\mathfrak{p}\in\mathrm{Ass}^\infty(I)} \alpha((I^k : \mathfrak{p})/I^k) \le \mathrm{v}(I^k) \le \min_{\mathfrak{p}\in\mathrm{Ass}^\infty(I)} \omega((I^k : \mathfrak{p})/I^k).$$

Setting $f(k) = \min_{\mathfrak{p}\in\mathrm{Ass}^\infty(I)} \alpha((I^k : \mathfrak{p})/I^k)$ and $g(k) = \min_{\mathfrak{p}\in\mathrm{Ass}^\infty(I)} \omega((I^k : \mathfrak{p})/I^k)$, by statement (b) it follows that $f(k)$ and $g(k)$ are the required eventually linear functions in k. Statement (c) follows.

To prove statement (b), we construct a suitable module that encodes the growth of the modules $(I^k : \mathfrak{p})/I^k$. Indeed, we define it in the following more general context. Let I be an ideal of a commutative Noetherian domain R and let $\mathfrak{p} \in \mathrm{Ass}^\infty(I)$. Then we set

$$\mathrm{Soc}_{\mathfrak{p}}(I) \;=\; \bigoplus_{k\ge 0}(I^k : \mathfrak{p})/I^k,$$

and $\mathrm{Soc}_{\mathfrak{p}}(I)_k = (I^k : \mathfrak{p})/I^k$ for all $k \ge 0$.

The symbol "Soc" is used, because when $R = S$ or R is local and $\mathfrak{p} = \mathfrak{m}$ is the (graded) maximal ideal, then $(I^k : \mathfrak{m})/I^k$ is the *socle module* of S/I^k, see [27].

The first step consists in showing that $\mathrm{Soc}_{\mathfrak{p}}(I)$ is a finitely generated graded module over a suitable ring. For this aim, we introduce the following ring,

$$\mathcal{F}_{\mathfrak{p}}(I) \;=\; \bigoplus_{k\ge 0}(I^k/\mathfrak{p}I^k),$$

and we set $\mathcal{F}_{\mathfrak{p}}(I)_k = I^k/\mathfrak{p}I^k$. We define addition in the obvious way and multiplication as follows. If $a \in I^k/\mathfrak{p}I^k$ and $b \in I^\ell/\mathfrak{p}I^\ell$, then $ab \in I^{k+\ell}/\mathfrak{p}I^{k+\ell}$. It is routine to check that this multiplication is well–defined.

As before, we note that if $R = S$ or R is local and $\mathfrak{p} = \mathfrak{m}$ is the maximal ideal, then $\mathcal{F}_{\mathfrak{m}}(I) = \bigoplus_{k\ge 0}(I^k/\mathfrak{m}I^k)$ is the well–known *fiber cone* of I.

With the notation introduced, we have

Theorem 10.2.2 *Let I be an ideal of a Noetherian commutative domain R and let $\mathfrak{p} \in \mathrm{Ass}^\infty(I)$. Then, $\mathrm{Soc}_{\mathfrak{p}}(I)$ is a finitely generated graded $\mathcal{F}_{\mathfrak{p}}(I)$-module.*

Proof. Firstly, we show that $\mathrm{Soc}_{\mathfrak{p}}(I)$ has the structure of a graded $\mathcal{F}_{\mathfrak{p}}(I)$-module. For this purpose, let $f \in I^\ell/\mathfrak{p}I^\ell$. It is clear that multiplication by f induces a map $(I^k : \mathfrak{p})/I^k \to (I^{k+\ell} : \mathfrak{p})/I^{k+\ell}$ for any $k \ge 0$. Hence $\mathcal{F}_{\mathfrak{p}}(I)_\ell\mathrm{Soc}_{\mathfrak{p}}(I)_k \subseteq \mathrm{Soc}_{\mathfrak{p}}(I)_{k+\ell}$.

To prove that $\mathrm{Soc}_{\mathfrak{p}}(I)$ is a finitely generated $\mathcal{F}_{\mathfrak{p}}(I)$-module, we consider

$$J = (0 :_{\mathrm{gr}_I(R)} \mathfrak{p}) = \{f \in \mathrm{gr}_I(R) : f\mathfrak{p} = 0\},$$

i.e., the annihilator of $\mathfrak{p}$ in the associated graded ring of I, $\mathrm{gr}_I(R) = \bigoplus_{k\ge 0}(I^k/I^{k+1})$. Recall that $\mathrm{gr}_I(R)$ is a Noetherian ring [126, Proposition (10.D)]. Thus, as an ideal of $\mathrm{gr}_I(R)$, J is a finitely generated graded $\mathrm{gr}_I(R)$-module. Since $\mathfrak{p}$ annihilates J, then J has also the structure of a finitely generated graded $\mathrm{gr}_I(R)/\mathfrak{p}\mathrm{gr}_I(R)$-module. But

$$\begin{aligned}
\mathrm{gr}_I(R)/\mathfrak{p}\mathrm{gr}_I(R) &= \frac{\bigoplus_{k\ge 0}(I^k/I^{k+1})}{\mathfrak{p}\bigoplus_{k\ge 0}(I^k/I^{k+1})} = \frac{\bigoplus_{k\ge 0}(I^k/I^{k+1})}{\bigoplus_{k\ge 0}(\mathfrak{p}I^k/I^{k+1})}\\
&= \bigoplus_{k\ge 0}\frac{I^k/I^{k+1}}{\mathfrak{p}I^k/I^{k+1}} = \bigoplus_{k\ge 0}(I^k/\mathfrak{p}I^k)\\
&= \mathcal{F}_{\mathfrak{p}}(I).
\end{aligned}$$

Consequently, J is a finitely generated graded $\mathcal{F}_{\mathfrak{p}}(I)$-module.

Let us show that $\mathrm{Soc}_{\mathfrak{p}}(I)_{k+1} = J_k$ for $k \gg 0$. For this purpose, we compute the kth graded component of J. We have

$$\begin{aligned} J_k &= \{f \in \mathrm{gr}_I(R)_k : f\mathfrak{p} = 0\} = \{f \in I^k/I^{k+1} \ : f\mathfrak{p} = 0\} \\ &= \{f \in I^k : f\mathfrak{p} \in I^{k+1}\}/I^{k+1} = (\{f \in R : f\mathfrak{p} \in I^{k+1}\} \cap I^k)/I^{k+1} \\ &= ((I^{k+1} : \mathfrak{p}) \cap I^k)/I^{k+1}. \end{aligned}$$

By Ratliff [139, Corollary 4.2], there exists r such that $(I^{k+1} : I) = I^k$ for all $k \geq r$. Whereas, by Brodmann [21], there exists b such that $\mathrm{Ass}(I^k) = \mathrm{Ass}^{\infty}(I)$ for all $k \geq b$. Let $k^* = \max\{r, b\}$. Next, we show that $\mathrm{Soc}_{\mathfrak{p}}(I)_{k+1} = J_k$ for $k \geq k^*$.

Let $k \geq k^*$. We claim that $\mathfrak{p}$ contains I. Indeed, $\mathfrak{p} \in \mathrm{Ass}(I^k)$, hence $I^k \subseteq \mathfrak{p}$. Let $a \in I$, then $a^k \in I^k \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is prime, actually $a \in \mathfrak{p}$ and so $I \subseteq \mathfrak{p}$. Therefore, $(I^{k+1} : \mathfrak{p}) \subseteq (I^{k+1} : I) = I^k$ by the Ratliff property. Hence,

$$J_k = ((I^{k+1} : \mathfrak{p}) \cap I^k)/I^{k+1} = (I^{k+1} : \mathfrak{p})/I^{k+1} = \mathrm{Soc}_{\mathfrak{p}}(I)_{k+1}.$$

Consequently, we obtain that $\mathrm{Soc}_{\mathfrak{p}}(I)_{\geq k^*+1} = J_{\geq k^*}$, where $M_{\geq \ell}$ denotes $\bigoplus_{k \geq \ell} M_\ell$ if $M = \bigoplus_{k \geq 0} M_k$ is graded. Since J is finitely generated as a $\mathcal{F}_{\mathfrak{p}}(I)$-module, it follows that $\mathrm{Soc}_{\mathfrak{p}}(I)$ is a finitely generated $\mathcal{F}_{\mathfrak{p}}(I)$-module as well. □

Now, we assume furthermore that $R = S = K[x_1, \dots, x_n]$ is the standard graded polynomial ring with K a field, that I is a graded ideal of S and $\mathfrak{p} \in \mathrm{Ass}^{\infty}(I)$ is a stable prime of I. Then, $I^k/\mathfrak{p}I^k$ is a graded S-module, for all $k \geq 0$. Therefore, $\mathcal{F}_{\mathfrak{p}}(I)$ is in a natural way a bigraded ring:

$$\mathcal{F}_{\mathfrak{p}}(I) \ = \ \bigoplus_{d,k \geq 0} (I^k/\mathfrak{p}I^k)_d.$$

In particular, we set $\mathcal{F}_{\mathfrak{p}}(I)_{(d,k)} = (I^k/\mathfrak{p}I^k)_d$ and $\mathrm{bideg}(f) = (d, k)$ for $f \in \mathcal{F}_{\mathfrak{p}}(I)_{(d,k)}$.

Note that each module $(I^k : \mathfrak{p})/I^k$ is a graded S-module. Thus, we can write

$$\mathrm{Soc}_{\mathfrak{p}}(I) \ = \ \bigoplus_{d,k \geq 0} \mathrm{Soc}_{\mathfrak{p}}(I)_{(d,k)}$$

where $\mathrm{Soc}_{\mathfrak{p}}(I)_{(d,k)} = ((I^k : \mathfrak{p})/I^k)_d$. Hence, $\mathrm{Soc}_{\mathfrak{p}}(I)$ is a bigraded $\mathcal{F}_{\mathfrak{p}}(I)$-module, because $\mathcal{F}_{\mathfrak{p}}(I)_{(d_1,\ell)}\mathrm{Soc}_{\mathfrak{p}}(I)_{(d_2,k)} \subseteq \mathrm{Soc}_{\mathfrak{p}}(I)_{(d_1+d_2,k+\ell)}$.

Therefore, we have proved that

Corollary 10.2.3 *Let I be a graded ideal of $S = K[x_1, \dots, x_n]$ with K a field and let $\mathfrak{p} \in \mathrm{Ass}^{\infty}(I)$. Then, $\mathrm{Soc}_{\mathfrak{p}}(I)$ is a finitely generated bigraded $\mathcal{F}_{\mathfrak{p}}(I)$-module.*

Let $u_1, \dots, u_m$ be a minimal system of homogeneous generators of I. It is well–known that the associated graded ring $\mathrm{gr}_I(S)$ has a presentation

$$\varphi : T = K[x_1, \dots, x_n, y_1, \dots, y_m] \to \mathrm{gr}_I(S)$$

defined by setting

$$\begin{aligned} \varphi(x_i) = x_i + I \in \mathrm{gr}_I(S)_0 = S/I, \ \text{ for } \ 1 \leq i \leq n, \\ \varphi(y_i) = u_i + I^2 \in \mathrm{gr}_I(S)_1 = I/I^2, \ \text{ for } \ 1 \leq i \leq m. \end{aligned}$$

Since I is graded, $\mathrm{gr}_I(S)$ is naturally bigraded, with $\mathrm{gr}_I(S)_{(d,k)} = (I^k/I^{k+1})_d$. Moreover, T can be made into a bigraded ring by setting $\mathrm{bideg}(x_i) = (1,0)$ for $1 \leq i \leq n$, and $\mathrm{bideg}(y_i) = (\deg(u_i), 1)$ for $1 \leq i \leq m$, where $\deg(u_i)$ is the degree of u_i in S. With these bigradings, φ is a bigraded surjective ring homomorphism.

In the proof of Theorem 10.2.2 we have seen that $\mathcal{F}_\mathfrak{p}(I) = \mathrm{gr}_I(S)/\mathfrak{p}\mathrm{gr}_I(S)$. Let $\pi : \mathrm{gr}_I(S) \to \mathcal{F}_\mathfrak{p}(I)$ be the canonical epimorphism. Then, the composition map $\psi = \pi \circ \varphi : T \to \mathcal{F}_\mathfrak{p}(I)$ is a surjective ring homomorphism. It is clear that ψ preserves the bigraded structure. Thus, $\mathrm{Soc}_\mathfrak{p}(I)$ has also the structure of a bigraded T-module, if we set

$$af = \psi(a)f \quad \text{for all} \quad a \in T \quad \text{and all} \quad f \in \mathrm{Soc}_\mathfrak{p}(I).$$

Since ψ is surjective and $\mathrm{Soc}_\mathfrak{p}(I)$ is a finitely generated $\mathcal{F}_\mathfrak{p}(I)$-module, it follows that $\mathrm{Soc}_\mathfrak{p}(I)$ is a finitely generated T-module, as well.

The following lemma is required. For a bigraded T-module $M = \bigoplus_{d,k} M_{d,k}$, we set $M_{(*,k)} = \bigoplus_d M_{(d,k)}$. Note that $M_{(*,k)}$ becomes a graded S-module.

Lemma 10.2.4 *Let $T = K[x_1, \dots, x_n, y_1, \dots, y_m]$ be a bigraded polynomial ring, with K a field,* $\mathrm{bideg}(x_i) = (1,0)$ *for $1 \leq i \leq n$ and* $\mathrm{bideg}(y_i) = (d_i, 1)$ *for $1 \leq i \leq m$. Let $\mathfrak{m} = (x_1, \dots, x_n)$ and $S = K[x_1, \dots, x_n] \subset T$. Let M be a finitely generated bigraded T-module. Then,*

$$\mathrm{Tor}_i^S(S/\mathfrak{m}, M_{(*,k)}) \cong \mathrm{Tor}_i^T(T/\mathfrak{m}, M)_{(*,k)}$$

for all i and k.

Proof. Let $\mathbb{F} : 0 \to \cdots \to F_j \to \cdots \to F_1 \to F_0 \to M \to 0$ be a minimal bigraded T-resolution of M. Then,

$$\mathbb{F}_k : 0 \to \cdots \to (F_j)_{(*,k)} \to \cdots \to (F_1)_{(*,k)} \to (F_0)_{(*,k)} \to M_{(*,k)} \to 0$$

is a graded (possibly non-minimal) free S-resolution of $M_{(*,k)} = \bigoplus_d M_{(d,k)}$. Since $\mathrm{Tor}_i^T(T/\mathfrak{m}, M) = H_i(\mathbb{F}/\mathfrak{m}\mathbb{F})$ we have that $\mathrm{Tor}_i^T(T/\mathfrak{m}, M)_{(*,k)} = H_i(\mathbb{F}_k/\mathfrak{m}\mathbb{F}_k)$ which in turn is isomorphic to $\mathrm{Tor}_i^S(S/\mathfrak{m}, M_{(*,k)})$. The desired conclusion follows. □

Note that $T/\mathfrak{m} = K[y_1, \dots, y_m]$ and that $\mathrm{Tor}_0^T(T/\mathfrak{m}, \mathrm{Soc}_\mathfrak{p}(I))$ is a finitely generated bigraded $T/\mathfrak{m}$-module. Therefore, by the above lemma, we have

$$\begin{aligned}\alpha((I^k : \mathfrak{p})/I^k) &= \alpha(\mathrm{Tor}_0^S(S/\mathfrak{m}, (I^k : \mathfrak{p})/I^k)) = \alpha(\mathrm{Tor}_0^S(S/\mathfrak{m}, \mathrm{Soc}_\mathfrak{p}(I)_{(*,k)})) \\ &= \alpha(\mathrm{Tor}_0^T(T/\mathfrak{m}, \mathrm{Soc}_\mathfrak{p}(I))_{(*,k)}).\end{aligned}$$

Similarly, $\omega((I^k : \mathfrak{p})/I^k) = \omega(\mathrm{Tor}_0^T(T/\mathfrak{m}, \mathrm{Soc}_\mathfrak{p}(I))_{(*,k)})$.

From this discussion, Theorem 10.2.1(b) follows from the next more general statement, which is a variation of [41, Theorem 3.4].

Proposition 10.2.5 *Let $T = K[y_1, \dots, y_s]$ a polynomial ring, with* $\mathrm{bideg}(y_i) = (d_i, 1)$ *for $1 \leq i \leq s$ and K a field, and let M be a finitely generated bigraded T-module. Then, $\alpha_M(k) = \min\{d : M_{(d,k)} \neq 0\}$ and $\omega_M(k) = \max\{d : M_{(d,k)} \neq 0\}$ are linear functions in k for $k \gg 0$.*

Proof. The claim about the linearity of $\omega_M(k)$ follows from [41, Theorem 3.4]. The proof of the claim of the linearity of $\alpha_M(k)$ is similar, but we include here all the details for the convenience of the reader.

For any exact sequence $0 \to M \to N \to P \to 0$ of finitely generated bigraded T-modules we have $\alpha_N(k) = \min\{\alpha_M(k), \alpha_P(k)\}$, for all k.

Since M is a finitely generated T-module and T is Noetherian, by the bigraded version of [47, Proposition 3.7] there exists a sequence of bigraded T-submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{i-1} \subset M_i = M$$

of M such that $M_j/M_{j-1} \cong T/\mathfrak{p}_j$, with $\mathfrak{p}_j$ a bigraded prime ideal of T, for all $1 \leq j \leq i$. Hence, we may suppose that $M = T/J$ with J a bigraded ideal of T. We show that J can be replaced by a monomial ideal. For this aim, let $>$ be a monomial order on T, and let $\mathrm{in}(J)$ be the initial ideal of J with respect to $>$. The natural K-basis of T/J consists of all residue classes (modulo J) of all monomials not belonging to $\mathrm{in}(J)$, see [89, Proposition 2.2.5.(a)]. The same residue classes modulo $\mathrm{in}(J)$ form a K-basis for $T/\mathrm{in}(J)$. Thus $\alpha_M(k) = \alpha_{T/J}(k) = \alpha_{T/\mathrm{in}(J)}(k)$, and we can assume that $M = T/J$ with J a monomial ideal of T.

Recall that $\mathrm{bideg}(y_i) = (d_i, 1)$ for $1 \leq i \leq s$. For later convenience, after a harmless relabeling of the variables, we may suppose that

$$d_1 \leq d_2 \leq \cdots \leq d_s. \tag{10.1}$$

Assume that J is minimally generated by the monomials $\mathbf{y}^{\mathbf{c}_i} = y_1^{c_{i,1}} y_2^{c_{i,2}} \cdots y_s^{c_{i,s}}$, for $1 \leq i \leq r$.

Let $\mathbf{a} = (a_1, a_2, \ldots, a_s) \in \mathbb{N}^s$, we denote by $\overline{\mathbf{y}^{\mathbf{a}}}$ the residue class of $\mathbf{y}^{\mathbf{a}} = y_1^{a_1} y_2^{a_2} \cdots y_s^{a_s}$ in T/J. Let $k \geq 0$, by B_k we denote the minimal basis of $(T/J)_k$. Then, we can write $\alpha_M(k) = \min\{v(\mathbf{a}) : \overline{\mathbf{y}^{\mathbf{a}}} \in B_k\}$, where $v(\mathbf{a}) = \sum_{i=1}^s a_i \deg(y_i) = \sum_{i=1}^s a_i d_i$.

Clearly, $\overline{\mathbf{y}^{\mathbf{a}}} \in B_k$ if and only if $\sum_{j=1}^s a_j = k$, and for all $i = 1, \ldots, s$, there exists j such that $a_j < c_{i,j}$. Denote by L the set of all maps $\{1, \ldots, r\} \to \{1, \ldots, s\}$. We can decompose the set B_k as the union $\bigcup_{f \in L} B_{k,f}$, where

$$B_{k,f} = \big\{\overline{\mathbf{y}^{\mathbf{a}}} \ : \ \sum_{j=1}^s a_j = k \text{ and } a_{f(i)} < c_{i,f(i)}, \ i = 1, \ldots, r\big\}.$$

With this in mind, we can write $\alpha_M(k) = \min_{f \in L} \alpha_f(k)$, where $\alpha_f(k)$ is defined as $\alpha_f(k) = \min\{v(\mathbf{a}) : \overline{\mathbf{y}^{\mathbf{a}}} \in B_{k,f}\}$. Hence, it is enough to prove that $\alpha_f(k)$ is a linear function with integer coefficients for all $f \in L$ and all $k \gg 0$.

Fix $f \in L$. Let $\{j_1 < j_2 < \cdots < j_t\}$ be the image of f. For $h = 1, \ldots, t$, we set $c_{j_h} = \min\{c_{i,j_h} : i = 1, \ldots, r\} - 1$. Then, we have that

$$B_{k,f} = \big\{\overline{\mathbf{y}^{\mathbf{a}}} \ : \ \sum_{j=1}^s a_j = k \text{ and } a_{j_h} \leq c_{j_h}, \ h = 1, \ldots, t\big\}.$$

Thus, $\alpha_f(k)$ is given by the maximum of the *functional* $v(\mathbf{a})$ on the following convex bounded set

$$C_{k,f} = \big\{\mathbf{a} \ : \ \sum_{j=1}^s a_j = k \text{ and } a_{j_h} \leq c_{j_h}, \ h = 1, \ldots, t\big\}.$$

Let ℓ be the smallest integer such that $j_1 = 1, \ldots, j_\ell = \ell$ and $j_{\ell+1} > \ell + 1$. Thus, for $\mathbf{a} = (a_1, a_2, \ldots, a_s) \in C_{k,f}$ we have $a_1 < c_{j_1}, \ a_2 < c_{j_2}, \ \ldots, \ a_\ell < c_{j_\ell}$ and no bound on $a_{j_{\ell+1}}$, except that $\sum_{j=1}^s a_j = k$. We distinguish the two possible cases.

Case 1. Suppose $\ell = s$. Then $\sum_{j=1}^s a_j$ can be at most $c_{j_1} + c_{j_2} + \cdots + c_{j_\ell}$. Thus, for all $k \gg 0$, $B_{k,f} = \varnothing$ and so $\alpha_f(k) = 0$.

Case 2. Suppose $\ell < s$. We let k such that $k \geq c_{j_1} + c_{j_2} + \cdots + c_{j_\ell}$. We claim that the functional v has its minimal value for $\mathbf{a}_* = (c_{j_1}, c_{j_2}, \dots, c_{j_\ell}, k - \sum_{p=1}^{\ell} c_{j_p}, 0, 0, \dots, 0)$. Then, for all large $k \gg 0$, we have that

$$\alpha_f(k) = v(\mathbf{a}_*) = \sum_{p=1}^{\ell} c_{j_p} d_{j_p} + d_{j_{\ell+1}}(k - \sum_{p=1}^{\ell} c_{j_p}),$$

which is a linear function in k with integer coefficients, as desired.

Let $\mathbf{a} = (a_1, a_2, \dots, a_s) \in C_{k,f}$. Assume that for some $1 \leq i < j \leq s$ we have $a_i < c_i$ and $a_j > 0$. Then, $\mathbf{a}' = (a_1, a_2 \dots, a_i + 1, \dots, a_j - 1, \dots, a_s)$ also belongs to $C_{k,f}$ and $v(\mathbf{a}') \leq v(\mathbf{a})$ because by equation (10.1) we have $d_i \leq d_j$. Thus, we see that the minimal value of v on $C_{k,f}$ is achieved when we fill up the first "boxes" of $\mathbf{a} \in C_{k,f}$ as much as possible. Finally, the functional v reaches its minimal value when $\mathbf{a} = \mathbf{a}_*$, completing the proof. □

Now, we come to our second fundamental result.

Theorem 10.2.6 *Let $I \subset S = K[x_1, \dots, x_n]$ be a graded ideal. Suppose that*

(a) *either $\mathrm{Ass}^\infty(I) = \mathrm{Max}^\infty(I)$ or*

(b) *for all $\mathfrak{p} \in \mathrm{Ass}^\infty(I)$ and all $k \gg 0$, $(I^k : \mathfrak{p})/I^k$ is generated in a single degree.*

Then, $\mathrm{v}_\mathfrak{p}(I^k)$, for all $\mathfrak{p} \in \mathrm{Ass}^\infty(I)$, and $\mathrm{v}(I^k)$ are linear functions in k for $k \gg 0$.

Proof. Under hypothesis (a), for all $\mathfrak{p} \in \mathrm{Ass}^\infty(I)$ and all $k \gg 0$, by Theorem 10.1.1(c) we have $\mathrm{v}_\mathfrak{p}(I^k) = \alpha((I^k : \mathfrak{p})/I^k)$. Thus, by Theorem 10.2.1(b), $\mathrm{v}_\mathfrak{p}(I^k)$ is a linear function in k for $k \gg 0$. By Theorem 10.1.1(d), $\mathrm{v}(I^k) = \min\{\alpha((I^k : \mathfrak{p})/I^k) : \mathfrak{p} \in \mathrm{Ass}^\infty(I)\}$ for $k \gg 0$. Thus $\mathrm{v}(I^k)$ is a linear function in k for $k \gg 0$.

Under hypothesis (b), for all $\mathfrak{p} \in \mathrm{Ass}^\infty(I)$ and all $k \gg 0$, we have $\alpha((I^k : \mathfrak{p})/I^k) = \omega((I^k : \mathfrak{p})/I^k)$. By Theorem 10.2.1(a)-(b), it follows that $\mathrm{v}_\mathfrak{p}(I^k) = \alpha((I^k : \mathfrak{p})/I^k)$ is a linear function in k for $k \gg 0$. The assertion about $\mathrm{v}(I^k)$ follows once again. □

Example 10.2.7 Let $I \subset S$ be a graded ideal without embedded primes. Thus $\mathrm{Ass}(I) = \mathrm{Max}(I)$. Recall that the *$k$th symbolic power* of $I \subset S$ is the ideal defined as $I^{(k)} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(I)} (I^k S_\mathfrak{p} \cap S)$. Suppose that $I^k = I^{(k)}$ for all $k \geq 1$. Since I does not have embedded primes, $I^k = I^{(k)} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(I)} (I^k S_\mathfrak{p} \cap S)$ is a primary decomposition of I^k, for all $k \geq 1$. Thus $\mathrm{Ass}^\infty(I) = \mathrm{Ass}(I) = \mathrm{Max}(I) = \mathrm{Max}^\infty(I)$. Hence, for such an ideal the conclusion of Theorem 10.2.6 holds. Next, we give some examples.

(i) Ideals of maximal minors [24, Corollary 3.5.3].

(ii) Binomial edge ideals of closed graphs [52, Corollary 3.4].

(iii) Normally torsionfree squarefree monomial ideals [89, Definition 1.4.5 and Theorem 1.4.6].

Theorem 10.1.1(c) combined with Theorem 10.2.1(b) yields

Corollary 10.2.8 *Let $I \subset S = K[x_1, \dots, x_n]$ be a graded ideal and let $\mathfrak{p} \in \mathrm{Max}^\infty(I)$. Then $\mathrm{v}_\mathfrak{p}(I^k)$ is a linear function in k for $k \gg 0$.*

We conclude this section with the following estimate on the growth of the functions $\mathrm{v}_\mathfrak{p}(I^k)$ for $k \gg 0$. For the proof, we recall the following basic rules. Let $I, I_1, I_2, \{J_i\}_i$ be ideals of a commutative Noetherian ring R and let $\mathfrak{p}$ be a prime ideal of R. Then,

(i) $(I : \sum_i J_i) = \bigcap_i (I : J_i)$,

(ii) $((I : I_1) : I_2) = (I : I_1 I_2)$,

(iii) if $\mathfrak{p} = \bigcap_i J_i$, then $\mathfrak{p} = J_i$ for some i.

Proposition 10.2.9 *Let $I \subset S = K[x_1, \dots, x_n]$ be a graded ideal and let $\mathfrak{p} \in \operatorname{Ass}^\infty(I)$. Then, we have*

$$\mathrm{v}_\mathfrak{p}(I^{k+1}) \le \mathrm{v}_\mathfrak{p}(I^k) + \omega(I), \quad \textit{for all } k \gg 0.$$

In particular, $\mathrm{v}(I^{k+1}) \le \mathrm{v}(I^k) + \omega(I)$ for all $k \gg 0$.

Proof. By Brodmann and Ratliff, there exists $k^* > 0$ such that $\operatorname{Ass}^\infty(I) = \operatorname{Ass}(I^k)$ and $(I^{k+1} : I) = I^k$ for all $k \ge k^*$. Fix $k \ge k^*$ and let $\mathfrak{p} \in \operatorname{Ass}^\infty(I)$.

Let $f \in S$ be a homogeneous element such that $(I^k : f) = \mathfrak{p}$ and $\deg(f) = \mathrm{v}_\mathfrak{p}(I^k)$. Let $f_1, \dots, f_m$ be a minimal homogeneous generating set of I. By rules (ii) and (i),

$$\begin{aligned}\mathfrak{p} = (I^k : f) &= (I^{k+1} : I) : f = (I^{k+1} : fI) \\ &= (I^{k+1} : \sum_{i=1}^m (ff_i)) = \bigcap_{i=1}^m (I^{k+1} : ff_i).\end{aligned}$$

Hence, by rule (iii), we have $\mathfrak{p} = (I^{k+1} : ff_i)$ for some i. By the definition of $\mathrm{v}_\mathfrak{p}(I^{k+1})$, this means that $\mathrm{v}_\mathfrak{p}(I^{k+1}) \le \deg(ff_i) = \deg(f) + \deg(f_i)$. The assertion follows, because $\deg(f) = \mathrm{v}_\mathfrak{p}(I^k)$ and $\deg(f_i) \le \omega(I)$. $\square$

Theorems 10.1.1(a) and 10.2.1(a) combined with the previous result give immediately

Corollary 10.2.10 *Let $I \subset S = K[x_1, \dots, x_n]$ be a graded ideal and let $\mathfrak{p} \in \operatorname{Ass}^\infty(I)$. Then, we have*

$$\alpha((I^{k+1} : \mathfrak{p})/I^{k+1}) \le \omega((I^k : \mathfrak{p})/I^k) + \omega(I), \quad \textit{for all } k \gg 0.$$

10.3 The v-number of monomial ideals in two variables

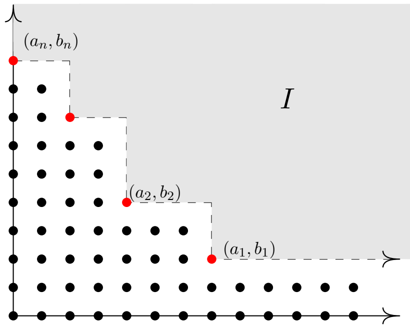
In this section, we consider monomial ideals of the polynomial ring in two variables $S = K[x, y]$. Let $I \subset S$ be a monomial ideal. As customary, we denote by $G(I)$ the unique minimal monomial generating set of I. Then

$$G(I) = \{x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, \dots, x^{a_m}y^{b_m}\},$$

where $\mathbf{a} : a_1 > a_2 > \dots > a_m \ge 0$ and $\mathbf{b} : 0 \le b_1 < b_2 < \dots < b_m$. Conversely, given any two such sequences $\mathbf{a}$ and $\mathbf{b}$, the set $\{x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, \dots, x^{a_m}y^{b_m}\}$ is the minimal monomial generating set of a monomial ideal of S.

Therefore, the monomial ideals of $S = K[x, y]$ are in bijection with all pairs $(\mathbf{a}, \mathbf{b})$ of sequences $\mathbf{a} : a_1 > a_2 > \dots > a_m \ge 0$ and $\mathbf{b} : 0 \le b_1 < b_2 < \dots < b_m$ as above. Hereafter, we write $I = I_{\mathbf{a},\mathbf{b}}$ for $I = (x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, \dots, x^{a_m}y^{b_m})$.

The natural K-basis of $S/I_{\mathbf{a},\mathbf{b}}$ consists of the residue classes (modulo $I_{\mathbf{a},\mathbf{b}}$) of the monomials not belonging to $I_{\mathbf{a},\mathbf{b}}$. These basis elements can be represented by the lattice points $(c, d) \in \mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 0}$ such that $x^c y^d \notin I_{\mathbf{a},\mathbf{b}}$, as in the next picture.



Our main goal in this section is to prove the following theorem.

Theorem 10.3.1 *Let $I \subset S = K[x, y]$ be a monomial ideal. Then, $\mathrm{v}_{\mathfrak{p}}(I^k)$, for all $\mathfrak{p} \in \mathrm{Ass}^\infty(I)$, and $\mathrm{v}(I^k)$ are linear functions in k for $k \gg 0$.*

The associated prime ideals of a monomial ideal I are monomial prime ideals, that is, ideals generated by a subset of the variables [89, Corollary 1.3.9]. Thus, in our case $\mathrm{Ass}(I) \subseteq \{(x), (y), (x, y)\}$. We set $\mathfrak{p}_x = (x)$, $\mathfrak{p}_y = (y)$ and $\mathfrak{m} = (x, y)$.

We can compute $\mathrm{Ass}(I_{\mathbf{a},\mathbf{b}})$ in terms of the sequences $\mathbf{a}$ and $\mathbf{b}$.

Proposition 10.3.2 *Let $I = I_{\mathbf{a},\mathbf{b}} \subset S$ be a monomial ideal. Then,*

(a) $\mathfrak{p}_x \in \mathrm{Ass}(I)$ *if and only if* $a_m > 0$.

(b) $\mathfrak{p}_y \in \mathrm{Ass}(I)$ *if and only if* $b_1 > 0$.

(c) $\mathfrak{m} \in \mathrm{Ass}(I)$ *if and only if* $m > 1$, *i.e., I is not a principal ideal.*

Proof. If $\mathfrak{p}_x \in \mathrm{Ass}(I)$ then $I \subseteq \mathfrak{p}_x = (x)$. Hence x divides all minimal monomial generators of I. In particular, x divides $x^{a_m}y^{b_m}$ and $a_m > 0$.

Conversely, let $a_m > 0$. Then x divides all minimal monomial generators of I. Hence $I \subseteq \mathfrak{p}_x = (x)$. Since $\mathfrak{p}_x$ is of height one, it follows that $\mathfrak{p}_x \in \mathrm{Ass}(I)$.

This proves (a), statement (b) can be proved similarly.

Finally, for the proof of (c), suppose $\mathfrak{m} \in \mathrm{Ass}(I)$. If I is a principal ideal, then for some c and d, $I = (x^c y^d) = (x^c) \cap (y^d) = \mathfrak{p}_x^c \cap \mathfrak{p}_y^d$ is the primary decomposition of I, which contradicts our assumption. Hence I is not principal.

Conversely, suppose I is not principal, but $\mathfrak{m} \notin \mathrm{Ass}(I)$. Then $\mathrm{Ass}(I) \subseteq \{\mathfrak{p}_x, \mathfrak{p}_y\}$. Since $\mathfrak{p}_x$ and $\mathfrak{p}_y$ are height one prime ideals, $I = \mathfrak{p}_x^c \cap \mathfrak{p}_y^d = (x^c) \cap (y^d) = (x^c y^d)$ for some c and d, against our assumption. The assertion follows. $\square$

The following lemma is required.

Lemma 10.3.3 *Let $I = I_{\mathbf{a},\mathbf{b}} \subset S$ be a monomial ideal. Then,*

$$x^{ka_1}y^{kb_1}, x^{ka_m}y^{kb_m} \in G(I^k), \quad \textit{for all} \quad k \geq 1.$$

Proof. Let $k \geq 1$. We know that

$$I^k \; = \; \Big(\prod_{i=1}^{m}(x^{a_i}y^{b_i})^{k_i} \; : \; \sum_{i=1}^{m} k_i = k\Big).$$

Let $x^r y^s$ be an arbitrary generator of I^k different from $x^{ka_1}y^{kb_1}$. Then, we have $r = k_1 a_1 + \dots k_m a_m$, $s = k_1 b_1 + \dots k_m b_m$, $\sum_{i=1}^{m} k_i = k$ and $k_i > 0$ for some $i \neq 1$. Thus,

$$ka_1 = k_1 a_1 + k_2 a_1 + \dots k_m a_1 > k_1 a_1 + k_2 a_2 + \dots k_m a_m = r$$

and
$$kb_1 = k_1b_1 + k_2b_1 + \dots k_mb_1 < k_1b_1 + k_2b_2 + \dots k_mb_m = s.$$
Therefore, x^ry^s does not divide $x^{ka_1}y^{kb_1}$. This shows that $x^{ka_1}y^{kb_1}$ is a minimal generator of I^k. By a similar argument we obtain that $x^{ka_m}y^{kb_m} \in G(I^k)$. ☐

Corollary 10.3.4 *Let $I = I_{\mathbf{a},\mathbf{b}} \subset S$ be a monomial ideal. Then $\operatorname{Ass}(I^k) = \operatorname{Ass}^\infty(I)$, for all $k \geq 1$. In particular, $\operatorname{astab}(I) = 1$.*

Proof. Let us prove that $\operatorname{Ass}(I) = \operatorname{Ass}(I^k)$ for all $k \geq 2$. By the previous proposition, $\mathfrak{p}_x \in \operatorname{Ass}(I)$ if and only if x divides all minimal monomial generators of I. Hence, if $\mathfrak{p}_x \in \operatorname{Ass}(I)$, then $\mathfrak{p}_x \in \operatorname{Ass}(I^k)$ for all $k \geq 2$, as well.

Now, suppose that $\mathfrak{p}_x \in \operatorname{Ass}(I^k)$ for some $k \geq 2$, but $\mathfrak{p}_x \notin \operatorname{Ass}(I)$. Then, by Proposition 10.3.2(a) we have $a_m = 0$. Hence $y^{b_m} \in G(I)$. By the previous corollary, $y^{kb_m} \in G(I^k)$, as well. But this is impossible, because $y^{kb_m} \notin \mathfrak{p}_x$, but by assumption $\mathfrak{p}_x$ contains I^k. Hence $a_m > 0$ and $\mathfrak{p}_x \in \operatorname{Ass}(I)$, as wanted.

The same reasoning can be applied to show that $\mathfrak{p}_y \in \operatorname{Ass}(I)$ if and only if $\mathfrak{p}_y \in \operatorname{Ass}(I^k)$, for any $k \geq 2$.

Finally, by the previous proposition, $\mathfrak{m} \in \operatorname{Ass}(I)$ if and only I is not principal. Lemma 10.3.3 implies that I is not principal if and only if I^k is not principal for any $k \geq 2$. Thus $\mathfrak{m} \in \operatorname{Ass}(I)$ if and only if $\mathfrak{m} \in \operatorname{Ass}(I^k)$ for any $k \geq 2$. ☐

Next, we compute the functions $\mathrm{v}_{\mathfrak{p}_x}(I^k_{\mathbf{a},\mathbf{b}})$, $\mathrm{v}_{\mathfrak{p}_y}(I^k_{\mathbf{a},\mathbf{b}})$.

Corollary 10.3.5 *Let $I = I_{\mathbf{a},\mathbf{b}} \subset S$ be a monomial ideal. The following holds.*

(a) *If $a_m > 0$, then $\mathrm{v}_{\mathfrak{p}_x}(I^k) = k(a_m + b_m) - 1$, for all $k \geq 1$.*

(b) *If $b_1 > 0$, then $\mathrm{v}_{\mathfrak{p}_y}(I^k) = k(a_1 + b_1) - 1$, for all $k \geq 1$.*

Proof. The only generator $\overline{u} \in (I : \mathfrak{p}_x)/I$ such that $(I : u) = \mathfrak{p}_x$ is $x^{a_m-1}y^{b_m}$, for it has the largest y-degree. For each $k \geq 1$, from Lemma 10.3.3, $x^{ka_m}y^{kb_m} \in G(I^k)$ and such generator has the highest y-degree. Thus, $\overline{u} = \overline{x^{ka_m-1}y^{kb_m}}$ is the only generator of $(I^k : \mathfrak{p}_x)/I^k$ such that $(I^k : u) = \mathfrak{p}_x$. Similarly, one can prove (b). ☐

We are in the position to prove our main result in the section.

Proof of Theorem 10.3.1. By Corollary 10.3.4, $\operatorname{Ass}(I^k) = \operatorname{Ass}^\infty(I)$ for all $k \geq 1$. If $\mathfrak{p}_x$ or $\mathfrak{p}_y$ belongs to $\operatorname{Ass}^\infty(I)$, then $\mathrm{v}_{\mathfrak{p}_x}(I^k)$ or $\mathrm{v}_{\mathfrak{p}_y}(I^k)$ is a linear function in k for $k \gg 0$, by Corollary 10.3.5. If $\mathfrak{m} \in \operatorname{Ass}^\infty(I)$, then $\mathrm{v}_{\mathfrak{m}}(I^k)$ is a linear function in k for $k \gg 0$ because $\mathfrak{m} \in \operatorname{Max}^\infty(I)$ (Corollary 10.2.8). Finally, it follows by definition that $\mathrm{v}(I^k)$ is a linear function in k for $k \gg 0$, as well. ☐

In the next proposition, we show how to compute $\mathrm{v}_{\mathfrak{m}}(I)$ for a non principal monomial ideal $I = I_{\mathbf{a},\mathbf{b}} \subset S$. For our convenience, if $c \geq 1$, in the proof of the next result we regard x^{-c} and y^{-c} as 1.

Proposition 10.3.6 *Let $I = I_{\mathbf{a},\mathbf{b}} \subset S$ be a non principal monomial ideal. Then,*
$$(I : \mathfrak{m})/I = \big(x^{a_j-1}y^{b_{j+1}-1} \;:\; 1 \leq j \leq m-1\big)/I. \tag{10.2}$$
In particular,
$$\mathrm{v}_{\mathfrak{m}}(I) = \min\{a_j + b_{j+1} - 2 \;:\; 1 \leq j \leq m-1\}.$$

Proof. Firstly, we compute $I:\mathfrak{m}$. We have

$$\begin{aligned} I:\mathfrak{m} &= (I:\mathfrak{p}_x)\cap(I:\mathfrak{p}_y)\\ &= (x^{a_1-1}y^{b_1},\dots,x^{a_m-1}y^{b_m})\cap(x^{a_1}y^{b_1-1},\dots,x^{a_m}y^{b_m-1})\\ &= \big(\mathrm{lcm}(x^{a_i-1}y^{b_i},x^{a_j}y^{b_j-1})\ :\ 1\le i\le m,\ 1\le j\le m\big) \qquad (10.3)\\ &= \big(\mathrm{lcm}(x^{a_i-1}y^{b_i},x^{a_{j+1}}y^{b_{j+1}-1})\ :\ 1\le i\le m,\ 0\le j\le m-1\big)\\ &= \big(x^{\max\{a_i-1,a_{j+1}\}}y^{\max\{b_i,b_{j+1}-1\}}\ :\ 1\le i\le m,\ 0\le j\le m-1\big). \end{aligned}$$

Fix $j\in\{0,\dots,m-1\}$ and let $i\in\{1,\dots,m\}$.

If $i\le j$, we have $a_i\ge a_j>a_{j+1}$ and $b_i\le b_j<b_{j+1}$. Therefore $x^{a_{j+1}}|x^{a_i-1}$ and $y^{b_i}|y^{b_{j+1}-1}$. Hence,

$$x^{a_j-1}y^{b_{j+1}-1}\in(I:\mathfrak{m})\ \text{ divides }\ x^{\max\{a_i-1,a_{j+1}\}}y^{\max\{b_i,b_{j+1}-1\}}\ \text{ for }\ i\le j. \qquad (10.4)$$

If $i>j$, we have $a_i-1\le a_{j+1}$ and $b_i\ge b_{j+1}-1$, so $x^{a_i-1}|x^{a_{j+1}}$ and $y^{b_{j+1}-1}|y^{b_i}$. Hence,

$$x^{a_{j+1}}y^{b_i}\in(I:\mathfrak{m})\ \text{ divides }\ x^{\max\{a_i-1,a_{j+1}\}}y^{\max\{b_i,b_{j+1}-1\}}\ \text{ for }\ i>j. \qquad (10.5)$$

Thus, by equations (10.3), (10.4) and (10.5) we have

$$I:\mathfrak{m} = \big(x^{a_j-1}y^{b_{j+1}-1},x^{a_{j+1}}y^{b_i}\ :\ 1\le j\le m-1,\ j+1\le i\le m\big).$$

Note that for each $i\ge j+1$ we have $x^{a_{j+1}}y^{b_i}\in I$. It is clear that $x^{a_j-1}y^{b_{j+1}-1}\notin I$, for all $j=1,\dots,m-1$. Hence, equation (10.2) follows.

The claim about $\mathrm{v}_{\mathfrak{m}}(I)$ follows from (10.2) and Theorem 10.1.1(c). □

As a consequence of our discussion, we obtain the next formula that shows us how to compute the v-number of $I_{\mathbf{a},\mathbf{b}}$ solely in terms of the sequences $\mathbf{a}$ and $\mathbf{b}$.

Theorem 10.3.7 *Let $I=I_{\mathbf{a},\mathbf{b}}\subset S$ be a monomial ideal. Then*

$$\mathrm{v}(I)=\begin{cases}\min\{a_i+b_{i+1}-2\ :\ 1\le i\le m-1\}, & \text{if } b_1=0 \text{ and } a_m=0,\\ \min\{a_1+b_1-1,a_i+b_{i+1}-2\ :\ 1\le i\le m-1\}, & \text{if } b_1\ne0 \text{ and } a_m=0,\\ \min\{a_m+b_m-1,a_i+b_{i+1}-2\ :\ 1\le i\le m-1\}, & \text{if } b_1=0 \text{ and } a_m\ne0,\\ \min\{a_1+b_1-1,a_m+b_m-1,a_i+b_{i+1}-2\ :\ 1\le i\le m-1\}, & \text{otherwise.}\end{cases}$$

Proof. Suppose I is non principal. Then, the statement follows by combining Proposition 10.3.2, Corollary 10.3.5 and Proposition 10.3.6. Now, if I is principal, the above formulas also hold. Indeed, in this case $m=1$ and in the above last three minimums one does not have to consider the terms $a_i+b_{i+1}-2$ because $m-1=0$. □

In all the examples we could check with *Macaulay2* [82], for a monomial ideal $I\subset S=K[x,y]$, if the v-function of I is $\mathrm{v}(I^k)=ak+b$, $k\gg0$, we always have that $b\ge-1$. At present, we do not know how to prove this lower bound. On the other hand, for any such linear function $f(k)=ak+b$, $(a\ge1,\ b\ge-1)$, there exists a monomial ideal $I\subset S=K[x,y]$ such that $\mathrm{v}(I^k)$ agrees with $f(k)$ for all $k\ge1$, as we show next.

Theorem 10.3.8 *Let $a\ge1$ and $b\ge-1$ be integers. Then, there exists a monomial ideal $I\subset S=K[x,y]$ such that*

$$\mathrm{v}(I^k)=ak+b,\quad \textit{for all}\quad k\ge1.$$

Proof. We claim that $I = (x^a, x^{a-1}y^{b+2}) = x^{a-1}(x, y^{b+2})$ satisfies our assertion. For this aim, let us show that $I^k = (x^{ka-i}y^{i(b+2)} : 0 \leq i \leq k)$ for all $k \geq 1$. Indeed,

$$\begin{aligned} I^k &= x^{k(a-1)}(x, y^{b+2})^k = x^{k(a-1)}\sum_{i=0}^{k}(x^{k-i}y^{i(b+2)}) \\ &= (x^{ka-i}y^{i(b+2)} : 0 \leq i \leq k). \end{aligned}$$

Since $ka > ka-1 > \cdots > ka-k$ and $b+2 < 2(b+2) < \cdots < k(b+2)$, it follows that $G(I^k) = \{x^{ka-i}y^{i(b+2)} : 0 \leq i \leq k\}$.

Note that $\text{Ass}^\infty(I) = \{\mathfrak{p}_x, \mathfrak{m}\}$ if $a > 1$ and $\text{Ass}^\infty(I) = \{\mathfrak{m}\}$ if $a = 1$.

By Corollary 10.3.5(a), if $a > 1$ we have

$$\text{v}_{\mathfrak{p}_x}(I^k) = k(a+b+1) - 1.$$

Whereas, by Proposition 10.3.6,

$$\begin{aligned} \text{v}_{\mathfrak{m}}(I^k) &= \min\{(ka-i) + (i+1)(b+2) - 2 : 0 \leq i \leq k-1\} \\ &= \min\{ka + (i+1)b + i : 0 \leq i \leq k-1\} \\ &= ak + b. \end{aligned}$$

If $a > 1$, then $\text{v}(I^k) = \min\{\text{v}_{\mathfrak{p}_x}(I^k), \text{v}_{\mathfrak{m}}(I^k)\} = \min\{k(a+b+1)-1, ak+b\} = ak+b$. Otherwise, if $a = 1$, then $\text{v}(I^k) = \text{v}_{\mathfrak{m}}(I^k) = ak+b$ once again. ☐

10.4 The v-number of ideals with linear powers

In this section, we consider several classes of graded ideals I arising from combinatorial contexts, with a particular focus on ideals having linear powers, and in some cases we compute explicitly the v-function $\text{v}(I^k)$.

Hereafter, S denotes the standard graded polynomial ring $K[x_1, \dots, x_n]$ with K a field, and $\mathfrak{m} = (x_1, \dots, x_n)$ denotes the unique graded maximal ideal of S.

We say that I has *linear powers* if I^k has a linear resolution, for all $k \geq 1$. Famous examples of ideals with linear powers are given in the following list.

(i) Edge ideals with linear resolution [89, Theorem 10.2.6].

(ii) Polymatroidal ideals [89, Corollary 12.6.4].

(iii) Hibi ideals [89, Corollary 10.2.9 and Theorem 9.1.13], or [35, Corollary 4.11].

In the following Theorems 10.4.4, 10.4.8 and 10.4.10, we show that for any ideal I in the above list, we have that $\text{v}(I^k) = \alpha(I)k - 1$, for all $k \geq 1$,

Due to these results, and experimental evidence, we expect that

Conjecture 10.4.1 *Let $I \subset S$ be a graded ideal with linear powers. Then*

$$\text{v}(I^k) = \alpha(I)k - 1, \quad \textit{for all } k \geq 1.$$

If I does not have linear powers, the conclusion of Conjecture 10.4.1 is no longer valid. Next example is due to Terai [30, Remark 3]. Let $\text{char}(K) \neq 2$, then the Stanley Reisner ideal $I = (abd, abf, ace, adc, aef, bde, bcf, bce, cdf, def)$ of the minimal triangulation of the projective plane has a linear resolution, while I^2 has not. By using *Macaulay2* [82], we have $\mathrm{v}(I) = \alpha(I) = 3$ and $\mathrm{v}(I^k) = 3k - 1$ for all $k \geq 2$.

Before verifying Conjecture 10.4.1 for the ideals listed in (i), (ii) and (iii), we state some useful results that will be needed later.

If $I \subset S$ is a monomial ideal, then all associated primes of I are monomial prime ideals, that is, ideals generated by a subset of the variables [89, Corollary 1.3.9]. Let A be a non empty subset of $[n]$. We denote by $\mathfrak{p}_A$ the monomial prime ideal $(x_i : i \in A)$.

The following result [140, Proposition 3.11] of Saha and Sengupta provides an useful general method to bound $\mathrm{v}(I)$ from above, when I is a monomial ideal.

Proposition 10.4.2 *Let $I \subset S$ be a monomial ideal and $f \in S \setminus I$ a monomial. Then,*

$$\mathrm{v}(I) \leq \mathrm{v}(I : f) + \deg(f).$$

On the other hand, one always has

Proposition 10.4.3 *Let $I \subset S$ be a monomial ideal. Then,*

$$\mathrm{v}_{\mathfrak{p}}(I) \geq \alpha(I) - 1, \quad \textit{for all} \ \ \mathfrak{p} \in \mathrm{Ass}(I).$$

Proof. Let $\mathfrak{p} \in \mathrm{Ass}(I)$ and let $u \in S$ be a monomial such that $(I : u) = \mathfrak{p}$ and $\deg(u) = \mathrm{v}_{\mathfrak{p}}(I)$. Then $\mathfrak{p} = \mathfrak{p}_A = (x_i : i \in A)$ for some $A \subseteq [n]$. Thus $x_i u \in I$ for all $i \in A$. In particular, $\deg(x_i u) \geq \alpha(I)$. Hence, $\deg(u) \geq \alpha(I) - 1$, as desired. □

10.4.1 Edge ideals with linear resolution

Let G be a finite simple graph with vertex set $V(G) = [n]$ and edge set $E(G)$.

As a consequence of Dirac and Fröberg theorems we have

Theorem 10.4.4 *Let $I(G)$ be the edge ideal of a graph G. Suppose that $I(G)$ has a linear resolution. Then,*

$$\mathrm{v}(I(G)^k) \ = \ 2k - 1, \quad \textit{for all} \ \ k \geq 1.$$

The proof is based upon the next lemma.

Lemma 10.4.5 *Let $I(G)$ be an edge ideal with linear resolution, and let $x_1 > x_2 > \cdots > x_n$ be a perfect elimination order of G^c. Then,*

$$(I(G) : x_1) \ = \ (x_j : j \in N_G(1)). \tag{10.6}$$

Proof. It is clear that

$$(x_j : j \in N_G(1)) \ \subseteq \ (I(G) : x_1).$$

Let us show the opposite inclusion. Let $x_k x_\ell \in I(G)$ and suppose that both k and ℓ are not in $N_G(1)$. Then $\{1,k\},\{1,\ell\} \in E(G^c)$, that is $k,\ell \in N_{G^c}(1)$. Since, $x_1 > x_2 > \cdots > x_n$ is a perfect elimination order of G^c, it follows that that $N_{G^c}(1)$ induces a complete subgraph of G_2^c, where G_2^c is the induced subgraph of G^c on the vertex set $\{2,\dots,n\}$. Since $k,\ell > 1$, it follows that $\{k,\ell\} \in E(G^c)$, in contradiction with $\{k,\ell\} \in E(G)$. Thus either $k \in N_G(1)$ or $\ell \in N_G(1)$ and formula (10.6) follows. □

As a consequence, we recover [111, Proposition 3.19].

Corollary 10.4.6 *Let $I(G)$ be the edge ideal of a graph G. Suppose that $I(G)$ has a linear resolution. Then,* $\mathrm{v}(I(G)) = 1$.

Proof. We proceed by induction on $|V(G)| \geq 2$ with the base case being trivial. Let $|V(G)| > 2$. Let $x_1 > x_2 > \cdots > x_n$ be a perfect elimination order of G^c. Then, by Lemma 10.4.5, equation (10.6) holds. Thus, by Proposition 10.4.2, $\mathrm{v}(I(G)) \leq \mathrm{v}(I(G) : x_1) + \deg(x_1) = 0 + 1 = 1$. On the other hand, by Proposition 10.4.3, $\mathrm{v}(I(G)) \geq \alpha(I(G)) - 1 = 1$. The assertion follows. □

Remark 10.4.7 Let $I \subset S$ be a graded ideal. Suppose that $(I^{k+1} : I) = I^k$ and $\mathfrak{p} \in \mathrm{Ass}(I^k)$ for all $k \geq 1$. Then, the proof of Proposition 10.2.9 shows that

$$\mathrm{v}_{\mathfrak{p}}(I^{k+1}) \leq \mathrm{v}_{\mathfrak{p}}(I^k) + \omega(I) \text{ for all } k \geq 1.$$

Now, we are in the position to prove Theorem 10.4.4.

Proof of Theorem 10.4.4. By the previous result, $\mathrm{v}(I(G)) = 1$. Therefore, for some $\mathfrak{p} \in \mathrm{Ass}(I(G))$, we have $\mathrm{v}_{\mathfrak{p}}(I(G)) = 1$. By [125, Theorem 2.15], we have

$$\mathrm{Ass}(I(G)) \subseteq \mathrm{Ass}(I(G)^2) \subseteq \cdots \subseteq \mathrm{Ass}(I(G)^k) \subseteq \cdots .$$

Hence, $\mathfrak{p} \in \mathrm{Ass}(I(G)^k)$ for all $k \geq 1$. By [125, Lemma 2.12], $(I(G)^{k+1} : I(G)) = I(G)^k$ for all $k \geq 1$. Since $\alpha(I(G)^{k+1}) = 2(k+1)$ and $\omega(I(G)) = 2$, by Remark 10.4.7 and Proposition 10.4.3, we have

$$2(k+1) - 1 \leq \mathrm{v}_{\mathfrak{p}}(I(G)^{k+1}) \leq \mathrm{v}_{\mathfrak{p}}(I(G)^k) + 2$$

for all $k \geq 1$. By induction on $k \geq 1$, we may assume that $\mathrm{v}_{\mathfrak{p}}(I(G)^k) = 2k - 1$. The above chain of inequalities gives $\mathrm{v}_{\mathfrak{p}}(I(G)^{k+1}) = 2(k+1) - 1 = \alpha(I(G)^{k+1}) - 1$. By Proposition 10.4.3 it follows that $\mathrm{v}(I(G)^k) = 2k - 1$ for all $k \geq 1$, as well. □

10.4.2 Polymatroidal ideals

In this section, we prove that

Theorem 10.4.8 *Let $I \subset S$ be a polymatroidal ideal. Then*

$$\mathrm{v}(I^k) = \alpha(I)k - 1, \quad \textit{for all} \quad k \geq 1.$$

The proof is based upon the next lemma.

Lemma 10.4.9 *Let $I \subset S$ be a polymatroidal ideal generated in degree $\alpha(I) \geq 2$. Then $(I : x_i)$ is a polymatroidal ideal generated in degree $\alpha(I) - 1$, for all $i \in [n]$.*

Proof. We may assume that $i = 1$. We can write $I = x_1 I_1 + I_2$, where I_1 and I_2 are the unique monomial ideals of S such that

$$\begin{aligned} G(x_1 I_1) &= \{u \in G(I) : x_1 \textit{ divides } u\}, \\ G(I_2) &= \{u \in G(I) : x_1 \textit{ does not divide } u\}. \end{aligned}$$

We claim that $I_2 \subset I_1$. It is enough to show that $G(I_2) \subseteq I_1$. Let $u \in G(I_2)$ and let $v \in G(x_1 I_1)$. Then $\deg_{x_1}(u) = 0 < \deg_{x_1}(v)$. Thus, by the dual exchange property, we can find j such that $\deg_{x_j}(u) > \deg_{x_j}(v)$ and $x_1(u/x_j) \in G(I)$. Hence $x_1(u/x_j) \in x_1 I_1$, and so $u/x_j \in I_1$. Consequently, $u \in I_1$ too, and thus $I_2 \subset I_1$.

By the previous paragraph, we have $(I : x_1) = I_1 + I_2 = I_1$. It is clear that I_1 is equigenerated in degree $\alpha(I) - 1$. It remains to prove that I_1 is polymatroidal. Let $u_1, v_1 \in G(I_1)$ and i such that $\deg_{x_i}(u_1) > \deg_{x_i}(v_1)$. Our job is to find j such that $\deg_{x_j}(u_1) < \deg_{x_j}(v_1)$ and $x_j(u_1/x_i) \in G(I_1)$. Set $u = x_1u_1$ and $v = x_1v_1$. Then $u, v \in G(x_1I_1) \subset G(I)$ and $\deg_{x_i}(u) > \deg_{x_i}(v)$. Since I is polymatroidal, there exists j such that $\deg_{x_j}(u) < \deg_{x_j}(v)$ and $x_j(u/x_i) \in G(I)$. We claim that x_1 divides $x_j(u/x_i)$. Indeed x_1 divides u. If $i \neq 1$, then x_1 divides $x_j(u/x_i)$ as well. If $i = 1$, since x_1 divides v and $\deg_{x_i}(u) > \deg_{x_i}(v) > 0$, it follows that x_1^2 divides u and so x_1 divides $x_j(u/x_i) = x_j(u/x_1)$. Therefore, in any case x_1 divides $x_j(u/x_i)$. Hence, $(x_j(u/x_i))/x_1 = x_j(u_1/x_i) \in G(I_1)$ and the proof is complete. □

We are ready for the proof of the theorem.

Proof of Theorem 10.4.8. Firstly, we show that $\mathrm{v}(I) = \alpha(I) - 1$. We proceed by strong induction on $\alpha(I) \geq 1$. If $\alpha(I) = 1$, then $I = \mathfrak{p}_A$ for some $A \subseteq [n]$, $\alpha(I) = 1$ and $\mathrm{v}(I) = \mathrm{v}_{\mathfrak{p}_A}(I) = 0$. Suppose $\alpha(I) > 1$. By the previous proposition, $(I : x_1)$ is a polymatroidal ideal and $\alpha(I : x_1) = \alpha(I) - 1$. By induction hypothesis, $\mathrm{v}(I : x_1) = \alpha(I : x_1) - 1 = \alpha(I) - 2$. Hence, by Proposition 10.4.2,

$$\mathrm{v}(I) \leq \mathrm{v}(I : x_1) + \deg(x_1) = \alpha(I) - 1.$$

By Proposition 10.4.3, $\mathrm{v}(I) \geq \alpha(I) - 1$. Equality follows.

Let $k > 1$. It is well–known that the product of polymatroidal ideals is polymatroidal [89, Theorem 12.6.3]. Hence, I^k is a polymatroidal ideal generated in degree $\alpha(I)k$. By what shown above, $\mathrm{v}(I^k) = \alpha(I^k) - 1 = \alpha(I)k - 1$. □

10.4.3 Hibi ideals

Let $(P, \succeq)$ be a finite poset with $P = \{p_1, \dots, p_n\}$. Recall that the Hibi ideal $H_P = (u_{\mathcal{I}} : \mathcal{I} \in \mathcal{J}(P))$ is equigenerated in degree $|P|$. It is well known that Hibi ideals have linear powers. Next, we calculate the v-function of H_P.

Theorem 10.4.10 *Let H_P be a Hibi ideal. Then,*

$$\mathrm{v}(H_P^k) = k|P| - 1, \quad \textit{for all} \quad k \geq 1.$$

Proof. By [89, Lemma 9.1.9], we have the minimal primary decomposition

$$H_P \;=\; \bigcap_{p_i \preceq p_j} (x_i, y_j).$$

Therefore $\mathrm{Ass}(H_P) = \{(x_i, y_j) : p_i \preceq p_j\}$. Let $p_i \in P$ be a minimal element of P with respect to $\succeq$. After a relabeling, we may assume that $i = 1$. Then $\{p_1\} \in \mathcal{J}(P)$ and $x_1y_2 \cdots y_n \in H_P$. Note that $y_1y_2 \cdots y_n \in H_P$ because $\varnothing \in \mathcal{J}(P)$. Thus,

$$(H_P : y_2 \cdots y_n) \supseteq (x_1, y_1).$$

Note that any generator of H_P is divided by either x_1 or y_1. Thus, any monomial $u \in (H_P : y_2 \cdots y_n)$ must be divided by either x_1 or y_1. Hence, we see that $(H_P : y_2 \cdots y_n) = (x_1, y_1)$. Therefore, $\mathrm{v}_{(x_1,y_1)}(H_P) \leq \deg(y_2 \cdots y_n) = |P| - 1$. By Proposition 10.4.3, it follows that $\mathrm{v}(H_P) = \mathrm{v}_{(x_1,y_1)}(H_P) = \alpha(H_P) - 1 = |P| - 1$.

By [88, Corollary 1.2] the Rees algebra $\mathcal{R}(H_P)$ is a normal domain. Hence, H_P is a normal ideal (see, also, [36, Corollary 3.5]). Thus, by [139, Proposition (4.7)],

$(H_P^{k+1} : H_P) = H_P^k$, for all $k \geq 1$. By [89, Theorem 9.1.13] H_P is the cover ideal $J(G)$ of a Cohen–Macaulay bipartite graph G. Next, by [137, Theorem 6.10] we have that $J(G)^k = J(G)^{(k)}$ for all $k \geq 1$, that is, ordinary and symbolic powers of $J(G) = H_P$ coincide. Since H_P is a squarefree monomial ideal, by [89, Proposition 1.4.4 and Corollary 1.3.6] we have

$$H_P^k \;=\; H_P^{(k)} \;=\; \bigcap_{p_i \preceq p_j} (x_i, y_j)^k.$$

Hence, $\mathrm{Ass}(H_P^k) = \mathrm{Ass}(H_P)$ for all $k \geq 1$. Thus, by Remark 10.4.7, for all $k \geq 1$,

$$\mathrm{v}_{(x_1,y_1)}(H_P^{k+1}) \leq \mathrm{v}_{(x_1,y_1)}(H_P^k) + |P|. \tag{10.7}$$

Now, we prove that $\mathrm{v}_{(x_1,y_1)}(H_P^k) = k|P| - 1$ for all $k \geq 1$. This is true for $k = 1$ as shown above. Assume $k > 1$ and that $\mathrm{v}_{(x_1,y_1)}(H_P^k) = k|P| - 1$. Then, Proposition 10.4.3, equation (10.7) and the inductive hypothesis give

$$(k+1)|P| - 1 = \alpha(H_P^{k+1}) - 1 \leq \mathrm{v}_{(x_1,y_1)}(H_P^{k+1}) \leq \mathrm{v}_{(x_1,y_1)}(H_P^k) + |P| = k|P| - 1 + |P|.$$

Hence, $\mathrm{v}_{(x_1,y_1)}(H_P^{k+1}) = (k+1)|P| - 1$, as wanted. Finally $\mathrm{v}(H_P^k) = k|P| - 1$, for all $k \geq 1$, as well. □

Notes

An essential part of the proofs of the results in this chapter relies on Proposition 10.2.5, which is based on an initial ideal trick to reduce to the monomial case and the subsequent linear programming argument. This is also a crucial step in the paper of Cutkosky, Herzog and Trung [41]. However, in 2005, Trung and Wang proved more generally that the Castelnuovo–Mumford regularity of I^kM is eventually a linear function in k [147, Theorem 3.2], where I is a graded ideal of a Noetherian commutative ring R and M is a finitely generated graded R-module. This result is remarkable because in this more general case, the initial ideal trick and the subsequent linear programming argument employed in the proof of Proposition 10.2.5 are not available. It would be nice to extend the concept of v-number and the results on the v-function in this more general situation.

At present, not much is known about the v-function of a graded ideal besides what proved here. On the other hand, the v-number of edge ideals $I(G)$, and more generally the v-number of edge ideals $I(\mathcal{C})$ of a clutter, was combinatorially computed in terms of G, respectively $\mathcal{C}$, by Jaramillo and Villarreal [111, Theorem 3.5]. That such a combinatorial description is possible is not that surprising. Indeed, both the the colon ideal of two monomial ideals and the primary decomposition of a monomial ideal are independent from the characteristic of the underlying field K. Thus, the v-functions, $\mathrm{v}_{\mathfrak{p}}(I)$ ($\mathfrak{p} \in \mathrm{Ass}^\infty(I)$) and $\mathrm{v}(I)$, of a monomial ideal I stay the same for any underlying field K of $S = K[x_1, \dots, x_n]$.

Due to the results above, to study the v-number of monomial ideals, one could expect, via polarization, that it is enough to consider squarefree monomial ideals. However, this is not the case. The behaviour of the v-number under polarization was studied by Saha and Sengupta [140]. They showed that in general $\mathrm{v}(I^\wp) \leq \mathrm{v}(I)$ with equality if I has no embedded primes. Such a condition, however, is not necessary to guarantee that $\mathrm{v}(I^\wp) = \mathrm{v}(I)$. The very useful Proposition 10.4.2 is also taken from their paper.

Chapter 11

Conclusions

We close this dissertation by discussing some open problems related to our work.

11.0.1 Vector–spread strongly stable ideals

Besides the case $\mathbf{t} = (1,\dots,1,0,\dots,0)$, the differentials of the minimal free resolution of a $\mathbf{t}$-spread strongly stable ideal are still unknown for a general choice of $\mathbf{t}$. Part of the difficulty of this problem stems from the fact that for a general choice of $\mathbf{t}$, a $\mathbf{t}$-spread strongly stable ideal needs not to have a monomial cycles basis as shown in the Notes at the ending of Chapter 4. Perhaps, a similar statement as in Theorem 4.3.2 also holds for each summand

$$(-1)^{\sigma(u;F)}\varepsilon(\mathbf{x}_F u'/\mathbf{x}_{F(u)})e_{\sigma\setminus F}\wedge e_{F(u)}\wedge e_{\max(u)},\qquad F\subseteq\sigma,$$

of the cycle $e(u;\sigma)$, $u\in G(I)$, $\sigma\subseteq[\max(u)-1]\setminus\mathrm{supp}_{\mathbf{t}}(u)$. Then, from these "local" pieces of $e(u;\sigma)$ perhaps one could determine the differentials? Maybe Betti splittings as used in Theorem 8.3.5 could also solve this problem.

In a discussion related to this kind of problems, Herzog and myself wondered if the following question could be true.

Question. *Let $I\subset S$ be a monomial ideal with linear quotients. Suppose that I has a regular decomposition function. Is it true that I has a monomial cycle basis? Or, conversely, assume that I has monomial cycle basis, is it true that the decomposition function of I is regular?*

Many other questions remain unsolved about vector–spread strongly stable ideals. To mention some, to describe the primary decomposition of t-spread lexsegment ideals and when is a t-spread lexsegment ideal sequentially Cohen–Macaulay are still open questions [32].

11.0.2 Homological shift ideals

Of course, the main question that still remains is to solve the Bandari–Bayati–Herzog conjecture. One could try to settle it for PLP–polymatroidal ideals and transversal polymatroidal ideals.

Since we know that $\mathrm{HS}_1(I)$ always has linear quotients if I has linear quotients, by symmetry one would expect that this is also true for $\mathrm{HS}_{n-1}(I)$, when I has linear quotients. However, this needs not to be the case. For instance, let

$$\begin{aligned} I \;=\; & (x_1^2x_2^2,\ x_1x_2^3,\ x_1x_2^2x_3,\ x_1x_2^2x_4,\ x_1x_2x_3^2,\ x_2^2x_3^2,\ x_2x_3^3,\\ & x_2x_3^2x_4,\ x_1x_2x_3x_4,\ x_2^2x_3x_4,\ x_2x_3^2x_4,\ x_2x_3x_4^2).\end{aligned}$$

By using [59], we checked that I has linear quotients and $\text{pd}(I) = 3 = n-1$. However,

$$\text{HS}_3(I) = x_1x_2x_3x_4 \cdot \text{soc}(I) = (x_1x_2^2x_3^2x_4^2, x_1x_2^2x_3^3x_4, x_1^2x_2^3x_3x_4)$$

does not has linear quotients, not even linear resolution.

Therefore, in general, if I is equigenerated with linear quotients and has maximal projective dimension, $\text{soc}(I)$ needs not to have linear quotients, not even linear resolution. Nonetheless, we still expect that for a polymatroidal ideal I, $\text{soc}(I)$ is polymatroidal. As I learned from Herzog, this would then imply that the saturation $I^{\text{sat}} = \bigcup_k (I : \mathfrak{m}^k)$ is a componentwise polymatroidal ideal, and thus a componentwise linear ideal. Here, componentwise polymatroidal means that $I_{\langle j\rangle}$ is polymatroidal for all j. Polymatroidal ideals are componentwise polymatroidal ideals, because $I_{\langle j\rangle} = \mathfrak{m}^{d-j}I$ where d is the initial degree of I, and products of polymatroidal ideals are polymatroidal.

11.0.3 The v-number

In Chapter 11, we showed under additional assumptions that the v-functions are linear. Of course it would be nice to prove this in general. One could ask if $\text{v}(I^{(k)})$ is also a (periodically) linear function in k for $k \gg 0$. For the Castelnuovo–Mumford regularity $\text{reg}(I^{(k)})$ of symbolic powers of a homogeneous ideal this is not the case in general, unless I is for example a monomial ideal [92, Corollary 3.3].

An interesting question in the context of Theorem 10.2.2 is to give criteria for when $\text{Soc}_{\mathfrak{p}}(I)$ is indeed an ideal of $\mathcal{F}_{\mathfrak{p}}(I)$. This is for sure the case, when $\text{Soc}_{\mathfrak{p}}(I)$ is equal, without any truncation, to $(0 :_{\text{gr}_I(R)} \mathfrak{p})$. In this latter situation, several questions arise. In which degrees is $\text{Soc}_{\mathfrak{p}}(I)$ minimally generated? And how is this information related to the function $\text{v}_{\mathfrak{p}}(I^k)$ for $k \gg 0$? When is $\text{Soc}_{\mathfrak{p}}(I)$ Cohen–Macaulay, Gorenstein, complete intersection?

In Section 10.4, we computed the v-number of edge ideals with linear resolution, polymatroidal ideals and Hibi ideals. For any ideal I of these classes, we have $\text{v}(I^k) = \alpha(I)k-1$ for all $k \geq 1$. It would also be interesting to compute combinatorially $\text{v}_{\mathfrak{p}}(I^k)$ for any $\mathfrak{p} \in \text{Ass}^{\infty}(I)$ and any I of the above classes. For instance, in the case of Hibi ideals, the following could be true.

Conjecture. *Let H_P be the Hibi ideal of the poset $(P, \succeq)$ with $P = \{p_1, \dots, p_n\}$. Then $\text{Ass}(H_P^k) = \text{Ass}^{\infty}(H_P) = \{(x_i, y_j) : p_i \preceq p_j\}$ for all $k \geq 1$. Let $\mathfrak{p} = (x_i, y_j)$ with $p_i \preceq p_j$ and $p_i \neq p_j$. Is it true that*

$$\text{v}_{\mathfrak{p}}(H_P^k) \;=\; \begin{cases} |P|k-1 & \textit{if } p_i = p_j, \\ |P|k + |\{p_\ell \in P : p_i \prec p_\ell \prec p_j\}| & \textit{otherwise.} \end{cases}$$

for all $k \geq 1$?

In [111], Jaramillo and Villarreal asked whether the inequality $\text{v}(I) \leq \text{reg}(I)$ holds for any squarefree monomial ideal I. Civan showed in [29] that this is not the case. Indeed, for any given integer a, there exists a graph G with $\text{v}(I(G)) = \text{reg}(I(G)) + a$ [29, Theorem 2]. Therefore, the number $\text{v}(I) - \text{reg}(I)$ can be arbitrarily large. On the other hand, assuming that the v-function of a given homogeneous ideal I is linear, a natural question arises.

Question. Let $I \subset S$ be a graded ideal and suppose that $\text{v}(I^k)$ is an eventually linear function. Is it true that $\text{v}(I^k) \leq \text{reg}(I^k)$ for all $k \gg 0$?

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