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On nilpotent Leibniz algebras, Lie biderivations and related topics

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in

Mathematics and Computational Sciences

Declaration of Authorship

I, Gianmarco La Rosa, declare that this thesis titled, "On nilpotent Leibniz algebras, Lie biderivations and related topics" and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
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"*I was within and without, simultaneously enchanted and repelled by the inexhaustible variety of life.*"

Fitzgerald F. Scott, *The Great Gatsby*

UNIVERSITY OF PALERMO

Abstract

Department of Mathematics and Computer Sciences

Doctor of Philosophy

On nilpotent Leibniz algebras, Lie biderivations and related topics

by Gianmarco La Rosa

This thesis classifies two-step nilpotent Leibniz algebras, with a specific emphasis on the real and complex cases of Heisenberg Leibniz algebras. It demonstrates a global integration property for nilpotent real Leibniz algebras, explores integration in specific scenarios and solves the coquecigrue problem by integrating into a Lie rack. It investigates Lie algebras of derivations in two-step nilpotent algebras and describes isotopism classes of nilpotent Leibniz algebras and introduces new invariants. Biderivations of complete Lie algebras are also described, with attention given to both symmetric and skew-symmetric cases. Furthermore, it provides isomorphism results for non-nilpotent non-Lie Leibniz algebras with a one-dimensional derived subalgebra.

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Dedicated to my family and to anyone who cares about me...

Introduction

This dissertation is the product of research conducted over the past three years. The aim was to gather articles and preprints produced during this period ([58], [57], [72], [31], and [31]), combining them to form a coherent thesis. The research primarily focuses on the structure of nilpotent Leibniz algebras, their derivations, and the biderivations of Lie algebras.

Leibniz algebras are a generalization of Lie algebras, first introduced by A. Blokh in 1965 (see [10]). The title of the work, "*A generalization of the concept of a Lie algebra*" is a clear and suggestive choice. The author introduces a new algebraic structure, called *D-algebras* (*distributive algebras*), which satisfy a certain identity. Roughly speaking, the concept requires multiplication on a vector space that acts on left as a derivation (*left differential identity*).

Subsequently, J.-L. Loday introduced the same algebraic structure in a paper from 1993 entitled "Une version non commutative des algèbres de Lie: les algèbres de Leibniz" ([62]). These algebraic structures emerged in homological algebra in order to construct new chain complexes. Previously, the chains of Chevalley-Eilenberg of Lie algebras involved the tensor product, but J.-L. Loday used the exterior product instead. For some time, these algebraic structures were known as *Loday algebras*, which may be due to the limited availability of A. Blokh's russian article. However, modern literature consistently uses the term Leibniz algebras to refer to these algebraic structures.

These introductory topics will be thoroughly covered in chapter 1. While some content needed to be trimmed and certain details skimmed over, the chapter is coherent and self-contained, and includes all the necessary definitions and results for proceeding with subsequent reading.

The first mentioned work is presented in chapter 2. The research centres on the examination of two-step nilpotent Leibniz algebras with a one-dimensional commutator ideal, and their subsequent integration. Three classifications of Leibniz algebras of this nature are introduced: *Heisenberg*, *Kronecker* and *Dieudonné*. A brief introduction to Lie racks is provided, and the correspondence between Leibniz algebras and Lie racks is explored in depth.

Chapter 3 presents a synthesis of the results derived from two distinct papers. In the initial section of the chapter derivations of the classes of Leibniz algebras found in the second chapter, while the subsequent part examines the isomorphisms of these classes. These maps, which can be thought as a generalization of the concept of isomorphism between two algebraic constructions, have been proven to be more advantageous in outlining the classes of Leibniz algebras found in the first of the aforementioned works.

The recent preprint seeks to complete the classification of Leibniz algebras with a commutator ideal of dimension one. It contains essential findings on this subject, forming chapter 4, which classifies non-nilpotent Leibniz algebras of this type. Additionally, a prior classification is extended to a generic field with a characteristic differing from two.

Finally, the last chapter examines biderivations of complete Lie algebras. They may be seen as a generalization of the concept of derivation. Therefore, we will describe all the biderivations of complete Lie algebras, i.e., Lie algebras with trivial center and inner derivations, such as the semisimple ones. Finally, we will describe the symmetric and skew-symmetric biderivations of a complete Lie algebra.

Chapter 1

Brief introduction to Leibniz algebras

This first chapter is devoted to the study of Leibniz algebras. We will give first definitions and fundamental results. Many of these (and much more) can be found in [6], a reference text for the composition of this chapter, to which we refer the reader for further details. Unless stated otherwise, every Leibniz algebra will be of finite dimension.

1.1 Basic definitions and examples

Definition 1.1.1. A *left Leibniz algebra L over a field* $\mathbb{F}$ is a vector space over $\mathbb{F}$, equipped with a $\mathbb{F}$-bilinear map $[-,-]: L \times L \to L$ satisfying the Leibniz identity

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]], \tag{1.1}$$

for all $x, y, x \in L$. L is said to be a *right Leibniz algebra over* $\mathbb{F}$ if

$$[[x,y],z] = [[x,z],y] + [x,[y,z]], \tag{1.2}$$

for all $x, y, x \in L$.

A Leibniz algebra that is both left and right is called *symmetric*.

According to the first definition given by A. Blokh in 1965 [10], the identity 1.1 is called *left differential identity*, while 1.2 is called *right differential identity*. Throughout this thesis we will be working with left Leibniz algebras unless otherwise specified. It is worth noting that results which are applicable to left Leibniz algebras also hold to right Leibniz algebras, given appropriate reformulation. Given a left Leibniz algebra $(L, [\cdot,\cdot])$, it is easy to define a new product on L, namely $\{\cdot,\cdot\}$, on the same vector space defined by $\{x,y\} = [y,x]$ (*opposite product*). In this way $(L, \{\cdot,\cdot\})$ is a right Leibniz algebra. In addition we note that the versions

$$[[x,y],z] = [x,[y,z]] - [y,[x,z]], \quad \text{for all } x,y,z \in L, \tag{1.3}$$

and

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \quad \text{for all } x,y,z \in L, \tag{1.4}$$

of the identities 1.1 and 1.2 are also often used.

Definition 1.1.2. Let L be a left Leibniz algebra and let K be a subspace of L. K is said to be a *Leibniz subalgebra* of L ($K \leq L$) if it is closed under the Leibniz bracket, i.e., if $x, y \in K$ then $[x, y] \in K$.

The following identity will prove useful later.

Lemma 1.1.3. *Let L be a left Leibniz algebra. Then, we have*

$$[[x, y], z] = -[[y, x], z],$$

for any $x, y, z \in L$

Proof. By Equation (1.3), for each $z \in L$ we have

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]] = -([y, [x, z]] - [x, [y, z]]) = -[[y, x], z].$$

□

Before we proceed with further definitions and initial properties of these algebraic structures, we shall look at some examples.

Example 1.1.3.1. The first example of a Leibniz algebra is a Lie algebra. Indeed, a Lie algebra $\mathfrak{g}$ is a Leibniz algebra with an alternative product, i.e , with $[x, x] = 0$, for each $x \in L$.

This example shows how a Leibniz algebra is a possible generalization of a Lie algebra, but it also hides deeper and non-trivial observations which we will discuss later.

Example 1.1.3.2. This example shows how to defined a Leibniz algebra structure on a Lie module with an equivariant linear map. Let $\mathfrak{g}$ be a Lie algebra and M be a $\mathfrak{g}$-module. Let $f\colon M \to \mathfrak{g}$ be a $\mathfrak{g}$-equivariant linear map, i.e.

$$f(x \cdot m) = [x, f(m)] \quad \text{for all } m \in M \text{ and } x \in \mathfrak{g}.$$

We now show that the bracket $[\cdot, \cdot]_M$ defined by $[l, m]_M := f(l) \cdot m$ provides a Leibniz algebra structure on M. Indeed, we have

$$\begin{aligned}
[l, [m, n]_M]_M &= [l, f(m) \cdot n]_M = f(l) \cdot (f(m) \cdot n) \\
&= f(m) \cdot (f(l) \cdot n) + [f(l), f(m)] \cdot n \\
&= [m, f(l) \cdot n]_M + f(f(l) \cdot m) \cdot n \\
&= [m, [l, n]_M]_M + [[l, m]_M, n]_M,
\end{aligned}$$

for every $l, m, n \in M$.

Example 1.1.3.3. In this example we define a Leibniz algebra structure over a direct sum of a Lie algebra and its Lie module. Let $\mathfrak{g}$ be a Lie algebra and M a $\mathfrak{g}$-module. We consider the multiplication

$$[x + l, y + m]_Q = [x, y] + x \cdot m,$$

where $x, y \in \mathfrak{g}$ and $l, m \in M$, defined on the vector space $Q = \mathfrak{g} \oplus M$. This multiplication provides a Leibniz algebra structure on Q. Indeed, we have

$$\begin{aligned}
\left[x+l,[y+m,z+n]_Q\right]_Q &= [x+l,[y,z]+y\cdot n]_Q = [x,[y,z]]+x\cdot(y\cdot n)\\
&= [[x,y],z]+[y,[x,z]]+[x,y]\cdot n+y\cdot(x\cdot n)\\
&= [[x,y],z]+[x,y]\cdot n+[y,[x,z]]+y\cdot(x\cdot n)\\
&= [[x,y]+x\cdot m,z+n]_Q+[y+m,[x,z]+x\cdot n]\\
&= \left[[x+l,y+m]_Q,z+n\right]_Q+\left[y+m,[x+l,z+n]_Q\right]_Q,
\end{aligned}$$

for every $x+l, y+m, z+n \in Q$.

Example 1.1.3.4. Unlike the previous one, where a Leibniz algebra was constructed using both a Lie algebra and a Lie module, this example demonstrates how such a construction can be achieved starting only from a Lie algebra. Let $\mathfrak{g}$ be a Lie algebra and let $L = \mathfrak{g} \otimes \mathfrak{g}$. We define on L the following bracket

$$[x\otimes y, a\otimes b]_L = [a,[x,y]]\otimes b + a\otimes[b,[x,y]]$$

where $x\otimes y, a\otimes b \in L$. We verify that L with this bracket is a Leibniz algebra, that is

$$[x\otimes y,[a\otimes b,s\otimes t]_L]_L = [[x\otimes y,a\otimes b]_L, s\otimes y]_L + [a\otimes b,[x\otimes y,s\otimes t]_L]_L,$$

for every $x\otimes y, a\otimes b, s\otimes t \in L$. On one hand,

$$\begin{aligned}
[x\otimes y,[a\otimes b,s\otimes t]_L] &= [x\otimes y,[s,[a,b]]\otimes t+s\otimes[t,[a,b]]]_L\\
&= [x\otimes y,[s,[a,b]]\otimes t]_L+[x\otimes y,s\otimes[t,[a,b]]]\\
&= [[s,[a,b]],[x,y]]\otimes t+[s,[a,b]]\otimes[t,[x,y]]\\
&+ [s,[x,y]]\otimes[t,[a,b]]+s\otimes[[t,[a,b]],[x,y]].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[[x\otimes y,a\otimes b]_L,s\otimes t]_L &= [[a,[x,y]]\otimes b+a\otimes[b,[x,y]],s\otimes t]_L\\
&= [[a,[x,y]]\otimes b,s\otimes t]_L+[a\otimes[b,[x,y]],s\otimes t]_L\\
&= [s,[[a,[x,y]],b]]\otimes t+s\otimes[t,[[a,[x,y]],b]]\\
&+ [s,[a,[b,[x,y]]]]\otimes t+s\otimes[t,[a,[b,[x,y]]]]
\end{aligned}$$

and

$$\begin{aligned}
[a\otimes b,[x\otimes y,s\otimes t]]_L &= [a\otimes b,[s,[x,y]]\otimes t+s\otimes[t,[x,y]]]_L\\
&= [a\otimes b,[s,[x,y]]\otimes t]_L+[a\otimes b,s\otimes[t,[x,y]]]_L\\
&= [[s,[x,y]],[a,b]]\otimes t+[s,[x,y]]\otimes[t,[a,b]]\\
&+ [s,[a,b]]\otimes[t,[x,y]]+s\otimes[[t,[x,y]],[a,b]].
\end{aligned}$$

Now we conclude by using anti-symmetry and the Jabobi identity of the Lie algebra $\mathfrak{g}$. Indeed, we obtain

$$\begin{aligned}[[s,[a,b]],[x,y]] &= [s,[[a,b],[x,y]]] + [[s,[x,y]],[a,b]] \\ &= [s,[a,[b,[x,y]]]] + [s,[[a,[x,y]],b]] + [[s,[x,y]],[a,b]]\end{aligned}$$

and

$$\begin{aligned}[[t,[a,b]],[x,y]] &= [t,[[a,b],[x,y]]] + [[t,[x,y]],[a,b]] \\ &= [t,[a,[b,[x,y]]]] + [t,[[a,[x,y]],b]] + [[t,[x,y]],[a,b]] .\end{aligned}$$

Example 1.1.3.5. Given an associative algebra A and a linear map $D\colon A \to A$ such that

$$D(aD(b)) = D(a)D(b) = D(D(a)b) \quad \text{for all } a, b \in A,$$

it is possible to define a Leibniz bracket in A as follows

$$[a,b]_D = bD(a) - D(a)b.$$

Indeed, we have

$$\begin{aligned}[a,[b,c]_D]_D &= [a, cD(b) - D(b)c]_D \\ &= (cD(b) - D(b)c)D(a) - D(a)(cD(b) - D(b)c) \\ &= cD(b)D(a) - D(b)cD(a) - D(a)cD(b) + D(a)D(b)c,\end{aligned}$$

$$\begin{aligned}[[a,b]_D,c]_D &= [bD(a) - D(a)b, c]_D \\ &= cD(bD(a) - D(a)b) - D(bD(a) - D(a)b)c \\ &= cD(bD(a)) - cD(D(a)b) - D(bD(a))c + D(D(a)b)c\end{aligned}$$

and

$$\begin{aligned}[b,[a,c]_D]_D &= [b, cD(a) - D(a)c]_D \\ &= (cD(a) - D(a)c)D(b) - D(b)(cD(a) - D(a)c) \\ &= cD(a)D(b) - D(a)cD(b) - D(b)cD(a) + D(b)D(a)c.\end{aligned}$$

Some examples of the map D are:

- a derivation D of an associative algebra such that $D^2 = 0$;
- an endomorphism D such that $D^2 = D$.

Example 1.1.3.6 (Direct sum of Leibniz algebras)**.** Let $(L_1, [\cdot,\cdot]_1)$ and $(L_2, [\cdot,\cdot]_2)$ be left Leibniz algebras. The *direct sum* of L_1 and L_2 is the left Leibniz algebra defined on the vector space $L = L_1 \times L_2$, denoted as $L_1 \oplus L_2$, with the following bracket

$$[(x_1,x_2),(y_1,y_2)] = ([x_1,y_1]_1, [x_2,y_2]_2),$$

for every $(x_1,x_2),(y_1,y_2) \in L$.

1.2 The tensor complex associated to a Leibniz algebra

Before delving into further definitions and basic results on Leibniz algebras, it is worth briefly explaining the context in which Leibniz algebras in the sense of Loday (*Loday algebras*) arose (again after Blokh). In 1993, J.-L. Loday introduced several classes of algebras in [62], which have captured the interest of researchers in the field for subsequent years, enabling ongoing research that continues to this day. New algebraic structures include associative, dendriform, and Zinbiel algebras, as well as Leibniz algebras. It has been observed that if a Chevalley-Eilenberg chain complex of a Lie algebra one replaces the exterior product with the tensor product, imposing the Leibniz identity alone is sufficient to prove that it is a chain complex. More precisely, given the Chevalley-Eilenberg chain complex of a Lie algebra $\mathfrak{g}$, that is the sequence of chain modules

$$\bigwedge *\mathfrak{g}\colon \; \cdots \xrightarrow{d_{n+2}} \wedge^{n+1}\mathfrak{g} \xrightarrow{d_{n+1}} \wedge^{n}\mathfrak{g} \xrightarrow{d_{n}} \wedge^{n-1}\mathfrak{g} \xrightarrow{d_{n-1}} \cdots$$

and buondary operator $d_n\colon \wedge^n \mathfrak{g} \to \wedge^{n-1}\mathfrak{g}$ classically defined by

$$d_n(x_1 \wedge x_2 \wedge \cdots \wedge x_n) := \sum_{i<j} (-1)^{i+j+1} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots x_n,$$

the property $d_n \circ d_{n+1} = 0$, wich makes this sequence a chain complex, is proved by using the antisymmetry $x \wedge y = -y \wedge x$ of the exterior product, the Jacobi identity $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ and the antisymmetry $[x,y] = -[y,x]$ of the Lie bracket given on $\mathfrak{g}$. We will construct a chain complex using the tensor product and define the Chevalley-Eilenberg chain complex and the boundary operator for an algebra A as follows:

$$\bigotimes *A\colon \; \cdots \xrightarrow{d_{n+2}} A^{\otimes(n+1)} \xrightarrow{d_{n+1}} A^{\otimes n} \xrightarrow{d_{n}} A^{\otimes(n-1)} \xrightarrow{d_{n-1}} \cdots$$

and

$$d_n(x_1 \otimes x_2 \otimes \cdots \otimes x_n) := \sum_{1 \le i < j \le n} (-1)^j x_1 \otimes \cdots \otimes x_j \cdot x_i \otimes \cdots \otimes \hat{x}_j \otimes \cdots x_n. \quad (1.5)$$

Note that the product $x_j \cdot x_i$ is at the place $\inf(i,j)$. This element appears as the first entry for $i = 1$, d_n is well-defined for every $n \in \mathbb{N}$. Our purpose is to prove that the boundary operator defined above satisfies the condition $d_n \circ d_{n+1} = 0$.

From this point onward, we assume that A is a left Leibniz algebra, thus Equation (1.5) holds and rather than having the product of two elements, we will have the bracket. Before we proceed, it is necessary to provide the following definition in order to enhance our understanding of Loday's construction and for our discussion to be comprehensive in general.

Definition 1.2.1. Let V be a vector space over a field $\mathbb{F}$ and let L be a left Leibniz algebra. A *Leibniz representation* of L is a triple (V, l, r), where

$l, r\colon L \to \mathfrak{gl}(V)$ are linear maps such that, for all $x, y \in L$, the following equalities hold:

i) $l_{[x,y]} = [l_x, l_y]$

ii) $r_{[x,y]} = [l_x, r_y]$

iii) $r_y l_x = -r_y r_x$.

Remark 1.2.1. It is noteworthy to highlight the contrast between the definitions of a Leibniz representation (of a Leibniz algebra) and that of a Lie algebra representation (or Lie representation). Indeed, if a Lie algebra and one of its Lie representation are given, we can construct a Leibniz representation of it in several ways. For instance, if a Lie algebra $\mathfrak{g}$ has a Lie representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$, we can obtain a Leibniz representation by setting $l = \rho$ and $r = -\rho$, or by setting $l = \rho$ and $r = 0$. This implies that, in this case, l is an homomorphism of Lie algebras, just as we would expect. However, while l is always an homomorphism of Leibniz algebras (by condition i)), r is not in general.

Definition 1.2.2. Let L be a Leibniz algebra over a field $\mathbb{F}$ and let V be a vector space over the same field. We define V to be an *L-module* equipped with two bilinear maps (*actions*)

$$[\cdot,\cdot] : L \times V \to V, \quad [\cdot,\cdot] : V \times L \to V$$

such that the identity

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$$

holds whenever one (any) of the variables is in V, and the other two are in L.

Just as in the case of Lie algebras (and not only), the definitions of a representation of L and an L-module are equivalent to each other. Indeed, the last definition says that the following equations must be true for every $x, y \in L, v \in V$:

$$[x, [y, v]] = [[x, y], v] + [y, [x, v]] \tag{1.6}$$
$$[x, [v, y]] = [[x, v], y] + [v, [x, y]] \tag{1.7}$$
$$[v, [x, y]] = [[v, x], y] + [x, [v, y]]. \tag{1.8}$$

Thus, if we have a representation (V, l, r) of L, we define

$$[x, v] = l_x(v) \quad [v, x] = r_x(v).$$

By computation one can easily obtain from Equations 1.6 and 1.7 respectively i) and ii). Equation (1.8) reads

$$r_{[x,y]} = r_y r_x(v) + l_x r_y(v).$$

By ii) we have $r_{[x,y]}(v) = (l_x r_y - r_y l_x)(v)$ and hence we obtain Equation (1.8).

Remark 1.2.2. Every left (right) Leibniz algebra L can be thought as a L-module. Indeed, just consider $l_x(y) = [x, y]$ and $r_x(y) = [y, x]$. By doing this, conditions $i)$-$iii)$ directly follow from the left (right) differential identity.

Now we are ready for the following observation. We can think of the tensor algebra $A^{\otimes n}$ as an A-module with the following left action:

$$[x, x_1 \otimes \cdots \otimes x_n] = \sum_{i=1}^{n} x_1 \otimes \cdots \otimes [x, x_i] \otimes \cdots \otimes x_n. \tag{1.9}$$

So the right linear map is naturally defined, and $A^{\otimes n}$ is indeed an A-module since A itself is (see the last Remark).

The next result directly follows from Equation (1.5) and the previous equation.

Lemma 1.2.3. *Let $d_{n+1}\colon A^{\otimes(n+1)} \to A^{\otimes n}$ be the boundary operator 1.5. Then for all $x_1, \dots, x_{n+1} \in A$ the following equation holds*

$$d_{n+1}(x_1 \otimes \cdots \otimes x_{n+1}) = d_n(x_1 \otimes \cdots \otimes x_n) \otimes x_{n+1} + (-1)^{n+1} [x_{n+1}, x_1 \otimes \cdots \otimes x_n] .$$

Lemma 1.2.4. *For all $x, x_1, \dots, x_{n+1} \in A$, we have*

$$[x, x_1 \otimes \cdots \otimes x_{n+1}] = [x, x_1 \otimes \cdots \otimes x_n] \otimes x_{n+1} + x_1 \otimes \cdots \otimes x_n \otimes [x, x_{n+1}] .$$

Proof. On one hand, by Equation (1.9), we have

$$[x, x_1 \otimes \cdots \otimes x_{n+1}] = \sum_{i=1}^{n+1} x_1 \otimes \cdots \otimes [x, , x_i] \otimes \cdots \otimes x_{n+1}.$$

On the other hand

$$[x, x_1 \otimes \cdots \otimes x_n] \otimes x_{n+1} = \sum_{i=1}^{n} x_1 \otimes \cdots \otimes [x_j, x_i] \otimes \cdots \otimes x_n \otimes x_{n+1}.$$

□

Now we need a last result.

Proposition 1.2.5. *Let $d_{n+1}\colon A^{\otimes(n+1)} \to A^{\otimes n}$ be the boundary operator 1.5. Then*

$$d_{n+1} [x, x_1 \otimes \cdots \otimes x_{n+1}] = [x, d_{n+1}(x_1 \otimes \cdots \otimes x_{n+1})] \tag{1.10}$$

holds for all $x, x_1, \dots, x_{n+1} \in A$.

Proof. We prove the above relation by induction on n. The equality is trivial for $n = 0$. For $n = 1$ it is precisely the equation Equation (1.1) since

$$\begin{aligned} d_2 [x, x_1 \otimes x_2] &= d_2([x, x_1] \otimes x_2 + x_1 \otimes [x, x_2]) \\ &= -[[x, x_1], x_2] - [x_1, [x, x_2]] \end{aligned}$$

and

$$[x, d_2(x_1 \otimes x_2)] = -[x, [x_1, x_2]] .$$

Suppose that Equation (1.10) is true for n. We prove the statement for $n+1$. By Lemma 1.2.3, Lemma 1.2.4, Equation (1.9), and induction on n we have that

$$d_{n+1}\left[x, x_1 \otimes \cdots \otimes x_{n+1}\right]$$

is equal to

$$\begin{aligned}
& d_{n+1}\left(\left[x, x_1 \otimes \cdots \otimes x_n\right] \otimes x_{n+1} + x_1 \otimes \cdots \otimes x_n \otimes \left[x, x_{n+1}\right]\right) \\
=\ & d_n\left(\left[x, x_1 \otimes \cdots \otimes x_n\right]\right) \otimes x_{n+1} + (-1)^{n+1}\left[x_{n+1}, \left[x, x_1 \otimes \cdots \otimes x_n\right]\right] \\
& + d_n(x_1 \otimes \cdots \otimes x_n) \otimes \left[x, x_{n+1}\right] + (-1)^{n+1}\left[\left[x, x_{n+1}\right], x_1 \otimes \cdots \otimes x_n\right] \\
=\ & d_n\left(\left[x, x_1 \otimes \cdots \otimes x_n\right]\right) \otimes x_{n+1} + d_n(x_1 \otimes \cdots \otimes x_n) \otimes \left[x, x_{n+1}\right] \\
& + (-1)^{n+1}\left(\left[x_{n+1}, \left[x, x_1 \otimes \cdots \otimes x_n\right]\right] + \left[\left[x, x_{n+1}\right], x_1 \otimes \cdots \otimes x_n\right]\right) \\
=\ & \left[x, d_n(x_1 \otimes \cdots \otimes x_n)\right] \otimes x_{n+1} + d_n(x_1 \otimes \cdots \otimes x_n) \otimes \left[x, x_{n+1}\right] \\
& + (-1)^{n+1}\left(\left[x_{n+1}, \left[x, x_1 \otimes \cdots \otimes x_n\right]\right] + \left[\left[x, x_{n+1}\right], x_1 \otimes \cdots \otimes x_n\right]\right) \\
=\ & \left[x, d_n(x_1 \otimes \cdots \otimes x_n) \otimes x_{n+1}\right] - d_n(x_1 \otimes \cdots \otimes x_n) \otimes \left[x, x_{n+1}\right] \\
& + d_n(x_1 \otimes \cdots \otimes x_n) \otimes \left[x, x_{n+1}\right] + (-1)^{n+1}\left[x, \left[x_{n+1}, x_1 \otimes \cdots \otimes x_n\right]\right] \\
=\ & \Big[x, d_n(x_1 \otimes \cdots \otimes x_n) \otimes x_{n+1} + (-1)^{n+1}\left[x_{n+1}, x_1 \otimes \cdots \otimes x_n\right]\Big] \\
=\ & \left[x, d_{n+1}(x_1 \otimes \cdots \otimes x_{n+1})\right],
\end{aligned}$$

for all $x \in A, x_1 \otimes \cdots \otimes x_{n+1} \in A^{\otimes(n+1)}$. □

Proposition 1.2.6. *Let $d_{n+1}\colon A^{\otimes(n+1)} \to A^{\otimes n}$ be the boundary operator 1.5. Then the condition*

$$d_n \circ d_{n+1} = 0$$

holds for all $x, x_1, \ldots, x_{n+1} \in A$.

Proof. We prove this relation by induction on n. For $n = 1$ is trivial. Suppose that Equation (1.10) is true for n. We shall now prove it for $n+1$. Thus by Lemma 1.2.3 and induction on n we have

$$\begin{aligned}
d_n \circ d_{n+1}(x_1 \otimes \cdots \otimes x_{n+1}) &= d_n(d_n(x_1 \otimes \cdots \otimes x_n) \otimes x_{n+1} \\
&\quad + (-1)^{n+1}\left[x_{n+1}, x_1 \otimes \cdots \otimes x_n\right]) \\
&= d_{n-1}d_n(x_1 \otimes \cdots \otimes x_n) \otimes x_{n+1} \\
&\quad + (-1)^n\left[x_{n+1}, d_n(x_1 \otimes \cdots \otimes x_n)\right] \\
&\quad + (-1)^{n+1}\left[x_{n+1}, d_n(x_1 \otimes \cdots \otimes x_n)\right] \\
&= 0.
\end{aligned}$$

□

This slight homological deviation indicates how the left differential identity alone is sufficient to construct a new chain complex using the tensor product instead of the exterior product. Moreover, we note that the tensor product's natural surjection to the exterior product induces the following commutative diagram for any Lie algebra $\mathfrak{g}$:

$$\begin{array}{ccc} \mathfrak{g}^{\otimes n+1} & \xrightarrow{d_{n+1}} & \mathfrak{g}^{\otimes n} \\ \downarrow & & \downarrow \\ \wedge^{n+1}\mathfrak{g} & \xrightarrow{d_{n+1}} & \wedge^{n}\mathfrak{g}. \end{array}$$

1.3 Leibniz kernel and centers of a Leibniz algebra

Now, we define a subset of crucial importance for Leibniz algebras.

Definition 1.3.1. Let L be a left Leibniz algebra. The *Leibniz kernel*, denoted by $\mathrm{Leib}(L)$, is the subspace of L spanned by squares of elements of the algebra L, i.e.

$$\mathrm{Leib}(L) = \langle [x, x] \mid x \in L \rangle.$$

Definition 1.3.2. Let I be a vector subspace of a Leibniz algebra L. I is a *left ideal* (*right ideal*) of L if, for every $i \in I$, $x \in L$, $[x, i] \in I$ ($[i, x] \in I$).

I is a *two-sided ideal* if it is a left ideal that is also a right ideal.

We note that, as in the Lie case, a left or a right ideal of L is in particular a subalgebra. From this point forward, the term two-sided ideal will be referred to as simply *ideal*.

Proposition 1.3.3. *Let I, J be left ideals of a left Leibniz algebra L. Then $I \cap J, I + J$ and $[I, J] = \mathrm{span}\{[x, y] \mid x \in I, y \in J\}$ are left ideals.*

Proof. $I \cap J$ and $I + J$ are trivially left ideal of L. Now, clearly $[I, J]$ is a subspace of L. Let $x \in L, i \in I$ and $j \in J$, then we have

$$[x, [i, j]] = [[x, i], j] + [i, [x, j]] \in [I, J].$$

This result also holds for a general linear combination of vectors in $[I, J]$. □

Remark 1.3.1. The last result is true only with left ideals (of a left Leibniz algebra). In the same way, for a right Leibniz algebra L, the constructions above are right ideals only if I, J are right ideals of L.

Proposition 1.3.4. *Let I, J be right ideal of a left Leibniz algebra L. If $[I, J] = [J, I]$, then $[I, J]$ is a right ideal.*

Proof. Let $x \in L, i \in I$ and $j \in J$, then we have

$$[[i, j], x] = \underbrace{[i, \underbrace{[j, x]}_{\in J}]}_{\in [I,J]} - \underbrace{[j, \underbrace{[i, x]}_{\in I}]}_{\in [J,I]}.$$

□

Corollary 1.3.5. *$L' = [L, L]$ is an ideal of L.*

Proposition 1.3.6. *Let L be a left Leibniz algebra. The subspace* $\mathrm{Leib}(L)$ *is an ideal of L.*

Proof. For every $x, y \in L$ we have

$$[[y,y],x] = [y,[y,x]] - [y,[y,x]] = 0 = [0,0].$$

Since $[x,x], [[y,y],[y,y]] \in \mathrm{Leib}(L)$ and $[[y,y],x] = 0$, we have

$$[x+[y,y], x+[y,y]] = [x,x] + [x,[y,y]] + [[y,y],x] + [[y,y],[y,y]]$$

and therefore

$$[x,[y,y]] = [x+[y,y], x+[y,y]] - [x,x] - [[y,y],[y,y]] \in \mathrm{Leib}(L).$$

□

Definition 1.3.7. Let L be a left Leibniz algebra and let I be an ideal of L. The *quotient* Leibniz algebra L/I is the left Leibniz algebra defined over the space $\{x+I \mid x \in L\}$, consisting of the cosets $x+I = \{x+i \mid i \in I\}$ with the following bracket:

$$[x+I, y+I] = [x,y] + I,$$

for all $x, y \in L$.

We must demonstrate that the bracket above is well-defined. Bilinearity and the left Leibniz differential identity ensue from those of L. Let $x, x', y, y' \in L$ such that

$$x + I = x' + I \quad \text{and} \quad y + I = y' + I,$$

then $x - x' \in I$ and $y - y' \in I$. Hence, we have

$$\begin{aligned} [x,y] &= [x' + (x-x'), y' + (y-y')] \\ &= [x',y'] + [x-x',y'] + [x',y-y'] + [x-x',y-y'], \end{aligned}$$

and since I is an ideal of L, $[x-x',y'], [x',y-y'], [x-x',y-y'] \in I$. Therefore, $[x+I, y+I] = [x',y'] + I$ and this proves that the bracket on the quotient is well-defined.

If L was a Lie algebra, the Leibniz kernel would be the null subspace. This demonstrates the trivial nature of defining such a subspace for a Lie algebra. Additionally, this concept can be referred to as a "liezator"[1], a subspace consisting only of the zero vector if the algebra is a Lie algebra. However, for non-Lie Leibniz algebras, this subspace would contain at least one nonzero element. Furthermore, the Leibniz kernel $\mathrm{Leib}(L)$ is the smallest ideal of L such that the quotient algebra becomes a Lie algebra. Indeed, it is easy to observe that this is a Lie algebra since, for every element in $L/\mathrm{Leib}(L)$, the bracket is alternating, and the Jacobi identity holds (which follows from alternativity and the left differential identity). This quotient algebra is frequently denoted as $L_{Lie} := L/\mathrm{Leib}(L)$. The upcoming elementary result holds generally for any

[1]Termed by Gorbatsevich in [43].

class of algebras. Nevertheless, for completeness, we will give a specific proof in the context of Leibniz algebras.

Theorem 1.3.8 (Correspondence Theorem)**.** *Let L be a left Leibniz algebra and let I be an ideal of L. Every subalgebra of the quotient algebra L/I is of the form $K/I = \{k + I \mid k \in K\}$, where $I \leq K \leq L$. Conversely, if $I \leq K \leq L$, then $K/I \leq L/I$.*

Proof. Let K' a subalgebra of L/I. We define the subset $f(K') \subseteq L$ as

$$f(K') = \{x \in L \mid x + I \in K'\}.$$

Clearly, $I \subseteq f(K')$. Thus, for every $x, y \in f(K')$ we have

$$x + I, y + I \in K' \Rightarrow [x + I, y + I] = [x, y] + I \in K' \Rightarrow [x, y] \in f(K').$$

Then $f(K')$ is a subalgebra of L that contains I and $K' = f(K')/L$.

Conversely, let K be a subalgebra of L such that $I \leq K \leq L$. Let $K/I = \{k + I \mid k \in K\} \subseteq L/I$. Since

$$[k_1 + I, k_2 + I] = [k_1, k_2] + I \in K/I,$$

for all $k_1, k_2 \in K$, then $K/I \leq L/I$. □

Notice that the correspondence theorem can be applied to either a left or a right ideal. Thus, each left (respectively right) ideal of the quotient Leibniz algebra L/I takes the form K/I, where K is a left (right) ideal that contains I, and vice versa.

Definition 1.3.9. Let $(L_1, [\cdot,\cdot]_1)$ and $(L_2, [\cdot,\cdot]_2)$ be two left Leibniz algebras over the same field $\mathbb{F}$. Then $\phi \colon L_1 \to L_2$ is a *homomorphism of left Leibniz algebras* if ϕ is a linear map and if it satisfies

$$\phi([x, y]_1) = [\phi(x), \phi(y)]_2 \text{ for all } x, y \in L_1.$$

The same isomorphism theorems that apply to Lie algebras (or groups) also hold for Leibniz algebras, and the proof is the classical one.

Theorem 1.3.10 (Isomorphism Theorems)**.**

1. *Let $\phi \colon L_1 \to L_2$ be a homomorphism of Leibniz algebras. Then $\ker\phi$ is an ideal of L_1, $\operatorname{Im}\phi$ is a subalgebra of L_2, and*

$$L_1/\ker\phi \cong \operatorname{Im}\phi.$$

2. *If I and J are two ideals of a Leibniz algebra, then $(I + J)/J \cong I/(I \cap J)$.*

3. *Suppose that I and J are ideals of the Leibniz algebra L such that $I \subseteq J$. Then J/I is an ideal of L/I and*

$$(L/I)/(J/I) \cong L/J.$$

The quotient mapping $\pi\colon L \to L_{Lie}$ is a (surjective) homomorphism of Leibniz algebras. In other words, one has an exact sequence of Leibniz algebras

$$0 \longrightarrow \mathrm{Leib}(L) \xrightarrow{\ i\ } L \xrightarrow{\ \pi\ } L_{Lie} \longrightarrow 0.$$

As we have already seen, many definitions provided for Leibniz algebras are not significantly different from those given for Lie algebras, and the next one is no exception.

Definition 1.3.11. A linear transformation $d\colon L \to L$ is a *derivation* if

$$d([x,y]) = [d(x),y] + [x,d(y)] \text{ for all} x,y \in L.$$

For a left Leibniz algebra L and for every $x \in L$ the operator $L_x\colon L \to L$,

$$L_x\colon y \mapsto [x,y]\,,$$

for every $y \in L$, is a derivation of L. Respectively, for a right Leibniz algebra L and for every $x \in L$ the operator $R_x\colon L \to L$,

$$R_x\colon y \mapsto [y,x]$$

is a derivation of L. So, left Leibniz algebras are characterized by this property, namely that every left (right) multiplication is a derivation. It is well-known that the set of derivations of a Lie algebra is still a Lie algebra. However, for a Leibniz algebra L, the set of derivations $\mathrm{Der}(L)$ (with the usual bracket, namely $[d_1,d_2] = d_1d_2 - d_2d_2$ for all $d_1,d_2 \in \mathrm{Der}(L)$) forms a specific type of Leibniz algebra, which is, in fact, a Lie algebra. We recall that this fact is true for any class of algebras. Left and right multiplications are called *inner derivations* on L, and the set of inner derivations is denoted as $\mathfrak{L}(L)$ (as $\mathfrak{R}(L)$ if L is a right Leibniz algebra). As in the Lie case, this set is an ideal of $\mathrm{Der}(L)$.

The center of a Lie algebra is defined as the subalgebra of elements that commute with every other element in the algebra, meaning that the Lie bracket of those elements with any element in the algebra is zero. Clearly, for Leibniz algebras, there is a similar notion that extends the concept of the center of a Lie algebra, with some predictable differences. Notice that in the following definition, it does not matter whether the algebra is on the left or on the right.

Definition 1.3.12. Let L be Leibniz algebra. The *left center* and the *right center* of L are defined, respectively, as

$$\mathrm{Z}_l(L) = \{x \in L \mid [x,y] = 0, \forall y \in L\} \text{ and } \mathrm{Z}_r(L) = \{x \in y \mid [y,x] = 0, \forall y \in L\}\,.$$

The *center* of L is the subspace $\mathrm{Z}(L) = \mathrm{Z}_l(L) \cap \mathrm{Z}_r(L)$.

Clearly, for a Lie algebra, the left and the right center coincide.

We immediately see a result that helps us understand how these two centers are slightly different from the center of a Lie algebra in general. In the case of Lie algebras, it is well-known that the center is an ideal. Here, however, we have the following.

Proposition 1.3.13. *Let L be a left Leibniz algebra. Then $\mathrm{Z}_l(L)$ is an ideal of L.*

Proof. By Equation (1.3) and Equation (1.1) we have, respectively,

$$[[x,z],y] = [x,[z,y]] - [z,[x,y]] = 0 - 0 = 0,$$

and

$$[[z,x],y] = [z,[x,y]] - [x,[z,y]] = 0 - 0 = 0,$$

for all $x, y \in L$ and $z \in Z_l(L)$ □

The surprising thing is that, for a left Leibniz algebra, the left center is a two-sided ideal, while the right center may not even be a subalgebra. For a right Leibniz algebra is exactly the opposite. Additionally, the left and right center center are different, even they might have different dimensions.

Example 1.3.13.1. Let $L = \langle e_1, e_2, e_3 \rangle$ with non-zero brackets $[e_1, e_1] = e_2$ and $[e_1, e_2] = e_3$. L is a left Leibniz algebra. From Leibniz brackets, we observe that the left center $\mathrm{Z}_l(L) = \langle e_2, e_3 \rangle$. The right center $\mathrm{Z}_r(L)$ is generated by e_3. Indeed $x = x_1e_1 + x_2e_2 + x_3e_3 \in \mathrm{Z}_r(L)$ if and only if, for every $y = y_1e_1 + y_2e_2 + y_3e_3 \in L$, we have $[y, x] = 0$, that is

$$[y_1e_1 + y_2e_2 + y_3e_3, x_1e_1 + x_2e_2 + x_3e_3] = y_1x_1e_2 + y_1x_2e_3 = 0.$$

Thus, for every $y_1 \in \mathbb{F}$, we have

$$\begin{cases} y_1x_1 = 0 \\ y_1x_2 = 0 \end{cases}$$

and then $x_1 = x_2 = 0$. Clearly, in this case $\mathrm{Z}_r(L)$ is a subalgebra of L since is abelian. Nevertheless, its dimension is different from that of the left center:

$$\dim \mathrm{Z}_r(L) = 1 < 2 = \dim \mathrm{Z}_l(L).$$

Proposition 1.3.14. *The Leibniz kernel* $\mathrm{Leib}(L)$ *is contained in the left center* $\mathrm{Z}_l(L)$*.*

Proof. For every $x, y \in L$, we have

$$[x,[x,y]] = [[x,x],y] + [x,[x,y]],$$

and then $[[x,x],y] = 0$. □

Corollary 1.3.15. *The Leibniz kernel* $\mathrm{Leib}(L)$ *is an abelian ideal of L:*

Recall that, for a left Leibniz algebra L, $\mathrm{Der}(L)$ is a Lie algebra. We denote with $\mathfrak{L}(L) = \{L_x \mid x \in L\}$ ($\mathfrak{R}(L) = \{R_x \mid x \in L\}$) the set of all left (right) multiplication operators.

Proposition 1.3.16. *Let L be a left (right) Leibniz algebra. $L/\,\mathrm{Z}_l(L)$ (respectively, $L/\,\mathrm{Z}_r(L)$) is (anti)isomorphic to $\mathfrak{L}(L)$ (respectively, $\mathfrak{R}(L)$).*

Proof. We define the linear map

$$\begin{aligned} \varphi\colon L/\operatorname{Z}_l(L) &\to \mathfrak{L}(L) \\ x + \operatorname{Z}_l(L) &\mapsto L_x. \end{aligned}$$

This map is well-defined since, for every $x + \operatorname{Z}_l(L), y + \operatorname{Z}_l(L) \in L/\operatorname{Z}_l(L)$, $x + \operatorname{Z}_l(L) = y + \operatorname{Z}_l(L)$ implies that $x - y \in \operatorname{Z}_l(L)$, then $[x - y, z] = 0$ for all $z \in L$, and $[x, z] = [y, z]$.

Surjectivity is trivial, and injectivity follows from

$$\varphi(x + \operatorname{Z}_l(L)) = 0 \Rightarrow [x, z] = 0, \forall z \in L \Rightarrow x \in \operatorname{Z}_l(L).$$

Finally, $\varphi([x, y] + \operatorname{Z}_l(L)) = L_{[x,y]}$ and, for every $z \in L$, by Equation (1.3) follows that

$$\begin{aligned} L_{[x,y]}(z) &= [[x, y], z] \\ &= [x, [y, z]] - [y, [x, z]] \\ &= (L_x L_y - L_y L_x)(z) \\ &= [L_x, L_y](z). \end{aligned}$$

If L is a right Leibniz algebra, the linear map is

$$\begin{aligned} \varphi\colon L/\operatorname{Z}_r(L) &\to \mathfrak{R}(L) \\ x + \operatorname{Z}_r(L) &\mapsto R_x. \end{aligned}$$

and by similar arguments one can prove that φ is well-defined, injective and surjective. To prove the anti-isomorphism we have to show that $\varphi([x, y] + \operatorname{Z}_r(L)) = [\varphi(y), \varphi(x)]$. Indeed, by Equation (1.4), we have

$$\begin{aligned} R_{[x,y]}(z) &= [z, [x, y]] \\ &= [[z, x], y] - [[z, y], x] \\ &= (R_y R_x - R_x R_y)(z) \\ &= [R_y, R_x](z), \end{aligned}$$

for all $z \in L$. □

1.4 Nilpotency and solvability

The definitions of nilpotency and solvability for Leibniz algebras are analogous to those for Lie algebras.

Definition 1.4.1. The *lower central series of* L is defined recursively as follows

$$L^1 = L, L^{k+1} = [L, L^k], k \geq 1.$$

L is said to be *nilpotent*, if there exists $n \in \mathbb{N}$ such that $L^n = 0$. The minimal number n with such property is said to be the *index of nilpotency* of the algebra L. Equivalently, the Leibniz algebra L is said n*-step nilpotent*.

Example 1.4.1.1.

- Every nilpotent Lie algebra is nilpotent as a Leibniz algebra.
- Let L be the left Leibniz algebra with basis $\{e_1, e_2, \ldots, e_n\}$ and non-zero brackets as below:

  i)
$$\begin{cases} [e_1, e_i] & = e_{i+1} \quad 1 \leq i \leq n-3, \\ [e_{n-1}, e_1] & = e_2 + e_n \\ [e_{n-1}, e_i] & = e_{i+1} \quad 2 \leq i \leq n-3, \end{cases}$$

  ii)
$$\begin{cases} [e_1, e_i] & = e_{i+1} \quad 1 \leq i \leq n-3, \\ [e_{n-1}, e_1] & = e_n. \end{cases}$$

  In each case L is nilpotent.

In order to prove that the direct sum of two nilpotent left Leibniz algebras is nilpotent, we will prove the following results.

Lemma 1.4.2. *Let L_1 and L_2 be left Leibniz algebras and $L = L_1 \oplus L_2$. Then, for all $k \geq 1$,*
$$L^k = L_1^k \oplus L_2^k.$$

Proof. We will prove the statement by induction on k. For $k = 1$ is trivial. Then we suppose the formula above holds for $k-1$. Thus, we have

$$\begin{aligned} L^k &= \left[L_1 \oplus L_2, (L_1 \oplus L_2)^{k-1}\right] \\ &= \left[L_1 \oplus L_2, L_1^{k-1} \oplus L_2^{k-1}\right] \\ &= \left[L_1, L_1^{k-1}\right] \oplus \left[L_2, L_2^{k-1}\right] \\ &= L_1^k \oplus L_2^k. \end{aligned}$$

□

Proposition 1.4.3. *Let L_1 and L_2 be nilpotent left Leibniz algebras. Then $L = L_1 \oplus L_2$ is nilpotent.*

Proof. Let k_1 and k_2 the indices of nilpotency respectively of L_1 and L_2. We set $k := \max\{k_1, k_2\}$. Hence, by the previous lemma, $L^k = L_1^k \oplus L_2^k = 0$. □

Definition 1.4.4. The *derived series of* L is defined recursively as follows

$$L^{(1)} = L, L^{(k+1)} = [L^{(k)}, L^{(k)}], k \geq 1.$$

L is said to be *solvable*, if there exists $n \in \mathbb{N}$ such that $L^n = 0$. The minimal number n with such property is said to be the *index of solvability* of the algebra L. Equivalently, the Leibniz algebra L is said *n-step solvable.*

Example 1.4.4.1.

- Every solvable Lie algebra is solvable as a Leibniz algebra.
- The direct sum of solvable Leibniz algebras is solvable.
- Let L be the left Leibniz algebra with basis $\{e_1, e_2, \dots, e_n, x\}$ and non-zero brackets as below:

$$\begin{cases} [e_1, e_i] & = e_{i+1} \qquad 1 \leq i \leq n-1, \\ [e_1, x] & = e_1 \\ [e_i, x] & = -ie_i \qquad 1 \leq i \leq n. \end{cases}$$

  L is solvable.

Obviously,

$$L^1 \supseteq L^2 \supseteq \cdots \supseteq L^i \supseteq \cdots,$$

and

$$L^{(1)} \supseteq L^{(2)} \supseteq \cdots \supseteq L^{(i)} \supseteq \cdots.$$

Moreover, as $L^{(i)} \subseteq L^i$, it follows that every nilpotent Leibniz algebra is solvable. As in the Lie case, the converse is not true.

The Engel's theorem is certainly one of the most classic criteria for the nilpotency of Lie algebras. This result has its version for Leibniz algebras. In [5], [43], and [68], the reader can find many results on Leibniz algebras, especially for the nilpotent ones. In order to prove Engel's theorem for Leibniz algebras, we look at some results that we need.

Lemma 1.4.5. *Let L be a left Leibniz algebra and let I be an ideal of L. Then, for all $k \geq 1$, we have*

$$(L/I)^k = (L^k + I)/I.$$

Proof. We prove this relation by induction on k. For $k = 1$, the result is trivial. Suppose that the statement is true for $k-1$. We shall prove it for k. Hence we have

$$(L/I)^k = \left[L/I, (L/I)^{k-1}\right] = \left[L/I, (L^{k-1} + I)/I\right].$$

Now let $y \in I$, $x \in L$, and $x_{k-1} \in L^{k-1}$. Thus,

$$[x + I, x_{k-1} + y + I] = [x, x_{k-1}] + [x, y] + [x, I].$$

Clearly, $[x, x_{k-1}] \in L^k$. Since I is an ideal of L, then $[x, y] \in I$ and $[x, I] \subseteq I$. □

Corollary 1.4.6. *For all $k \geq 1$, we have*

$$(L/\operatorname{Z}_l(L))^k = (L^k + \operatorname{Z}_l(L))/\operatorname{Z}_l(L).$$

Lemma 1.4.7. *Let I be an ideal of L such that $I \subseteq \operatorname{Z}_l(L)$. If L/I is nilpotent, then L is nilpotent.*

Proof. By the previous lemma, if $(L/I)^m = 0$, then

$$(L^m + I)/I = 0 \Rightarrow L^m \subseteq I \subseteq \operatorname{Z}_l(L) \Rightarrow L^{m+1} = 0.$$

□

This result is equivalent to the following.

Lemma 1.4.8. *A central extension of a nilpotent Leibniz algebra by a nilpotent Leibniz algebra is nilpotent.*

Theorem 1.4.9. *Let L be a left Leibniz algebra. If all operators L_x of left multiplication are nilpotent, then the algebra L is nilpotent.*

Proof. Since $L/\operatorname{Leib}(L)$ is a Lie algebra, then the result holds. For every $x \in L$, L_x is nilpotent by hypothesis, then $L_{x+\operatorname{Leib}(L)}$ is nilpotent. Then, every left multiplications of the Lie algebra $L/\operatorname{Leib}(L)$ is nilpotent and by Engel's theorem we have that $L/\operatorname{Leib}(L)$ is a nilpotent Lie algebra. Since $\operatorname{Leib}(L) \subseteq \mathrm{Z}_l(L)$, by Lemma 1.4.7 we conclude the proof.

□

Definition 1.4.10. The maximal nilpotent ideal of a Leibniz algebra is said to be *the nilradical* of L, further denoted by $\mathcal{N}$.

Proposition 1.4.11. *For any left Leibniz algebra L, there exists a maximal nilpotent ideal containing all nilpotent ideals of L.*

Proof. Since $L/\operatorname{Leib}(L)$ is a Lie algebra, then the result holds and there exists a maximal nilpotent ideal $\mathcal{N}' \leq L/\operatorname{Leib}(L)$. By the Correspondence Theorem 1.3.8, there exists an ideal $\operatorname{Leib}(L) \subseteq \mathcal{N} \subseteq L$ such that $\mathcal{N}/\operatorname{Leib}(L) = \mathcal{N}'$. Since $\mathcal{N}'$ is nilpotent and $\operatorname{Leib}(L) \subseteq \mathrm{Z}_l(L)$, then $\mathcal{N}$ is nilpotent. This is the maximal ideal of L with this property. Indeed, if an ideal $\mathcal{M}$ existed in L with $\mathcal{M} \supsetneqq \mathcal{N}$, then $\mathcal{M}/I \supsetneqq \mathcal{N}/I$ would hold, where $\mathcal{N}/I$ is the maximal nilpotent ideal of L/I, resulting in an absurdity.

□

Proposition 1.4.12. *Let L be a left Leibniz algebra. If all left multiplication operators are nilpotent, then all right multiplication operators are also nilpotent.*

Proof. We will prove this result by proving the following equation

$$R_x^n = (-1)^{n-1} R_x L_x^{n-1},$$

for every $x \in L$ and $n \geq 2$. By induction, for $n = 2$ we have $R_x^2(y) = -R_x L_x(y)$, for every $x, y \in L$, since $[[x,y],x] = -[[y,x],x]$ (Lemma 1.1.3 with $z = x$). We now suppose the formula above is true for $n-1$, i.e.

$$R_x^{n-1} = (-1)^{n-2} R_x L_x^{n-2},$$

and we will prove it for n. Then, for every $x, y \in L$, by induction and the Lemma 1.1.3, we have

$$\begin{aligned}
R_x^n(y) = \left[R_x^{n-1}(y), x\right] &= \left[(-1)^{n-2} R_x L_x^{n-2}(y), x\right] \\
&= (-1)^{n-2} \left[\left[L_x^{n-2}(y), x\right], x\right] \\
&= (-1)^{n-1} \left[\left[x, L_x^{n-2}(y)\right], x\right] \\
&= (-1)^{n-1} R_x L_x^{n-1}(y).
\end{aligned}$$

□

For the purpose of demonstrating the Lie's theorem for Leibniz algebras as well, let us explore some results on solvability.

Lemma 1.4.13. *Let $\varphi\colon L_1 \to L_2$ be an epimorphism of left Leibniz algebras, Then*

$$\varphi(L_1^{(k)}) = L_2^{(k)},$$

for every $k \geq 1$.

Proof. We prove this relation by induction on k. For $k = 1$ is trivial. Suppose the result holds for $k-1$, i.e. $\varphi(L_1^{(k-1)}) = L_2^{(k-1)}$. Hence, we have

$$\begin{aligned}\varphi(L_1^{(k)}) &= \varphi\left(\left[L_1^{(k-1)}, L_1^{(k-1)}\right]\right)\\ &= \left[\varphi(L_1^{(k-1)}), \varphi(L_1^{(k-1)})\right]\\ &= \left[L_2^{(k-1)}, L_2^{(k-1)}\right] = L_2^{(k)}.\end{aligned}$$

□

Lemma 1.4.14. *Let L a left Leibniz algebra.*

i) If L is solvable, then every subalgebra and every homomorphic image of L are sovable.

ii) Suppose that L has an ideal I such that I and L/I are solvable. Then L is solvable.

iii) If I and J are solvable ideals of L, then $I + J$ is a solvable ideal of L.

Proof.

i) If L_1 is a subalgebra of L, then for each k it is clear that $L_1^{(k)} \subseteq L^{(k)}$, so if $L^{(k)} = 0$, then also $L_1^{(k)} = 0$. The second statement follows by Lemma 1.4.13.

ii) Applying the same lemma with the canonical homomorphism $\pi : L \to L/I$, we obtain $(L/I)^{(k)} = (L^{(k)} + I)/I$. If L/I is solvable, then for some $m \geq 1$ we have $(L/I)^{(m)} = 0$, that is $L^{(m)} + I = I$ and therefore $L^{(m)} \subseteq I$. If I is also solvable, then $I^{(k)} = 0$ for some $k \geq 1$ and hence $(L^m)^{(k)} \subseteq I^{(k)} = 0$. By definition, we have $(L^{(m)})^{(k)} = L^{(m+k)}$ and this concludes the proof.

iii) By the second isomorphism theorem $(I + J)/I \cong J/I \cap J$, so it is solvable by $i)$, Since I is also solvable, $ii)$ implies that $I + J$ is solvable.

□

Corollary 1.4.15. *For any left Leibniz algebra L, there exists a maximal solvable ideal containing all solvable ideals of L.*

Proof. Let R be a solvable ideal of largest possible dimension. Suppose that I is any solvable ideal. By $iii)$ of the previous lemma, we know that $R + I$ is a solvable ideal. Now $R \subseteq R + I$ and hence $\dim R \leq \dim(R + I)$ and hence $R = R + I$. □

Definition 1.4.16. The maximal solvable ideal of a Leibniz algebra is said to be *the radical* of L (or *solvable radical*), further denoted by $\mathcal{R}$ or $\mathrm{Rad}(L)$.

Theorem 1.4.17. *Let L be a left Leibniz algebra. Then there exists a complete flag of subspaces wich is invariant under the left multiplication operator L_x, for every $x \in L$.*

Proof. For Lie algebras this theorem is well known. Now let L be an arbitrary solvable Leibniz algebra. Since the Leibniz kernel $\mathrm{Leib}(L)$ is central, then the solvability of the Leibniz algebra L is equivalent to the solvability in the quotient space $L/\mathrm{Leib}(L)$ and by lifting and considering the flag space of ideal, one may present L itself as a full flag space. By the classical Lie theorem $L/\mathrm{Leib}(L)$ has a complete flag, which is invariant under the multiplication (both left and right). Let x be an element of L. Its right action on $\mathrm{Leib}(L)$ is trivial and therefore in $\mathrm{Leib}(L)$ there exists a complete flag, which is invariant under left multiplication. Extending it to the full invariant flag in the quotient space until the full flag in L, we obtain a complete flag in L, which is invariant under the left multiplication operator. □

This result is equivalent to say that there is a basis of L such that the matrix associated with L_x with respect to such a basis is upper triangular, for every $x \in L$. Indeed, suppose that

$$\{0\} \subset L_1 \subset L_2 \subset \cdots \subset L_n = L$$

is a complete flag of subspaces of L, with $\dim L_i = i$, for every $1 \leq i \leq n$. L_i is invariant under the left multiplications and without loss of generality we suppose that $\{e_1, e_2, \ldots, e_i\}$ is a basis of L_i and therefore $\mathcal{B} = \{e_1, e_2, \ldots, e_n\}$ is a basis of L. Hence the matrix associated with L_x with respect to $\mathcal{B}$ is upper triangular. The converse is trivial.

Now, let us examine the following connection between the radical and the nilradical.

Proposition 1.4.18. *Let $\mathcal{R}$ be the radical of a left Leibniz algebra L and let $\mathcal{N}$ be the nilradical of L. Then $[L, \mathcal{R}] \subseteq \mathcal{N}$.*

Proof. The assertion is true for Lie algebras. Let L be a left Leibniz algebra. Then $\mathrm{Leib}(L) \subseteq \mathcal{N} \subseteq \mathcal{R}$ and $\mathrm{Leib}(L) \subseteq \mathrm{Z}_l(L)$. Consider the Lie algebra $L_{Lie} = L/\mathrm{Leib}(L)$. From the definitions of the radical and nilradical it follows that $\mathcal{R}_{Lie} = \mathcal{R}/\mathrm{Leib}(L)$ and $\mathcal{N}_{Lie} = \mathcal{N}/\mathrm{Leib}(L)$ are the radical and nilradical of the Lie algebra L_{Lie}, respectively. Since L_{Lie} is a Lie algebra we have $[L_{Lie}, \mathcal{R}_{Lie}] \subset \mathcal{N}_{Lie}$. Then, by the inclusions $\mathrm{Leib}(L) \subseteq \mathcal{N} \subseteq \mathcal{R}$, we obtain $[L, \mathcal{R}] \subseteq \mathcal{N}$. □

Corollary 1.4.19. $[\mathcal{R}, \mathcal{R}] \subset \mathcal{N}$. *In particular, $[\mathcal{R}, \mathcal{R}]$ is nilpotent.*

Corollary 1.4.20. *A left Leibniz algebra L is solvable if and only if $[L, L]$ is nilpotent.*

Proof. In one direction the statement is given in the last corollary. The converse is proved as follows. If $L = \mathcal{R}$, we are done. Next suppose that $\mathcal{R} \subsetneq L$. As $[L, L]$ is nilpotent, the ideal $[L, L] + \mathcal{R}$ is solvable and

$$(L/\mathcal{R})' = ([L, L] + \mathcal{R})\,/\mathcal{R}$$

is solvable. Now, the Lie algebra $L/\mathcal{R}$ is semisimple because it does not contain any non-zero solvable ideal. Thus $([L, L] + \mathcal{R})\,/\mathcal{R}$ is the zero ideal of $L/\mathcal{R}$, that is $[L, L] \subseteq \mathcal{R}$ and $L/\mathcal{R}$ is a non-zero abelian Lie algebra. This is a contradiction to the fact that $L/\mathcal{R}$ is semisimple.

□

1.5 Classification of low-dimensional Leibniz algebras

Before we delve into the significance of classifying Lie and Leibniz algebras, it is essential to outline one of the fundamental theorems in the field of Lie algebras.

Theorem 1.5.1 (Levi's Theorem)**.** *Every Lie algebra $\mathfrak{g}$ is the semi-direct sum of its radical* $\mathrm{Rad}(\mathfrak{g})$ *and a semisimple subalgebra $\mathfrak{s}$ such $\mathfrak{g} = \mathrm{Rad}(L) \oplus \mathfrak{s}$ as a vector space.*

This result was first conjectured by W. Killing and É. Cartan, and later proven by E. Levi in 1905 (see [60]). This result can be extended to Leibniz algebras (see [7]).

Theorem 1.5.2. *Let L be a finite-dimensional Leibniz algebra over a field of characteristic* 0 *and let $\mathcal{R}$ be its soluble radical. There exists a semi-simple subalgebra $\mathcal{S}$ of L such that $L = \mathcal{S} + \mathcal{R}$ and $\mathcal{S} \cap \mathcal{R} = 0$.*

The classification of Lie algebras and Leibniz algebras is of fundamental and crucial interest for a better understanding of these algebraic structures. However, along with the strength of these results comes a high level of difficulty. To better understand, the problem of classifying Lie algebras can be divided into three parts:

1. classification of nilpotent Lie algebras;

2. description of solvable Lie algebras with given nilradical;

3. description of Lie algebras with given radical.

The initial problem is undoubtedly the most challenging of the three. It should be noted that, currently, the classification of nilpotent Lie algebras is only known up to dimension 7. (see [27]). This section is devoted to the classification of Leibniz algebras in low dimensions. In dimension 3, just a summary table is given showing all isomorphism classes for complex Leibniz algebras of dimension 3.

1.5.1 One-dimensional Leibniz algebra

Theorem 1.5.3. *Let L be a (left or right) Leibniz algebra over a field $\mathbb{F}$. If $\dim_{\mathbb{F}} L = 1$, then L is abelian.*

Proof. Let $x \in L$ be a non-zero element of L such that $L = \langle x \rangle$. Suppose that L is not abelian, thus $[x, x] \neq 0$ and there exists $\alpha \in \mathbb{F}^*$ such that $[x, x] = \alpha x$. If L is a left Leibniz algebra, by Equation (1.1) we have

$$\begin{aligned}
[x,[x,x]] &= [[x,x],x] + [x,[x,x]] \\
\alpha\,[x,x] &= \alpha\,[x,x] + \alpha\,[x,x] \\
\alpha\,[x,x] &= 0.
\end{aligned}$$

This would imply that $\alpha = 0$, which is a contradiction. If L is a right Leibniz algebra, by Equation (1.2) we obtain the same and this completes the proof. □

1.5.2 Two dimensional Leibniz algebras

C. Cuvier classified two-dimensional Leibniz algebras in [28] in 1994. In this theorem, we summarize this classification. We also propose a proof that is different from the one found in the original work of C. Cuvier.

Theorem 1.5.4. *Let L be a left Leibniz algebra over a field $\mathbb{F}$. If $\dim_{\mathbb{F}} L = 2$, with $L = \langle e_1, e_2 \rangle$, then L is isomorphic to one of the following algebras:*

$$\begin{aligned}
&L_1\colon\ \textit{abelian algebra} \\
&L_2\colon\ [e_1,e_2] = e_2,\ \textit{Lie algebra} \\
&L_3\colon\ [e_2,e_2] = e_1 \\
&L_4\colon\ [e_2,e_1] = [e_2,e_2] = e_1
\end{aligned}$$

Proof. If L is abelian, then $L = L_1$. So, let L be a left a non-abelian Leibniz algebra, with $L = \langle e_1, e_2 \rangle$ and the following brackets:

$$\begin{aligned}
[e_1,e_1] &= \alpha_1 e_1 + \alpha_2 e_2 & \qquad [e_1,e_2] &= \beta_1 e_1 + \beta_2 e_2, \\
[e_2,e_1] &= \gamma_1 e_1 + \gamma_2 e_2, & \qquad [e_2,e_2] &= \delta_1 e_1 + \delta_2 e_2,
\end{aligned}$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{F}$, $i = 1, 2$. By Proposition 1.3.14 we have $\mathrm{Leib}(L) \subseteq \mathrm{Z}_l(L)$ and $\dim_{\mathbb{F}} \mathrm{Z}_l(L) \leq 1$ since L is not abelian. If $\dim_{\mathbb{F}} \mathrm{Z}_l(L) = 0$, hence $\mathrm{Leib}(L) = 0$ and L is the two-dimensional non-abelian Lie algebra L_2. Now let $\dim_{\mathbb{F}} \mathrm{Z}_l(L) = 1$, then $\mathrm{Leib}(L) \neq 0$. If not, L would be a two-dimensional Lie algebra with center $\mathrm{Z}(L)$ of dimension 1 but this algebra does not exist. Without loss of generality, let $\mathrm{Leib}(L) = \mathrm{Z}_l(L) = \langle e_1 \rangle$, hence $\alpha_i = \beta_i = 0, i = 1, 2$ and we have

$$[e_2, e_1] = \gamma_1 e_1 + \gamma_2 e_2, \quad [e_2, e_2] = \delta_1 e_1.$$

By Equation (1.1) we have

$$\begin{aligned} [e_2,[e_1,e_2]] &= [[e_2,e_1],e_2] + [e_1,[e_2,e_2]] \\ [[e_2,e_1],e_2] &= 0 \\ \gamma_2\delta_1 e_1 &= 0. \end{aligned}$$

If $\delta_1 = 0$, then $\mathrm{Leib}(L) = 0$ and this is a contradiction. Hence we must have $\gamma_2 = 0$ and

$$[e_2,e_1] = \gamma_1 e_1, \quad [e_2,e_2] = \delta_1 e_1.$$

If $\gamma_1 = 0$ we obtain the symmetric Leibniz algebra$L_3 = \langle e'_1, e'_2\rangle$ with non-zero bracket $[e'_2, e'_2] = e'_1$ by the basis transformation $e'_1 = \delta_1 e_1$, and $e'_2 = e_2$. Otherwise, if $\gamma_1 \neq 0$, by the basis transformation $e'_1 = \frac{\delta_1}{\gamma_1^2} e_1$, and $e'_2 = \frac{1}{\gamma_1} e_2$ we obtain the left Leibniz L_4 with non-zero brackets

$$[e'_2,e'_1] = e'_1, \text{ and } [e'_2,e'_2] = e'_1.$$

□

Remark 1.5.1. The Leibniz algebra L_4 is only left, unlike of the symmetric Leibniz algebra L_3. Indeed, by a direct computation of right differential identity 1.2 for the triple $\{e_2, e_2, e_2\}$, we have

$$\begin{aligned} [[e_2,e_2],e_2] &= [[e_2,e_2],e_2] + [e_2,[e_2,e_2]] \\ [e_1,e_2] &= [e_1,e_2] + [e_2,e_1] \\ 0 &\neq e_1. \end{aligned}$$

1.5.3 Three-dimensional Leibniz algebras

A list of complex Leibniz algebras of dimension 3, up to isomorphism, can be found in [4] or [19]. A classification of these algebras over an arbitrary field of different characteristic than two is also known (see [70] and [6]). The latter classification is accomplished utilizing case-by-case considerations, with respect to isomorphism invariants such as $\dim_{\mathbb{F}} Z_l(L)$ and $\dim_{\mathbb{F}} L^k$.

Algebra	Multiplication Table	Classification
$L_1(\alpha)$ $\alpha \neq 0$ $\alpha \in \mathbb{C}$	$[e_1, e_1] = e_3, [e_1, e_2] = e_2 + e_3$ $[e_1, e_3] = \alpha e_3$	Solvable Leibniz
L_2	$[e_1, e_1] = e_3,\ [e_1, e_2] = e_2 + e_3$	Solvable Leibniz
L_3	$[e_1, e_2] = e_3, [e_1, e_3] = -2e_3$ $[e_2, e_3] = 2e_3$	Simple Lie
$L_4(\alpha)$	$[e_1, e_1] = e_3, [e_1, e_2] = -e_2$ $[e_1, e_3] = \alpha e_3, [e_2, e_1] = e_2$	Solvable Leibniz
L_5	$[e_1, e_1] = e_3, [e_1, e_3] = e_3$ $[e_1, e_2] = e_3$	Solvable Leibniz
L_6	$[e_1, e_3] = e_2, [e_1, e_1] = e_3$	Nilpotent Leibniz
L_7	$[e_1, e_2] = e_3, [e_1, e_3] = e_3$ $[e_2, e_2] = e_3, [e_2, e_3] = e_3$	Solvable Leibniz
L_8	$[e_1, e_1] = e_2, [e_1, e_2] = e_2$	Solvable Leibniz
$L_9(\alpha)$ $\alpha \neq 0, 1$ $\alpha \leftrightarrow \alpha^{-1}$	$[e_1, e_2] = e_2, [e_1, e_3] = \alpha e_3$	Solvable Lie
L_{10}	$[e_1, e_2] = e_2$	Solvable Lie
L_{11}	$[e_1, e_2] = e_2, [e_1, e_3] = e_2 + e_3$	Solvable Lie
$L_{12}(\alpha)$ $\alpha \in \mathbb{C}$	$[e_1, e_1] = \alpha e_3, [e_1, e_2] = e_3$ $[e_2, e_2] = e_3$	Nilpotent Leibniz
L_{13}	$[e_2, e_2] = e_1, [e_2, e_3] = e_1$ $[e_3, e_2] = e_1$	Nilpotent Leibniz
L_{14}	$[e_1, e_1] = e_3, [e_1, e_2] = e_2$ $[e_1, e_3] = e_3$	Solvable Leibniz
L_{15}	$[e_1, e_1] = e_2$	Composable, Nilpotent Leibniz
L_{16}	$[e_1, e_2] = e_2, [e_1, e_3] = e_3$	Solvable Lie
L_{17}	$[e_1, e_2] = e_3$	Nilpotent Lie
L_{18}	-	Abelian

TABLE 1.1: Isomorphism classes of three-dimensional Leibniz algebras.

Chapter 2

Two-step nilpotent algebras and their integration

The content of this chapter is predominantly derived from a collaborative paper in which I have co-authored [58].

Let us begin this chapter by demonstrating how it is possible to associate bilinear forms with the Leibniz bracket and how to reduce them to their canonical representation. Let L be a two-step nilpotent left Leibniz algebra and suppose that $\dim_{\mathbb{F}}[L, L] = t$, then let $\{z_1, \ldots, z_t\}$ be a basis of $[L, L]$. For any $x, y \in L$ we have

$$[x, y] = \alpha_1 z_1 + \cdots + \alpha_t z_t,$$

for suitable $\alpha_1, \ldots, \alpha_t \in \mathbb{F}$. We can thus associate, for all $i = 1, \ldots, t$, a bilinear form $\phi_i \colon L \times L \to \mathbb{F}$ for any $x, y \in L$, which maps (x, y) to the scalar α_i. It is noteworthy that if L were a Lie algebra, the bilinear forms defined above would be skew-symmetric. Let now L be a nilpotent Leibniz algebra with $\dim_{\mathbb{F}}[L, L] = 1$ with $[L, L] = \operatorname{span}_{\mathbb{F}}\{z\}$, where $z \in L$ is fixed. According to the above considerations, for any $x, y \in L$, $[x, y] = \phi(x, y)z$, where $\phi \colon L \times L \to \mathbb{F}$ is a bilinear form. We note that, if L is not a Lie algebra, then $\operatorname{Leib}(L) = [L, L] \subseteq \mathrm{Z}_l(L)$.

The bilinear form ϕ can be decomposed into its symmetric and skew-symmetric parts, that are respectively

$$\sigma = \frac{\phi + \phi^t}{2} \quad \text{and} \quad \alpha = \frac{\phi - \phi^t}{2},$$

where ϕ^t is the transpose of ϕ, that is $\phi^t(x, y) = \phi(y, x)$ for every $x, y \in L$. By fully classifying this pair of bilinear forms, it is possible to obtain a classification of this class of Leibniz algebras. Now our claim is to reduce simultaneously such pair of bilinear forms into a canonical form. Valuable contributions regarding this have been provided by Kronecker in [56] and Dieudonné in [34]. We recall here some definitions and some results from these works. A short overview on this subject can be found in [37].

2.1 Kronecker Modules

The problem of simultaneously reducing a pair of symmetric bilinear forms over a given field is a classical one. In 1868, K. Weierstrass initially solved this problem for fields of characteristic zero under the assumption that both forms are non-degenerate. K. Kronecker ([56], 1890) and L. E. Dickson ([33], 1909) have provided a complete answer for fields of characteristic zero. J. Williamson ([83], 1935 and [82], 1945) extended these results to fields of characteristic not equal to 2, requiring the condition that at least one form is non-degenerate. W. Waterhouse ([80], 1976) solved the cases where both forms are degenerate, in addition to the case of symmetric bilinear forms (even degenerate) over a field with characteristic 2 ([81], 1977).

P. Gabriel [42] and R. Scharlau [75] published papers in the 1970s highlighting the fundamental role J. Dieudonné's ([34], 1946) classification of pairs of linear mappings or Kronecker modules plays in the study of pairs of bilinear forms. R. Scharlau provided a complete answer for a pair of alternating bilinear forms, as noted by W. Waterhouse. The question of one form being symmetric and the other alternating has been addressed by various authors, including C. Riehm [71] and P. Gabriel [42], but their arguments no longer involve the theory of Kronecker modules.

Definition 2.1.1. A *Kronecker module over a field* $\mathbb{F}$ is a quadruple $\Phi = (V_1, V_2; f_1, f_2)$, where V_1, V_2 are vector spaces over $\mathbb{F}$ and $f_1, f_2 \colon V_1 \to V_2$ are linear maps. An *isomorphism* $\iota \colon \Phi \to \Psi$ from Φ onto the Kronecker module $\Psi = (W_1, W_2; g_1, g_2)$ is a pair of bijective linear mappings $\iota = (\iota_1, \iota_2)$, where $\iota_k \colon V_k \to W_k$, such that $\iota_2 f_k = g_k \iota_1$, with $k = 1, 2$.

Starting from a Kronecker module, it is possible to obtain two further Kronecker modules. The *opposite* Kronecker module of Φ is

$$\Phi^\circ := (V_1, V_2; f_2, f_1)$$

and the *transpose* Kronecker module of Φ

$${}^t\Phi := (V_2^*, V_1^*; {}^tf_1, {}^tf_2)$$

where we denote by ${}^tf$ the *transpose* of the linear map $f \colon V_1 \to V_2$, a linear map from the dual vector space V_2^* of V_2 into the dual vector space V_1^* of V_1, defined by

$${}^tf(x_2^*)(x_1) = x_2^*(f(x_1))$$

for all $x_1 \in V_1$ and $x_2^* \in V_2$. A Kronecker module Φ is *self-transpose* if there exists an isomorphism $\Phi \to {}^t\Phi$.

Decomposing a Kronecker module into the direct sum of indecomposable submodules allows for multiple possibilities, and the Krull-Remak-Schmidt Theorem holds true for two distinct decompositions (see [48] p.83, Theorem 3.3). This means that $\Phi = \Phi_1 \bigoplus \cdots \bigoplus \Phi_t$ for a fixed number t of indecomposable submodules Φ_i, determined up to permutations and isomorphisms. As previously mentioned, Kronecker and Dieudounné classified the Kronecker modules respectively in [56] and [34]. Let

$$\Phi_f(\mathbb{F}^n, \mathbb{F}^n; \mathrm{id}, f), n \geq 0, f \in \mathrm{End}_{\mathbb{F}}\mathbb{F}^n$$
$$\Phi_n(\mathbb{F}^n, \mathbb{F}^{n+1}; f_1, f_2), n \geq 0,$$

where

$$f_1 : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0) \quad (2.1)$$
$$f_2 : (x_1, \dots, x_n) \mapsto (0, x_1, \dots, x_n). \quad (2.2)$$

Then, for a suitable endomorphism $f \in \mathrm{End}_{\mathbb{F}}\mathbb{F}^n$ wich makes $\mathbb{F}^n$ an indecomposable $\mathbb{F}[f]$-module, an indecomposable Kronecker module is isomorphic to one of Φ_n, ${}^t\Phi_n$, Φ_f or Φ°. Note that Φ_f is self-transpose, while Φ_n is not. Therefore, Φ is self-transpose if and only if $\dim_{\mathbb{F}} V_1 = \dim_{\mathbb{F}} V_2$.

The Krull–Remak–Schmidt Theorem has the following useful corollary.

Corollary 2.1.2 (Exchange Theorem)**.** *Let* $\Phi = \Gamma_1 \bigoplus \Gamma_2$ *and* $\Phi = \overline{\Gamma}_1 \bigoplus \overline{\Gamma}_2$ *be two decompositions of* Φ*. If no indecomposable component* Γ_1 *(resp.* $\overline{\Gamma}_1$*) is isomorphic to any indecomposable component of* Γ_2 *(resp.* $\overline{\Gamma}_2$*), then* $\Phi = \Gamma_1 \bigoplus \overline{\Gamma}_2 = \Phi = \overline{\Gamma}_1 \bigoplus \Gamma_2$.

2.1.1 Kronecker modules associated to nilpotent Leibniz algebras L with $\dim_{\mathbb{F}} L' = 1$

Before associating a Kronecker module with a Leibniz algebra and, consequently, with the bilinear forms α and σ, it is necessary to see how this is done for two generic symmetric or skew-symmetric forms. Furthermore, we will present some helpful results for our purpose.

Let $f_1, f_2 \colon V \times V \to \mathbb{F}$ be a pair of bilinear forms defined on a $\mathbb{F}$-vector space V, each being symmetric or skew-symmetric. We can associate to the triple $\mathcal{F} = (V; f_1, f_2)$ the self-transpose Kronecker module $\Phi(\mathcal{F}) = (V, V^*; \bar{f}_1, \bar{f}_2)$, where, for $x \in V$, $\bar{f}_i(y)$ is the mapping $y \mapsto f_i(x, y)$, for $i = 1, 2$. For a nontrivial subspace U of V we can set

$$U^\perp = \{v \in V \mid f_1(u, v) = f_2(u, v) = 0 \text{ for any } u \in U\},$$

that is the orthogonal space of U with respect to both f_1 and f_2.

Definition 2.1.3. The triple $\mathcal{F}$ is *decomposable* if $V = U \oplus U^\perp$.

Manifestly, any decomposition of V into the direct sum of two subspaces U_1 and U_2, orthogonal with respect to both f_1 and f_2, provides a decomposition of $\Phi(\mathcal{F})$. The converse, in general, is not true. The canonical identification of $V = V^{**}$ implies the subsequent identification $\Phi(\mathcal{F}) = {}^t\Phi(\mathcal{F})$. Therefore, the number of components of $\Phi(\mathcal{F})$ isomorphic to Φ_n is the same as the ones isomorphic to ${}^t\Phi_n$. This leads to a decomposition of ${}^t\Phi$ into self-transpose submodules, with no isomorphic components in common. Consequently, this provides an orthogonal decomposition of V, as claimed by the following lemma.

Lemma 2.1.4. *Let* $\Phi(\mathcal{F}) = \Gamma_1 \bigoplus \Gamma_2$ *with self-transpose* Γ_i, $i = 1, 2$*. Assume that no component* Γ_1 *is isomorphic to any component of* Γ_2*, then* $\mathcal{F}$ *decomposes.*

In view of the above lemma, indecomposable $\mathcal{F}$ correspond to Kronecker modules $\Phi(\mathcal{F})$ isomorphic to either $(\Phi_f)^r$ or $(\Phi_f^\circ)^r$, or $(\Phi_n)^s \bigoplus ({}^t\Phi_n)^s$. Moreover, according to [80], [81], [71], and [75], direct computations on the bases show that $s = 1$ and

- $r = 1$ for an indecomposable pair of symmetric forms,
- $r = 1, 2$ for an indecomposable pair, where one is symmetric and the other is alternating,
- $r = 1$ for an indecomposable pair of alternating bilinear forms.

There we have the following result.

Theorem 2.1.5. *Let $\mathcal{F}$ be indecomposable. Then the Kronecker module $\Phi(\mathcal{F})$ is isomorphic to either Φ_f or Φ_f°, or $\Phi_n \bigoplus {}^t\Phi_n$.*

Finally, we present a result proven by both Scharlau [75] and Waterhouse [80], [81]. The former proved it for pairs of alternating forms, while the latter, using different techniques, proved it for pairs of symmetric forms.

Theorem 2.1.6. *Let $\mathcal{F}$ be an indecomposable pair of degenerate bilinear forms on a $\mathbb{F}$-vector space V, each being symmetric or alternating. Then, V has odd dimension $2n+1$ over $\mathbb{F}$ and $\mathcal{F}$ has representation*

$$\begin{pmatrix} D_1 & J_1 \\ (\pm 1)J_1^t & 0 \end{pmatrix}, \begin{pmatrix} D_2 & J_2 \\ (\pm 1)J_2^t & 0 \end{pmatrix}$$

for suitable diagonal matrices D_1, D_2, where

$$J_1 = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{pmatrix} \text{ and } J_2 = \begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

Moreover, if the characteristic of $\mathbb{F}$ is not 2, there exists a representation with $D_1 = D_2 = 0$.

Now let σ and α be the pair of bilinear forms defined above. We observe that, if $v \in U^\perp$ and the characteristic of $\mathbb{F}$ is not 2, then $\phi(u, v) = 0$. Generally, the converse is not true.

We will now demonstrate how to associate a Kronecker module to a Leibniz algebra L, considering the aforementioned results. Let L^* the dual vector space of L and, for every $x, y \in L$, let $\bar{\alpha}(x)$ and $\bar{\sigma}(y)$ be the linear maps defined by

$$\bar{\alpha}(x) : z \mapsto \alpha(x, z), \quad \bar{\sigma}(y) : z \mapsto \sigma(y, z), \ \forall z \in L.$$

From the previous results, the indecomposable module $(L, L^*, \bar{\alpha}, \bar{\sigma})$ turns out to be one of the following three pairs:

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix} \tag{2.3}$$

$$\begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \tag{2.4}$$

$$\begin{pmatrix} 0 & J_1 \\ -J_1^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & J_2 \\ J_2^t & 0 \end{pmatrix} \tag{2.5}$$

where $A \in \mathrm{M}_n(\mathbb{F})$, and J_1, J_2 are the matrices associated with the linear applications 2.1. The following outcome provides a helpful instrument in the classification of indecomposable nilpotent Leibniz algebras featuring a one-dimensional derived subalgebra.

Proposition 2.1.7. *Let L_1 and L_2 be Leibniz algebras of dimension n with one-dimensional commutator ideals $[L_1, L_1] = \mathbb{F}z_1$ and $[L_2, L_2] = \mathbb{F}z_2$, and let ϕ_1 and ϕ_2 be the bilinear forms associated with L_1 and L_2 respectively. One can fix bases of L_1 and L_2 such that, if Φ_1 and Φ_2 are the matrices of ϕ_1 and ϕ_2 respectively, then L_1 is isomorphic to L_2 if and only if Φ_1 is congruent to Φ_2.*

Proof. Let $\varphi \colon L_1 \to L_2$ be a Leibniz algebras isomorphism and let $\{e_1, \dots, e_{n-1}, z_1\}$ be a basis of L_1. Then $\varphi(z_1) = kz_2$, for some $k \in \mathbb{F}^*$, and $\{\varphi(e_1), \dots, \varphi(e_{n-1}), kz_2\}$ is a basis of L_2 such that the associated matrix is Φ_1. Then there exists a matrix $P \in \mathrm{GL}_n(\mathbb{F})$ such that $P\Phi_2P^t = \Phi_1$.

Conversely, we suppose that there exists $P \in \mathrm{GL}_n(\mathbb{F})$ such that $P\Phi_1P^t = \Phi_2$. Φ_1 and Φ_2 are matrices associated with bilinear forms, so P induces a change of basis $\{e_1, \dots, e_{n-1}, z_1\} \to \{\overline{e_1}, \dots, \overline{e_{n-1}}, kz_1\}$, with $k \in \mathbb{F}^*$, of L_1. Thus the isomorphism between L_1 and L_2 is given by the linear map

$$\varphi(\overline{e_i}) = e'_i, \ \forall i = 1, \dots, n-1, \ \varphi(kz_1) = z_2,$$

where $\left\{e'_1, \dots, e'_{n-1}, z_2\right\}$ is a basis of L_2. □

Lemma 2.1.8. *If $A, B \in \mathrm{M}_n(\mathbb{F})$ are similar matrices, then there exists a change of basis that transforms the pairs 2.3 and 2.4 respectively in*

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & B \\ -B^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

Proof. Let $X \in \mathrm{GL}_n(\mathbb{F})$ such that $B = XAX^{-1}$. Then the matrix

$$\begin{pmatrix} X & 0 \\ 0 & \left(X^{-1}\right)^t \end{pmatrix}$$

induces a change of basis that transforms the canonical pairs 2.3 and 2.4 respectively in

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \begin{pmatrix} 0 & XAX^{-1} \\ \left(XAX^{-1}\right)^t & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & XAX^{-1} \\ -\left(XAX^{-1}\right)^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

□

The first two canonical pairs showed above may be presented in a simpler form. If $A \in \mathrm{M}_n(\mathbb{F})$ is not the zero matrix and L is indecomposable, A is the matrix of a power of a monic irreducible polynomial $f(x) \in \mathbb{F}[x]$ (see [50] or Appendix A).

The next result states that, under a certain condition, the canonical pairs 2.3 and 2.4 can be transformed into each other. Before proceeding, we should recall a definition of the congruence of n-tuples of matrices ([65], Chapter VI).

Definition 2.1.9. Let $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_k)$ be two k-tuples of matrices, with $A_j, B_j \in \mathrm{M}_n(\mathbb{F})$ for all $j = 1, \ldots, k$. A and B are called *congruent* provided there is a non-singular matrix P with $B_j = PA_jP^t$ for $j = 1, \ldots, k$.

Lemma 2.1.10. *Let $A \in \mathrm{M}_n(\mathbb{F})$ be a non-singular matrix. Then the pair 2.3 is congruent to the pair 2.4.*

Proof. It is sufficient to show that

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & I_n \end{pmatrix}^t = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$
$$\begin{pmatrix} A^{-1} & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & I_n \end{pmatrix}^t = \begin{pmatrix} 0 & A^{-1} \\ (A^{-1})^t & 0 \end{pmatrix}.$$

□

Otherwise, if we represent a singular matrix as an $n \times n$ Jordan block with an eigenvalue zero, we obtain a unique Kronecker module of type 2.4 up to isomorphisms.

Definition 2.1.11. Let $f(x) \in \mathbb{F}[x]$ be a monic irreducible polynomial. Let $k \in \mathbb{N}$ and let A be the companion matrix of $f(x)^k$. We define the *Heisenberg* Leibniz algebra $\mathfrak{l}_{2n+1}^A$ as the $(2n+1)$-dimensional indecomposable Leibniz algebra with associated Kronecker module of type 2.3.

In general, for $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{F})$ and suitable basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n, h\}$ of $\mathfrak{l}_{2n+1}^A$ the list of non-trivial commutators between basis elements amounts to

$$[e_i, f_j] = (\delta_{ij} + a_{ij})\, h, \;\; [f_j, e_i] = (-\delta_{ij} + a_{ij})\, h, \;\; \forall i, j = 1, \ldots, n,$$

so we can associate with $\mathfrak{l}_{2n+1}^A$ the following structure matrix

$$\left(\begin{array}{c|c|c} 0 & I_n + A & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline -I_n + A^t & 0 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & 0 \cdots 0 & 0 \end{array}\right)$$

Notice that, if (a_{ij}) is the zero matrix, then we obtain the classical Heisenberg algebra $\mathfrak{h}_{2n+1}$.

Definition 2.1.12. Let $n \in \mathbb{N}$ and let A be the companion matrix of the polynomial x^n. We define the *Kronecker* Leibniz algebra $\mathfrak{k}_n$ as the $(2n+1)$-dimensional indecomposable Leibniz algebra with associated Kronecker module of type 2.4.

Definition 2.1.13. We define the *Dieudonné* Leibniz algebra $\mathfrak{d}_n$ to be the $(2n+2)-$dimensional Leibniz algebra with associated Kronecker module of type 2.5.

Remark 2.1.1. For every $n \in \mathbb{N}$, the Kronecker Leibniz algebra $\mathfrak{k}_n$ and the Dieudonné Leibniz algebra $\mathfrak{d}_n$ are not Lie algebras and they are unique up to isomorphism, because of the unicity of the Kronecker modules of type 2.4 and 2.5.

2.2 Complex and Real Heisenberg Leibniz algebras

Now we want to describe in detail the indecomposable Heisenberg Leibniz algebras in the case the field $\mathbb{F}$ is $\mathbb{C}$ or $\mathbb{R}$.

2.2.1 The case $\mathbb{F} = \mathbb{C}$

Let $k \in \mathbb{N}$ and let $f(x) = x - a \in \mathbb{C}[x]$. Then the companion matrix of $f(x)^k$ is

$$A = \begin{pmatrix} 0 & \cdots & \cdots & 0 & -c_k \\ 1 & \ddots & & \vdots & -c_{k-1} \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -c_1 \end{pmatrix} \in \mathrm{M}_k(\mathbb{C}),$$

where $c_j = \binom{k}{j-1}(-a)^{k-j+1}$, for every $j = 1, \ldots, k$. In this case, however, it is more convenient to use the Jordan canonical form. Indeed, it is well known that the matrix A is similar to the $k \times k$ Jordan block of eigenvalue a

$$J_a = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 1 & a \end{pmatrix}.$$

Thus $\mathfrak{l}_{2k+1}^{A} \cong \mathfrak{l}_{2k+1}^{J_a}$ and the Leibniz bracket is given by

$$\begin{aligned}
[e_1, f_j] =& \delta_{1,j}(1+a)h \\
[e_i, f_j] =& (\delta_{i,j}(1+a) + \delta_{i-1,j})h, \ \forall i = 2, \dots, k; \\
[f_j, e_i] =& (\delta_{i,j}(-1+a) + \delta_{i,j+1})h, \ \forall j = 1, \dots, k-1 \\
[f_k, e_i] =& \delta_{i,k}(-1+a)h,
\end{aligned}$$

where $\{e_1, \dots, e_n, f_1, \dots, f_n, h\}$ is a basis of $\mathfrak{l}_{2k+1}^{J_a}$.

Proposition 2.2.1. *Let $a \in \mathbb{C}$. The Heisenberg-Leibniz algebras $\mathfrak{l}_{2k+1}^{J_a}$ and $\mathfrak{l}_{2k+1}^{J_{-a}}$ are isomorphic.*

Proof. The algebras $\mathfrak{l}_{2k+1}^{J_a}$ and $\mathfrak{l}_{2k+1}^{-J_a^t}$ are isomorphic via the linear map φ defined by

$$\varphi(e_i) = f_i', \quad \varphi(f_i) = e_i', \quad \varphi(h) = -h', \quad \forall i = 1, \cdots, n$$

where $\{e_1, \cdots, e_n, f_1, \cdots, f_n, h\}$ and $\{e_1', \cdots, e_n', f_1', \cdots, f_n', h'\}$ are bases of $\mathfrak{l}_{2k+1}^{J_a}$ and $\mathfrak{l}_{2k+1}^{-J_a^t}$ respectively. Moreover, the matrix $-J_a^t$ is similar to the $n \times n$ Jordan block J_{-a}.

Thus $\mathfrak{l}_{2k+1}^{J_a} \cong \mathfrak{l}_{2k+1}^{J_{-a}}$. □

Regarding the quest for a necessary and sufficient condition for the isomorphism class of Heisenberg Leibniz algebras, the problem remains unsolved. In Section 3.2, we will delve into recent achievements made in this direction. However, it is crucial to emphasize that when $k = 1$, the problem has been completely resolved.

Proposition 2.2.2. *Let $a, a' \in \mathbb{C}$. The Heisenberg Leibniz algebras $\mathfrak{l}_3^{a}$ and $\mathfrak{l}_3^{a'}$ are isomorphic if and only if $a' = \pm a$.*

Proof. It is easy to check that the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

defines a Leibniz algebras isomorphism φ between $\mathfrak{l}_3^{a}$ and $\mathfrak{l}_3^{-a}$. Indeed, the determinant of the above matrix is 1,

$$\begin{aligned}
\varphi([e_1, f_1]) &= \varphi((1+a)z) = -(1+a)z' \\
[\varphi(e_1), \varphi(f_1)] &= [f_1', e_1'] = (-1-a)z'
\end{aligned}$$

and

$$\begin{aligned}
\varphi([f_1, e_1]) &= \varphi((-1+a)z) = (1-a)z' \\
[\varphi(f_1), \varphi(e_1)] &= [e_1', f_1'] = (1-a)z'.
\end{aligned}$$

Conversely, let $\varphi : \mathfrak{l}_3^{a} \to \mathfrak{l}_3^{a'}$ be a Leibniz algebra isomorphism defined by

$$\varphi(x) = \alpha x' + \beta y' + \gamma z', \ \varphi(y) = \alpha' x' + \beta' y' + \gamma' z', \ \varphi(z) = kz',$$

where $\{x, y, z\}$ and $\{x', y', z'\}$ are basis of $\mathfrak{l}_3^a$ and $\mathfrak{l}_3^{a'}$ respectively. Thus

$$0 = \varphi([x, x]) = [\varphi(x), \varphi(x)] = \alpha\beta\,(1 + a' - 1 + a')\, z' = 2\alpha\beta a' z',$$

$$0 = \varphi([y, y]) = [\varphi(y), \varphi(y)] = \alpha'\beta'\,(1 + a' - 1 + a')\, z' = 2\alpha'\beta' a' z',$$

$$k\,(1 + a)\, z' = \varphi([x, y]) = [\varphi(x), \varphi(y)] = (\alpha'\beta\,(1 + a') + \alpha\beta'\,(-1 + a'))\, z',$$

$$k\,(-1 + a)\, z' = \varphi([y, x]) = [\varphi(y), \varphi(x)] = (\alpha\beta'\,(1 + a') + \alpha'\beta\,(-1 + a'))\, z'.$$

We have that $(\alpha', \beta) = (0, 0)$ or $(\alpha, \beta') = (0, 0)$. In the first case φ is the identity map and $a = a'$. In the second case φ is defined by $\varphi(x) = y'$, $\varphi(y) = x'$ and $\varphi(z) = -z'$, thus $a' = -a$. □

2.2.2 The case $\mathbb{F} = \mathbb{R}$

Irreducible polynomials in $\mathbb{R}[x]$ have degree one or two. Let $f(x) \in \mathbb{R}[x]$ be an irreducible monic polynomial. If $f(x) = x - a$, then we obtain the same results of the previous case. So we suppose that $f(x) = x^2 + bx + c$, with $b^2 - 4c < 0$.

Let $z = \alpha + i\beta \in \mathbb{C}$ be a root of $f(x)$. Then $f(x) = (x - z)(x - \bar{z})$ and the companion matrix A of $f(x)^k$ in similar to the $2k \times 2k$ real block matrix

$$J_R = \begin{pmatrix} R & 0 & \cdots & 0 \\ I_2 & R & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I_2 & R \end{pmatrix},$$

where

$$R = R_{\alpha,\beta} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

is the realification of the complex number z (see Appendix A). Thus $\mathfrak{l}_{4k+1}^A \cong \mathfrak{l}_{4k+1}^{J_R}$ and the structure matrix is given by

$$\left(\begin{array}{c|c|c} 0 & I_n + J_R & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline -I_n + J_R^t & 0 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & 0 \cdots 0 & 0 \end{array}\right)$$

In the case that $k = 1$, the real Heisenberg Leibniz algebra $\mathfrak{l}_5^R$ is the realification of the complex algebra $\mathfrak{l}_3^z$. Thus we can conclude that

Proposition 2.2.3. *Let $f(x), g(x) \in \mathbb{R}[x]$ be two irreducible monic polynomials of degree two and let $z, z' \in \mathbb{C}$ be roots of $f(x)$ and $g(x)$ respectively. Let $R, R' \in \mathrm{M}_2(\mathbb{R})$ be the realification of the complex numbers z and z'. Then $\mathfrak{l}_5^R \cong \mathfrak{l}_5^{R'}$ if and only if $R' = \pm R$.*

Proof. The algebras $\mathfrak{l}_5^R$ and $\mathfrak{l}_5^{R'}$ are the realification of the complex Heisenberg Leibniz algebras $\mathfrak{l}_3^z$ and $\mathfrak{l}_3^{z'}$ respectively. From *Proposition 3.2* we know that $\mathfrak{l}_3^z \cong \mathfrak{l}_3^{z'}$ if and only if $z = \pm z'$. Moreover, these are $\mathbb{R}$-linear isomorphisms because the matrix associated with the isomorphism $\varphi : \mathfrak{l}_3^z \leftrightarrows \mathfrak{l}_3^{-z}$ is the rotation

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathrm{SO}(3).$$

Thus $\mathfrak{l}_5^R \cong \mathfrak{l}_5^{R'}$ if and only if $R = \pm R'$. □

2.3 Integration of two-step nilpotent Leibniz algebra

The "coquecigrue problem" was initially proposed by J.-L. Loday in [62]. Its aim is to find algebraic structures that play a similar role for Leibniz algebras as Lie groups play for Lie algebras. In essence, the goal is to extend the validity of the Lie's third theorem to Leibniz algebras. Here we present a modern and global version of the Lie's third theorem. It is stated that for each finite-dimensional real Lie algebra $\mathfrak{g}$, there is a Lie group G with a Lie algebra isomorphic to $\mathfrak{g}$ (refer to [45] for further details).

Theorem 2.3.1 (Lie's third theorem)**.** *If $\mathfrak{g}$ is any finite-dimensional, real Lie algebra, there exists a connected Lie subgroup G of $\mathrm{GL}_n(\mathbb{C})$ whose Lie algebra is isomorphic to $\mathfrak{g}$.*

The group-like objects will be referred to as "coquecigrues" with precise definition, but their properties remain largely unknown [62]. Past efforts to address the coquecigrue problem have involved the study of non-associative multiplications defined on reductive homogeneous spaces associated with Leibniz algebras [55]. However, these methods did not offer a satisfactory solution, especially when the Leibniz algebra is a Lie algebra. After studying the proof that the tangent space at the unit element of a Lie group forms a Lie algebra, it becomes clear that the essential properties related to the Jacobi identity lie in conjugation rather than in group multiplication. This observation leads to the study of *Lie racks*, which are manifolds equipped with a smooth left distributive binary operation. These algebraic structures extend the concept of Lie groups. Unless stated otherwise, the real numbers are considered as the underlying field of any vector space.

2.3.1 Lie Racks

The concept of a *rack* was initially presented in 1992 by R. Fenn and C. Rourke [41] to provide a thorough and sophisticated algebraic structure for studying links, knots and 3-manifolds. Numerous scholars have since studied racks, frequently employing diverse terminology to describe them.

In the following discussion, we will present only those concepts that are relevant to racks; for the sake of conciseness, definitions of operator group, associated group, orbits, stabiliser, and the like will be excluded.

Definition 2.3.2. A (left) *rack* is a set X with a binary operation $\rhd : X \times X \to X$ such that

- for all $x, y, z \in X$, $x \rhd (y \rhd z) = (x \rhd y) \rhd (x \rhd z)$ (left distributivity)
- for all $x, z \in X$ there is a unique $y \in X$ such that $z = x \rhd y$.

A rack is said to be *pointed* if there exists an element $1 \in X$, called the unit, such that $1 \rhd x = x$ and $x \rhd 1 = 1$ for all $x \in X$.

The two preceding conditions can be substituted with equivalent versions that may be more useful, depending on the context in which they are used. The second axiom is equivalent to saying that the function $x \rhd - : X \times X \to X$ (left multiplication) is a bijection for all $x \in X$. Moreover, the two axioms together are equivalent to the statement that the multiplication on the left is an automorphism. To be more precise, let $\mathrm{Aut}(X)$ denote the set of permutations of a magma $(X, \rhd)$ that preserve $\rhd$. A permutation ψ of X is in $\mathrm{Aut}(X)$ if and only if $\psi(x \rhd y) = \psi(x) \rhd \psi(y)$ for all $x, y \in X$. If we denote the left translations in a rack $(X, \rhd)$ by $\phi(x)y := x \rhd y$, then the left distributive axiom simply asserts that $\phi(x) \in \mathrm{Aut}(X)$ for all $x \in X$. When defining new algebraic structures, their morphisms are also defined.

Definition 2.3.3. A *pointed rack homomorphism* is a map $f : X \to Y$ such that $f(x \rhd y) = f(x) \rhd f(y)$, for all $x, y \in X$ and such that $f(1_X) = 1_Y$.

Below are a few examples of racks. The first of these examples, the *conjugation rack*, is the most prevalent and the motivations lie in what was mentioned above regarding the properties related to the Jacobi identity.

Example 2.3.3.1 (*Conjugation Rack*)**.** Let G be a group. The conjugation G defines a rack operation on G. That is, we set $x \rhd y := x^{-1}yx$ and the left multiplication is manifestly an automorphism. Then $(G, \rhd)$ is the conjugation rack of G and it is denoted by $\mathrm{Conj}(G)$. In other words, every group endowed with the conjugation is a pointed rack, so there is a functor $\mathrm{Conj} : \mathbf{Grp} \to \mathbf{Rack}$, between the category $\mathbf{Grp}$ of groups and the category $\mathbf{Rack}$ of racks.

This functor has a left adjoint $\mathrm{As} : \mathbf{Rack} \to \mathbf{Grp}$ defined by

$$\mathrm{As}(X) = \mathrm{F}(X)/\overline{\langle\{xyx^{-1}\,(x \rhd y^{-1}) \mid xy \in X\}\rangle},$$

where $\mathrm{F}(X)$ is the free group generated by X.

Example 2.3.3.2 (*Dihedral Rack*)**.** Let $R_n = \{\rho_i\}_{i=1,\ldots,n-1}$ be the set of the reflections in the dihedral group D_n, the group of symmetries of a regular n-gon. That is, $D_n = \{r_0 = 1, r_1, \ldots, r_{n-1}, s, sr_1, \ldots, sr_{n-1}\}$ where, for every $i = 0, \ldots, n-1$, r_i are the rotations, s is the reflection across a line through a vertex and $\rho_i = sr_i$ are the reflections of the n-gon. Then R_n forms a rack of order n. Since $\rho_i\rho_j = r_{i-j \pmod n}$ for all $i, j = 1, \ldots, n-1$, then

$$\begin{aligned}\rho_i \triangleright \rho_j &= \rho_i^{-1}\rho_j\rho_i = \rho_i^{-1}r_{i-j} = r_i^{-1}s^{-1}r_{i-j \pmod n} \\ &= r_i^{-1}sr_{i-j \pmod n} = sr_is^{-1}rr_{i-j \pmod n} \\ &= sr_isr_{i-j \pmod n} = \rho_i\rho_{i-j \pmod n} = \rho_{2i-j \pmod n}.\end{aligned}$$

For this reason, the dihedral rack is often referred to as $R_n = \{0, 1, \ldots, n-1\}$ with $i \triangleright j = 2i - j \pmod n$ ([49], [73]).

Example 2.3.3.3 (*Reflection Rack*)**.** Let P, Q be points of the plane and define $Q \triangleright P$ to be the point P reflected in Q. Since Q is the midpoint of the segment from P to $Q \triangleright P$, in vector notation we have $Q \triangleright P = 2Q - P$. Then, for all Q, P_1, P_2 on the plane, we have on one hand

$$Q \triangleright (P_1 \triangleright P_2) = Q \triangleright (2P_1 - P_2) = 2Q - (2P_1 - P_2) = 2(Q - P_1) + P_2$$

and,on the other hand,

$$\begin{aligned}(Q \triangleright P_1) \triangleright (Q \triangleright P_2) &= (2Q - P_1) \triangleright (2Q - P_2) = 2(2Q - P_1) - (2Q - P_2) \\ &= 4Q - 2P_1 - 2Q + P_2 = 2(Q - P_1) + P_2.\end{aligned}$$

The Figure 2.1 shown below proves the same statement.

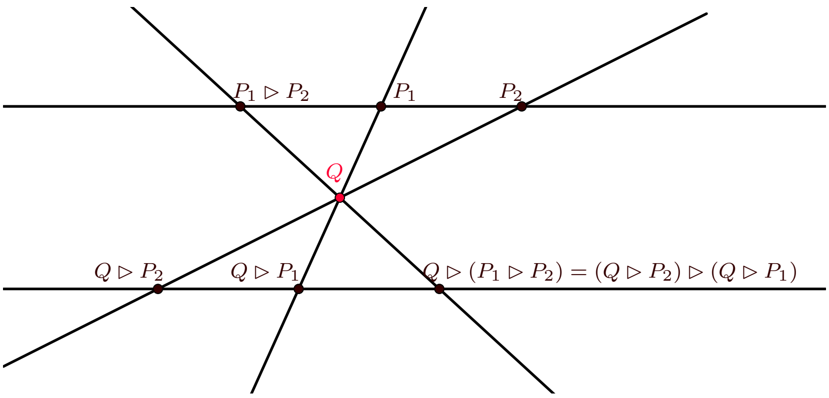


FIGURE 2.1: Reflection rack.

Example 2.3.3.4 (*Alexander Rack*)**.** Let R be the ring of Laurent polynomials $\mathbb{Z}[t, t^{-1}]$ in the variable t. Given elements $a, b \in M$, where M is an R-module, we can define $a \triangleright b$ as $tb + (1-t)a$, where t is the variable in the Laurent polynomial ring $R = \mathbb{Z}[t, t^{-1}]$. This operation makes M into a rack. For example, letting M be the plane and the action of t multiplication by -1, yelds the reflection rack of the previous example.

Now let us provide a definition that will be useful down the line.

Definition 2.3.4. A rack $(X, \rhd)$ is a *quandle* if $x \rhd x = x$, for every $x \in X$.

Informally, a rack is a quandle if the rack multiplication $\rhd$ is idempotent. In the few examples above, each rack shown is a quandle. This term is due to D. Joyce [52]. In order to solve the "coquecigrue problem", the next definition extends the notion of Lie group, as racks extend the notion of groups.

Definition 2.3.5. A *Lie rack* is a pointed rack $(X, \rhd, 1)$ such that X is a smooth manifold, $\rhd$ is a smooth map and such that for all $x \in X$ $x \rhd -$ is a diffeomorphism.

If G is a Lie group in Example 2.3.3.1, then $(G, \rhd)$ is a Lie rack. Here is another example.

Example 2.3.5.1 (*Linear Lie Rack*)**.** Let H be a Lie group and let V be an H-module. On $X := V \times H$, define a binary operation $\rhd$ by

$$(u, A) \rhd (v, B) := (Av, ABA^{-1}) \tag{2.7}$$

for all $u, v \in V$, $A, B \in H$. Setting $\mathbf{1} = (1, 0)$, we have that $(X, \rhd, \mathbf{1})$ is Lie rack. Indeed, the binary operation $\rhd : X \times X \to X$ is manifestly a smooth map. Then for any $(u, A) \in X$, the map $\phi(u, A) : X \to X$ which sends (v, B) to $(u, A) \rhd (v, B)$ is invertible with smooth inverse[1] $\phi^{-1}(u, A) = \phi(u, A^{-1})$.
We will demonstrate that the right self-distributivity condition holds true. Let $(u, A), (v, B), (w, C) \in X$. Then we have

$$\begin{aligned}(u, A) \rhd ((v, B) \rhd (w, C)) &= (u, A) \rhd (Bw, BCB^{-1})\\ &= (A(Bw), A(BCB^{-1})A^{-1}),\end{aligned}$$

and

$$\begin{aligned}((u, A) \rhd (v, B)) \rhd ((u, A) \rhd (w, C)) &= (Av, ABA^{-1}) \rhd (Aw, ACA^{-1})\\ &= (ABA^{-1}(Aw), ABA^{-1}ACA^{-1}(ABA^{-1})^{-1})\\ &= (A(Bw), A(BCB^{-1})A^{-1}).\end{aligned}$$

M. K. Kinyon showed in [54] that for every Lie rack X there exists a distinct tangent space T_1Q with its unique algebraic structure. In his approach, the basic concept involves differentiating the conjugation operation to obtain the adjoint

[1]Let $x = (u, A) \rhd (v, B) \in X$. Indeed, we have

$$\begin{aligned}\phi(u, A^{-1})(x) &= (u, A^{-1}) \rhd ((u, A) \rhd (v, B))\\ &= (u, A^{-1}) \rhd (Av, ABA^{-1})\\ &= (A^{-1}(Av), A^{-1}(ABA^{-1})A)\\ &= (v, B),\end{aligned}$$

and then

$$\phi^{-1}(u, A)(x) = (v, B).$$

representation of the group, and then performing a secondary differentiation to establish a mapping, which is then used to define the Lie bracket. This process leads to the adjoint representation of the Lie algebra. We will now try to summarize this process.

Suppose that R is a Lie rack. Let $\phi(x)$ the automorphism of R, for each $x \in R$, defined by $\phi(x)y = x \rhd y$, for all $y \in R$. Then $\phi(x)1 = x \rhd 1 = 1$ since that rack R is pointed. So we may apply the tangent functor T_1 to $\phi(x)\colon R \to R$ to obtain a linear mapping $\Phi(x) := T_1\phi(x)\colon T_1R \to T_1R$. Since $\phi(x)$ is invertible for each $x \in R$, we have each $\Phi(x) \in \mathrm{GL}(T_1R)$. Now the mapping $\Phi\colon R \to \mathrm{GL}(T_1R)$ satisfies $\Phi(1) = I$, where $I \in \mathrm{GL}(T_1R)$ is the identity mapping. Thus we may differentiate again to obtain a mapping $\mathrm{ad}\colon T_1R \to \mathfrak{gl}(T_1R)$, that is

$$\mathrm{ad}(X)Y = \left.\frac{d}{dt}\right|_{t=0} \Phi(\gamma(t))(Y),$$

for all $X, Y \in T_1R$, where $\gamma\colon]-\epsilon, \epsilon[\to R$ is a smooth path in R such that $\gamma(0) = 1$ and $\gamma'(0) = X$. As usual we identify the tangent space at the identity element of $\mathrm{GL}(V)$ for a vector space V with the general linear algebra $\mathfrak{gl}(V)$.

Now we set

$$[X, Y] := \mathrm{ad}(X)Y$$

for all $X, Y \in T_1R$. In terms of the left multiplication $\phi(x)$, the left distributive property of racks can be expressed by the equation

$$\phi(x)\phi(y)z = \phi(\phi(x)y)\phi(x)z. \tag{2.8}$$

Let $X, Y, Z \in T_1R$, and let γ_X, γ_Y, and γ_Z smooth paths in R such that

$$\gamma_X(0) = \gamma_Y(0) = \gamma_Z(0) = 1$$

and

$$\left.\frac{\partial}{\partial r}\right|_{r=0} \gamma_X(r) = X, \left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_Y(s) = Y, \left.\frac{\partial}{\partial t}\right|_{t=0} \gamma_Z(t) = Z.$$

In order to study the algebraic structure of T_1R we differentiate Equation (2.8) at $1 \in R$, first respect to z (then respect to t in $t = 0$), then with respect to y (then respect to s in $s = 0$) and finally respect to x (then respect to r in $r = 0$). Let us start with the left side of Equation (2.8), then we have

$$\begin{aligned}\left.\frac{\partial}{\partial t}\right|_{t=0} \gamma_X(r) \rhd (\gamma_Y(s) \rhd \gamma_Z(t)) &= \left.\frac{\partial}{\partial t}\right|_{t=0} \phi(\gamma_X(r))\phi(\gamma_Y(s))\gamma_Z(t) \\ &= \Phi(\gamma_X(r))\Phi(\gamma_Y(s))Z.\end{aligned}$$

By differentiating with respect to y we have

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \Phi(\gamma_X(r))\Phi(\gamma_Y(s))Z = \Phi(\gamma_x(r))\mathrm{ad}(Y)Z,$$

and by differentiating with respect to x we finally have

$$\left.\frac{\partial}{\partial r}\right|_{r=0} \Phi(\gamma_X(r))\mathrm{ad}(Y)Z = \mathrm{ad}(X)\mathrm{ad}(Y)(Z). \tag{2.9}$$

Now, if we consider the right side of Equation (2.8) equation, we have

$$\begin{aligned}\left.\frac{\partial}{\partial t}\right|_{t=0} (\gamma_X(r) \triangleright \gamma_Y(s)) \triangleright (\gamma_X(r) \triangleright \gamma_Z(t)) &= \left.\frac{\partial}{\partial t}\right|_{t=0} \phi(\gamma_X(r) \triangleright \gamma_Y(s))\phi(\gamma_X(r))\gamma_Z(t) \\ &= \Phi(\gamma_X(r) \triangleright \gamma_Y(s))\Phi(\gamma_X(r))Z \\ &= \Phi(\phi(\gamma_X(r))\gamma_Y(s))\Phi(\gamma_X(r))Z.\end{aligned}$$

By differentiating with respect to y we have

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \Phi(\phi(\gamma_X(r))\gamma_Y(s))\Phi(\gamma_X(r))Z = \mathrm{ad}(\Phi(\gamma_X(r))Y)\Phi(\gamma_X(r))Z.$$

We remind that if $r = 0$, then $\gamma_X(0) = 1$ and $\Phi(1) = 1$. By differentiating with respect to x we finally have

$$\left.\frac{\partial}{\partial r}\right|_{r=0} \mathrm{ad}(\Phi(\gamma_X(r))Y)\Phi(\gamma_X(r))Z = \mathrm{ad}(\mathrm{ad}(X)Y)Z + \mathrm{ad}(Y)\mathrm{ad}(X)Z \tag{2.10}$$

Therefore, by comparing equations 2.10 and 2.9 we obtain

$$\mathrm{ad}(X)\mathrm{ad}(Y)(Z) = \mathrm{ad}(\mathrm{ad}(X)Y)Z + \mathrm{ad}(Y)\mathrm{ad}(X)Z,$$

that is equivalent to

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]].$$

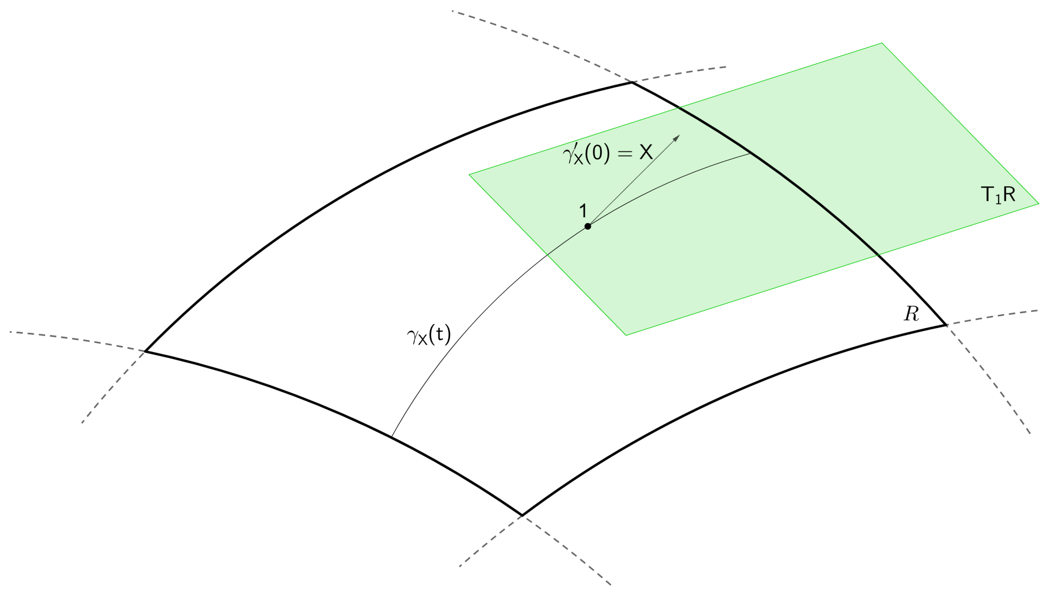


FIGURE 2.2: Tangent space of a rack R in its identity.

In summary, we have shown the following.

Theorem 2.3.6. *Let* $(R, \triangleright, 1)$ *be a Lie rack, and let* $\mathfrak{g} = T_1R$*. Then there exists a bilinear mapping* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *such that*

- $(\mathfrak{g}, [\cdot,\cdot])$ *is a left Leibniz algebra;*
- *for each* $x \in R$*, the tangent mapping* $\Phi(x) = T_1\phi(x)$ *is an automorphism of* $(\mathfrak{g}, [\cdot,\cdot])$*;*
- *if* $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *is defined by* $X \mapsto \mathrm{ad}(X)$*, where* $\mathrm{ad}(X)\colon Y \mapsto [X, Y]$*, then* $\mathrm{ad} = T_1\Phi$*.*

From now on, we will say $\mathfrak{g} = T_1R$ is the tangent Leibniz algebra of the rack R, that is the Leibniz algebra structure on the tangent space at the distinguished element 1 of the Lie rack R.

The converse problem, i.e. to find a manifold endowed with a smooth operation such that the tangent space at the distinguished point, endowed with the differential of this operation, gives a Leibniz algebra isomorphic to the given one, is the *coquecigrue* problem mentioned above. In [54] M. K. Kinyon also provides a solution of the coquecigrue problem for the class of *split* Leibniz algebras. We will now describe it in more detail.

Definition 2.3.7. Let $\mathfrak{g}$ be a Leibniz algebra, let $\mathcal{S} = \mathrm{Leib}(\mathfrak{g})$ be the Leibniz kernel of $\mathfrak{g}$, and let $\mathcal{E} \subseteq \mathfrak{g}$ be an ideal such that $\mathcal{S} \subseteq \mathcal{E} \subseteq \mathrm{Z}_l(L)$. Then $\mathfrak{g}$ *splits* over $\mathcal{E}$ if there exits a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathcal{E} \oplus \mathfrak{h}$, a direct sum of vector spaces.

In this case, for $u, v \in \mathcal{E}$, $X, Y \in \mathfrak{h}$, we have

$$[u + X, v + Y] = [X, v] + [X, Y] \tag{2.11}$$

since $\mathcal{E} \subseteq Z_l(L)$, i.e. $[u, \cdot] \equiv 0$ for all $u \in \mathcal{E}$.

In order to present a theorem on split Lie algebras, we first provide a definition.

Definition 2.3.8. [53] Let $\mathfrak{h}$ be a Lie algebra, let V be an $\mathfrak{h}$-module and set $\mathfrak{g} := V \oplus \mathfrak{h}$. With the bracket

$$[u + X, v + Y] = Xv + [X, Y], \tag{2.12}$$

$\mathfrak{g}$ becomes a Leibniz algebra called *demisemidirect product* of V and $\mathfrak{h}$.

Theorem 2.3.9. *Let $\mathfrak{g}$ be a left Leibniz algebra, let $\mathfrak{h}$ be a Lie algebra and let V be an $\mathfrak{h}$-module. Let $\mathcal{E}$ be an ideal of $\mathfrak{g}$ such that $\mathcal{S} \subseteq \mathcal{E} \subseteq Z_l(L)$. Then $\mathfrak{g}$ splits over $\mathcal{E}$. Conversely, $\mathfrak{g}$ is a demisemidirect product of V with a Lie algebra $\mathfrak{h}$.*

Proof. One of the two directions is proved above. It remains to prove that if $\mathfrak{g}$ is a demisemidirect product of V and $\mathfrak{h}$ as above, then $\mathfrak{g}$ splits over V. Indeed, the ideal $\mathcal{S}$ of squares of $\mathfrak{g}$, generated by squares, agrees with $\mathcal{S} = \mathfrak{h} + V$ since $[u + X, u + X] = Xu + [X, X] = Xu$, for all $u \in V$ and $X \in \mathfrak{h}$. The kernel of ad is

$$Z_l(L) = V \oplus \{X \in \mathfrak{h} \mid Xv = 0, \forall v \in V\} \cap Z(\mathfrak{h}).$$

Finally, $V \cong V \oplus \{0\}$ is an ideal of $\mathfrak{g}$ such that $\mathcal{S} \subseteq V \subseteq Z_l(L)$. We have $\mathfrak{g}/V \cong \mathfrak{h}$, and so $\mathfrak{g}$ splits over V. □

Remark 2.3.1. If a Leibniz algebra $\mathfrak{g}$ splits over $\mathcal{S}$ with its complementary Lie subalgebra being $\mathfrak{h}$, then, as shown in the identity 2.12, $\mathfrak{h}\mathcal{S} = \mathcal{S}$. Conversely, if $\mathfrak{g}$ is a demisemidirect product given by $\mathfrak{g} = V \oplus \mathfrak{h}$ and $\mathfrak{h}V = V$, then it follows that $\mathcal{S}$ coincides with V.

A Leibniz algebra may split over more than one ideal. Let us illustrate this with an example.

Example 2.3.9.1. Let $V := \mathbb{R}^n$ and on $\mathfrak{g} := V \oplus \mathsf{gl}(V) \oplus \mathsf{gl}(V)$, define

$$[u + X + Y, v + U + V] := Yv + [X, U] + [Y, V]$$

for $u, v \in V$, $X, Y, U, V \in \mathsf{gl}(V)$. Then $S \cong V$ and $\ker(\mathrm{ad}) = V \oplus \{aI \mid a \in \mathbb{R}\} \oplus \{0\}$, where $I \in \mathsf{gl}(V)$ denotes the identity matrix. Here, $\mathfrak{g}$ splits over S with complement $\mathsf{gl}(V) \oplus V$, and $\mathfrak{g}$ also splits over $\ker(\mathrm{ad})$ with complement $\mathsf{sl}(V) \oplus \mathsf{gl}(V)$.

We note that a Leibniz algebra may split over $\ker(\mathrm{ad})$ without splitting over its ideal generated by squares.

Example 2.3.9.2. Let $V := \mathbb{R}^2$, let $\mathfrak{h} := \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{R} \right\}$, and let $\mathfrak{g} = V \oplus \mathfrak{h}$ be the demisemidirect product. Then $Z_l(L) = V$ and $\mathcal{S} \cong \mathbb{R} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. While $\mathfrak{g}$

splits over ker(ad), it easy to check that $\mathfrak{g}$ does not split over $\mathcal{S}$. Indeed, if $\mathfrak{g}$ splits over $\mathcal{S}$, then no element of the form $\begin{pmatrix} x \\ y \end{pmatrix} + 0$, with $y \neq 0$, would belong to $\mathfrak{g}$.

We next present M.K. Kinyon's result (see Theorem 3.5 in [54]) on split Leibniz algebras.

Theorem 2.3.10. *Let H be a Lie group with Lie algebra* $\mathrm{Lie}(H) = \mathfrak{h}$*, let V be an H-module, and let $(X, \rhd, \mathbf{1})$ be the linear Lie rack defined by the Equation (2.7), where $X = V \times H$. Then the tangent Leibniz algebra T_1X of X is the demisemidirect product $\mathfrak{g} = V \oplus \mathfrak{h}$ with bracket given by the Equation (2.12).*

On the other hand, let $\mathfrak{g}$ be a split Leibniz algebra. Then there exists a linear Lie rack X with tangent Leibniz algebra isomorphic to $\mathfrak{g}$.

Proof. We prove the first part of the statement. Let $\mathfrak{h}$ be the Lie algebra of the Lie group H. Then we may identify T_1X with $\mathfrak{g} := V \oplus \mathfrak{h}$. For $u \in V, A \in H$, the tangent mapping $\Phi(u, A) = T_1\phi(u, A) \colon \mathfrak{g} \to \mathfrak{g}$ is given by

$$\Phi(u, A)(v + X) = Av + \mathrm{Ad}(A)X$$

for all $v \in V, X \in \mathfrak{h}$, where $\mathrm{Ad}\colon A \to \mathrm{GL}(V)$ is the adjoint representation. Differentiating this, we find that

$$[u + X, v + Y] = Xv + [X, Y]$$

for all $u, v \in V, X, Y \in \mathrm{Z}_l(L)$. By comparing with the identity 2.12, we obtain that the tangent Leibniz algebra T_1X for the Lie rack $X = V \times H$ is exactly the demisemidirect product of V with $\mathfrak{h}$.

Now, in order to prove the second statement let $\mathfrak{g} = \mathcal{E} \oplus \mathfrak{h}$ be a splitting over $\mathfrak{g}$, where $\mathcal{E}$ is an ideal with $\mathcal{S} \subseteq \mathcal{E} \subseteq \ker(\mathrm{ad})$ and $\mathfrak{h}$ is a Lie subalgebra. Remembering that $\mathfrak{g}$ is then a demisemidirect product of $\mathcal{E}$ with $\mathfrak{h}$, let H be a connected Lie group with Lie algebra $\mathfrak{h}$. Set $Q = \mathcal{E} \times H$, and note that we may identify $\mathfrak{g}$ with T_1X. Give X the Lie rack structure $(X, \rhd, \mathbf{1})$ where $\rhd$ is given by the identity 2.7. Finally, the result follows from the first part of the proof we shown above. □

More recently, S. Covez in [26] gives a solution to this problem, which is in general only local: he shows how to integrate any Leibniz algebra into a local Lie rack. The central point of his result is to see every Leibniz algebra $\mathfrak{g}$ as an abelian extension of its left centre $\mathrm{Z}_l(\mathfrak{g})$ and to explicitly integrate the corresponding Leibniz algebra 2-cocycle into a local Lie rack 2-cocycle. However, M. Bordemann and F. Wagemann (see [11]) and J. Mostovoy (see [67]) independently give two different answers to the general coquecigrue problem: Bordemann and Wagemann's solution is not functorial (nor is Covez's method, since the left centre does not depend functorially on the Leibniz algebra in general); Mostovoy's solution is global, but does not generalize the classical Lie solution. The general coquecigrue problem is still open.

The aim of this subsection is to use the Leibniz algebras - Lie local racks correspondence proposed by S. Covez to show that the integration of the two-step nilpotent Leibniz algebras is global.

In [26] S. Covez gives the definition of *smooth rack modules*, *rack cohomology* and *cohomolgy theory* for Leibniz algebras. In particular, for X a Lie rack and A a smooth $X-$module, he defines a cochain complex $\{\mathrm{CR}^n(X,A), d_R^n\}_{n\in\mathbb{N}}$ by setting

$$\begin{aligned}\mathrm{CR}^n(X,A) = \{f : X^n \to A \mid & f(x_1,\dots,1,\dots,x_n)=0, \\ & f \text{ is smooth in a neighborhood of } (1,\dots,1)\in X^n\}\end{aligned}$$

and d_R^n is the differential operator. Moreover, for $\mathfrak{g}$ a left Leibniz algebra and M a $\mathfrak{g}$-module, he defines a cochain complex $\{\mathrm{CL}^n(\mathfrak{g},M), dL^n\}_{n\in\mathbb{N}}$ by setting

$$\mathrm{CL}^n(\mathfrak{g},M) = \mathrm{Hom}(\mathfrak{g}^{\otimes n}, M)$$

and dL^n is the differential operator.

Any Leibniz algebra $\mathfrak{g}$ can be turned in several ways into an abelian extension of a Lie subalgebra $\mathfrak{g}_0 \subseteq \mathfrak{gl}(V)$ by a $\mathfrak{g}_0-$module $\mathfrak{a}$. For example we can take $\mathfrak{a} = \mathrm{Z}_l(\mathfrak{g})$ and $\mathfrak{g}_0 = \mathfrak{g}/\,\mathrm{Z}_l(\mathfrak{g})$. Thus we can associate with $\mathfrak{g}$ a short exact sequence

$$0 \to \mathrm{Z}_l(\mathfrak{g}) \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_0 \to 0.$$

in the category **LeibAlg**. Because $\mathrm{Z}_l(\mathfrak{g})$ is a $\mathfrak{g}_0-$module in the sense of Lie algebras, there is a Leibniz algebra $2-$cocycle $\omega \in \mathrm{ZL}^2(\mathfrak{g}_0, \mathrm{Z}_l(\mathfrak{g}))$ such that $\mathfrak{g} = \mathfrak{g}_0 \oplus_\omega \mathrm{Z}_l(\mathfrak{g})$. The Leibniz bracket in $\mathfrak{g}$ can be written as follows

$$[(x,a),(y,b)] = \left([x,y]_{\mathfrak{g}_0}, \rho_x(b) + \omega(x,y)\right),$$

where $\rho : \mathfrak{g}_0 \times \mathrm{Z}_l(\mathfrak{g}) \to \mathrm{Z}_l(\mathfrak{g})$ is the action induced by the $\mathfrak{g}_0$-module structure of $\mathrm{Z}_l(\mathfrak{g})$. Now, we report some results from S. Covez, which are proven in [26] and will be useful later on.

Theorem 2.3.11. *Every Leibniz algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus_\omega \mathfrak{a}$ can be integrated into a local Lie rack of the form*

$$G_0 \times_f \mathfrak{a}$$

with operation defined by

$$(g,a) \rhd (h,b) = \left(ghg^{-1}, \phi_g(b) + f(g,h)\right)$$

and unit $(1,0)$, where G_0 is a Lie group such that $\mathrm{Lie}(G_0) = \mathfrak{g}_0$, ϕ is the exponentiation of the action ρ,
$f : G_0 \times G_0 \to \mathfrak{a}$ is the Lie local racks $2-$cocycle defined by

$$f(g,h) = \int_{\gamma_h}\left(\int_{\gamma_g} \tau^2(\omega)^{eq}\right)^{eq}, \quad \forall g,h \in G_0$$

and $\tau^2(\omega) \in \mathrm{ZL}^1(\mathfrak{g}_0, Hom(\mathfrak{g}_0,\mathfrak{a}))$ is defined by $\tau^2(\omega)(x)(y) = \omega(x,y)$, for all $x,y \in \mathfrak{g}_0$.

Corollary 2.3.12. *Let X be a Lie rack. Then any Lie rack integrating the Leibniz algebra T_1X is isomorphic to X.*

We finally can answer the question whether a Lie rack integrating a Leibniz algebra can be the quandle $\mathrm{Conj}(G)$, for a suitable Lie group G. The answer is no in general, as the following theorem shows.

Theorem 2.3.13. *Let R be a Lie rack integrating a Leibniz algebra $\mathfrak{g}$. R is a quandle if and only if $\mathfrak{g}$ is a Lie algebra. In particular $R = \mathrm{Conj}(G)$, where $\mathrm{Lie}(G) = \mathfrak{g}$.*

Proof. If $\mathfrak{g}$ is a Lie algebra, then it is clear that $R = Conj(G)$, where $\mathrm{Lie}(G) = \mathfrak{g}$. Conversely, we suppose that R is a Lie quandle. Again we can write $\mathfrak{g} = \mathfrak{g}_0 \oplus_\omega \mathrm{Z}_l(\mathfrak{g})$, thus R is of the form $G_0 \times_f \mathrm{Z}_l(\mathfrak{g})$, with multiplication

$$(g,a) \rhd (h,b) = \left(ghg^{-1}, \phi_g(b) + f(g,h)\right),$$

where f is the Lie racks $2-$cocycle integrating ω. To prove that $\mathfrak{g}$ is a Lie algebra, we have to show that $[(x,a),(x,a)] = (0,0)$, for all $(x,a) \in \mathfrak{g}$.
The condition $(g,a) \rhd (g,a) = (g,a)$ implies that $f(g,g) = 0$, for all $g \in G_0$, and then $\phi_g(a) = a$, for all $a \in Z_l(\mathfrak{g})$. Indeed the action ρ of $\mathfrak{g}_0$ on $\mathrm{Z}_l(\mathfrak{g})$ is trivial and $\omega(x,x) = 0$ for all $x \in \mathfrak{g}_0$. Finally $R = \mathrm{Conj}(G)$, where $G = G_0 \times_F \mathrm{Z}_l(\mathfrak{g})$ is the Lie group with operation

$$(g,a)(h,b) = (gh, a + b + F(g,h)),$$

and $F : G_0 \times G_0 \to \mathrm{Z}_l(\mathfrak{g})$ is a Lie group $2-$cocycle such that

$$f(g,h) = F(g,h) - F(g,g^{-1}) + F(gh,g^{-1}) \quad \forall g,h \in G_0.$$

In fact with this condition we have that

$$(g,a) \rhd (h,b) = (g,a)(h,b)(g,a)^{-1} \quad \forall (g,a),(h,b) \in G_0 \times \mathrm{Z}_l(\mathfrak{g})$$

and the Lie algebra of the Lie group G is clearly $\mathfrak{g}$. □

Now we will claim a result about the integration of nilpotent Leibniz algebras. We will show that, for this class of Leibniz algebras, the integration proposed by S. Covez is global.

Theorem 2.3.14. *Every nilpotent real Leibniz algebra $\mathfrak{g}$ has a global integration into a Lie rack.*

Proof. Let $\mathfrak{g}$ be a nilpotent real Leibniz algebra $\mathfrak{g}$, seen as the abelian extension of $\mathfrak{g}_0 = \mathfrak{g}/\mathrm{Z}_l(\mathfrak{g})$ by its left center $\mathrm{Z}_l(\mathfrak{g})$. Thus $\mathfrak{g}_0$ is a nilpotent Lie algebra and for every $x \in \mathfrak{g}_0$ the action ρ_x defined by

$$\rho_x(a) = \mathrm{ad}_x(a), \quad \forall a \in \mathrm{Z}_l(\mathfrak{g}),$$

can be represented as $m \times m$ strictly lower triangular matrix (see [36]), where $m = \dim_{\mathbb{R}} \mathrm{Z}_l(\mathfrak{g})$. If G_0 is the simply connected Lie group integrating $\mathfrak{g}_0$, the automorphism $\phi_{\exp(x)} = \exp(\rho_x) \in \mathrm{Aut}(\mathrm{Z}_l(\mathfrak{g}))$ is a unitriangular matrix whose entries are polynomial expressions of the coordinates of the vector $x \in \mathfrak{g}_0$, with respect to a fixed basis. Thus, for every $g,h \in G_0$ and fixed smooth paths

$\gamma_g(s) = g^s$ and $\gamma_h(t) = h^t$ in G_0 from 1 to g and from 1 to h respectively, we have that the Lie racks 2−cocycle

$$f(g,h) = \int_{\gamma_h} \left(\int_{\gamma_g} \tau^2(\omega)^{eq} \right)^{eq}, \quad \forall g, h \in G_0$$

is everywhere defined because it involves the integration of matrices with polynomial entries in $\mathbb{R}[s]$ and $\mathbb{R}[t]$. Then the vector space supporting $G_0 \times_f \mathrm{Z}_l(\mathfrak{g})$ has a Lie global rack structure integrating the nilpotent Leibniz algebra $\mathfrak{g}$. □

In the case that $\mathfrak{g}$ is a two-step nilpotent Leibniz algebra, a Lie rack integrating $\mathfrak{g}$ can be defined without integrating the Leibniz algebras 2−cocycle associated with $\mathfrak{g}$, a fact that we note explicitly in the following Theorem.

Theorem 2.3.15. *Let* $(\mathfrak{g}, [-,-])$ *be a two-step nilpotent Leibniz algebra and let* $\omega : \mathfrak{g}_0 \times \mathfrak{g}_0 \to [\mathfrak{g}, \mathfrak{g}]$*, where* $\mathfrak{g}_0 = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$*, be the Leibniz algebras* 2−*cocycle associated with the short exact sequence*

$$0 \to [\mathfrak{g}, \mathfrak{g}] \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_0 \to 0.$$

Then the multiplication

$$(x,a) \rhd (y,b) = (y, b + \omega(x,y)), \quad \forall (x,a), (y,b) \in \mathfrak{g}_0 \times [\mathfrak{g}, \mathfrak{g}]$$

defines a Lie global rack structure on $\mathfrak{g}_0 \times [\mathfrak{g}, \mathfrak{g}]$*, such that* $T_{(0,0)}(\mathfrak{g}_0 \times_\omega [\mathfrak{g}, \mathfrak{g}], \rhd)$ *is a Leibniz algebra isomorphic to* $\mathfrak{g}$*.*

Proof. We have $[\mathfrak{g}, \mathfrak{g}] \subseteq Z(\mathfrak{g})$, so we can see $\mathfrak{g}$ as an abelian extension of $[\mathfrak{g}, \mathfrak{g}]$ by the quotient $\mathfrak{g}_0 = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ via a Leibniz algebras 2−cocycle $\omega \in \mathrm{ZL}^2(\mathfrak{g}_0, [\mathfrak{g}, \mathfrak{g}])$. Thus $\mathfrak{g} = \mathfrak{g}_0 \oplus_\omega [\mathfrak{g}, \mathfrak{g}]$ with bracket

$$[(x,a), (y,b)] = (0, \omega(x,y)).$$

Actually, the condition $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathrm{Z}_l(\mathfrak{g}) \cap \mathrm{Z}_r(\mathfrak{g})$ implies that the action of $\mathfrak{g}_0$ on $[\mathfrak{g}, \mathfrak{g}]$ is trivial. Moreover $\mathfrak{g}_0$ is an abelian Lie algebra, thus a Lie group integrating $\mathfrak{g}_0$ is $G_0 = \mathfrak{g}_0$. Then we can define a Lie rack structure on the cartesian product $\mathfrak{g}_0 \times [\mathfrak{g}, \mathfrak{g}]$ by setting

$$(x,a) \rhd (y,b) = (y, b + \omega(x,y)) \quad \forall (x,a), (y,b) \in \mathfrak{g}_0 \times [\mathfrak{g}, \mathfrak{g}],$$

with unit element $(0,0)$. Finally, the tangent space $T_{(0,0)}(\mathfrak{g}_0 \times [\mathfrak{g}, \mathfrak{g}])$ has a Leibniz algebra structure isomorphic to $\mathfrak{g}$. In fact

$$\left.\frac{\partial^2}{\partial s \partial t}\right|_{s,t=0} (sx, sa) \rhd (ty, tb) = \left.\frac{\partial^2}{\partial s \partial t}\right|_{s,t=0} (ty, tb + \omega(sx, ty)) = (0, \omega(x,y)) = [(x,a), (y,b)].$$

□

Remark 2.3.2. The main point of the strategy in proving Theorem 2.3.15 was to choose $G_0 = \mathfrak{g}_0$ as a Lie group integrating the abelian Lie algebra $\mathfrak{g}_0$. If we change the Lie group G_0, then the integration may not be global, as the following example illustrates.

Example 2.3.15.1. Let $a \in \mathbb{R}$ and let $\mathfrak{g} = \mathfrak{l}_3^a$ be the three-dimensional Heisenberg Lebiniz algebra. Then $[\mathfrak{g}, \mathfrak{g}] = \mathrm{Z}(\mathfrak{g}) \cong \mathbb{R}$ and we can see $\mathfrak{g}$ as an abelian extension of the Lie algebra $\mathfrak{g}_0 = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \cong \mathbb{R}^2$ by $\mathbb{R}$. The corresponding Leibniz algebras 2-cocycle is

$$\omega((x, y), (x', y')) = (1 + a)xy' + (-1 + a)x'y.$$

Now we can choose

$$G_0 = \mathrm{SO}(2) \times \mathrm{SO}(2) \cong \{(e^{ix}, e^{iy}) \mid x, y \in \mathbb{R}\}$$

as a Lie group integrating $\mathfrak{g}_0$. In this case a Lie local rack integrating $\mathfrak{g}$ is $(G_0 \times \mathrm{SO}(2), \rhd)$ with multiplication

$$(e^{ix}, e^{iy}, e^{iz}) \rhd (e^{ix'}, e^{iy'}, e^{iz'}) = (e^{ix'}, e^{iy'}, e^{i(z'+\omega((\log(e^{ix}),\log(e^{iy})),(\log(e^{ix'}),\log(e^{iy'}))))}),$$

that is defined only for $(x, y), (x', y') \in [0, 2\pi[\times[0, 2\pi[$, where we choose $[0, 2\pi[$ as the domain of the principal value of the function log. Thus the integration is not global.

In order to show that Theorem 2.3.15 provides an effective tool for the construction of a global rack integrating a Leibniz algebra $\mathfrak{g}$ with $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathrm{Z}(\mathfrak{g})$, we can reformulate an example proposed by S. Covez in [26].

Example 2.3.15.2. Let $\mathfrak{g} = (\mathbb{R}^4, [-, -])$ with basis $\{e_1, e_2, e_3, e_4\}$ and nonzero brackets

$$[e_1, e_1] = [e_1, e_2] = [e_2, e_2] = [e_3, e_3] = e_4,$$
$$[e_2, e_1] = -e_4.$$

It is easy to see that $\mathfrak{g}$ is a left Leibniz algebra with $[\mathfrak{g}, \mathfrak{g}] = \mathrm{Z}(\mathfrak{g}) = \mathbb{R}e_4$. We have that $\mathfrak{g} = \mathfrak{g}_0 \oplus_\omega \mathbb{R}e_4$, where $\mathfrak{g}_0 \cong \mathrm{Span}_\mathbb{R}\{e_1, e_2, e_3\}$, and the Leibniz 2-cocycle is given by

$$\omega(x, y) = [(x_1, x_2, x_3, 0), (y_1, y_2, y_3, 0)] = (0, 0, 0, x_1y_1 + x_1y_2 - x_2y_1 + x_2y_2 + x_3y_3).$$

Thus, by Theorem 2.3.15, a Lie global rack integrating $\mathfrak{g}$ is $(\mathfrak{g}_0 \times_\omega \mathbb{R}e_4, \rhd)$ with multiplication given by

$$(x_1, x_2, x_3, x_4) \rhd (y_1, y_2, y_3, y_4) = (y_1, y_2, y_3, y_4 + \omega(x, y)).$$

Now we can globally integrate all the indecomposable nilpotent real Leibniz algebras with one-dimensional commutator ideal classified in the previous sections. For the Heisenberg Leibniz algebras, we will obtain Lie racks that are "perturbations" of the conjugation of the Heisenberg Lie group H_{2n+1}. From now on we suppose that $A \in \mathrm{M}_n(\mathbb{R})$ is the companion matrix of the power of an irreducible monic polynomial $f(x) \in \mathbb{R}[x]$. Thus A is a $n \times n$ Jordan block of eigenvalue $a \in \mathbb{R}$ or $A = J_R$, where $R \in \mathrm{M}_2(\mathbb{C})$ is the realification of some complex number $z = \alpha + i\beta$.

Example 2.3.15.3. Let $\mathfrak{g} = \mathfrak{l}_{2n+1}^A$ and let $\{e_1, \ldots, e_n, f_1, \ldots, f_n, h\}$ be a basis of $\mathfrak{g}$. Then $[\mathfrak{g}, \mathfrak{g}] = \mathbb{R}h \subseteq \mathrm{Z}(\mathfrak{g})$, thus we can use Theorem 2.3.15 to find the Lie global rack integrating $\mathfrak{g}$. The Leibniz bracket of $\mathfrak{g}$ is given by

$$[(x_1,\ldots,x_n,y_1,\ldots,y_n,z),(x'_1,\ldots,x'_n,y'_1,\ldots,y'_n,z')] =$$
$$\left(0,\ldots,0,\sum_{i,j=1}^{n}(\delta_{ij}+a_{ij})\,x_iy'_j+\sum_{i,j=1}^{n}(-\delta_{ij}+a_{ij})\,x'_iy_j\right),$$

so we obtain a Lie rack $R^A_{2n+1}=(\mathfrak{g}_0\times_f\mathbb{R}h,\rhd)$ with multiplication

$$(x_1,\ldots,x_n,y_1,\ldots,y_n,z)\rhd(x'_1,\ldots,x'_n,y'_1,\ldots,y'_n,z') =$$
$$\left(x'_1,\ldots,x'_n,y'_1,\ldots,y'_n,z'+\sum_{i,j=1}^{n}\Big[(\delta_{ij}+a_{ij})\,x_iy'_j+(-\delta_{ij}+a_{ij})\,x'_iy_j\Big]\right)$$

and $T_{(0,0)}R^A_{2n+1}=\mathfrak{l}^A_{2n+1}$.

Definition 2.3.16. We define $(R^A_{2n+1},\rhd)$ as the *Heisenberg rack*.

We want to make explicit that the multiplication $\rhd$ in R^A_{2n+1} is a *perturbation* of the conjugation of the Heisenberg Lie group H_{2n+1}. To do this, we will use the canonical matrix representation

$$H_{2n+1}=\left\{\begin{pmatrix}1 & x & z\\ 0 & I_n & y^t\\ 0 & 0 & 1\end{pmatrix}\;\middle|\; x=(x_1,\ldots,x_n),y=(y_1,\ldots,y_n)\in\mathbb{R}^n, z\in\mathbb{R}\right\}\leq \mathrm{GL}_{n+2}(\mathbb{R}).$$

The conjugation formula for two matrices in H_{2n+1} is given by

$$\begin{pmatrix}1 & x & z\\ 0 & I_n & y^t\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & x' & z'\\ 0 & I_n & y'^t\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & x & z\\ 0 & I_n & y^t\\ 0 & 0 & 1\end{pmatrix}^{-1}=\begin{pmatrix}1 & x' & z'+\sum_{i=1}^{n}(x_iy'_i-y_ix'_i)\\ 0 & I_n & y'^t\\ 0 & 0 & 1\end{pmatrix}$$

With the same representation, the multiplication $\rhd$ of R^A_{2n+1} turns into

$$\begin{pmatrix}1 & x & z\\ 0 & I_n & y^t\\ 0 & 0 & 1\end{pmatrix}\rhd\begin{pmatrix}1 & x' & z'\\ 0 & I_n & y'^t\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & x' & z'+\sum_{i,j=1}^{n}\Big[(\delta_{ij}+a_{ij})\,x_iy'_j+(\delta_{ij}-a_{ij})\,x'_iy_j\Big]\\ 0 & I_n & y'^t\\ 0 & 0 & 1\end{pmatrix},$$

hence for $A=0_{n\times n}$, it holds $R^{0_{n\times n}}_{2n+1}=\mathrm{Conj}(H_{2n+1})$.

Example 2.3.16.1. Let $\mathfrak{g}=\mathfrak{k}_n$ and let $\{e_1,\ldots,e_n,f_1,\ldots,f_n,h\}$ be a basis of $\mathfrak{g}$. Then the Leibniz bracket of $\mathfrak{g}$ is given by

$$[(x_1,\ldots,x_n,y_1,\ldots,y_n,z),(x'_1,\ldots,x'_n,y'_1,\ldots,y'_n,z')] =$$
$$\left(0,\ldots,0,x_1y'_1+\sum_{i=2}^{n}(x_iy'_i+x_iy'_{i-1}+x'_{i-1}y_{i-1}-x'_iy_{i-1})+x'_ny_n\right),$$

so we obtain a Lie global rack $K_n = (\mathfrak{g}_0 \times_f \mathbb{R}h, \rhd)$ with multiplication

$$(x_1, \ldots, x_n, y_1, \ldots, y_n, z) \rhd (x'_1, \ldots, x'_n, y'_1, \ldots, y'_n, z') =$$

$$\left(x'_1, \ldots, x'_n, y'_1, \ldots, y'_n, z' + x_1 y'_1 + \sum_{i=2}^{n} (x_i y'_i + x_i y'_{i-1} + x'_{i-1} y_{i-1} - x'_i y_{i-1}) + x'_n y_n\right)$$

and $T_{(0,0)} K_n = \mathfrak{k}_n$.

Definition 2.3.17. We define $(K_n, \rhd)$ as the *Kronecker rack.*

Example 2.3.17.1. Let $\mathfrak{g} = \mathfrak{d}_n$ and let $\{e_1, \ldots, e_{2n+1}, h\}$ be a basis of $\mathfrak{g}$. Then the Leibniz bracket of $\mathfrak{g}$ is given by

$$[(x_1, \ldots, x_{2n+1}, z), (x'_1, \ldots, x'_{2n+1}, z')] = (0, \ldots, 0, \bar{z}),$$

where

$$\bar{z} = x_1 x'_{n+2} + \sum_{i=2}^{n} x_i (x'_{i+n} + x'_{i+n+1}) + x_{n+1} x'_{2n+1} + \sum_{i=n+2}^{2n+1} x_i (x'_{i-n} - x'_{i-n-1}),$$

thus a Lie global rack integrating $\mathfrak{g}$ is $D_n = (\mathfrak{g}_0 \times_f \mathbb{R}h, \rhd)$ with multiplication

$$(x_1, \ldots, x_n, y_1, \ldots, y_n, z) \rhd (x'_1, \ldots, x'_n, y'_1, \ldots, y'_n, z') = (x'_1, \ldots, y'_1, \ldots, y'_n, z' + \bar{z})$$

and $T_{(0,0)} D_n = \mathfrak{d}_n$.

Definition 2.3.18. We call $(D_n, \rhd)$ the *Dieudonné rack.*

We want to conclude this section with the following example. The realification of an indecomposable nilpotent Leibniz algebra with one-dimensional commutator ideal over the field $\mathbb{C}$ is a nilpotent real Leibniz algebra with two-dimensional commutator ideal. In the following example, we integrate the realification of the complex indecomposable Heisenberg Leibniz algebra $\mathfrak{l}_{2n+1}^{J_a}$, where $J_a \in \mathrm{M}_n(\mathbb{C})$ is the Jordan block of eigenvalue $a \in \mathbb{C}$.

Example 2.3.18.1. Let $\mathfrak{h} = \mathfrak{l}_{2n+1}^{J_a}$ and let $\{e_1, \ldots, e_n, f_1, \ldots, f_n, h\}$ be a basis of $\mathfrak{h}$ over $\mathbb{C}$. Then $dim_{\mathbb{R}} \mathfrak{h} = 4n+2$ and $\{e_1, ie_1, \ldots, e_n, ie_n, f_1, if_1, \ldots, f_n, if_n, h, ih\}$ is a basis of $\mathfrak{h}$ over $\mathbb{R}$. For every $(x_1, \ldots, x_n, y_1, \ldots, y_n, z), (x'_1, \ldots, x'_n, y'_1, \ldots, y'_n, z') \in \mathbb{C}^{2n+1}$, the Leibniz bracket of $\mathfrak{h}$ over $\mathbb{R}$ is given by

$$[(x_1, \ldots, x_n, y_1, \ldots, y_n, z), (x'_1, \ldots, x'_n, y'_1, \ldots, y'_n, z')] = (0, \ldots, 0, \Re(\bar{z}), \Im(\bar{z})),$$

where $\Re(a + ib) = a, \Im(a + ib) = b$ and

$$\bar{z} = \sum_{i=1}^{n} [(1 + a) x_i y'_i + (-1 + a) x'_i y_i] + \sum_{i=2}^{n} (x_i y'_{i-1} + x'_i y_{i-1}).$$

Thus a Lie global rack integrating $(\mathfrak{h}, [-,-])$ is $(\mathfrak{h}_0 \times_f \mathrm{Span}_{\mathbb{R}}\{h, ih\}, \rhd)$ with multiplication

$$(x_1, \ldots, x_n, y_1, \ldots, y_n, z) \rhd (x'_1, \ldots, x'_n, y'_1, \ldots, y'_n, z') =$$

$$(\Re(x'_1), \Im(x'_1), \ldots, \Re(x'_n), \Im(x'_n), \Re(y'_1), \Im(y'_1), \ldots, \Re(y'_n), \Im(y'_n), \Re(z' + \bar{z}), \Im(z' + \bar{z}))\,.$$

Chapter 3

Derivations and isotopisms

3.1 Derivations

The derivations and isotopism classes of two-step nilpotent (Leibniz) algebras are outlined in chapter three. The main sources for the composition of this chapter are two papers that were co-authored by me ([57] and [72]).

We study the Lie algebra of derivations of the three classes of indecomposable nilpotent Leibniz algebras with one-dimensional commutator ideal become outlined in the present chapter. We observe that, given a derivation d of a Leibniz algebra L, we have

$$d([L,L]) \subseteq [L,L],$$

thus, if $[L,L] = \mathbb{F}z$, it follows that $d(z) = \gamma z$, for some $\gamma \in \mathbb{F}$.

3.1.1 Derivations of the Heisenberg Leibniz algebras $\mathfrak{l}_{2n+1}^{A}$

Now we want to study in details the Lie algebras of derivations of the Heisenberg Leibniz algebras in the case the field $\mathbb{F}$ is $\mathbb{C}$ or $\mathbb{R}$.

3.1.2 The complex case

Let $n \in \mathbb{N}$ and let $f(x) = x - a \in \mathbb{C}[x]$. Then the companion matrix A of $f(x)^n$ is similar to the $n \times n$ Jordan block J_a of eigenvalue a. Thus $\mathfrak{l}_{2n+1}^{A} \cong \mathfrak{l}_{2n+1}^{J_a}$ and the Leibniz brackets are given by

$$\begin{aligned} &[e_i, f_i] = (1+a)z, \quad [f_i, e_i] = (-1+a)z, \quad \forall i = 1, \ldots, n, \\ &[e_i, f_{i-1}] = [f_{i-1}, e_i] = z, \quad \forall i = 2, \ldots, n, \end{aligned}$$

where $\{e_1, \ldots, e_n, f_1, \ldots, f_n, z\}$ is a basis of $\mathfrak{l}_{2n+1}^{J_a}$. Moreover $\mathfrak{l}_{2n+1}^{J_a} \cong \mathfrak{l}_{2n+1}^{J_{-a}}$.

Now let $d \colon \mathfrak{l}_{2n+1}^{J_a} \to \mathfrak{l}_{2n+1}^{J_a}$ be a linear endomorphism such that

$$d(e_i) = \sum_{j=1}^{n} a_{ji} e_j + \sum_{k=1}^{n} b_{ki} f_k + a_i z, \quad \forall i = 1, \ldots, n,$$

$$d(f_i) = \sum_{j=1}^{n} c_{ji} e_j + \sum_{k=1}^{n} d_{ki} f_k + b_i z, \quad \forall i = 1, \ldots, n$$

and

$$d(z) = \gamma z.$$

Then d is a derivation of the Heisenberg algebra $\mathfrak{l}_{2n+1}^{J_a}$ if and only if

$$\begin{aligned}
0 = d([e_i, e_j]) &= [d(e_i), e_j] + [e_i, d(e_j)] \\
&= (b_{ji}(-1+a) + b_{ij}(1+a) + b_{j-1,i} + b_{i-1,j})z, \\
0 = d([f_i, f_j]) &= [d(f_i), f_j] + [f_i, d(f_j)] \\
&= (c_{ji}(1+a) + c_{ij}(-1+a) + c_{j+1,i} + c_{i+1,j})z, \\
(1+a)\gamma z = d([e_i, f_i]) &= [d(e_i), f_i] + [e_i, d(f_i)] \\
&= ((a_{ii} + d_{ii})(1+a) + a_{i+1,i} + d_{i-1,i})z, \\
(-1+a)\gamma z = d([f_i, e_i]) &= [d(f_i), e_i] + [f_i, d(e_i)] \\
&= ((a_{ii} + d_{ii})(-1+a) + a_{i+1,i} + d_{i-1,i}), \\
\gamma z = d([e_{i+1}, f_i]) &= [d(e_{i+1}), f_i] + [e_{i+1}, d(f_i)] \\
&= ((a_{i,i+1} + d_{i+1,i})(1+a) + a_{i+1,i+1} + d_{ii})z, \\
\gamma z = d([f_i, e_{i+1}]) &= [d(f_i), e_{i+1}] + [f_i, d(e_{i+1})] \\
&= ((a_{i,i+1} + d_{i+1,i})(-1+a) + a_{i+1,i+1} + d_{ii})z, \\
0 = d([e_i, f_j]) &= [d(e_i), f_j] + [e_i, d(f_j)] \\
&= ((a_{ji} + d_{ij})(1+a) + a_{j+1,i} + d_{i-1,j})z, \quad j \neq i-1, i, \\
0 = d([f_j, e_j]) &= [d(f_j), e_i] + [f_j, d(e_i)] \\
&= ((a_{ji} + d_{ij})(-1+a) + a_{j+1,i} + d_{i-1,j})z, \quad j \neq i-1, i,
\end{aligned}$$

for every $i, j = 1, \ldots, n$. So, for $a \neq 0$ the linear endomorphism d is a derivation of the Heisenberg algebra $\mathfrak{l}_{2n+1}^{J_a}$ if and only if it has the following form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & D & 0 \\ \mu & \nu & \gamma \end{pmatrix}$$

where

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_n \\ 0 & \alpha_1 & \alpha_2 & \ldots & \alpha_{n-1} \\ 0 & 0 & \alpha_1 & \ldots & \alpha_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \alpha_1 \end{pmatrix}, \quad D = \begin{pmatrix} \beta_1 & 0 & 0 & \ldots & 0 \\ -\alpha_2 & \beta_1 & 0 & \ldots & 0 \\ -\alpha_3 & -\alpha_2 & \beta_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \ldots & \beta_1 \end{pmatrix},$$

$$\mu = (\mu_1, \mu_2, \mu_3, \ldots, \mu_n), \quad \nu = (\nu_1, \nu_2, \nu_3, \ldots, \nu_n), \quad \gamma = \alpha_1 + \beta_1.$$

Thus, $\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a})$ is the Lie subalgebra of $\mathrm{gl}(2n+1, \mathbb{C})$ of dimension $3n+1$ with basis

$$\{x, y, E_1, \ldots, E_{n-1}, A_1, \ldots, A_n, B_1, \ldots, B_n\},$$

where

$$x = \sum_{k=1}^{n} e_{k,k} + e_{2n+1,2n+1}, \quad y = \sum_{k=1}^{n} e_{n+k,n+k} + e_{2n+1,2n+1},$$
$$E_i = \sum_{k=1}^{n-i} (e_{k,k+i} - e_{n+i+k,n+k}), \quad \forall i = 1, \dots, n-1,$$
$$A_i = e_{2n+1,i}, \quad B_i = e_{2n+1,n+i}, \quad \forall i = 1, \dots, n,$$

and e_{ij} are matrix units and non-trivial commutators given by

$$[x, B_i] = B_i, \ [y, A_i] = A_i, \quad \forall i = 1, \dots, n,$$
$$[E_i, B_k] = B_{k-i}, \quad 1 \leq i < k \leq n,$$
$$[E_i, A_k] = -A_{i+k}, \quad 1 \leq i \leq n-1, \ 1 \leq k \leq n-i.$$

Remark 3.1.1. With the change of basis

$$\{e_1, \dots, e_n, f_1, \dots, f_n, z\} \mapsto \{e_1, f_1, \dots, e_n, f_n, z\},$$

a derivation of $\mathfrak{l}_{2n+1}^{J_a}$ is represented by the $(2n+1) \times (2n+1)$ matrix

$$\left(\begin{array}{ccccccc}
M_1 & -\widetilde{M_2} & -\widetilde{M_3} & -\widetilde{M_4} & \cdots & -\widetilde{M_n} & 0 \\
M_2 & M_1 & -\widetilde{M_2} & -\widetilde{M_3} & \cdots & -\widetilde{M}_{n-1} & 0 \\
M_3 & M_2 & M_1 & -\widetilde{M_2} & \cdots & -\widetilde{M}_{n-2} & 0 \\
M_4 & M_3 & M_2 & M_1 & \cdots & -\widetilde{M}_{n-3} & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
M_n & M_{n-1} & M_{n-2} & M_{n-3} & \cdots & M_1 & 0 \\
\hline
v_1 & v_2 & v_3 & v_4 & \cdots & v_n & \mathrm{tr}(M_1)
\end{array}\right)$$

where

$$M_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}, \quad M_i = \begin{pmatrix} 0 & 0 \\ 0 & \alpha_i \end{pmatrix}, \quad \widetilde{M_i} = \begin{pmatrix} \alpha_i & 0 \\ 0 & 0 \end{pmatrix}, \quad \forall i = 2, \dots, n,$$

$v_k = (\mu_k, \nu_k)$, for any $k = 1, \dots, n$ and $\mathrm{tr}(M_1) = \alpha_1 + \beta_1$.

In this case $\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a})$ has basis

$$\{x, y, E_1, \dots, E_{n-1}, A_1, \dots, A_n, B_1, \dots, B_n\},$$

where

$$x = \sum_{k=1}^{n} e_{2k-1,2k-1} + e_{2n+1,2n+1}, \quad y = \sum_{k=1}^{n} e_{2k,2k} + e_{2n+1,2n+1},$$
$$E_i = \sum_{k=0}^{n-i+1} (e_{2(k+i+1),2(k+1)} - e_{2k+1,2(k+i)+1}), \quad \forall i = 1, \dots, n-1,$$
$$A_i = e_{2n+1,2i-1}, \quad B_i = e_{2n+1,2i}, \quad \forall i = 1, \dots, n$$

and the Lie brackets are given by

$$\begin{aligned}
&[x, B_i] = B_i,\ [y, A_i] = A_i, \quad \forall i = 1, \dots, n,\\
&[E_i, B_k] = -B_{k-2i}, \quad k > 2i,\\
&[E_i, A_k] = A_{k+2i}, \quad k + 2i \leq 2n.
\end{aligned}$$

With this representation, one can check that

$$\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a}) \subseteq \mathrm{Der}(\mathfrak{h}_{2n+1}),$$

where $\mathrm{Der}(\mathfrak{h}_{2n+1})$ are the derivations of the $(2n+1)-$dimensional Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$ with respect to the symplectic basis $\{e_1, f_1, \dots, e_n, f_n, z\}$ of $\mathfrak{h}_{2n+1}$. The description of these derivations have been well-established in the mathematical literature for a significant period (see [63] and [44]), and continue to be studied and applied today (as evidenced by a recent article in control theory, [29]). Later we present the derivations of the Heisenberg Leibniz algebra $\mathfrak{l}_{2n+1}^{J_0}$ and the ones of the Kronecker Leibniz algebra $\mathfrak{k}_n$ with respect to this basis, in order to compare them with the corresponding ones of the Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$.

The commutator ideal of $\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a})$ is the abelian algebra of dimension $2n$ with basis

$$\{A_1, \dots A_n, B_1, \dots, B_n\},$$

thus $\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a})$ is a two-step solvable Lie algebra. Moreover the lower central series is

$$\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a}) \supseteq \langle A_1, \dots A_n, B_1, \dots, B_n\rangle \supseteq \langle A_1, \dots A_n, B_1, \dots, B_n\rangle \supseteq \dots.$$

so $\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a})$ is not nilpotent and its nilradical is the ideal

$$N = \langle E_1, \dots, E_{n-1}, A_1, \dots, A_n, B_1, \dots, B_n\rangle.$$

Finally the center $\mathrm{Z}(\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a}))$ is trivial and the algebra of inner derivations of $\mathfrak{l}_{2n+1}^{J_a}$ is

$$\mathrm{Inn}(\mathfrak{l}_{2n+1}^{J_a}) = \langle A_h, \dots, A_n, B_1, \dots, B_k\rangle,$$

with $h = 1$ and $k = n$ if $a \neq \pm 1$; $h = 2$ and $k = n$ if $a = 1$; and $h = 1$ and $k = n - 1$ if $a = -1$. Indeed

$$\mathrm{ad}_{e_i} = B_{i-1} + (1 + a)B_i, \quad \forall i = 2, \dots, n,$$

$$\mathrm{ad}_{f_j} = A_{j+1} + (-1 + a)A_j, \quad \forall j = 1, \dots, n - 1$$

and

$$\mathrm{ad}_{e_1} = (1 + a)B_1,\ \mathrm{ad}_{f_n} = (-1 + a)A_n,$$

thus for $a = 1$ we have

$$\mathrm{ad}_{f_n} = 0,\ \mathrm{ad}_{f_j} = A_{j+1}, \quad \forall j = 1, \dots, n - 1$$

and the matrix A_1 does not represent an inner derivations. In the same way, if $a=-1$, then $B_n \notin \mathrm{Inn}(\mathfrak{l}_{2n+1}^{J_1})$. We will show later that

$$\mathrm{AIDer}(\mathfrak{l}_{2n+1}^{J_a}) = \langle A_1, \ldots, A_n, B_1, \ldots, B_n \rangle$$

for every $a \in \mathbb{C}$.

When $a=0$, a derivation of the Heisenberg Leibniz algebra $\mathfrak{l}_{2n+1}^{J_0}$ has the form

$$\begin{pmatrix} A & C & 0 \\ B & D & 0 \\ \mu & \nu & \gamma \end{pmatrix}$$

where A, D, μ, ν and γ are as above,

$$B = \begin{pmatrix}
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & -b_{n+2} \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & b_{n+2} & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & -b_{n+2} & 0 & -b_{n+4} \\
\vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & b_{2n-6} & 0 & b_{2n-4} & 0 \\
0 & 0 & 0 & -b_{n+2} & \ldots & 0 & -b_{2n-4} & 0 & -b_{2n-2} \\
0 & 0 & b_{n+2} & 0 & \ldots & b_{2n-4} & 0 & b_{2n-2} & 0 \\
0 & -b_{n+2} & 0 & -b_{n+4} & \ldots & 0 & -b_{2n-2} & 0 & -b_{2n}
\end{pmatrix},$$

$$C = \begin{pmatrix}
c_2 & 0 & c_4 & 0 & \ldots & c_{n-2} & 0 & c_n & 0 \\
0 & -c_4 & 0 & -c_6 & \ldots & 0 & -c_n & 0 & 0 \\
c_4 & 0 & c_6 & 0 & \ldots & c_n & 0 & 0 & 0 \\
0 & -c_6 & 0 & -c_8 & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots \\
c_{n-1} & 0 & c_n & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & -c_n & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
c_n & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0
\end{pmatrix},$$

if n is even,

$$B = \begin{pmatrix}
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & b_{n+1} \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & -b_{n+1} & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & b_{n+1} & 0 & b_{n+3} \\
0 & 0 & 0 & 0 & \ldots & -b_{n+1} & 0 & -b_{n+3} & 0 \\
\vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & -b_{n+1} & \ldots & -b_{2n-3} & 0 & -b_{2n-2} & 0 \\
0 & 0 & b_{n+1} & 0 & \ldots & 0 & b_{2n-3} & 0 & b_{2n-2} \\
0 & -b_{n+1} & 0 & -b_{n+3} & \ldots & -b_{2n-2} & 0 & -b_{2n} & 0 \\
b_{n+1} & 0 & b_{n+3} & 0 & \ldots & 0 & b_{2n-2} & 0 & b_{2n}
\end{pmatrix},$$

$$C=\begin{pmatrix}
c_2 & 0 & c_4 & 0 & \dots & 0 & c_{n-1} & 0 & c_{n+1}\\
0 & -c_4 & 0 & -c_6 & \dots & -c_{n-1} & 0 & -c_{n+1} & 0\\
c_4 & 0 & c_6 & 0 & \dots & 0 & c_{n+1} & 0 & 0\\
0 & -c_6 & 0 & -c_8 & \dots & -c_{n+1} & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots\\
0 & -c_{n-1} & 0 & -c_{n+1} & \dots & 0 & 0 & 0 & 0\\
c_{n-1} & 0 & c_{n+1} & 0 & \dots & 0 & 0 & 0 & 0\\
0 & -c_{n+1} & 0 & 0 & \dots & 0 & 0 & 0 & 0\\
c_{n+1} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0
\end{pmatrix},$$

if n is odd.

If we reorder the basis as in Remark 3.1.1, then a derivation of $\mathfrak{l}_{2n+1}^{J_0}$ is represented by

$$\left(\begin{array}{ccccccccccccc}
\alpha_1 & c_2 & -\alpha_2 & 0 & -\alpha_3 & c_4 & \cdots & -\alpha_{n-1} & c_n & -\alpha_n & 0 & 0\\
0 & \beta_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \alpha_1 & -c_4 & -\alpha_2 & 0 & \cdots & -\alpha_{n-2} & 0 & -\alpha_{n-1} & 0 & 0\\
0 & \alpha_2 & 0 & \beta_1 & 0 & 0 & \cdots & 0 & 0 & -b_{n+2} & 0 & 0\\
0 & c_4 & 0 & 0 & \alpha_1 & c_6 & \cdots & -\alpha_{n-3} & 0 & -\alpha_{n-2} & 0 & 0\\
0 & \alpha_3 & 0 & \alpha_2 & 0 & \beta_1 & \cdots & b_{n+2} & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & c_n & 0 & 0 & 0 & 0 & \cdots & \alpha_1 & 0 & -\alpha_2 & 0 & 0\\
0 & \alpha_{n-1} & 0 & \alpha_{n-2} & b_{n+2} & \alpha_{n-3} & \cdots & b_{2n-2} & \beta_1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \alpha_1 & 0 & 0\\
0 & \alpha_n & -b_{n+2} & \alpha_{n-1} & 0 & \alpha_{n-2} & \cdots & 0 & \alpha_2 & b_{2n} & \beta_1 & 0\\
\hline
\mu_1 & \nu_1 & \mu_2 & \nu_2 & \mu_3 & \nu_3 & \cdots & \mu_{n-1} & \nu_{n-1} & \mu_n & \nu_n & \alpha_1+\beta_1
\end{array}\right)$$

if n is even, and

$$\left(\begin{array}{ccccccccccccc}
\alpha_1 & c_2 & -\alpha_2 & 0 & -\alpha_3 & c_4 & \cdots & -\alpha_{n-1} & 0 & -\alpha_n & c_{n+1} & 0\\
0 & \beta_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{n+1} & 0 & 0\\
0 & 0 & \alpha_1 & -c_4 & -\alpha_2 & 0 & \cdots & -\alpha_{n-2} & -c_{n+1} & -\alpha_{n-1} & 0 & 0\\
0 & \alpha_2 & 0 & \beta_1 & 0 & 0 & \cdots & -b_{n+1} & 0 & 0 & 0 & 0\\
0 & c_4 & 0 & 0 & \alpha_1 & c_6 & \cdots & -\alpha_{n-3} & 0 & -\alpha_{n-2} & 0 & 0\\
0 & \alpha_3 & 0 & \alpha_2 & 0 & \beta_1 & \cdots & 0 & 0 & b_{n+3} & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & -c_{n+1} & 0 & 0 & \cdots & \alpha_1 & 0 & -\alpha_2 & 0 & 0\\
0 & \alpha_{n-1} & -b_{n+1} & \alpha_{n-2} & 0 & \alpha_{n-3} & \cdots & -b_{2n-2} & \beta_1 & 0 & 0 & 0\\
0 & c_{n+1} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \alpha_1 & 0 & 0\\
b_{n+1} & \alpha_n & 0 & \alpha_{n-1} & b_{n+3} & \alpha_{n-2} & \cdots & 0 & \alpha_1 & b_{2n} & \beta_1 & 0\\
\hline
\mu_1 & \nu_1 & \mu_2 & \nu_2 & \mu_3 & \nu_3 & \cdots & \mu_{n-1} & \nu_{n-1} & \mu_n & \nu_n & \alpha_1+\beta_1
\end{array}\right)$$

if n is odd. We can now study in details these two cases.

If n is even, then $\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_0})$ is a Lie algebra of dimension $4n+1$ with basis

$$\{x, y, E_1, \dots, E_{n-1}, c_2, c_4, \dots, c_n, b_{n+2}, b_{n+4} \dots, b_{2n}, A_1, \dots, A_n, B_1, \dots, B_n\},$$

where

$$\begin{aligned}
x &= \sum_{k=1}^{n} e_{2k-1,2k-1} + e_{2k+1,2k+1},\\
y &= \sum_{k=1}^{n} e_{2k,2k} + e_{2k+1,2k+1},\\
E_i &= \sum_{k=0}^{n-i+1} (e_{2(k+i+1),2(k+1)} - e_{2k+1,2(k+i)+1}), \quad \forall i = 1, \dots, n-1,\\
A_i &= e_{2n+1,2i-1}, \quad B_i = e_{2n+1,2i}, \quad \forall i = 1, \dots, n,\\
c_h &= \sum_{i=0}^{h-2} (-1)^i e_{2(h-i-1)-1,2(1+i)}, \quad \forall h = 2, 4, \dots, n,\\
b_h &= \sum_{i=0}^{2n-h} (-1)^i e_{2(n-i),2(h-n+i)-1}, \quad \forall h = n+2, n+4, \dots, 2n
\end{aligned}$$

and commutators

$$\begin{aligned}
&[x, B_i] = B_i, \ [y, A_i] = A_i, \quad \forall i = 1, \dots, n,\\
&[E_i, B_k] = -B_{k-2i}, \quad k > 2i,\\
&[E_i, A_k] = A_{k+2i}, \quad k + 2i \leq 2n,\\
&[x, c_h] = c_h, \ [y, c_h] = -c_h, \quad \forall h = 2, 4, \dots, n,\\
&[x, b_h] = -b_h, \ [y, b_h] = b_h, \quad \forall h = n+2, n+4, \dots, 2n,\\
&[A_i, c_k] = (-1)^{i+1} B_{k-i}, \ [B_i, b_k] = (-1)^i A_{k-i}, \quad 1 \leq k - i \leq n,\\
&[c_k, b_h] = E_{h-k}, \quad h - k \geq 1,\\
&[\alpha_{2i}, c_h] = -2c_{h-2i}, \quad h - 2i > 0,\\
&[\alpha_{2i}, b_h] = 2b_{h+2i}, \quad h + 2i \leq 2n.
\end{aligned}$$

Then the commutator ideal of $\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_0})$ has basis

$$\{E_2, E_4, \dots, E_{n-2}, c_2, c_4, \dots, c_n, b_{n+2}, b_{n+4} \dots, b_{2n}, A_1, \dots, A_n, B_1, \dots, B_n\}$$

and we have a $(\frac{n}{2}+1)-$step solvable Lie algebra with derived series

$$\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_0}) \supseteq [\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_0}), \mathrm{Der}(\mathfrak{l}_{2n+1}^{J_0})] \supseteq$$

$$\supseteq \langle E_2, E_4, \dots, E_{n-2}, c_2, c_4, \dots, c_{n-2}, b_{n+4}, \dots, b_{2n}, A_2, \dots, A_n, B_1, \dots, B_{n-1}\rangle \supseteq \dots$$

$$\dots \supseteq \langle c_2, b_{2n}, A_{\frac{n}{2}+1}, \dots, A_n, B_1, \dots, B_{\frac{n}{2}}\rangle \supseteq 0$$

Moreover, $\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_0})$ is not nilpotent and its nilradical is the ideal

$$N = \langle E_1, \dots, E_{n-1}, c_2, c_4, \dots, c_n, b_{n+2}, b_{n+4} \dots, b_{2n}, A_1, \dots, A_n, B_1, \dots, B_n\rangle.$$

If n is odd, then the algebra of derivations of $\mathfrak{l}_{2n+1}^{J_0}$ has dimension $4n+2$ and it is generated by

$$\{x, y, E_1, \dots, E_{n-1}, c_2, c_4, \dots, c_{n+1}, b_{n+1}, b_{n+3} \dots, b_{2n}, A_1, \dots, A_n, B_1, \dots, B_n\}.$$

The Lie brackets are the same of the ones listed for the case n even, except for the facts that $[B_i, b_k] = (-1)^{i+1} A_{k-i}$, for any $1 \leq k-i \leq n$, and $[c_{n+1}, b_{n+1}] = x - y$. Then the commutator ideal is the subspace generated by

$$\{x-y, E_2, E_4, \dots, E_{n-1}, c_2, c_4, \dots, c_{n+1}, b_{n+1}, b_{n+3} \dots, b_{2n}, A_1, \dots, A_n, B_1, \dots, B_n\}.$$

In this case the Lie algebra of derivations is not solvable since

$$[[\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_0}), \mathrm{Der}(\mathfrak{l}_{2n+1}^{J_0})], [\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_0}), \mathrm{Der}(\mathfrak{l}_{2n+1}^{J_0})]] = [\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_0}), \mathrm{Der}(\mathfrak{l}_{2n+1}^{J_0})]$$

and the Levi decomposition is given by

$$\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_0}) = R \rtimes S,$$

where the radical of the Lie algebra is

$$R = \langle x+y, E_1, \dots, E_{n-1}, c_2, c_4, \dots, c_{n-1}, b_{n+3}, b_{n+5} \dots, b_{2n}, A_1, \dots, A_n, B_1, \dots, B_n\rangle$$

and the Levi complement is

$$S = \langle x - y, c_{n+1}, b_{n+1}\rangle.$$

Finally the nilradical is the ideal

$$N = \langle E_1, \dots, E_{n-1}, c_2, c_4, \dots, c_{n-1}, b_{n+3}, b_{n+5} \dots, b_{2n}, A_1, \dots, A_n, B_1, \dots, B_n\rangle.$$

In both cases n is even or odd, we have that $\mathrm{Z}(\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_0})) = 0$ and the Lie algebra of inner derivations is represented by the matrices of the type

$$\left(\begin{array}{c|c} \multirow{3}{*}{$\mathbf{0}$} & 0 \\ & \vdots \\ & 0 \\ \hline \mu_1 \ \nu_1 \dots \mu_n \ \nu_n & 0 \end{array}\right),$$

thus $\mathrm{Inn}(\mathfrak{l}_{2n+1}^{J_0})$ is an abelian algebra of dimension $2n$. Moreover, for every $n \in \mathbb{N}$ and for every $a \in \mathbb{C}^*$, we observe that

$$\mathrm{Der}(\mathfrak{h}_{2n+1}) \supseteq \mathrm{Der}(\mathfrak{l}_{2n+1}^{J_0}) \supseteq \mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a}).$$

3.1.3 The real case

Irreducible polynomials in $\mathbb{R}[x]$ have degree one or two. Let $f(x) \in \mathbb{R}[x]$ be an irreducible monic polynomial. If $f(x) = x - a$, then we obtain the same results of the complex case. So we suppose that $f(x) = x^2 + Cx + D$, with $C^2 - 4D < 0$.

Let $z = a + ib \in \mathbb{C} \setminus \mathbb{R}$ be a root of $f(x)$. Then $f(x) = (x - z)(x - \bar{z})$ and the companion matrix A of $f(x)^n$ in similar to the $2n \times 2n$ real block matrix

$$J_R = \begin{pmatrix} R & 0 & \cdots & 0 \\ I_2 & R & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I_2 & R \end{pmatrix},$$

where

$$R = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

is the realification of the complex number z. Thus $\mathfrak{l}^{A}_{4n+1} \cong \mathfrak{l}^{J_R}_{4n+1}$ and $\mathfrak{l}^{J_R}_{4n+1}$ is the realification of the complex algebra $\mathfrak{l}^{J_z}_{2n+1}$. In [9] the derivations of the realification of the $(2n+1)-$dimensional Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$ were studied. We want to find the conditions such that the realification of a derivation of the complex algebra $\mathfrak{l}^{J_z}_{2n+1}$, with $z = a + ib \in \mathbb{C} \setminus \mathbb{R}$, is a derivation of the real algebra $\mathfrak{l}^{J_R}_{4n+1}$. We will investigate the case $n = 1$.

Let $\{e_1, f_1, e_2, f_2, z\}$ be a basis of the real algebra $\mathfrak{l}^{R}_{5}$. The non-trivial commutators are

$$[e_i, f_i] = (1 + a)z, \; [f_i, e_i] = (-1 + a)z, \quad \forall i = 1, 2,$$
$$[e_1, f_2] = [f_2, e_1] = bz, \; [e_2, f_1] = [f_1, e_2] = -bz$$

and it comes out that a general derivation of $\mathfrak{l}^{R}_{5}$ is represented by the matrix

$$\left(\begin{array}{cccc|c} \alpha_1 & 0 & \alpha_2 & 0 & 0 \\ 0 & \beta_1 & 0 & \alpha_2 & 0 \\ -\alpha_2 & 0 & \alpha_1 & 0 & 0 \\ 0 & -\alpha_2 & 0 & \beta_1 & 0 \\ \hline \mu_1 & \nu_1 & \mu_2 & \nu_2 & \alpha_1 + \beta_1 \end{array}\right)$$

if $a \neq 0$ and by

$$\left(\begin{array}{cccc|c} \alpha_1 & \delta & \alpha_2 & 0 & 0 \\ \delta' & \beta_1 & 0 & \alpha_2 & 0 \\ -\alpha_2 & 0 & \alpha_1 & \delta & 0 \\ 0 & -\alpha_2 & \delta' & \beta_1 & 0 \\ \hline \mu_1 & \nu_1 & \mu_2 & \nu_2 & \alpha_1 + \beta_1 \end{array}\right)$$

if $a = 0$. Then

- if $a \neq 0$, $\mathrm{Der}(\mathfrak{l}^{R}_{5})$ is generated by the set

$$\{x, y, E, A_1, A_2, B_1, B_2\},$$

where $x = e_{11} + e_{33} + e_{55}$, $y = e_{22} + e_{44} + e_{55}$, $E = e_{13} + e_{24} - e_{31} - e_{42}$, $A_i = e_{5,2i-1}$ and $B_i = e_{5,2i}$, for every $i = 1, 2$, and the non-trivial Lie brackets are

$$\begin{aligned}&[x, B_i] = B_i,\ [y, A_i] = A_i,\ \ \forall i = 1, 2,\\&[E, A_1] = -A_2,\ [E, A_2] = -A_1,\\&[E, B_1] = -B_2,\ [E, B_2] = -B_1.\end{aligned}$$

Then we have a solvable Lie algebra with abelian commutator ideal generated by

$$\{A_1, A_2, B_1, B_2\},$$

which coincides with the ideal $\mathrm{Inn}(\mathfrak{l}_5^R)$ and with the nilradical of the Lie algebra itself. Moreover the center $\mathrm{Z}(\mathrm{Der}(\mathfrak{l}_5^R))$ is trivial.

- if $a = 0$, a basis of $\mathrm{Der}(\mathfrak{l}_5^R)$ is

$$\{x, y, E, F, G, A_1, A_2, B_1, B_2\},$$

where $x, y, E, A_1, A_2, B_1, B_2$ are defined as above, $F = e_{12} + e_{34}$, $G = e_{21} + e_{43}$ and the non-trivial Lie brackets are given by the ones above and by

$$\begin{aligned}&[x, F] = F,\ [x, G] = -G,\ [y, F] = -F,\ [y, G] = -G,\\&[F, G] = x - y,\ [F, A_i] = -B_i,\ [F, B_i] = -A_i,\ \forall i = 1, 2.\end{aligned}$$

It follows that $\mathrm{Z}(\mathrm{Der}(\mathfrak{l}_5^R)) = 0$ and the Lie algebra is not solvable. Its radical is given by the ideal

$$R = \langle x + y, E, A_1, A_2, B_1, B_2\rangle,$$

a Levi complement is the semisimple Lie algebra

$$S = \langle x - y, F, G\rangle$$

and the nilradical of $\mathrm{Der}(\mathfrak{l}_5^R)$ is the abelian four-dimensional algebra

$$N = \langle A_1, A_2, B_1, B_2\rangle \cong \mathbb{R}^4$$

and again it coincides with the set of inner derivations of $\mathfrak{l}_5^R$.

Now let

$$\begin{pmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ \mu & \nu & \alpha+\beta\end{pmatrix}$$

be a derivation of the complex Heinseberg algebra $\mathfrak{l}_3^z$, where $z = a + ib$. Then its realification is represented by the matrix

$$\left(\begin{array}{cccc|c} \Re(\alpha) & \Im(\alpha) & 0 & 0 & 0 \\ -\Im(\alpha) & \Re(\alpha) & 0 & 0 & 0 \\ 0 & 0 & \Re(\beta) & \Im(\beta) & 0 \\ 0 & 0 & -\Im(\beta) & \Re(\beta) & 0 \\ \hline \mu_1 & \nu_1 & \mu_2 & \nu_2 & \gamma \end{array}\right)$$

and this is a derivation of the real Heisenberg Leibniz algebra $\mathfrak{l}_5^R$ if and only if

$$\alpha = \beta \in \mathbb{R}$$

in both cases that $a \neq 0$ or $a = 0$. Then the set of realifications of the derivations of $\mathfrak{l}_3^z$ that are derivations of the real algebra $\mathfrak{l}_5^R$ form the proper Lie subalgebra of the matrices of the form

$$\left(\begin{array}{cccc|c} \alpha & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 \\ \hline \mu_1 & \nu_1 & \mu_2 & \nu_2 & 2\alpha \end{array}\right)$$

of $\mathrm{Der}(\mathfrak{l}_5^R)$.

3.1.4 Derivations of the Kronecker Leibniz algebra $\mathfrak{k}_n$

Now we return to the case that $\mathbb{F}$ is a field with $\mathrm{char}(\mathbb{F}) \neq 2$. Let $n \in \mathbb{N}$ and let $\mathfrak{k}_n$ be the *Kronecker Leibniz algebra*. We fix the basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n, z\}$ of $\mathfrak{k}_n$.

A linear endomorphism $d\colon \mathfrak{k}_n \to \mathfrak{k}_n$ defined by

$$d(e_i) = \sum_{j=1}^{n} a_{ji} e_j + \sum_{k=1}^{n} b_{ki} f_k + a_i z, \quad \forall i = 1, \ldots, n,$$

$$d(f_i) = \sum_{j=1}^{n} c_{ji} e_j + \sum_{k=1}^{n} d_{ki} f_k + b_i z, \quad \forall i = 1, \ldots, n$$

and

$$d(z) = \gamma z,$$

is a derivation if and only if the following equations hold

$$\begin{aligned}
0 = d([e_i, e_j]) &= [d(e_i), e_j] + [e_i, d(e_j)] \\
&= (b_{ji} + b_{ij} - b_{j-1,i} + b_{i-1,j})z, \\
0 = d([f_i, f_j]) &= [d(f_i), f_j] + [f_i, d(f_j)] \\
&= (c_{ji} + c_{ij} - c_{j+1,i} + c_{i+1,j})z, \\
\gamma z = d([e_i, f_i]) &= [d(e_i), f_i] + [e_i, d(f_i)] \\
&= (a_{ii} + d_{ii} + a_{i+1,i} + d_{i-1,i})z, \\
\gamma z = d([f_i, e_i]) &= [d(f_i), e_i] + [f_i, d(e_i)] \\
&= (a_{ii} + d_{ii} - a_{i+1,i} - d_{i-1,i}), \\
\gamma z = d([e_{i+1}, f_i]) &= [d(e_{i+1}), f_i] + [e_{i+1}, d(f_i)] \\
&= (a_{i,i+1} + d_{i+1,i} + a_{i+1,i+1} + d_{ii})z, \\
-\gamma z = d([f_i, e_{i+1}]) &= [d(f_i), e_{i+1}] + [f_i, d(e_{i+1})] \\
&= (a_{i,i+1} + d_{i+1,i} - a_{i+1,i+1} - d_{ii})z, \\
0 = d([e_i, f_j]) &= [d(e_i), f_j] + [e_i, d(f_j)] \\
&= (a_{ji} + d_{ij} + a_{j+1,i} + d_{i-1,j})z, \quad j \neq i-1, i, \\
0 = d([f_j, e_j]) &= [d(f_j), e_i] + [f_j, d(e_i)] \\
&= (a_{ji} + d_{ij} - a_{j+1,i} - d_{i-1,j})z, \quad j \neq i-1, i,
\end{aligned}$$

for every $i, j = 1, \ldots, n$ and we have the following.

Theorem 3.1.1. *A linear endomorphism $d\colon \mathfrak{k}_n \to \mathfrak{k}_n$ is a derivation of the Kronecker algebra $\mathfrak{k}_n$ if and only if it has the form*

$$\begin{pmatrix} A & C & 0 \\ B & D & 0 \\ \mu & \nu & \gamma \end{pmatrix}$$

where

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_n \\ 0 & \alpha_1 & \alpha_2 & \ldots & \alpha_{n-1} \\ 0 & 0 & \alpha_1 & \ldots & \alpha_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \alpha_1 \end{pmatrix}, \quad D = \begin{pmatrix} \beta_1 & 0 & 0 & \ldots & 0 \\ -\alpha_2 & \beta_1 & 0 & \ldots & 0 \\ -\alpha_3 & -\alpha_2 & \beta_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \ldots & \beta_1 \end{pmatrix},$$

$$B=\begin{pmatrix}
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & b_{n+1}\\
0 & 0 & 0 & 0 & \dots & 0 & 0 & -b_{n+1} & 0\\
0 & 0 & 0 & 0 & \dots & 0 & b_{n+1} & 0 & b_{n+3}\\
0 & 0 & 0 & 0 & \dots & -b_{n+1} & 0 & -b_{n+3} & 0\\
\vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & b_{n+1} & \dots & 0 & b_{2n-5} & 0 & b_{2n-3}\\
0 & 0 & -b_{n+1} & 0 & \dots & -b_{2n-5} & 0 & -b_{2n-3} & 0\\
0 & b_{n+1} & 0 & b_{n+3} & \dots & 0 & b_{2n-3} & 0 & b_{2n-1}\\
-b_{n+1} & 0 & -b_{n+3} & 0 & \dots & -b_{2n-3} & 0 & -b_{2n-1} & 0
\end{pmatrix},$$

$$C=\begin{pmatrix}
0 & c_3 & 0 & c_5 & \dots & 0 & c_{n-1} & 0 & c_{n+1}\\
-c_3 & 0 & -c_5 & 0 & \dots & -c_{n-1} & 0 & -c_{n+1} & 0\\
0 & c_5 & 0 & c_7 & \dots & 0 & c_{n+1} & 0 & 0\\
-c_5 & 0 & -c_7 & 0 & \dots & -c_{n+1} & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots\\
0 & c_{n-1} & 0 & c_{n+1} & \dots & 0 & 0 & 0 & 0\\
-c_{n-1} & 0 & -c_{n+1} & 0 & \dots & 0 & 0 & 0 & 0\\
0 & c_{n+1} & 0 & 0 & \dots & 0 & 0 & 0 & 0\\
-c_{n+1} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0
\end{pmatrix},$$

if n is even,

$$B=\begin{pmatrix}
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -b_{n+2}\\
0 & 0 & 0 & 0 & \dots & 0 & 0 & b_{n+2} & 0\\
0 & 0 & 0 & 0 & \dots & 0 & -b_{n+2} & 0 & -b_{n+4}\\
\vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \dots & 0 & -b_{2n-5} & 0 & -b_{2n-3}\\
0 & 0 & 0 & b_{n+2} & \dots & b_{2n-5} & 0 & b_{2n-3} & 0\\
0 & 0 & -b_{n+2} & 0 & \dots & 0 & -b_{2n-3} & 0 & -b_{2n-1}\\
0 & b_{n+2} & 0 & b_{n+4} & \dots & b_{2n-3} & 0 & b_{2n-1} & 0
\end{pmatrix},$$

$$C=\begin{pmatrix}
0 & c_3 & 0 & c_5 & \dots & c_{n-2} & 0 & c_n & 0\\
-c_3 & 0 & -c_5 & 0 & \dots & 0 & -c_n & 0 & 0\\
0 & c_5 & 0 & c_7 & \dots & c_n & 0 & 0 & 0\\
-c_5 & 0 & -c_7 & 0 & \dots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots\\
-c_{n-2} & 0 & -c_n & 0 & \dots & 0 & 0 & 0 & 0\\
0 & c_n & 0 & 0 & \dots & 0 & 0 & 0 & 0\\
-c_n & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0
\end{pmatrix},$$

if n is odd and

$$\mu=(\mu_1,\mu_2,\mu_3,\dots,\mu_n),\quad \nu=(\nu_1,\nu_2,\nu_3,\dots,\nu_n),\quad \gamma=\alpha_1+\beta_1.$$

Remark 3.1.2. Again, with the change of basis

$$\{e_1, \dots, e_n, f_1, \dots, f_n, z\} \mapsto \{e_1, f_1, \dots, e_n, f_n, z\},$$

a derivation of $\mathfrak{k}_n$ is represented by the $(2n+1) \times (2n+1)$ matrix

$$\left(\begin{array}{cccccccccccc}
\alpha_1 & 0 & -\alpha_2 & c_3 & -\alpha_3 & 0 & \cdots & -\alpha_{n-1} & 0 & -\alpha_n & c_{n+1} & 0 \\
0 & \beta_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{n+1} & 0 & 0 \\
0 & -c_3 & \alpha_1 & 0 & -\alpha_2 & -c_5 & \cdots & -\alpha_{n-2} & -c_{n+1} & -\alpha_{n-1} & 0 & 0 \\
0 & \alpha_2 & 0 & \beta_1 & 0 & 0 & \cdots & -b_{n+1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_5 & \alpha_1 & 0 & \cdots & -\alpha_{n-3} & 0 & -\alpha_{n-2} & 0 & 0 \\
0 & \alpha_1 & 0 & \alpha_2 & 0 & \beta_1 & \cdots & 0 & 0 & -b_{n+3} & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & c_{n+1} & 0 & 0 & \cdots & \alpha_1 & 0 & -\alpha_2 & 0 & 0 \\
0 & \alpha_{n-1} & b_{n+1} & \alpha_{n-2} & 0 & \alpha_{n-3} & \cdots & 0 & \beta_1 & -b_{2n-1} & 0 & 0 \\
0 & -c_{n+1} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \alpha_1 & 0 & 0 \\
-b_{n+1} & \alpha_n & 0 & \alpha_{n-1} & b_{n+3} & \alpha_{n-2} & \cdots & b_{2n-1} & \alpha_2 & 0 & \beta_1 & 0 \\
\hline
\mu_1 & \nu_1 & \mu_2 & \nu_2 & \mu_3 & \nu_3 & \cdots & \mu_{n-1} & \nu_{n-1} & \mu_n & \nu_n & \alpha_1 + \beta_1
\end{array}\right)$$

if n is even and

$$\left(\begin{array}{cccccccccccc}
\alpha_1 & 0 & -\alpha_2 & c_3 & -\alpha_3 & 0 & \cdots & -\alpha_{n-1} & c_n & -\alpha_n & 0 & 0 \\
0 & \beta_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & -c_3 & \alpha_1 & 0 & -\alpha_2 & -c_5 & \cdots & -\alpha_{n-2} & 0 & -\alpha_{n-1} & 0 & 0 \\
0 & \alpha_2 & 0 & \beta_1 & 0 & 0 & \cdots & 0 & 0 & -b_{n+2} & 0 & 0 \\
0 & 0 & 0 & -c_5 & \alpha_1 & 0 & \cdots & -\alpha_{n-3} & 0 & -\alpha_{n-2} & 0 & 0 \\
0 & \alpha_3 & 0 & \alpha_2 & 0 & \beta_1 & \cdots & b_{n+2} & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & c_n & 0 & 0 & 0 & 0 & \cdots & \alpha_1 & 0 & -\alpha_2 & 0 & 0 \\
0 & \alpha_{n-1} & 0 & \alpha_{n-2} & -b_{n+2} & \alpha_{n-3} & \cdots & 0 & \beta_1 & -b_{2n-1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \alpha_1 & 0 & 0 \\
0 & \alpha_n & b_{n+2} & \alpha_{n-1} & 0 & \alpha_{n-2} & \cdots & b_{2n-1} & \alpha_2 & 0 & \beta_1 & 0 \\
\hline
\mu_1 & \nu_1 & \mu_2 & \nu_2 & \mu_3 & \nu_3 & \cdots & \mu_{n-1} & \nu_{n-1} & \mu_n & \nu_n & \alpha_1 + \beta_1
\end{array}\right)$$

if n is odd.

We can now describe the main properties of the Lie algebra $\mathrm{Der}(\mathfrak{k}_n)$.

- If n is even, then $\mathrm{Der}(\mathfrak{k}_n)$ has basis

  $$\{x, y, E_1, \dots, E_{n-1}, c_3, c_5, \dots, c_{n+1}, b_{n+1}, b_{n+3} \dots, b_{2n-1}, A_1, \dots, A_n, B_1, \dots, B_n\},$$

where

$$
\begin{aligned}
x &= \sum_{k=1}^{n} e_{2k-1,2k-1} + e_{2k+1,2k+1},\\
y &= \sum_{k=1}^{n} e_{2k,2k} + e_{2k+1,2k+1},\\
E_i &= \sum_{k=0}^{n-i+1} (e_{2(k+i+1),2(k+1)} - e_{2k+1,2(k+i)+1}), \quad \forall i = 1,\dots,n-1,\\
A_i &= e_{2n+1,2i-1}, \quad B_i = e_{2n+1,2i}, \quad \forall i = 1,\dots,n,\\
c_h &= \sum_{i=0}^{h-2} (-1)^{i+1} e_{2(h-i-1)-1,2(1+i)}, \quad \forall h = 3,5,\dots,n+1,\\
b_h &= \sum_{i=0}^{2n-h} (-1)^{i+1} e_{2(n-i),2(h-n+i)-1}, \quad \forall h = n+1,n+3,\dots,2n-1,
\end{aligned}
$$

and

$$
\begin{aligned}
&[x,B_i] = B_i,\ [y,A_i] = A_i, \quad \forall i = 1,\dots,n,\\
&[E_i,B_k] = -B_{k-2i}, \quad k > 2i,\\
&[E_i,A_k] = A_{k+2i}, \quad k+2i \le 2n,\\
&[x,c_h] = c_h,\ [y,c_h] = -c_h, \quad \forall h = 3,5,\dots,n+1,\\
&[x,b_h] = -b_h,\ [y,b_h] = b_h, \quad \forall h = n+1,n+3,\dots,2n-1,\\
&[A_i,c_k] = (-1)^{i+1}B_{k-i},\ [B_i,b_k] = (-1)^{i+1}A_{k-i}, \quad 1 \le k-i \le n,\\
&[c_k,b_h] = E_{h-k}, \quad h-k \ge 1,\\
&[c_{n+1},b_{n+1}] = -x+y,\\
&[E_{2i},c_h] = -2c_{h-2i}, \quad h-2i > 0,\\
&[E_{2i},b_h] = 2b_{h+2i}, \quad h+2i \le 2n.
\end{aligned}
$$

The commutator ideal of $\mathrm{Der}(\mathfrak{k}_n)$ has basis

$$\{x-y, E_2, E_4, \dots, E_{n-2}, c_3, c_5, \dots, c_{n+1}, b_{n+1}, b_{n+3} \dots, b_{2n-1}, A_1, \dots, A_n, B_1, \dots, B_n\}$$

and, as in the case of the Heisenberg Leibniz algebra $\mathfrak{l}^{J_0}_{2n+1}$ with n odd, we have that the Lie algebra of derivations is not solvable. The Levi decomposition is

$$\mathrm{Der}(\mathfrak{k}_n) = R \rtimes S,$$

where

$$R = \langle x+y, E_1, \dots, E_{n-1}, c_3, c_5, \dots, c_{n-1}, b_{n+3}, b_{n+5} \dots, b_{2n-1}, A_1, \dots, A_n, B_1, \dots, B_n\rangle$$

is the radical and

$$S = \langle x-y, c_{n+1}, b_{n+1}\rangle$$

is a Levi complement. Moreover the nilradical of $\mathrm{Der}(\mathfrak{k}_n)$ is

$$N = \langle E_1, \dots, E_{n-1}, c_3, c_5, \dots, c_{n-1}, b_{n+3}, b_{n+5} \dots, b_{2n-1}, A_1, \dots, A_n, B_1, \dots, B_n\rangle.$$

- If n is odd, then the algebra of derivations of $\mathfrak{k}_n$ has dimension $4n$ and it is generated by

  $$\{x, y, E_1, \ldots, E_{n-1}, c_3, c_5, \ldots, c_n, b_{n+2}, b_{n+4} \ldots, b_{2n-1}, A_1, \ldots, A_n, B_1, \ldots, B_n\}.$$

  The Lie brackets are the same of the ones listed before when n is even, except for the facts that $x-y$ does not belong to the commutator ideal and $[B_i, b_k] = (-1)^i A_{k-i}$, for any $1 \leq k-i \leq n$. In this case the commutator ideal is the subspace generated by

  $$\{E_2, E_4, \ldots, E_{n-1}, c_3, c_5, \ldots, c_n, b_{n+2}, b_{n+4} \ldots, b_{2n-1}, A_1, \ldots, A_n, B_1, \ldots, B_n\}$$

  and we have a $(\frac{n+1}{2}+1)-$step solvable Lie algebra with nilradical

  $$N = \langle E_1, \ldots, E_{n-1}, c_3, c_5, \ldots, c_n, b_{n+2}, b_{n+4} \ldots, b_{2n-1}, A_1, \ldots, A_n, B_1, \ldots, B_n\rangle.$$

In both cases n is odd or even, the center $\mathrm{Z}(\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_0}))$ is trivial, the Lie algebra of inner derivations is

$$\mathrm{Inn}(\mathfrak{k}_n) = \langle A_1, \ldots, A_n, B_1, \ldots, B_n\rangle \cong \mathbb{F}^{2n},$$

since

$$\mathrm{ad}_{e_i} = B_{i-1} + B_i,\ \mathrm{ad}_{f_i} = A_i + A_{i+1},\ \ \forall i = 1, \ldots, n$$

and

$$\mathrm{Der}(\mathfrak{h}_{2n+1}) \supseteq \mathrm{Der}(\mathfrak{k}_n) \supseteq \mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a}),$$

for any $a \neq 0$. More precisely

$$\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_0}) \cap \mathrm{Der}(\mathfrak{k}_n) = \mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a}).$$

3.1.5 Derivations of the Dieudonné Leibniz algebra $\mathfrak{d}_n$

Finally we study the derivations of the *Dieudonné Leibniz algebra* $\mathfrak{d}_n$. We fix the basis $\{e_1, \ldots, e_{2n+1}, z\}$ of $\mathfrak{d}_n$.

If

$$D = \left(\begin{array}{c|c} A & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline a_1\ a_2 \ldots a_{2n+1} & \gamma \end{array}\right)$$

is a derivation of $\mathfrak{d}_n$, then it turns out that the entries of $A = (a_{ij})_{i,j} \in M_{2n+1}(\mathbb{F})$ must satisfy the following set of equations

$$\begin{cases} a_{j+n,1} - a_{j+n+1,1} + a_{n+2,j} = 0, & \forall j = 2, \ldots, n, \\ a_{2n+1,1} + a_{n+2,n+1} = 0, & \\ a_{j-n,1} + a_{j-n-1,1} + a_{n+2,j} = 0, & \forall j = n+3, \ldots, 2n+1, \\ a_{11} + a_{21} + a_{n+2,n+2} = \gamma, & \\ -a_{n+2,i} + a_{n+i,1} + a_{n+i+1,1} = 0, & \\ a_{n+j,i} - a_{n+j+1,i} + a_{n+i,j} + a_{n+i+1,j} = 0, & \forall j = 2, \ldots, n, \\ a_{2n+1,i} + a_{n+i,n+1} + a_{n+i+1,n+1} = 0, & \\ a_{j-n,i} + a_{j-n-1,i} + a_{n+i,j} + a_{n+i+1,j} = 0, & \forall j = n+2, \ldots, 2n+1,\ j \neq n+i, n+i+1, \\ a_{ii} + a_{i-1,i} + a_{n+i,n+i} + a_{n+i+1,n+i} = \gamma, & \\ a_{i+1,i} + a_{ii} + a_{n+i,n+i+1} + a_{n+i+1,n+i+1} = \gamma & \end{cases} \tag{3.1}$$

for every $i = 2, \ldots, n$ and

$$\begin{cases} -a_{n+2,n+1} + a_{2n+1,1} = 0, & \\ a_{n+j,n+1} - a_{n+j+1,n+1} + a_{2n+1,j} = 0, & \forall j = 2, \ldots, n, \\ a_{2n+1,n+1} = 0, & \\ a_{j-n,n+1} + a_{j-n-1,n+1} + a_{2n+1,j} = 0, & \forall j = n+2, \ldots, 2n, \\ a_{n,n+1} + a_{n+1,n+1} + a_{2n+1,2n+1} = \gamma, & \\ -a_{n+2,k} + a_{k-n,1} - a_{k-n-1,1} = 0, & \text{if } k \neq n+2, \\ a_{n+j,k} - a_{n+j+1,k} + a_{k-n,j} - a_{k-n-1,j} = 0, & \forall j = 2, \ldots, n,\ j \neq k-n, k-n-1, \\ a_{2n+1,k} + a_{k-n,n+1} - a_{k-n-1,n+1} = 0, & \\ a_{j-n,k} + a_{j-n-1,k} + a_{k-n,j} - a_{k-n-1,j} = 0, & \forall j = n+2, \ldots, 2n+1, \\ a_{kk} - a_{k+1,k} + a_{k-n,k-n} - a_{k-n-1,k-n} = \gamma, & \text{if } k \neq 2n+1, \\ a_{2n+1,2n+1} + a_{n+1,n+1} - a_{n,n+1} = \gamma, & \\ -a_{n+2,n+2} + a_{21} - a_{11} = -\gamma, & \\ a_{k-1,k} - a_{kk} + a_{k-n,k-n-1} - a_{k-n-1,k-n-1} = -\gamma, & \text{if } k \neq n+2. \end{cases} \tag{3.2}$$

for every $k = n+2, \ldots, 2n+1$.

Theorem 3.1.2. *A derivation of $\mathfrak{d}_n$ is represented by a $(2n+2) \times (2n+2)$ matrix*

$$\left(\begin{array}{c|c|c} \alpha I_{n+1} & C & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 & \beta I_n & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \mu_1 \ldots \mu_{n+1} & \nu_1 \ldots \nu_n & \alpha + \beta \end{array}\right)$$

where the $(n+1)\times n$ *matrix* C *is*

$$\begin{pmatrix}
\alpha_1 & 0 & \alpha_2 & 0 & \alpha_3 & 0 & \cdots & \alpha_{\frac{n}{2}} & 0\\
0 & -\alpha_2 & 0 & -\alpha_3 & 0 & -\alpha_4 & \cdots & 0 & -\alpha_{\frac{n}{2}+1}\\
\alpha_2 & 0 & \alpha_3 & 0 & \alpha_4 & 0 & \cdots & \alpha_{\frac{n}{2}+1} & 0\\
0 & -\alpha_3 & 0 & -\alpha_4 & 0 & -\alpha_5 & \cdots & 0 & -\alpha_{\frac{n}{2}+2}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\
0 & -\alpha_{\frac{n}{2}+1} & 0 & -\alpha_{\frac{n}{2}+2} & 0 & -\alpha_{\frac{n}{2}+3} & \cdots & 0 & -\alpha_n\\
\alpha_{\frac{n}{2}+1} & 0 & \alpha_{\frac{n}{2}+2} & 0 & \alpha_{\frac{n}{2}+3} & 0 & \cdots & \alpha_n & 0
\end{pmatrix}$$

if n *is even and*

$$\begin{pmatrix}
\alpha_1 & 0 & \alpha_2 & 0 & \alpha_3 & 0 & \cdots & 0 & \alpha_{\frac{n+1}{2}}\\
0 & -\alpha_2 & 0 & -\alpha_3 & 0 & -\alpha_4 & \cdots & -\alpha_{\frac{n+1}{2}} & 0\\
\alpha_2 & 0 & \alpha_3 & 0 & \alpha_4 & 0 & \cdots & 0 & \alpha_{\frac{n+1}{2}+1}\\
0 & -\alpha_3 & 0 & -\alpha_4 & 0 & -\alpha_5 & \cdots & -\alpha_{\frac{n+1}{2}+1} & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\
\alpha_{\frac{n+1}{2}} & 0 & \alpha_{\frac{n+1}{2}+1} & 0 & \alpha_{\frac{n+1}{2}+2} & 0 & \cdots & 0 & \alpha_n\\
0 & -\alpha_{\frac{n+1}{2}+1} & 0 & -\alpha_{\frac{n+1}{2}+2} & 0 & -\alpha_{\frac{n+1}{2}+3} & \cdots & -\alpha_n & 0
\end{pmatrix}$$

if n *is odd.*

The Lie algebra $\mathrm{Der}(\mathfrak{d}_n)$ has dimension $3n+3$ and basis

$$\{x, y, E_1, \ldots, E_n, A_1, \ldots, A_{2n+1}\},$$

where

$$\begin{aligned}
x &= \sum_{i=1}^{n+1} e_{ii} + e_{2n+1,2n+1},\\
y &= \sum_{i=n+2}^{2n+2} e_{ii},\ A_i = e_{2n+1,i},\quad \forall i = 1,\ldots,2n+1,\\
E_i &= \sum_{k=1}^{2i-1}(-1)^{k+1}e_{k,n+2i+1-k},\quad \forall i = 1,\ldots,\left\lfloor\frac{n+1}{2}\right\rfloor,\\
E_{\frac{n}{2}+j} &= \sum_{k=1}^{n+2-2j}(-1)^{k+1}e_{n+2-k,n+2j-1+k},\quad \forall j = 1,\ldots,\frac{n}{2},
\end{aligned}$$

if n even, and

$$E_{\frac{n+1}{2}+j} = \sum_{k=1}^{n+1-2j}(-1)^{k}e_{n+2-k,n+2j+k},\quad \forall j = 1,\ldots,\frac{n-1}{2},$$

if n is odd. The non-zero Lie brackets are

$$\begin{aligned}
&[x, E_i] = [E_i, y] = \alpha_i, \quad \forall i = 1, \dots, n,\\
&[y, A_h] = A_h, \quad \forall h = 1, \dots, n+1,\\
&[x, A_k] = A_k, \quad \forall k = n+2, \dots, 2n+1,\\
&[A_i, E_k] = \varepsilon_j A_j, \quad \forall i = 1, \dots, n+1,
\end{aligned}$$

where $\varepsilon_j = \pm 1$ is the only entry different than zero in the i^{th} row of the matrix α_k and $j \in \{n+2, \dots, 2n+1\}$ is its column. Thus $\mathrm{Der}(\mathfrak{d}_n)$ is a $3-$step solvable Lie algebra with commutator ideal consisting of the matrices

$$\left(\begin{array}{c|c|c}
\begin{matrix} \\ 0 \\ \\ \end{matrix} & \begin{matrix} \\ C \\ \\ \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\
\hline
\begin{matrix} \\ 0 \\ \\ \end{matrix} & \begin{matrix} \\ 0 \\ \\ \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\
\hline
\mu_1 \dots \mu_{n+1} & \nu_1 \dots \nu_n & 0
\end{array}\right)$$

The derived series is

$$\mathrm{Der}(\mathfrak{d}_n) \supseteq \langle E_1, \dots, E_n, A_1, \dots, A_{2n+1} \rangle \supseteq \langle A_{n+2}, \dots, A_{2n+1} \rangle \supseteq 0$$

and the nilradical coincides with the commutator ideal (which is a two-step nilpotent Lie algebra). Finally $\mathrm{Z}(\mathrm{Der}(\mathfrak{d}_n))$ is trivial and the left adjoint maps are

$$\mathrm{ad}_{e_1} = A_{n+2}, \ \mathrm{ad}_{e_i} = A_{n+i} + A_{n+i+1}, \quad \forall i = 2, \dots, n,$$
$$\mathrm{ad}_{e_{n+1}} = A_{2n+1}, \ \mathrm{ad}_{e_j} = A_{j-n} - A_{j-n-1}, \quad \forall j = n+2, \dots, 2n+1,$$

thus the inner derivations of the Dieudonné algebra $\mathfrak{d}_n$ are represented by the matrices of the form

$$\left(\begin{array}{c|c}
\begin{matrix} \\ \mathbf{0} \\ \\ \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\
\hline
\mu_1 \ \mu_2 \dots \mu_n \ \mu \ \nu_1 \dots \nu_n & 0
\end{array}\right)$$

where $\mu = -\sum_{k=1}^{n} \mu_k$. More precisely, the matrix

$$\left(\begin{array}{c|c} & 0 \\ \mathbf{0} & \vdots \\ & 0 \\ \hline 0\dots 0\ \mu_{n+1}\ 0\dots 0 & 0 \end{array}\right)$$

does not represent an inner derivation, for every $a_{n+1} \neq 0$.

For example, we study $\mathrm{Der}(\mathfrak{d}_n)$ in the case that $n \leq 3$.

Example 3.1.2.1. If $n = 1$, then

$$D = \left(\begin{array}{cccc} \alpha & 0 & \alpha_1 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ \hline \mu_1 & \mu_2 & \nu_1 & \alpha+\beta \end{array}\right)$$

thus $\mathrm{Der}(\mathfrak{d}_1)$ is the six-dimensional solvable Lie algebra with basis

$$\{x, y, E, A_1, A_2, A_3\},$$

where $x = e_{11} + e_{22} + e_{44}, y = e_{33} + e_{44}, E = e_{1,3}$ and $A_i = e_{4,i}$, for any $i = 1, 2, 3$, and with non-trivial commutators

$$[x, E] = [E, y] = E,\ [x, A_3] = A_3,\ [y, A_1] = A_1,\ [y, A_2] = A_2,\ [A_1, E] = A_3.$$

Example 3.1.2.2. If $n = 2$, the derivations of $\mathfrak{d}_2$ are of the form

$$\left(\begin{array}{cccccc} \alpha & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \alpha & 0 & 0 & -\alpha_2 & 0 \\ 0 & 0 & \alpha & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 \\ \hline \mu_1 & \mu_2 & \mu_3 & \nu_1 & \nu_2 & \alpha+\beta \end{array}\right)$$

and $\mathrm{Der}(\mathfrak{d}_2)$ is a nine-dimensional Lie algebra with commutator ideal consisting of the matrices

$$\left(\begin{array}{cccccc} 0 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \mu_1 & \mu_2 & \mu_3 & \nu_1 & \nu_2 & 0 \end{array}\right)$$

Example 3.1.2.3. If $n = 3$, then

$$D = \left(\begin{array}{c|c|c} \alpha I_4 & C & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ \hline 0 & \beta I_3 & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline \mu_1\ \mu_2\ \mu_3\ \mu_4 & \nu_1\ \nu_2\ \nu_3 & \alpha+\beta \end{array}\right)$$

where

$$C = \begin{pmatrix} \alpha_1 & 0 & \alpha_2 \\ 0 & -\alpha_2 & 0 \\ \alpha_2 & 0 & \alpha_3 \\ 0 & -\alpha_3 & 0 \end{pmatrix}$$

and the Lie algebra $\mathrm{Der}(\mathfrak{d}_3)$ has dimension 9 with generators

$$\{x, y, E_1, E_2, E_3, A_1, \ldots, A_8\},$$

where

$$\begin{aligned} & x = e_{11} + e_{22} + e_{33} + e_{44} + e_{88},\ y = e_{55} + e_{66} + e_{77} + e_{88}, \\ & E_1 = e_{1,5},\ E_2 = e_{1,7} - e_{2,6} + e_{3,5}, E_3 = e_{3,7} - e_{2,8}, \\ & A_i = e_{8,i},\ \ \forall i = 1, \ldots, 7 \end{aligned}$$

and with Lie brackets

$$\begin{aligned} & [x, E_i] = [E_i, y] = E_i,\ \ \forall i = 1, 2, 3, \\ & [y, A_h] = A_h,\ [x, A_k] = A_k\ \ \forall h = 1, 2, 3, 4,\ \forall k = 5, 6, 7, \\ & [A_1, E_1] = A_5,\ [A_1, E_2] = A_7,\ [A_2, E_2] = -A_6, \\ & [A_3, E_2] = A_5,\ [A_3, E_3] = A_7,\ [A_4, E_3] = -A_6. \end{aligned}$$

3.1.6 Almost inner derivations of nilpotent Leibniz algebras with one-dimensional commutator ideal

We recall that a derivation d of a left Leibniz algebra L is an *almost inner derivation* if $d(x) \in [L, x]$, for every $x \in L$. The set of all almost inner derivations of L forms a Lie subalgebra of $\mathrm{Der}(L)$, denoted by $\mathrm{AIDer}(L)$, containing the ideal $\mathrm{Inn}(L)$ of inner derivations of L.

Derivations of two-step nilpotent Lie algebras were studied in [18] and [17] by D. Burde, K. Dekimpe and B. Verbeke. They proved that every almost inner derivations of a Lie algebra of *genus* 1 (i.e. with one-dimensional commutator ideal) is an inner derivation. We want to generalize this result in the frame of Leibniz algebras.

Proposition 3.1.3. *Let L be a complex nilpotent Leibniz algebra with $[L, L] = \mathbb{C}z$, such that $L \ncong \mathfrak{l}^{J_{\pm1}}_{2n+1}$ and $L \ncong \mathfrak{d}_n$. Then every almost inner derivation*

$d \in \mathrm{AIDer}(L)$ *is an inner derivation.*

Proof. Let $d \in \mathrm{AIDer}(L)$. Then $d(y) \in [L, y] \subseteq [L, L] = \mathbb{C}z$, for any $y \in L$ and $d([L, L]) = 0$. Fixed a basis $\{e_1, \ldots, e_{t-1}, z\}$ of L, where $t = \dim_{\mathbb{C}} L$, we have that

$$d(e_i) = a_i z, \quad \forall i = 1, \ldots, t-1,$$

with $a_i \in \mathbb{F}$ and $d(z) = 0$, thus $d \in \mathrm{Inn}(L)$. □

For the Heisenberg Leibniz algebras $\mathfrak{l}_{2n+1}^{J_a}$ with $a = \pm 1$ (in [58] it was proved that these two algebras are isomorphic) and for the Dieudonné Leibniz algebra $\mathfrak{d}_n$, it is possible to define an almost inner derivation d which is not inner. For instance, if $a = 1$ and we fixed the basis $\{e_1, f_1, \ldots, e_n, f_n, z\}$ of $\mathfrak{l}_{2n+1}^{J_1}$, then the matrix

$$\left(\begin{array}{c|c} & 0 \\ \mathbf{0} & \vdots \\ & 0 \\ \hline 1\ 0 \ldots 0\ 0 & 0 \end{array}\right)$$

defines a derivation $d \in \mathrm{AIDer}(\mathfrak{l}_{2n+1}^{J_1}) \setminus \mathrm{Inn}(\mathfrak{l}_{2n+1}^{J_1})$. In the same way

$$\left(\begin{array}{c|c} & 0 \\ \mathbf{0} & \vdots \\ & 0 \\ \hline 0\ 0 \ldots 0\ 1 & 0 \end{array}\right)$$

is an almost inner but non-inner derivation of $\mathfrak{l}_{2n+1}^{J_{-1}}$. More precisely every almost inner derivation of $\mathfrak{l}_{2n+1}^{J_{\pm 1}}$ is of the form

$$\left(\begin{array}{c|c} & 0 \\ \mathbf{0} & \vdots \\ & 0 \\ \hline \mu_1\ \nu_1 \ldots \mu_n\ \nu_n & 0 \end{array}\right)$$

with $a_1, b_1, \ldots, a_n, b_n \in \mathbb{C}$, meanwhile the inner derivations are represented by the set of matrices

$$\left(\begin{array}{c|c} & 0 \\ \mathbf{0} & \vdots \\ & 0 \\ \hline 0\ \nu_1\ \mu_2\ \nu_2 \ldots \mu_n\ \nu_n & 0 \end{array}\right)$$

for $\mathfrak{l}_{2n+1}^{J_1}$, and by

$$\left(\begin{array}{c|c} \multirow{3}{*}{$\mathbf{0}$} & 0 \\ & \vdots \\ & 0 \\ \hline \mu_1\ \nu_1\ \mu_2\ \nu_2 \dots \mu_n\ 0\ 0 & 0 \end{array}\right)$$

for the Leibniz algebra $\mathfrak{l}_{2n+1}^{J_{-1}}$. Finally AIDer$(\mathfrak{d}_n)$ consists of the matrices of the type

$$\left(\begin{array}{c|c} \multirow{3}{*}{$\mathbf{0}$} & 0 \\ & \vdots \\ & 0 \\ \hline \mu_1 \dots \mu_{n+1}\ \nu_1 \dots \nu_n & 0 \end{array}\right)$$

and an example of almost inner but non-inner derivations is given by the linear map $d \in \mathrm{gl}(\mathfrak{d}_n)$ defined by $d(e_{n+1}) = z$.

3.2 Isotopisms

The concept of isotopism between two algebraic structures was introduced in [1] by A. A. Albert (see also [2] and [38]) in order to classify non-associative algebras by generalizing the notion of isomorphism. Before defining isotopisms for Leibniz algebras and racks, we recall some results and properties of isotopisms in the context of non-associative algebras.

3.2.1 Isotopisms of algebras

The aim of this subsection is to introduce the concept of *isotopism* according to the ideas of A.A. Albert as presented in [1]. More specifically, Albert discusses *isotopy* in both associative and non-associative algebras, while also considering their inclusion of units. In the forthcoming introduction, we will focus exclusively on the non-commutative context we are about to explore, particularly in the context of Leibniz algebras, and will therefore not discuss algebras containing units.

Let A be an algebra over a field $\mathbb{F}$. Denote with $R(A)$ and $L(A)$ the vector spaces, respectively, of all right and left multiplications. Namely,

$$R(A) = \{R_x : A \to A \mid R_x(a) = ax, \forall a, x \in A\}$$
$$L(A) = \{L_x : A \to A \mid R_x(a) = xa, \forall a, x \in A\}$$

Given an algebra A, we can determine the vector space $R(A)$ and establish the $x \mapsto R_x$ mapping that goes from A to $R(A)$ and vice versa (the same can be done for left multiplications). Suppose we have a second algebra A_0 over $\mathbb{F}$ with $\dim_{\mathbb{F}} A_0 = \dim_{\mathbb{F}} A$ and then a corresponding linear mapping $x \mapsto R_x^{(0)}$. So the product $*$ of A_0 can be written as follows

$$a * x = R_x^{(0)}(a).$$

Definition 3.2.1. Two algebras A and A_0 over a field $\mathbb{F}$ are *isotopic* if there exist non-singular linear transformations $P, Q, C \colon A_0 \to A$ such that

$$R_x^{(0)} = P^{-1} R_{Q(x)} C \tag{3.14}$$

for every $x \in A_0$. We shall call Equation (3.14) an *isotopy* (or *isotopism*) of A and A_0.

We remind that in this case the two algebras A_0 A have the same dimension, so here the notions of non-singular linear map and bijective linear map (then invertible) are equivalent.

Proposition 3.2.2. *The relation of isotopy is an equivalence relation.*

Proof. We shall make use the Equation (3.14). Reflexivity follows by selecting $P = Q = C$ the identity, in order to see that $R_x^{(0)}$ is the identity as well. If $R_x^{(0)} = P^{-1} R_{Q(x)} C$ is an isotopy of A and A_0, then putting $y = Q(x)$ we have

$$R_y = P R^{(0)}_{Q^{-1}(y)} C^{-1}.$$

This is an isotopy of A_0 and A. Now let $R_x^{(1)} = P_1^{-1} R^{(0)}_{Q_1(x)} C_1$ be an isotopy of A_0 and A_1. If we put $P_2 = PP_1$, $Q_2 = QQ_1$ and $C_2 = CC_1$, we obtain and isotopism of A_1 and A by

$$R_x^{(1)} = P_2^{-1} R_{Q_2(x)} C_2.$$

Indeed we have

$$\begin{aligned} R_x^{(0)} &= P^{-1} R_{Q(x)} C \\ \Rightarrow R^{(0)}_{Q_1(x)} &= P^{-1} R_{Q(Q_1(x))} C \\ \Rightarrow R_x^{(1)} &= P_1^{-1} \left(P^{-1} R_{Q(Q_1(x))} C \right) C_1. \end{aligned}$$

□

Is it reasonable to ask what happens if, instead of examining right multiplications in an algebra, we consider left multiplications? Before showing an isotopism in these terms, let us make a few remarks beforehand. The following relations hold in an algebra A

$$\begin{aligned} ax &= R_x(a) = L_a(x) \\ xa &= L_x(a) = R_a(x) \end{aligned}$$

for every $x, a \in A$.

Theorem 3.2.3. *Let A and A_0 be algebra over a field $\mathbb{F}$. If A and A_0 are isotopic by Equation (3.14), then this is equivalent to*

$$L_x^{(0)} = P^{-1} L_{C(x)} Q, \tag{3.15}$$

for every $x \in A_0$.

Proof. For $x, a \in A_0$, we put $z := C(x)$. Then, by Equation (3.14) we have

$$\begin{aligned} L_x^{(0)}(a) &= R_a^{(0)}(x) \\ &= P^{-1} R_{Q(a)} C(x) \\ &= P^{-1} R_{Q(a)} z \\ &= P^{-1}(zQ(a)) \\ &= P^{-1} L_z Q(a) \\ &= P^{-1} L_{C(x)} Q(a). \end{aligned}$$

□

With the same arguments used in Proposition 3.2.2 the reader can prove that to be isotopic by Equation (3.15) is still a relation of equivalence. For the sake of completeness and clarity, let us state the result without providing the proof.

Proposition 3.2.4. *The relation of isotopy by Equation (3.15) is an equivalence relation.*

Two algebra $(A, \cdot)$ and $(A_0, *)$ over the same field are said to be *isomorphic* if there exists a bijective map (i.e. non-singular) $\phi : A_0 \to A$ that preserves algebra multiplication. Thus, for every $x, a \in A_0$

$$\phi(a * x) = \phi(a)\phi(x),$$

that is

$$\phi R_x^{(0)}(a) = R_{\phi(x)} \phi(a)$$

and then

$$R_x^{(0)}(a) = \phi^{-1} R_{\phi(x)} \phi(a). \tag{3.16}$$

The last equations is equivalent by Theorem 3.2.3 to the following

$$L_x^{(0)} = \phi^{-1} L_{\phi(x)} \phi. \tag{3.17}$$

It is often more practical to employ simplified versions of equations 3.14 and 3.15, which can be achieved by substituting A_0 with an isomorphic algebra. Thus we may apply Equation (3.16) to Equation (3.14). To be more precise we indicate with A_1 the algebra A with a different multiplication $R^{(1)}$, $\phi\colon A_1 \to A_0$ is an algebra isomorphism and $\phi = Q^{-1}$. Thus we have

$$R_x^{(1)} = \phi^{-1} R_{\phi(x)}^{(0)} \phi = \phi^{-1} P^{-1} R_{Q(\phi(x))} C \phi.$$

This result may be stated as follows.

Proposition 3.2.5. *Let A be an algebra over a field $\mathbb{F}$. Every isotopism of A is equivalent to an isotopism defined by*

$$R_x^{(1)} = P^{-1}R_xC, \quad \forall x \in A,$$

for some non-singular linear transformations $P, C\colon A_1 \to A$.

In light of this latest result, by Theorem 3.2.3 we have the following result.

Proposition 3.2.6. *Let A be an algebra over a field $\mathbb{F}$. Every isotopism of A is equivalent to an isotopism defined by*

$$L_x^{(1)} = P^{-1}L_{C(x)}$$

for some algebra A_0 and for some non-singular linear transformations $P, C\colon A_0 \to A$.

Clearly, the proposition above has the advantage of mapping x to $P^{-1}R_xC$. Even though it is already a simpler isotopism compared to Equation (3.14), it can be further simplified by setting $\phi = C^{-1}$ and obtain

$$R_x^{(1)} = P^{-1}CR_{C^{-1}(x)}.$$

Definition 3.2.7. An isotopism of two algebra A and A_0 is *principal* if $C = \mathrm{id}$.

We give the following result to summarize the results showed above.

Theorem 3.2.8. *Every isotopism of an algebra A is equivalent to a principal isotopism of A_0, that is, an isotopism with*

$$R_x^{(0)} = P^{-1}R_{Q(x)}, \quad L_x^{(0)} = Q^{-1}L_{P(x)},$$

for every $x \in A_0$ and for some non-singular linear transformations $P, C\colon A_0 \to A$.

Before concluding this subsection, we would like to take a moment to reflect on how the results we have just seen clarify the manner in which the notion of isotopism can be thought of as a generalization or a broader concept compared to that of isomorphism.

Example 3.2.8.1. Let $A = \langle e_1, e_2, e_3\rangle$ be a nilpotent 3-dimensional algebra over a field $\mathbb{F}$ such that $e_1e_2 = -e_2e_1 = e_3$ and all other products zero[1]. Now, for every $x \in A$ we can write this element as a linear combinations of e_1, e_2, e_3, namely $x = \xi_1e_1 + \xi_2e_2 + \xi_3e_3$ for some $\xi_1, \xi_2, \xi_3 \in \mathbb{F}$. Thus we have

$$R_x = -L_x = \begin{pmatrix} 0 & 0 & \xi_2 \\ 0 & 0 & -\xi_1 \\ 0 & 0 & 0 \end{pmatrix}.$$

[1]This is the Heisenberg Lie algebra $\mathfrak{h}_3$.

Let

$$\Lambda = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and define an algebra A_0 with the same basis of A which multiplication is

$$R^{(0)} = \Lambda R_x = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \xi_2 \\ 0 & 0 & -\xi_1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \xi_1 \\ 0 & 0 & \xi_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

A and A_0 are isotopic but they are not isomorphic. Indeed,

$$e_1 e_1 = R^{(0)}_{e_1}(e_1) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = e_3$$

and in A the square of every element is zero, while in A_0 we have

$$\phi(e_1 e_1) = 0 \neq e_3 = e_1 e_1 = \phi(e_1)\phi(e_1).$$

To be thorough and comprehensive, there is much more to discuss regarding the isotopisms of algebras. However, as mentioned earlier, we would also need to consider associative algebras with or without units. This would extend our discussion considerably and take us further away from our main focus, which is on Leibniz algebras. Nevertheless, we refer the reader to the articles by A.A. Albert ([1] and [2]) for a broader overview of this topic.

3.2.2 Isotopisms of Leibniz algebras and racks

Definition 3.2.9. Let $(\mathfrak{g}, [-,-])$ and $(\mathfrak{h}, [-,-]_{\mathfrak{h}})$ be left Leibniz algebras over $\mathbb{F}$. An *isotopism* between $\mathfrak{g}$ and $\mathfrak{h}$ is a triple of linear isomorphisms

$$(f, g, h) \colon \mathfrak{g} \leftrightarrows \mathfrak{h}$$

such that

$$[f(x), g(y)]_{\mathfrak{h}} = h([x, y]), \quad \forall x, y \in \mathfrak{g}.$$

We say that $\mathfrak{g}$ and $\mathfrak{h}$ are *isotopic* Leibniz algebras.

An isotopism is said to be a *right isotopism* if $f = h$ and a *left isotopism* if $g = h$. We observe that, if $f \colon \mathfrak{g} \leftrightarrows \mathfrak{h}$ is an isomorphism of Leibniz algebras, then the triple (f, f, f) is a (left and right) isotopism.

Definition 3.2.10. Let $(X, \triangleright, 1)$ and $(Y, \triangleright_Y, 1_Y)$ be left pointed racks. An *isotopism* between X and Y is a triple of bijections

$$(f, g, h) \colon X \leftrightarrows Y$$

such that $f(1) = g(1) = h(1) = 1_Y$ and

$$f(x) \triangleright_Y g(y) = h(x \triangleright y), \quad \forall x, y \in X.$$

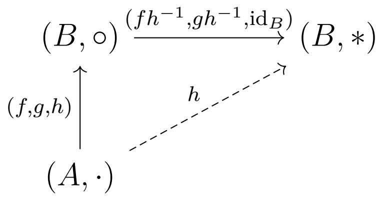
Also in this case, every isomorphism of racks can be seen as a left and right isotopism. Moreover, one can define the notion of *Lie rack isotopism* by asking that f, g, h are diffeomorphisms.

The following is the generalization of a result proved by A. A. Albert in [2] for isotopisms of *quasigroups*.

Remark 3.2.1. Every isotopism (f, g, h) between two algebraic structures is isomorphic to a *principal isotopism*, i.e. an isotopism of the form $(\tilde{f}, \tilde{g}, \mathrm{id})$. To be more precise, let $(A, \cdot)$ and $(B, \circ)$ be two algebraic structures and let $(f, g, h)\colon A \leftrightarrows B$ be an isotopism. We define a new binary operation $*\colon B \times B \to B$ by

$$h(x) * h(y) = h(xy), \quad \forall x, y \in A.$$

In this way $(A, \cdot)$ and $(B, *)$ are isomorphic and $h(x) * h(y) = f(x) \circ g(y)$. Thus $(hf^{-1})(\bar{x}) * (hg^{-1})(\bar{y}) = \bar{x} \circ \bar{y}$, where $\bar{x} = f(x)$, $\bar{y} = g(y)$ and we obtain the **principal isotopism** $(fh^{-1}, gh^{-1}, \mathrm{id}_B)\colon (B, \circ) \leftrightarrows (B, *)$.



In the case that $(A, \cdot)$ and $(B, \circ)$ are finite dimensional non-associative algebras over a field $\mathbb{F}$, such as Leibniz algebras, we can identify the underlying vector space of A and B with $\mathbb{F}^n$, where $n = \dim_{\mathbb{F}} A = \dim_{\mathbb{F}} B$, and $\mathrm{id}_B = \mathrm{id}_{\mathbb{F}^n}$.

From Definition 3.2.9, it turns out that a difference between isomorphisms and isotopisms is that one can find a Lie algebra that is isotopic to a Leibniz non-Lie algebra, as the following example shows.

Example 3.2.10.1. Let $\mathfrak{h}_3$ be the three-dimensional Heisenberg Lie algebra, let $A = (a_{ij})_{i,j} \in \mathrm{GL}_2(\mathbb{F})$, $B = (b_{ij})_{i,j} = A \cdot \mathrm{diag}\{\lambda, \mu\}$, with $\lambda, \mu \in \mathbb{F}^*$ and let $\mathfrak{g}^A_{\lambda,\mu}$ be the nilpotent Leibniz algebra with one-dimensional commutator ideal with structure matrix

$$\begin{pmatrix} 0 & a_{11}b_{22} - a_{21}b_{12} & 0 \\ a_{12}b_{21} - a_{22}b_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the triple of linear isomoprhisms $(f, g, \mathrm{id}_{\mathbb{F}^3})\colon \mathfrak{g}^A_{\lambda,\mu} \leftrightarrows \mathfrak{h}_3$, where f and g are defined respectively by the matrices

$$\begin{pmatrix} a_{11} & a_{21} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & \alpha \end{pmatrix} \text{ and } \begin{pmatrix} b_{11} & b_{21} & 0 \\ b_{12} & b_{22} & 0 \\ 0 & 0 & \beta \end{pmatrix},$$

with $\alpha, \beta \in \mathbb{F}^*$, is a principal isotopism. Indeed, for every $x \equiv (x_1, x_2, x_3), y \equiv (y_1, y_2, y_3) \in \mathbb{F}^3$, we have

$$\begin{aligned}[f(x), g(y)]_{\mathfrak{h}_3} &= (0, 0, (a_{11}x_1 + a_{12}x_2)(b_{21}y_1 + b_{22}y_2) - (a_{21}x_1 + a_{22}x_2)(b_{11}y_1 + b_{12}y_2)) \\ &= (0, 0, (a_{11}b_{22} - a_{21}b_{12})x_1y_2 - (a_{12}b_{21} - a_{22}b_{11})x_2y_1) = [x, y]_{\mathfrak{g}^A_{\lambda,\mu}},\end{aligned}$$

since $a_{11}b_{21} - b_{11}a_{21} = a_{12}b_{22} - b_{12}a_{22} = 0$.

If $\lambda = \dfrac{2}{\det(A)} - \mu$ and $\alpha = \mu \det(A) - 1$, then $\mathfrak{g}^A_{\lambda,\mu} = \mathfrak{l}^\alpha_3$. Indeed $\mathfrak{g}^A_{\lambda,\mu}$ and $\mathfrak{l}^\alpha_3$ have the same structure matrix, that is

$$\begin{pmatrix} 0 & \mu\det(A) & 0 \\ -\lambda\det(A) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1+\alpha & 0 \\ -1+\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Conversely, for every $\alpha \in \mathbb{F}^* \setminus \{\pm 1\}$, if we choose $A = \mathrm{I}_2$, $\lambda = 1-\alpha$ and $\mu = 1+\alpha$, then $\mathfrak{g}^{\mathrm{I}_2}_{\lambda,\mu} = \mathfrak{l}^\alpha_3$. Thus $\mathfrak{h}_3$ and $\mathfrak{l}^\alpha_3$ are isotopic for every $\alpha \in \mathbb{F}$, $\alpha \neq \pm 1$. We note that this result is not true if $\alpha = \pm 1$, in fact in the next section we will show that $\mathfrak{l}^{\pm 1}_3$ and $\mathfrak{h}_3$ are not isotopic.

In a similar way, if $\mu = -\lambda = \dfrac{1}{\det(A)}$, then the structure matrix of $\mathfrak{g}^A_{\lambda,\mu}$ is

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

i.e. $\mathfrak{g}^A_{\lambda,\mu} = \mathfrak{k}_1$ is the Kronecker Leibniz algebra. Thus $\mathfrak{l}^\alpha_3$ is isotopic to $\mathfrak{k}_1$, for every $\alpha \neq \pm 1$.

3.2.3 Isotopisms of two-step nilpotent Leibniz algebras and Lie racks

The notion of isotopism was used in [39] and [40] for classifying some families of Lie algebras, such as *filiform Lie algebras* over finite fields. The authors also determined some isotopism invariants for these algebras.

The main goal of this section is to investigate the isotopism classes of indecomposable nilpotent Leibniz algebras with one-dimensional commutator ideal over a field $\mathbb{F}$, with $\mathrm{char}(\mathbb{F}) \neq 2$. Moreover we will show that, when $\mathbb{F} = \mathbb{R}$, two Leibniz algebras of this type are isotopic if and only if the Lie racks integrating them are isotopic. For doing this, we need to find new isotopism invariants for Leibniz algebras and Lie racks.

We start with the following definition, which was given by M. Elhamdadi and E. M. Moutuou in [35].

Definition 3.2.11. Let X be a left rack. The *center* of X is

$$\mathrm{Z}(X) = \{x \in X \mid x \rhd y = y,\ \forall y \in X\}.$$

We observe that $\mathrm{Z}(X)$ is a *subrack* of X since $1 \rhd y = y$ and

$$(x \rhd x') \rhd y = (x \rhd x') \rhd (x \rhd y) = x \rhd (x' \rhd y) = x \rhd y = y,$$

for every $x, x' \in \mathrm{Z}(X)$ and for every $y \in X$. In [35] the center of X is called the *stabilizer* of X and it is denoted by $\mathrm{Stab}(X)$. We prefer the designation "center" for the following reasons:

- If G is a group, then $\mathrm{Z}(\mathrm{Conj}(G)) = \mathrm{Z}(G)$;
- If X is a Lie rack and $\mathfrak{g} = \mathrm{T}_1 X$, then $\mathrm{T}_1 \mathrm{Z}(X) = \mathrm{Z}_l(\mathfrak{g})$.

Proposition 3.2.12.

(i) Let $\mathfrak{g}, \mathfrak{h}$ be Leibniz algebras and let $(f, g, h)\colon \mathfrak{g} \leftrightarrows \mathfrak{h}$ be an isotopism. Then

$$f(\mathrm{Z}_l(\mathfrak{g})) = \mathrm{Z}_l(\mathfrak{h}), \quad g(\mathrm{Z}_r(\mathfrak{g})) = \mathrm{Z}_r(\mathfrak{h}), \quad h([\mathfrak{g}, \mathfrak{g}]) = [\mathfrak{h}, \mathfrak{h}]$$

and the dimensions of the left center, of the right center and of the commutator ideal are isotopism invariants*.*

(ii) Let X, Y be left racks and let $(f, g, g)\colon X \leftrightarrows Y$ be a left isotopism*. Then*

$$f(\mathrm{Z}(X)) = \mathrm{Z}(Y).$$

In the case of Lie racks, the dimension of the center is a left isotopism inviariant*.*

Proof.

(i) Let $x, y \in \mathfrak{h}$ such that $x = f(\bar{x})$, $y = g(\bar{y})$, with $\bar{x}, \bar{y} \in \mathfrak{g}$. If $\bar{x} \in \mathrm{Z}_l(\mathfrak{g})$, then

$$[x, y]_{\mathfrak{h}} = [f(\bar{x}), g(\bar{y})]_{\mathfrak{h}} = h([\bar{x}, \bar{y}]) = h(0) = 0$$

hence $x \in \mathrm{Z}_l(\mathfrak{h})$. Conversely, if $x \in \mathrm{Z}_l(\mathfrak{h})$, then

$$0 = [\bar{x}, \bar{y}]_{\mathfrak{h}} = [f(\bar{x}), g(\bar{y})]_{\mathfrak{h}} = h([x, y]).$$

Thus $[\bar{x}, \bar{y}] \in \mathrm{Ker}(h) = 0$, i.e. $\bar{x} \in \mathrm{Z}_l(\mathfrak{g})$ and $f(\mathrm{Z}_l(\mathfrak{g})) = \mathrm{Z}_l(\mathfrak{h})$. In a similar way, one can check that $g(\mathrm{Z}_r(\mathfrak{g})) = \mathrm{Z}_r(\mathfrak{h})$. Finally, for every $\bar{x}, \bar{y} \in \mathfrak{g}$

$$h([\bar{x}, \bar{y}]) = [f(\bar{x}), g(\bar{y})]_{\mathfrak{h}} \in [\mathfrak{h}, \mathfrak{h}].$$

Coversely, for any $x, y \in \mathfrak{h}$, with $x = f(\bar{x})$ and $y = g(\bar{y})$, we have

$$[x, y]_{\mathfrak{h}} = h([\bar{x}, \bar{y}]) \in h([\mathfrak{g}, \mathfrak{g}])$$

and $h([\mathfrak{g}, \mathfrak{g}]) = [\mathfrak{h}, \mathfrak{h}]$.

(ii) Let $x, y \in Y$ such that $x = f(\bar{x})$, $y = g(\bar{y})$, with $\bar{x}, \bar{y} \in X$. If $\bar{x} \in \mathrm{Z}(X)$, then

$$x \rhd y = f(\bar{x}) \rhd_Y g(\bar{y}) = g(\bar{x} \rhd \bar{y}) = g(\bar{y}) = y$$

hence $x \in \mathrm{Z}(Y)$. Conversely, if $x \in \mathrm{Z}(Y)$, then

$$g(\bar{y}) = y = x \rhd_Y y = f(\bar{x}) \rhd_Y g(\bar{y}) = g(\bar{x} \rhd \bar{y})$$

and the statement is proved, since g is injective.

□

The last proposition allows us to conclude that the algebras $\mathfrak{l}_3^{\pm 1}$ and $\mathfrak{h}_3$ are not isotopic, since $\dim_{\mathbb{F}} \mathrm{Z}_l(\mathfrak{l}_3^1) = 2$ (see Section 2.2) and $\dim_{\mathbb{F}} \mathrm{Z}(\mathfrak{h}_3) = 1$. Moreover we recall that $\mathfrak{l}_3^1$ and $\mathfrak{l}_3^{-1}$ are isomorphic, and $\mathfrak{h}_3$ is isotopic to $\mathfrak{l}_3^{\alpha}$, for every $\alpha \neq \pm 1$, thus $\mathfrak{l}_3^{\alpha}$ and $\mathfrak{l}_3^{\pm 1}$ are not isotopic. This fact, combined with the one we illustrated at the end of the previous section (Example 3.2.10.1), leeds to the following.

Proposition 3.2.13. *Let $\mathfrak{g}$ be a three-dimensional indecomposable nilpotent Leibniz algebra with one-dimensional commutator ideal over a field $\mathbb{F}$, with $\mathrm{char}(\mathbb{F}) \neq 2$. Then either $\mathfrak{g}$ is isomorphic to the Heisenberg Leibniz algebra $\mathfrak{l}_3^1$, or $\mathfrak{g}$ is isotopic to the Heisenberg Lie algebra $\mathfrak{h}_3$.*

We want now to extend this result to the case that $\dim_{\mathbb{F}} \mathfrak{g} > 3$.

Theorem 3.2.14. *Let $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) \neq 2$ and let $\mathfrak{g}$ be a nilpotent Leibniz algebra with $\dim_{\mathbb{F}} \mathfrak{g} = t \geq 3$ and $\dim_{\mathbb{F}}[\mathfrak{g}, \mathfrak{g}] = 1$.*

(i) If $t = 2n+2$, then $\mathfrak{g}$ is isomorphic to the Dieudonné *Leibniz algebra $\mathfrak{d}_n$;*

(ii) If $t = 2n+1$, then either $\mathfrak{g}$ is isomorphic to the Heisenberg Leibniz algebra $\mathfrak{l}_{2n+1}^{J_1}$, where J_1 is the $n \times n$ Jordan block of eigenvalue 1, or $\mathfrak{g}$ is isotopic to the Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$;

(iii) The Heisenberg Leibniz algebras $\mathfrak{l}_{2n+1}^{J_1}$ and $\mathfrak{l}_{2n+1}^{J_{-1}}$ are not isotopic to any Lie algebra.

Proof.

(i) If $\dim_{\mathbb{F}} \mathfrak{g}$ is even, then $\mathfrak{g}$ must be isomorphic to the Dieudonné algebra $\mathfrak{d}_n$, where $t = 2n+2$;

(ii) If $\dim_{\mathbb{F}} \mathfrak{g} = 2n+1$ and $\mathfrak{g}$ is not isomorphic to $\mathfrak{l}_{2n+1}^{J_1}$, then $\mathfrak{g} \cong \mathfrak{k}_n$ or $\mathfrak{g} \cong \mathfrak{l}_{2n+1}^{A}$, where A is the companion matrix of the power of an irreducible monic polynomial $p(x)^k \in \mathbb{F}[x]$ and it is not similar to the Jordan blocks J_1 adn J_{-1}. We want to show that both these Leibniz algebras are isotopic to the Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$. The Leibniz algebra $\mathfrak{k}_n$ is isotopic to the $(2n+1)$-dimensional Heisenberg Lie algebra via the left principal isotopism $(f, \mathrm{id}_{\mathbb{F}^t}, \mathrm{id}_{\mathbb{F}^t})$, where

$$f(x_1, \dots, x_n, y_1, \dots, y_n, z) =$$
$$(x_1 + x_2, x_2 + x_3, \dots, x_{n-1} + x_n, x_n, -y_1, y_1 - y_2, y_2 - y_3, \dots, y_{n-1} - y_n, z),$$

for every $(x_1, \ldots, x_n, y_1, \ldots, y_n, z) \in \mathbb{F}^t$. The Heisenberg Leibniz algebra $\mathfrak{l}_{2n+1}^A$ is isotopic to $\mathfrak{h}_{2n+1}$ via the triple $(f, \mathrm{id}_{\mathbb{F}^t}, \mathrm{id}_{\mathbb{F}^t})$, where f is the linear isomorphism defined by the matrix

$$\left(\begin{array}{c|c|c} I_n + A & \mathbf{0} & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ \hline \mathbf{0} & I_n - A^T & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ \hline 0 \ \cdots\cdots \ 0 & 0 \ \cdots\cdots \ 0 & 1 \end{array}\right)$$

We observe that f is a bijection since the matrices $I_n + A$ and $I_n - A^T$ are invertible. Indeed one has

$$\det(xI_n + A) = (-1)^n f(x),$$

$$\det(xI_n - A^T) = \det(xI_n - A) = f(x)$$

and for $x = 1$

$$\det(I_n + A) = (-1)^n f(-1),$$
$$\det(I_n - A^T) = f(1).$$

Thus f is not invertible if and only if $f(1) = 0$ or $f(-1) = 0$. Since $f(x)$ is irreducible over $\mathbb{F}$, this would imply that $f(x) = x - 1$ or $f(x) = x + 1$ and A would be similar to the $n \times n$ Jordan block J_1 or J_{-1}. We have a contradiction since we supposed that $\mathfrak{g}$ is not isomorphic to the Heisenberg Leibniz algebra $\mathfrak{l}_{2n+1}^{J_1}$. Finally, by Proposition 3.2.12, we have that $\mathfrak{l}_{2n+1}^{J_1}$ and $\mathfrak{l}_{2n+1}^{J_{-1}}$ are not isotopic to $\mathfrak{h}_{2n+1}$ since

$$\dim_{\mathbb{F}} \mathrm{Z}_l(\mathfrak{l}_{2n+1}^{J_{\pm 1}}) = 2,$$

$$\dim_{\mathbb{F}} \mathrm{Z}(\mathfrak{h}_{2n+1}) = 1$$

and the dimension of the left center is an isotopism invariant.

(iii) If we suppose that there exists a Lie algebra $\mathfrak{p}$ and an isotopism

$$(f, g, h) \colon \mathfrak{l}_{2n+1}^{J_1} \leftrightarrows \mathfrak{p},$$

then, by Proposition 3.2.12, $\dim_{\mathbb{F}}[\mathfrak{p}, \mathfrak{p}] = 1$ and $\dim_{\mathbb{F}} \mathrm{Z}(\mathfrak{p}) = 2$. For the classification of Lie algebras with one-dimensional commutator ideal (see Section 3 of [36]), $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$, where $\mathfrak{p}_1$ is an abelian Lie algebra and $\mathfrak{p}_2$ is either the non-abelian Lie algebra of dimension 2 (i.e. $\mathfrak{p}_2$ has basis $\{e_1, e_2\}$ and the the Lie bracket is defined by $[e_1, e_2] = e_1$), or it is an Heisenberg algebra $\mathfrak{h}_{2k+1}$. In the first case, $\dim_{\mathbb{F}} \mathfrak{p}_1$ must be odd. Thus $\dim_{\mathbb{F}} \mathrm{Z}(\mathfrak{p})$ is also odd, since $\mathrm{Z}(\mathfrak{p}) = \mathfrak{p}_1$. In the second case, $\dim_{\mathbb{F}} \mathfrak{p}_1$ must be even and

$$\dim_{\mathbb{F}} \mathrm{Z}(\mathfrak{p}) = \dim_{\mathbb{F}} \mathfrak{p}_1 + \dim_{\mathbb{F}} \mathrm{Z}(\mathfrak{h}_{2k+1}) = \dim_{\mathbb{F}} \mathfrak{p}_1 + 1$$

is odd. Hence, in both cases we have a contradiction and $\mathfrak{l}_{2n+1}^{J_1}$ cannot be isotopic to a Lie algebra. The statement is also valid for the Leibniz algebra $\mathfrak{l}_{2n+1}^{J_{-1}}$, since it is isomorphic to $\mathfrak{l}_{2n+1}^{J_1}$.

□

We just showed that an isotopism between two nilpotent Leibniz algebras with one-dimensional commutator ideal $\mathfrak{g}, \mathfrak{h}$ can be always chosen of the form $(f, \mathrm{id}_{\mathbb{F}^t}, \mathrm{id}_{\mathbb{F}^t})$. Next we show that for $\mathbb{F}$ the reals, such an isotopism turns out to be also an isotopism between the Lie racks integrating $\mathfrak{g}$ and $\mathfrak{h}$. Recall from Theorem 2.3.15 that we can identify the underlying vector space of a Lie rack X integrating a two step nilpotent Leibniz algebra, with its tangent space $\mathrm{T}_1 X$.

Lemma 3.2.15. *Let $\mathbb{F} = \mathbb{R}$ and let $\mathfrak{g}, \mathfrak{h}$ be two-step nilpotent Leibniz algebras with $\dim_{\mathbb{R}} \mathfrak{g} = \dim_{\mathbb{R}} \mathfrak{h} = t$. Let X, Y be the Lie racks integrating $\mathfrak{g}$ and $\mathfrak{h}$ respectively.*

- *(i) If $(f, g, g)\colon \mathfrak{g} \leftrightarrows \mathfrak{h}$ is a left Leibniz algebra isotopism, then it is also a Lie rack isotopism between X and Y;*
- *(ii) If $(f, g, h)\colon X \leftrightarrows Y$ is a Lie rack isotopism, then $g = h$ and it turns out to be a Leibniz algebra isotopism between $\mathfrak{g}$ and $\mathfrak{h}$;*
- *(iii) Let $\dim_{\mathbb{R}}[\mathfrak{g}, \mathfrak{g}] = \dim_{\mathbb{R}}[\mathfrak{h}, \mathfrak{h}] = 1$. Then $\mathfrak{g}$ is isotopic to $\mathfrak{h}$ if and only if X is isotopic to Y.*

Proof. The multiplications of the Lie racks X and Y can be written as

$$x \rhd y = y + [x, y]_{\mathfrak{g}}, \quad x' \rhd_Y y' = y' + [x', y']_{\mathfrak{h}}, \quad \forall x, y \in X, \ \forall x', y' \in Y.$$

(i) If $(f, g, g)\colon \mathfrak{g} \leftrightarrows \mathfrak{h}$ is an isotopism of Leibniz algebra, then

$$f(x) \rhd_Y g(y) = g(y) + [f(x), g(y)]_{\mathfrak{h}} = g(y) + g([x, y]) = g(x \rhd y),$$

for every $x, y \in X$;

(ii) If $(f, g, h)\colon X \leftrightarrows Y$ is an isotopism of Lie racks, then

$$g(y) = 1_Y \rhd_Y g(y) = f(1_X) \rhd_Y g(y) = h(1_X \rhd y) = h(y), \quad \forall y \in X,$$

thus $g = h$ and the triple (f, g, g) becomes an isotopism between $\mathfrak{g}$ and $\mathfrak{h}$, since

$$g(y) + [f(x), g(y)]_{\mathfrak{h}} = f(x) \rhd_Y g(y) = g(x \rhd y) = g(y) + g([x, y]),$$

for every $x, y \in X$, and then

$$[f(x), g(y)]_{\mathfrak{h}} = g([x, y]).$$

(iii) If X and Y are isotopic Lie racks, then from (ii) $\mathfrak{g}$ and $\mathfrak{h}$ are isotopic Leibniz algebra. Conversely, if $\mathfrak{g}$ and $\mathfrak{h}$ are isotopic nilpotent Leibniz algebras with one-dimensional commutator ideal, then by Theorem 3.2.14, an isotopism between them can be chosen of the form $(f, \mathrm{id}_{\mathbb{F}^t}, \mathrm{id}_{\mathbb{F}^t})$ and, from (ii), it becomes an isotopism between X and Y.

□

The last result allows us to describe the isotopism classes of Lie global racks integrating the indecomposable nilpotent Leibniz algebras with one-dimensional commutator ideal.

Theorem 3.2.16. *Let $\mathbb{F} = \mathbb{R}$ and let X be a Lie rack integrating a nilpotent Leibniz algebra $\mathfrak{g}$ with $\dim_{\mathbb{F}} \mathfrak{g} = t$ and $\dim_{\mathbb{F}}[\mathfrak{g}, \mathfrak{g}] = 1$.*

(i) If $t = 2n + 2$, then X is isomorphic to the Dieudonné *rack D_n;*

(ii) If $t = 2n + 1$, then either X is isomorphic to the Heisenberg rack $R^{J_1}_{2n+1}$, where J_1 is the $n \times n$ Jordan block of eigenvalue 1, or X is isotopic to the conjugation of the Heisenberg Lie group $\mathrm{Conj}(H_{2n+1})$*;*

(iii) The Heisenberg racks $R^{J_1}_{2n+1}$ and $R^{J_{-1}}_{2n+1}$ are not isotopic to any Lie quandle.

Proof.

(i) If t is even, then $\mathfrak{g} \cong \mathfrak{d}_n$ and X is isomorphic to the Dieudonné rack D_n;

(ii) If $t = 2n + 1$ and $\mathfrak{g} \cong \mathfrak{l}^{J_1}_{2n+1}$, then X is isomorphic to the Heisenberg rack $R^{J_1}_{2n+1}$. If this is not the case, then $\mathfrak{g}$ is isotopic to $\mathfrak{h}_{2n+1}$ and X is isomorphic either to the Kronecker rack K_n, or to R^A_{2n+1}, where $A = J_\alpha$ with $\alpha \in \mathbb{R} \setminus \{\pm 1\}$, or $A = J_R$ and $R = R_{\alpha,\beta}$ as in Section 2.2. Note that the last case is admissible if and only if n in even. Since both, $\mathfrak{k}_n$ and $\mathfrak{l}^A_{2n+1}$, are isotopic to the Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$ with a left principal isotopism $(f, \mathrm{id}_{\mathbb{F}^t}, \mathrm{id}_{\mathbb{F}^t})$, by Lemma 3.2.15 we can conclude that both K_n and R^A_{2n+1} are isotopic to $\mathrm{Conj}(H_{2n+1})$. Thus X is isotopic to the conjugation of the Heisenberg Lie group. Finally $R^{J_1}_{2n+1}$ and $\mathrm{Conj}(H_{2n+1})$ are not isotopic. Indeed, if we suppose that there exists an isotopism

$$(f, g, h)\colon R^{J_1}_{2n+1} \leftrightarrows \mathrm{Conj}(H_{2n+1}),$$

then, by Lemma 3.2.15, $g = h$ and we obtain a contradiction, since (f, g, g) would become an isotopism between the Leibniz algebras $\mathfrak{l}^{J_1}_{2n+1}$ and $\mathfrak{h}_{2n+1}$. Another way to show that $R^{J_1}_{2n+1}$ and $\mathrm{Conj}(H_{2n+1})$ are not isotopic racks is to observe that

$$\mathrm{Z}(R^{J_1}_{2n+1}) = \{(0, y, z) \mid y, z \in \mathbb{R}\},$$

$$\mathrm{Z}(\mathrm{Conj}(H_{2n+1})) = \{(0, 0, z) \mid z \in \mathbb{R}\},$$

where we use the same notation of Example 2.3.15.3 and we proved in Proposition 3.2.12 that the dimension of the center of a Lie rack is a left isotopism invariant.

(iii) If $R^{J_1}_{2n+1}$ were isotopic to a Lie quandle X, then $\mathfrak{l}^{J_1}_{2n+1}$ would be isotopic to the Leibniz algebra $\mathfrak{g} = \mathrm{T}_1 X$. By Theorem 2.3.13, $\mathfrak{g}$ is a Lie algebra and $X = \mathrm{Conj}(G)$, where $\mathrm{Lie}(G) = \mathfrak{g}$. Thus $\mathfrak{l}^{J_1}_{2n+1}$ would be isotopic to a Lie algebra and, by (iii) of Theorem 3.2.14, this is a contradiction. The same argument can be used for the Lie rack $R^{J_{-1}}_{2n+1}$.

□

In the last section we saw that, though the problem of finding isomorphism classes of the complex Heisenberg Leibniz algebras $\mathfrak{l}^{J_a}_{2n+1}$ is still open, the notion of isotopism allow us to classify all the possible non-isotopic nilpotent Leibniz algebras with one-dimensional commutator ideal. Moreover, this induces also a classification of the non-isotopic Lie racks integrating such Leibniz algebras.

Chapter 4

Non-nilpotent Leibniz algebras with one-dimensional derived subalgebra

Here, we finalize the classification of Leibniz algebras with one-dimensional derived subalgebras, considering both nilpotent and non-nilpotent cases. This research is a joint work, to which I have also contributed, together with my co-supervisor ([32]).

4.1 Preliminaries

Let L be a non-nilpotent left Leibniz algebra over $\mathbb{F}$ with $\dim_{\mathbb{F}} L = n$ and $\dim_{\mathbb{F}} [L, L] = 1$. We observe that such an algebra is two-step solvable since the derived subalgebra $[L, L]$ is abelian.

It is well known that a non-nilpotent Lie algebra with one-dimensional derived subalgebra is isomorphic to the direct sum of the two-dimensional non-abelian Lie algebra and an abelian algebra (see Section 3 of [36]). Thus we are interested in the classification of non-Lie Leibniz algebras with these properties.

In Theorem 2.6 of [30] the authors prove that a *complex* non-split non-nilpotent non-Lie Leibniz algebra with one-dimensional derived subalgebra is isomorphic to the two-dimensional algebra with basis $\{e_1, e_2\}$ and multiplication table $[e_2, e_1] = [e_2, e_2] = e_1$. Here we generalize this result when $\mathbb{F}$ is a general field with $\mathrm{char}(\mathbb{F}) \neq 2$.

Proposition 4.1.1. *Let L be a non-nilpotent left Leibniz algebra over $\mathbb{F}$ with $\dim_{\mathbb{F}} [L, L] = 1$. Then L has a two-dimensional bilateral ideal S which is isomorphic to one of the following Leibniz algebras:*

(i) $S_1 = \langle e_1, e_2 \rangle$ *with* $[e_2, e_1] = -[e_1, e_2] = e_1$*;*

(ii) $S_2 = \langle e_1, e_2 \rangle$ *with* $[e_2, e_1] = [e_2, e_2] = e_1$*.*

Proof. Let $[L, L] = \mathbb{F}z$. L is not nilpotent, then

$$[L, [L, L]] \neq 0,$$

i.e. $z \notin \mathrm{Z}_r(L)$. Since $[L, L]$ is an abelian algebra, there exists a vector $x \in L$, which is linearly independent than z, such that $[x, z] \neq 0$. Thus

$$[x, z] = \gamma z,$$

for some $\gamma \in \mathbb{F}^*$. The subspace $S = \langle x, z \rangle$ is an ideal of L and it is not nilpotent. In fact

$$0 \neq \gamma z = [x, z] \in [S, [S, S]] \,.$$

Thus S is a non-nilpotent Leibniz algebra. Using the classification of two-dimensional Leibniz algebras given by C. Cuvier in [28], S is isomorphic either to S_1 or to S_2. □

Remark 4.1.1. One can see the algebras S_1 and S_2 are respectively the Leibniz algebras L_2 and L_4 of Section 3.1 in [6]. We observe that S_1 is a Lie algebra, meanwhile S_2 is a non-right left Leibniz algebra.

One can see L as an extension of the abelian algebra $L_0 = L/S \cong \mathbb{F}^{n-2}$ by S (see [61])

$$0 \longrightarrow S \xrightarrow{\ i\ } L \underset{s}{\overset{\pi}{\rightleftarrows}} L_0 \longrightarrow 0 \,. \tag{4.1}$$

It turns out that there exists an equivalence of Leibniz algebra extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S & \xrightarrow{\ i_2\ } & L_0 \ltimes_\omega S & \underset{i_1}{\overset{\pi_1}{\rightleftarrows}} & L_0 & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \mathrm{id}_S} & & \Big\downarrow{\scriptstyle \theta} & & \Big\downarrow{\scriptstyle \mathrm{id}_B} & & \\
0 & \longrightarrow & S & \xrightarrow{\ i\ } & L & \underset{\sigma}{\overset{\pi}{\rightleftarrows}} & L_0 & \longrightarrow & 0
\end{array}$$

where $L_0 \ltimes_\omega S$ is the Leibniz algebra defined on the direct sum of vector spaces $L_0 \oplus S$ with the bilinear operation given by

$$[(x, a), (y, b)]_{(l,r,\omega)} = (0, [a, b] + l_x(b) + r_y(a) + \omega(x, y)),$$

where

$$\omega(x, y) = [\sigma(x), \sigma(y)]_L - \sigma([x, y]_{L_0}) = [\sigma(x), \sigma(y)]_L$$

is the Leibniz algebra 2-cocycle associated with (4.1) and

$$l_x(b) = [\sigma(x), i(b)]_L, \quad r_y(a) = [i(a), \sigma(y)]_L$$

define the action of L_0 on S; i_1, i_2, π_1 are the canonical injections and projection. The Leibniz algebra isomorphism θ is defined by $\theta(x, a) = \sigma(x) + i(a)$, for every $(x, a) \in L_0 \oplus S$.

By Proposition 4.2 of [61], the 2-cocycle $\omega \colon L_0 \times L_0 \to S$ and the linear maps $l, r \colon L_0 \to \mathrm{gl}(S)$ must satisfy the following set of equations

(L1) $l_x([a, b]) = [l_x(a), b] + [x, l_x(b)]$;

(L2) $r_x([a, b]) = [a, r_x(b)] - [b, r_x(a)]$;

(L3) $[l_x(a) + r_x(a), b] = 0$;

(L4) $[l_x, l_y]_{\mathrm{gl}(S)} - l_{[x,y]_{L_0}} = \mathrm{ad}_{\omega(x,y)}$;

(L5) $[l_x, r_y]_{\mathrm{gl}(S)} - r_{[x,y]_{L_0}} = \mathrm{Ad}_{\omega(x,y)}$;

(L6) $r_y(r_x(a) + l_x(a)) = 0$;

(L7) $l_x(\omega(y,z)) - l_y(\omega(x,z)) - r_z(\omega(x,y)) =$
$= \omega([x,y]_{L_0}, z) - \omega(x, [y,z]_{L_0}) + \omega(y, [x,z]_{L_0})$

for any $x, y \in L_0$ and for any $a, b \in S$. Notice that these equations where also studied in [25] in the case of Leibniz algebra *split extensions*.

Remark 4.1.2. The first three equations state that the pair (l_x, r_x) is a biderivation of the Leibniz algebra S, for any $x \in L_0$. Biderivations of low-dimensional Leibniz algebras were classified in [66] and it turns out that

- $\mathrm{Bider}(S_1) = \{(d, -d) \mid d \in \mathrm{Der}(S_1)\}$ and

  $$\mathrm{Der}(S_1) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \middle| \, \alpha, \beta \in \mathbb{F} \right\};$$

- $$\mathrm{Bider}(S_2) = \left\{ \left(\begin{pmatrix} \alpha & \alpha \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \right) \middle| \, \alpha, \beta \in \mathbb{F} \right\}.$$

We study now in detail the non-abelian extension (4.1) in both cases that S is isomorphic either to S_1 or to S_2.

4.1.1 S is a Lie algebra

When $S \cong S_1$, we have that $r_y = -l_y$, for any $y \in L_0$ and the bilinear operation of $L_0 \ltimes_\omega S_1$ becomes

$$[(x,a),(y,b)]_{(l,\omega)} = (0, [a,b] + l_x(b) - l_y(a) + \omega(x,y)).$$

The linear map l_x is represented by a 2×2 matrix

$$\begin{pmatrix} \alpha_x & \beta_x \\ 0 & 0 \end{pmatrix}$$

with α_x,$\beta_x \in \mathbb{F}$. From equations (L4)-(L5) it turns out that

$$\omega(x,y) = (\alpha_x \beta_y - \alpha_y \beta_x) e_1, \quad \forall x, y \in L_0$$

and the 2-cocycle ω is skew-symmetric. Moreover, equations (L6)-(L7) are automatically satisfied and the resulting algebra $L_0 \ltimes_\omega S_1 \cong L$ is a Lie algebra. We conclude that L is isomorphic to the direct sum of S_1 and $L_0 \cong \mathbb{F}^{n-2}$.

4.1.2 S is not a Lie algebra

With the change of basis $e_2 \mapsto e_2 - e_1$, S_2 becomes the Leibniz algebra with basis $\{e_1, e_2\}$ and the only non-trivial bracket given by $[e_2, e_1] = e_1$. Now a biderivation of S_1 is represented by a pair of matrices

$$\left(\begin{pmatrix}\alpha & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & \beta\\ 0 & 0\end{pmatrix}\right)$$

with $\alpha, \beta \in \mathbb{F}$ and the pair $(l_x, r_x) \in \mathrm{Bider}(S_2)$ is defined by $l_x(e_1) = \alpha_x e_1$ and $r_x(e_2) = \beta_x e_1$, for any $x \in L_0$.

Equation (L4) states that $[l_x, l_y]_{\mathrm{gl}(S_2)} = [\omega(x,y), -]$, with

$$\begin{aligned}[l_x, l_y]_{\mathrm{gl}(S_2)} = l_x \circ l_y - l_y \circ l_x &= \begin{pmatrix}\alpha_x & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\alpha_y & 0\\ 0 & 0\end{pmatrix} - \begin{pmatrix}\alpha_y & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\alpha_x & 0\\ 0 & 0\end{pmatrix} = \\ &= \begin{pmatrix}\alpha_x\alpha_y & 0\\ 0 & 0\end{pmatrix} - \begin{pmatrix}\alpha_x\alpha_y & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},\end{aligned}$$

for any $x, y \in L_0$. Thus $\omega(x,y) \in \mathrm{Z}_l(S_2) = \mathbb{F}e_1$.

From equation (L5) we have $[l_x, r_y]_{\mathrm{gl}(S_2)} = [-, \omega(x,y)]_{S_2}$, with

$$[l_x, r_y]_{\mathrm{gl}(S_2)} = l_x \circ r_y - r_y \circ l_x = \begin{pmatrix}0 & \alpha_x\beta_y\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & \alpha_x\beta_y\\ 0 & 0\end{pmatrix}.$$

Thus, for every $a = a_1e_1 + a_2e_2 \in S_2$ and for every $x, y \in L_0$, we have

$$[a, \omega(x,y)] = [l_x, r_y](a) = \alpha_x\beta_y a_2 e_1$$

i.e. $\omega(x,y) = \alpha_x\beta_y e_1$. Finally, equations (L6) and (L7) are identically satisfied.

Summarizing we have

$$\begin{cases} l_x \equiv \begin{pmatrix}\alpha_x & 0\\ 0 & 0\end{pmatrix} \\ r_y \equiv \begin{pmatrix}0 & \beta_y\\ 0 & 0\end{pmatrix} \\ \omega(x,y) = \alpha_x\beta_y e_1 \end{cases}$$

for every $x, y \in L_0$ and the bilinear operation $[-,-]_{(l,r,\omega)}$ becomes

$$[(x,a),(y,b)]_{(l,r,\omega)} = (0, (a_2b_1 + \alpha_x b_1 + \beta_y a_2 + \alpha_x\beta_y)e_1),$$

for any x, $y \in L_0$ and for any $a = a_1e_1 + a_2e_2$, $b = b_1e_1 + b_2e_2 \in S_2$.

If we fix a basis $\{f_3, \ldots, f_n\}$ of L_0 and we denote by

$$\alpha_i = \alpha_{f_i}, \quad \beta_i = \beta_{f_i}, \quad \forall i = 3, \ldots, n$$

then L is isomorphic to the Leibniz algebra with basis $\{e_1, e_2, f_3, \ldots, f_n\}$ and non-zero brackets

$$\begin{aligned}
&[e_2, e_1] = e_1\\
&[e_2, f_i] = \beta_i e_1, \quad \forall i = 3, \ldots, n\\
&[f_i, e_1] = \alpha_i e_1, \quad \forall i = 3, \ldots, n\\
&[f_i, f_j] = \alpha_i \beta_j e_1, \quad \forall i, j = 3, \ldots, n.
\end{aligned}$$

With the change of basis $f_i \mapsto f_i' = \dfrac{f_i}{\beta_i} - e_1$, if $\beta_i \neq 0$, we obtain that

$$\begin{aligned}
&[e_2, f_i'] = e_1 - [e_2, e_1] = 0,\\
&[f_i', e_1] = \gamma_i e_1, \quad \text{where } \gamma_i = \frac{\alpha_i}{\beta_i},\\
&[f_i, f_j'] = \alpha_i e_1 - [f_i, e_1] = 0,\\
&[f_i', f_j'] = \gamma_i e_1 - \frac{1}{\beta_i}[f_i, e_1] = 0.
\end{aligned}$$

If we denote again $f_i \equiv f_i'$ and $\alpha_i \equiv \gamma_i$ when $\beta_i \neq 0$, then L has basis $\{e_1, e_2, f_3, \ldots, f_n\}$ and non-trivial brackets

$$[e_2, e_1] = e_1, \quad [f_i, e_1] = \alpha_i e_1, \quad \forall i = 3, \ldots, n.$$

Finally, when $\alpha_i \neq 0$, we can operate the change of basis

$$f_i \mapsto \frac{f_i}{\alpha_i} - e_2.$$

One can check that the only non-trivial bracket now is $[e_2, e_1] = e_1$ and L is isomorphic to the direct sum of S_2 and the abelian algebra $L_0 \cong \mathbb{F}^{n-2}$. This allows us to conclude with the following.

Theorem 4.1.2. *Let $\mathbb{F}$ be a field with $\operatorname{char}(\mathbb{F}) \neq 2$. Let L be a non-nilpotent non-Lie left Leibniz algebra over $\mathbb{F}$ with $\dim_{\mathbb{F}} L = n$ and $\dim_{\mathbb{F}}[L, L] = 1$. Then L is isomorphic to the direct sum of the two-dimensional non-nilpotent non-Lie Leibniz algebra S_2 and an abelian algebra of dimension $n-2$. We denote this algebra by L_n.* □

If we suppose that L is a *non-split* algebra, i.e. L cannot be written as the direct sum of two proper ideals, then we obtain the following result, that is a generalization of Theorem 2.6 of [30] and which is valid over a general field $\mathbb{F}$ with $\operatorname{char}(\mathbb{F}) \neq 2$.

Corollary 4.1.3. *Let L be a non-split non-nilpotent non-Lie left Leibniz algebra over $\mathbb{F}$ with $\dim_{\mathbb{F}} L = n$ and $\dim_{\mathbb{F}}[L, L] = 1$. Then $n = 2$ and $L \cong S_2$.* □

Now we study in detail the algebra $L_n = S_2 \oplus \mathbb{F}^{n-2}$ by describing the Lie algebra of derivations, its Lie group of automorphisms and the Leibniz algebra of biderivations. Moreover, when $\mathbb{F} = \mathbb{R}$, we solve the *coquegigrue problem* (see [26] and [54]) for L_n by integrating it into a Lie rack.

4.2 Derivations, automorphisms and biderivations of L_n

Let $n \geq 2$ and let $L_n = S_2 \oplus \mathbb{F}^{n-2}$. We fix the basis $\mathcal{B}_n = \{e_1, e_2, f_3, \ldots, f_n\}$ of L_n and we recall that the only non-trivial commutator is $[e_2, e_1] = e_1$. A straightforward application of the algorithm proposed in [66] for finding derivations and anti-derivations of a Leibniz algebra as pair of matrices with respect to a fixed basis produces the following.

Theorem 4.2.1.

(i) A derivation of L_n is represented, with respect to the basis $\mathcal{B}_n$, by a matrix

$$\left(\begin{array}{cc|cccc} \alpha & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & a_3 & & & & \\ 0 & a_4 & & & & \\ \vdots & \vdots & & & A & \\ 0 & a_n & & & & \end{array}\right)$$

where $A \in \mathrm{M}_{n-2}(\mathbb{F})$.

(ii) The group of automorphisms $\mathrm{Aut}(L_n)$ is the Lie subgroup of $\mathrm{GL}_n(\mathbb{F})$ of matrices of the form

$$\left(\begin{array}{cc|cccc} \beta & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & b_3 & & & & \\ 0 & b_4 & & & & \\ \vdots & \vdots & & & B & \\ 0 & b_n & & & & \end{array}\right)$$

where $\beta \neq 0$ and $B \in \mathrm{GL}_{n-2}(\mathbb{F})$.

(iii) The Leibniz algebra of biderivations of L_n consists of the pairs (d, D) of linear endomorphisms of L_n which are represented by the pair of matrices

$$\left(\left(\begin{array}{cc|cccc} \alpha & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & a_3 & & & & \\ 0 & a_4 & & & & \\ \vdots & \vdots & & & A & \\ 0 & a_n & & & & \end{array}\right), \left(\begin{array}{cc|cccc} 0 & \alpha' & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & a'_3 & & & & \\ 0 & a'_4 & & & & \\ \vdots & \vdots & & & A' & \\ 0 & a'_n & & & & \end{array}\right)\right)$$

where $A,A' \in \mathrm{M}_{n-2}(\mathbb{F})$.

□

4.3 The integration of the Leibniz algebra L_n

Our aim here is to solve the coquecigrue problem for the non-nilpotent Leibniz algebra $L_n = S_2 \oplus \mathbb{F}^{n-2}$. One can check that S_2 is a split Leibniz algebra, in the sense of M. K. Kinyon, with $I = \mathrm{Z}_l(S_2) \cong \mathbb{R}$ and $M \cong \mathbb{R}$. Thus $L \cong (\mathbb{R}^2, \{-,-\})$ with the bilinear operation defined by

$$\{(x_1,x_2),(y_1,y_2)\} = (0, \rho_{x_1}(y_2))$$

and $\rho_{x_1}(y_2) = x_1 y_2$, for any $x_1, y_2 \in \mathbb{R}$. It turns out that a Lie rack integrating S_2 is $(\mathbb{R}^2, \rhd)$, where

$$(x_1,x_2) \rhd (y_1,y_2) = (y_1, y_2 + e^{x_1} y_2).$$

and the unit element is $(0,0)$. Finally, one can check that the binary operation

$$(x_1,x_2,x_3,\ldots,x_n) \rhd (y_1,y_2,y_3,\ldots,y_n) = (y_1, y_2 + e^{x_1} y_2, y_3, \ldots, y_n)$$

defines on $\mathbb{R}^n$ a Lie rack structure with unit element $1 = (0,\ldots,0)$, such that $(\mathrm{T}_1 \mathbb{R}^n, \rhd)$ is a Leibniz algebra isomorphic to L_n. This result, combined with the ones of Section 4 of [58], completes the classification of Lie racks whose tangent space at the unit element gives a Leibniz algebra with one-dimensional derived subalgebra.

Chapter 5

Biderivations of complete Lie algebras

This chapter is based on a published paper of which I am co-author with my co-supervisor ([31]).

Let R be a ring and let X be a subset of R. A map $f\colon X \to R$ is called *commuting (on X)* if

$$f(x)x = xf(x), \tag{5.1}$$

for every $x \in X$. If we write the commutator $xy - yx$ of x, y in R as $[x, y]$, then the equation above can be written as $[f(x), x] = 0$. The typical goal when dealing with a commuting map is to characterize its form. For a better understanding, consider just two examples: the identity map and any map with its range in the centers $\mathrm{Z}(R)$ of A. In addition, the sum and the dot product of commuting maps also give commuting maps. For instance, the map

$$f(x) = \lambda_0(x)x^n + \lambda_1(x)x^{n-1} + \cdots + \lambda_{n-1}(x)x + \lambda_n(x),$$

where $\lambda_i\colon R \to \mathrm{Z}(R)$, is commuting for any choice of central maps λ_i. The first important result on commuting maps is Posner's theorem [69] from 1957. His theorem says that if there exists a nonzero commuting derivation on a prime ring R, then R is commutative.

The first appearance of biderivation dates back to 1980 when G. Maksa studied in [64] symmetric biadditive maps with non-negative diagonalization in a different context from ours (Hilbert spaces), which have been revealed to be biderivations. From the 1980s the study of biderivations on rings and algebras has had a great increase. The first of these works that initiated the study of the aforementioned biderivations in a purely algebraic context dates back to 1989 by J. Vukman [77]. He demonstrated that if there exists a nonzero symmetric biderivation, denoted as B, on a prime ring R with a characteristic different from 2 such that $B(x, x)x = xB(x, x)$ for every $x \in R$, then R is commutative.

The history of the development of these results can be found in a special survey written by M. Brešar in 2004 [12], who contributed (and continues to contribute) significantly with his own findings. Now, what do these commuting maps have to do with biderivations? Let $f\colon R \to R$ be an additive commuting map. By Equation (5.1), we linearize with respect to x and y, meaning:

$$[f(x), y] = [x, f(y)] \quad \text{for all } x, y \in R.$$

The map $(x, y) \mapsto [f(x), y]\,(= [x, f(y)])$ is an inner derivation of R in each argument, that is, a biderivation.

The definition of biderivation for a Lie algebra was given in 2011 by D. Wang *et al.* in [79]. A considerable number of articles appeared in the literature since then, where biderivations of Lie algebras have been studied (see [78], [46], [23], [20],[21], [22] to name a few). The main goal of this section is to prove some results related to biderivations of Lie algebras in terms of matrices associated to these particular bilinear maps. This approach offers an opportunity to deepen certain aspects of this topic and use linear algebraic tools to study and expose the results obtained.
In the next subsection, we will introduce the matrix approach we will use to study biderivations on Lie algebras. In Section 5.1 we will prove the main result which describes all biderivations of complete Lie algebras (in particular, if L is semisimple). It also extends a well known result on simple Lie algebras obtained by X. Tang in 2018 [76]. In conclusion, in the last subsection, we will show some results about symmetric and skew-symmetric biderivations.

Let A be an associative algebra over the commutative ring R. A *derivation* $d\colon A \to A$ is a linear map that satisfies the *Leibniz identity*

$$d(xy) = d(x)y + xd(y),$$

for every $x, y \in A$. A similar definition can be given for a Lie algebra. Denote by L a Lie algebra over a field $\mathbb{F}$. An example of derivation on L is the linear map $\mathrm{ad}_x\colon L \to L$, with $x \in L$, that maps every $y \in L$ in $[x, y]$. These derivations are called *inner*. In order to generalize the definition of derivation on Lie algebras, some researchers gave the definitions of *generalized derivation*, *quasiderivation*, *near derivation*, etc. and a lot of them studied this maps in several different cases (see for example [59] and [13]). Another way to generalize the definition of derivation is to consider bilinear map instead of a linear map and require that this bilinear map is a derivation in each of its arguments. More precisely, referring to the article [14] of M. Brešar, W. Martindale and C. Miers, a bilinear map $f\colon A \times A \to A$ is called a *biderivation* of A if

$$f(xy, z) = xf(y, z) + f(x, z)y$$
$$f(x, yz) = f(x, y)z + yf(x, z),$$

for all $x, y, z \in A$. Suppose now that A is a noncommutative algebra and let $[x, y] = xy - yx$ be the Lie product of the elements $x, y \in A$. Then, for all $x, y \in A$ and $\lambda \in \mathrm{Z}(A)$ (the center of A), the map

$$f(x, y) = \lambda\,[x, y]$$

is the main example of biderivation on A. The biderivations of this form are called *inner biderivations*. In [15] it was proved that all biderivations on noncommutative prime rings are of this type. D. Benkonvič in [8] proved furthermore that, under certain conditions, all biderivations on a triangular algebra is a sum of an extremal and an inner biderivation. This result extends that obtained by J. Zhang et al. in [84], which stated that biderivations of nest algebras are usually inner (they showed by some examples that, in some special cases, there exist non-inner biderivations). Biderivations have many applications to other field (see [12] for more details). Motivated by this, D. Wang, X. Zu and Z. Chen gave in [79] the definition of biderivation on Lie algebras to study these latter on parabolic subalgebras of Lie algebras.

Definition 5.0.1. [79] Let L be a Lie algebra over a field $\mathbb{F}$. A bilinear map $B\colon L \times L \to L$ is called *biderivation* if it satisfies

$$B([x,y],z) = [x, B(y,z)] + [B(x,z), y] \tag{5.2}$$
$$B(x,[y,z]) = [B(x,y), z] + [y, B(x,z)], \tag{5.3}$$

for all $x, y, z \in L$.

An equivalent way to approach to a biderivation of a Lie algebra it is the following. We note the same could be done for other algebraic structures.

Definition 5.0.2. Let L be a Lie algebra over a field $\mathbb{F}$. A bilinear map $B\colon L \times L \to L$ is called *biderivation* if the maps $B(x,-)$ and $B(-,y)$ are derivations of L, for all $x, y \in L$.

5.0.1 A matricial approach for the study of biderivations of Lie algebras

To make the calculations more clear we use the following matricial approach. Let L be a Lie algebra of dimension n over a field $\mathbb{F}$ and let $\{e_1, \ldots, e_n\}$ be a basis of L. Consider now the vector space product $\mathrm{M}_n(\mathbb{F})^n$ of the n-tuples of matrices in $\mathrm{M}_n(\mathbb{F})$. A biderivation $B\colon L \times L \to L$ can be thought as an element of $\mathrm{M}_n(\mathbb{F})^n$. In general, a biderivation B can be written as

$$B(x,y) = \beta_1(x,y)e_1 + \cdots + \beta_n(x,y)e_n,$$

where $\beta_1, \ldots, \beta_n\colon L \times L \to \mathbb{F}$ are bilinear forms. Let B_i be the matrix associated with the bilinear form β_i, for every $i = 1, \ldots, n$. We denote with BiDer(L) the set of all biderivations of the Lie algebra L. Now we are ready to define the following map

$$\begin{aligned} F\colon \mathrm{BiDer(L)} &\to M_n(\mathbb{F})^n \\ B &\mapsto (B_1, \ldots, B_n)\,. \end{aligned}$$

We denote with b_{ij}^k the (i,j)-th entry of the matrix B_k, i.e. the scalar $\beta_k(e_i, e_j)$, with $i, j, k = 1, \ldots, n$.
Let U, V and W be vector spaces over $\mathbb{F}$ and let $\mathrm{B}(U,V;W)$ be the set of all bilinear maps from $U \times V$ to W. By definition, we have that BiDer(L) is a subset of $\mathrm{B}(L,L;L)$. Now, since $\mathrm{B}(U,V;W)$ is a vector space (see Chapter 5 in [74]), it is natural to wonder whether BiDer(L) is a subspace of $\mathrm{B}(L,L;L)$ or whether it is just a subset.

Proposition 5.0.3. *Let L be a Lie algebra over a field $\mathbb{F}$. The set* BiDer(L) *is a subspace of* $\mathrm{B}(L,L;L)$.

Proof. We want to show that BiDer(L) is a subspace of $B(L, L; L)$, so we show that, for all $B_1, B_2, B \in$ BiDer(L) and $\lambda \in \mathbb{F}$, $B_1 + B_2$ and λB belongs to BiDer(L). In particular, since the biderivations are bilinear maps, $B_1 + B_2$ and λB are bilinear maps, for every $B_1, B_2, B \in$ BiDer(L) and $\lambda \in \mathbb{F}$. To prove that $B_1 + B_2$ is a biderivation we have to show that $B_1 + B_2$ satisfies equations (5.2) and (5.3). The same applies to λB. Therefore

$$\begin{aligned}(B_1 + B_2)([x, y], z) &= B_1([x, y], z) + B_2([x, y], z)\\ &= [x, B_1(y, z)] + [B_1(x, z), y] + [x, B_2(y, z)] + [B_2(x, z), y]\\ &= [x, B_1(y, z) + B_2(y, z)] + [B_1(x, z) + B_2(x, z), y]\\ &= [x, (B_1 + B_2)(y, z)] + [(B_1 + B_2)(x, z), y],\end{aligned}$$

for all $B_1, B_2 \in$ BiDer(L) and $x, y, z \in L$;

$$\begin{aligned}(\lambda B)([x, y], z) = \lambda B([x, y], z) &= \lambda ([x, B(y, z)] + [B(x, z), y])\\ &= \lambda [x, B(y, z)] + \lambda [B(x, z), y]\\ &= [x, (\lambda B)(y, z)] + [(\lambda B)(x, z), y],\end{aligned}$$

for all $\lambda \in \mathbb{F}$ and $B \in$ BiDer(L). In a similar manner we could show that $B_1 + B_2$ and λB satisfy the Equation (5.3). □

Proposition 5.0.4. *The map*

$$\begin{aligned}F\colon \text{BiDer(L)} &\to \mathrm{M}_n(\mathbb{F})^n\\ B &\mapsto (B_1, \dots, B_n).\end{aligned}$$

is a monomorphism of vector spaces.

Proof. The map F is linear since the biderivations are bilinear maps. Furthermore $F(B) = (0, 0, \dots, 0)$ if and only if $B = 0$, since $B_1 = B_2 = \cdots = B_n = 0_n \in \mathrm{M}_n(\mathbb{F})$. □

In the following proposition we will rewrite the the equations (5.2) and (5.3) in terms of bilinear forms associated with a biderivation of a Lie algebras and its structure constants.

Proposition 5.0.5. *Let L be a Lie algebra over a field $\mathbb{F}$ and $\{e_1, \dots, e_n\}$ a basis of L. Let $\left\{c_{ij}^k\right\}$ be the structure constants of L, that is, $[e_i, e_j] = \sum_{k=1}^n c_{ij}^k e_k$ for every $i, j, k = 1, \dots, n$. Then $B\colon L \times L \to L$ is a biderivation of L if and only if*

$$\sum_{t=1}^n c_{jk}^t b_{it}^r = \sum_{t=1}^n \left(c_{tk}^r b_{ij}^t + c_{jt}^r b_{ik}^t\right) \text{ and } \sum_{t=1}^n c_{ij}^t b_{tk}^r = \sum_{t=1}^n \left(c_{tj}^r b_{ik}^t + c_{jt}^r b_{jk}^t\right),$$

for every $i, j, k, r = 1, \dots, n$.

Proof. From Equation (5.2) we have

$$B([e_i, e_j], e_k) = [B(e_i, e_k), e_j] + [e_i, B(e_j, e_k)],$$

for every $i, k, k = 1, \dots, n$. Then

$$B([e_i, e_j], e_k) = B(\sum_{t=1}^{n} c_{ij}^t e_t, e_k) = \sum_{t=1}^{n} c_{ij}^t B(e_t, e_k) = \sum_{r=1}^{n}\sum_{t=1}^{n} c_{ij}^t b_{tk}^r e_r;$$

on the other hand,

$$\begin{aligned} &[B(e_i, e_k), e_j] + [e_i, B(e_j, e_k)] = \\ &= [\sum_{t=1}^{n} b_{ik}^t e_t, e_j] + [e_i, \sum_{t=1}^{n} b_{jk}^t e_t] = \sum_{t=1}^{n} b_{ik}^t [e_t, e_j] + \sum_{t=1}^{n} b_{jk}^t [e_i, e_t] \\ &= \sum_{r=1}^{n}\sum_{t=1}^{n} c_{tj}^r b_{ik}^t e_r + \sum_{r=1}^{n}\sum_{t=1}^{n} c_{it}^r b_{jk}^t e_r = \sum_{r=1}^{n}(\sum_{t=1}^{n} c_{tj}^r b_{ik}^t + c_{it}^r b_{jk}^t) e_r. \end{aligned}$$

Since $\{e_i, \dots, e_n\}$ is a basis of L, by comparing both equations we obtain the first equation. With similar computations we obtain the second equation. □

In light of this result one can asks if, in matricial terms, a relation between biderivations and derivations exists and, if so, what kind of relation there is. The next result proves that there is an affirmative answer in this sense and, besides, describes this relation. But first we introduce some notations. Let $(B_1, \dots, B_n)$ the n-tuple of matrices associated to a biderivation B of a n-dimensional Lie algebra L respect to a fixed basis. We denote with $(B_i)^k$ the k-th row of the matrix B_i and with $(B_i)_j$ the j-th column of (B_i), for all $i = 1, \dots, n$.

Proposition 5.0.6. *Let $B\colon L \times L \to L$ be a bilinear map of a n- dimensional Lie algebra L and let $B_1, \dots, B_n \in \mathrm{M}_n(\mathbb{F})$ the matrices associated to B. B is a biderivation L if and only if, for every $i = 1, \dots, n$,*

$$\begin{pmatrix} (B_1)^i \\ (B_2)^i \\ \vdots \\ (B_n)^i \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} (B_1)_i & (B_2)_i & \cdots & (B_n)_i \end{pmatrix}$$

are matrices associated to two derivations of L.

Proof. The "if" directions follows directly by the definition of derivation. The other direction is not trivial. Let $\delta^x := B(x, -)$ be the linear function that maps every $y \in L$ in $B(x, y)$, for every $x \in L$. Since B is a biderivation, δ^x is a derivation of L for every $x \in L$. Let $\mathcal{B} = \{e_1, \dots, e_n\}$ be a basis of L. For $i = 1, \dots, n$, we have

$$\begin{aligned} \delta^{e_j}(e_i) = B(e_j, e_i) &= \beta_1(e_j, e_i)e_1 + \cdots + \beta_n(e_j, e_i)e_n \\ &= b_{ji}^1 e_1 + \cdots + b_{ji}^n e_n, \end{aligned}$$

where $\beta_1, \ldots, \beta_n$ are the bilinear forms associated to the biderivation B respect to the basis $\mathcal{B}$. The matrix associated to the derivation δ^{e_j} is

$$\begin{pmatrix} b^1_{ji} & b^1_{j2} & \cdots & b^1_{jn} \\ b^2_{ji} & b^2_{j2} & \cdots & b^2_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ b^n_{ji} & b^n_{j2} & \cdots & b^n_{jn} \end{pmatrix}.$$

It is clear that the i-th row of the matrix above is the j-th row of B_i, the i-th matrix associated to the biderivation B. If we define the linear function $\delta_x := B(-, x)$, the same arguments may be used to prove that the i-th row of the matrix associated to the derivation δ_{e_j} is the j-th column of the matrix B_i. □

Recall that the set BiDer(L) is a vector space, as we have seen before. In addition, every biderivation B has an image in $\mathrm{M}_n(\mathbb{F})^n$ via the linear map φ. It is clear that not all n-tuple of matrices in $\mathrm{M}_n(\mathbb{F})$ correspond to a biderivation of L. For example:

Example 5.0.6.1. Let $L = L_{2,2} = \langle e_1, e_2 \rangle_{\mathbb{F}}$ be a Lie algebra of dimension 2 over $\mathbb{F}$ such that $[e_1, e_2] = e_1$. The pair of matrices

$$\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

is not image of a biderivation B because $B(e_1, e_1) = e_2$, $B(e_2, e_2) = e_1$ and

$$B([e_1, e_2], e_1) = B(e_1, e_1) = e_2$$
$$[e_1, B(e_2, e_1)] + [B(e_1, e_1), e_2] = [e_1, 0] + [e_2, e_2] = 0.$$

This is just an example of how hard it could be to define a Lie bracket on BiDer(L) such that φ is a Lie monomorphism. In general the Lie bracket of $\mathrm{M}_n(\mathbb{F})$ is not closed in general. In the next proposition we show how a Lie bracket could be defined on the set of biderivations.

Proposition 5.0.7. *If* $F(\mathrm{BiDer(L)})$ *is a Lie subalgebra of* $\mathrm{M}_n(\mathbb{F})^n$, *then* $\{\mathrm{BiDer(L)}, \{-,-\}\}$ *is a Lie algebra, where*

$$\{-,-\} = F^{-1} \circ [-,-]_{\mathrm{M}_n(\mathbb{F})^n}.$$

Proof. Since $J := F(\mathrm{BiDer(L)})$ is a Lie subalgebra of $\mathrm{M}_n(\mathbb{F})^n$, then J is closed under the Lie bracket. In addition, $F\colon \mathrm{BiDer} \to J$ is a linear isomorphism. Thus, for every $A, B \in \mathrm{BiDer(L)}$, we have

$$\{A, B\} = F^{-1}([A, B]) = F^{-1}\left([A_1, B_1]_{\mathrm{M}_n(\mathbb{F})}, \ldots, [A_n, B_n]_{\mathrm{M}_n(\mathbb{F})}\right).$$

Since $[-,-]_{\mathrm{M}_n(\mathbb{F})^n}$ is a Lie bracket and F^{-1} is a linear isomorphism, the bilinear map $\{-,-\}$ defined above is a Lie bracket on BiDer(L). □

We conclude this first section with a couple of results regarding biderivations on two-step nilpotent Lie algebras. Remind that the *lower central series* defined recursively as the series $L^1 = L'$ and $L^k = \left[L, L^{k-1}\right]$, for $k \geq 2$. L is *nilpotent* if exists $k \geq 1$ such that $L^{k-1} \neq 0$ and $L^k = 0$. A Lie algebra L is *two-step nilpotent* if $k = 2$. For such Lie algebras the ideal commutator ideal L' is contained in the center of L. On the other hand, the condition $L' \subseteq \mathrm{Z}(L)$ implies that L is a two-step nilpotent Lie algebra.

Proposition 5.0.8. *Let L be a two-step nilpotent Lie algebra over a field $\mathbb{F}$ and B a biderivation of L. Then, for every $x \in L$ and $z \in L'$, $B(x,z), B(z,x) \in L' \subseteq Z(L)$.*

Proof. Let $\{e_1, \dots, e_n,\}$ be a basis of L. For every $z \in L'$, $z = \sum_{i,j=1}^{n} \alpha_{ij} [e_i, e_j]$ for some $\alpha_{ij} \in \mathbb{F}$. Thus we have

$$B(x,z) = B\left(x, \sum_{i,j=1,i<j}^{n} \alpha_{ij} [e_i, e_j]\right)$$
$$= \sum_{i,j=1}^{n} \alpha_{ij} B(x, [e_i, e_j]) = \sum_{i,j=1}^{n} \alpha_{ij} ([B(x,e_i), e_j] + [e_i, B(x, e_j)]) \in L'.$$

With similar computations we obtain $B(z,x)$ and these results prove the statement. ☐

Corollary 5.0.9. *Let L be a two-step nilpotent Lie algebra and B a biderivation of L. Then, for every $z, z' \in L'$, $B(z,z') = 0$.*

5.1 Biderivations of complete Lie algebras

X. Tang in [76] proved that all biderivations of a complex simple Lie algebra are inner. We will extend this result. We begin this section with a result that makes it easier the study of biderivations of complete Lie algebras (in particular, if the Lie algebra is semisimple).
In 1962 N. Jacobson gave in [51] the definition of *complete* Lie algebra, that is a Lie algebra L with $\mathrm{Z}(L) = 0$ and $\mathrm{Der(L)} = \mathrm{ad(L)}$. The next result makes easier the study of biderivations of this class of Lie algebras.

Proposition 5.1.1. *Let L be a complete Lie algebra over a field $\mathbb{F}$. B is a biderivation of L if and only if exist two linear maps $\varphi, \psi \in \mathrm{End(L)}$ such that, for every $x, y \in L$,*

$$B(x,y) = [\varphi(x), y] = [x, \psi(y)].$$

Proof. The "if" direction is trivial to prove. To prove the other direction we recall that, by Definition 5.0.2, $B(x,-)$ and $B(-,x)$ are derivations of L, for every $x \in L$. Since L is complete, all derivations of L are inner. Thus, for every $x \in L$, there exist $u, v \in L$ such that $B(x,-) = \mathrm{ad}_u(-) = [u,-]$ and $B(-,x) = \mathrm{ad}_v(-) = [v,-]$. So we can define two maps $\varphi, \psi : L \to L$ in the following way

$$\varphi \colon x \mapsto u \quad \text{and} \quad \psi \colon x \mapsto -v.$$

Now we have to prove that φ and ψ are linear. By the definition of φ we have

$$\varphi(x+y)(t) = \mathrm{ad}_{\varphi(x+y)}(t) = [\varphi(x+y), t]\,,$$

for every $t \in L$. Moreover,

$$\varphi(x)(t)+\varphi(y)(t) = \mathrm{ad}_{\varphi(x)}(t)+\mathrm{ad}_{\varphi(y)}(t) = [\varphi(x), t]+[\varphi(y), t] = [\varphi(x)+\varphi(y), t]\,.$$

Since B is a bilinear map, for every $t \in L$ we have $[\varphi(x+y), t] = [\varphi(x)+\varphi(y), t]$ and this implies that $\varphi(x+y) - \varphi(x) - \varphi(y) \in Z(L)$. The center of L is zero because L is complete, hence $\varphi(x+y) = \varphi(x)+\varphi(y)$ for all $x, y \in L$. By similar computations we can affirm that ψ is also linear. To conclude the proof we observe that $[\varphi(x), y]$ and $[x, \psi(y)]$ are biderivations of L because the Lie bracket is skew-symmetric and verifies the Jacoby identity. □

Under the same assumptions the following holds:

Corollary 5.1.2. *If B is a biderivation of L and $\varphi = \lambda\,\mathrm{id}_L$ for some $\lambda \in \mathbb{F}$, then $\varphi = \psi$.*

Proof. For any $x, y \in L$ we have

$$B(x, y) = [\varphi(x), y] = [\lambda x, y] = \lambda\,[x, y]\,.$$

On the other hand $B(x, y) = [x, \psi(y)]$, then $[x, \lambda y - \psi(y)] = 0$, for all $x, y \in L$. This implies that $\lambda y - \psi(y)$ belongs to the center of L, that is trivial because L is complete and this conlcude the proof. □

It is well known that all derivations of a simple Lie algebra are inner (see [36]) and this happens also for semisimple Lie algebras (see [47]). Biderivations of complex simple Lie algebras are studied by X. Tang in [76] where he proved the following result.

Theorem 5.1.3. *[76] Suppose that L is a finite-dimensional complex simple Lie algebra. Then B is a biderivation of L if and only if it is inner, i.e. there is a complex number λ such that*

$$B(x, y) = \lambda\,[x, y]\,,\ \forall x, y \in L.$$

One can asks if something like that happens to semisimple Lie algebras and the answer is affirmative. We would to remind that every semisimple Lie algebra has trivial center and all derivations on it are inner. Thus every semisimple Lie algebra is complete. While it is straightforward to demonstrate that all semisimple Lie algebras are complete, proving that the converse (that all complete Lie algebras are semisimple) is false, is not as obvious. E. Angelopoulos constructed in [3] a class of *sympathetic* Lie algebras, i.e. complete Lie algebras with $[L, L] = L$, which are not semisimple. Notably, there exists a counter-example within this class of minimal dimension, namely a Lie algebra of dimension 35, whose Levi subalgebra is isomorphic to $\mathfrak{sl}(2)$. Now we are ready to prove the main result of this section. From now on we indicate with P' the transpose matrix of a matrix P.

Theorem 5.1.4. *Let $L = L_1 \oplus \cdots \oplus L_t$ be an n-dimensional complex semisimple Lie algebra, where L_i is a complex simple Lie algebra with $\dim_{\mathbb{C}} L_i = n_i$ for every $i = 1, \ldots, t$ and $n_1 + \cdots + n_t = n$. A bilinear map $B\colon L \times L \to L$ is a biderivation of L if and only if exist $\lambda_1, \ldots, \lambda_t \in \mathbb{C}$ such that*

$$B(x_1 + \cdots + x_t, y_1 + \cdots + y_t) = \lambda_1 [x_1, y_1] + \cdots + \lambda_t [x_t, y_t],$$

with $x_i, y_i \in L_i$, $i = 1, \ldots, t$.

Proof. For every $i = 1, \ldots, t$ let $\mathcal{B}_i = \left\{e_{i_1}, \ldots, e_{i_{n_i}}\right\}$ be a basis of L_i and let $\left\{(c_i)^k_{lm}\right\}_{l,m,k=1,\ldots,n_i}$ be the structure constants of L_i, i.e. $[e_{i_l}, e_{i_m}] = \sum_{k=1}^{n_i} (c_i)^k_{lm} e_{i_k}$ for every $e_{i_l}, e_{i_m} \in \mathcal{B}_i$. Then the structure matrices of L_i forms an n_i-tuples of $n_i \times n_i$ matrices and let this tuple be $\left(C_{i_1} \ldots, C_{i_{n_i}}\right)$, where $C_{i_j} = (c_i)^j_{lm}$ with $j = 1, \ldots, n_i$. Thus the structure matrices of L are γ_{ij}, with $i = 1, \ldots, t$ and, for any i, $j = 1, \ldots, n_i$, where

$$\gamma_{ij} = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ & \ddots & & & \\ \vdots & & C_{i_j} & & \vdots \\ & & & \ddots & \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}$$

is the diagonal block matrix where the i-th diagonal block is the matrix $C_{i_j} \in \mathrm{M}_{n_i}(\mathbb{C})$. By Proposition 5.1.1 there exist $\varphi, \psi \in \mathrm{End(L)}$ such that

$$B(x, y) = [\varphi(x), y] = [x, \psi(y)] \tag{5.4}$$

for every $x, y \in L$. Thus there exist two matrices $P, Q \in \mathrm{M}_n(\mathbb{C})$ associated with, respectively, φ and ψ with respect to the basis $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_t$ of L. With this assumptions equations (5.4) hold if and only if $P'\gamma_{ij} = \gamma_{ij} Q$, for any $i = 1, \ldots, t$ and $j = 1, \ldots, n_i$. The matrices $P = (P_{ij})_{i,j=1,\ldots,t}$ e $Q = (Q_{ij})_{i,j=1,\ldots,t}$ are block matrices, where $P_{ij}, Q_{ij} \in \mathrm{M}_{n_i \times n_j}(\mathbb{C})$. The condition $P'\gamma_{ij} = \gamma_{ij} Q$ implies that $P'_{ik} C_{i_j} = 0 = C_{i_j} Q_{ik}$, for every $k \in \{1, \ldots, t\} \setminus \{i\}$, because

$$P'\gamma_{ij} = \begin{pmatrix} 0 & \cdots & 0 & P'_{i1} C_{i_j} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & P'_{i2} C_{i_j} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & P'_{it} C_{i_j} & 0 & \cdots & 0 \end{pmatrix}$$

and

$$\gamma_{ij}Q = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ C_{i_j}Q_{i1} & C_{i_j}Q_{i2} & \cdots & C_{i_j}Q_{it} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then, for every $x \in L_k, y \in L_i$, we have

$$\begin{aligned} &x'P'_{ik}C_{i_j}y = 0 \Rightarrow (P_{ik}x)'C_{i_j}y = 0, \ \forall j = 1, \ldots, n_i \Rightarrow \\ &\Rightarrow \big((P_{ik}x)'C_{i_1}y, \ldots, (P_{ik}x)'C_{i_{n_i}}y\big) = (0, \ldots, 0) \cong 0_{L_i}. \end{aligned}$$

The matrix $P_{ik} \in \mathrm{M}_{n_i \times n_k}(\mathbb{C})$ is associated to a linear map $f\colon L_k \to L_i$ with respect to the basis $\mathcal{B}_k$ of L_k and $\mathcal{B}_i$ of L_i. Thus $[f(x), y]_{L_i} = 0_{L_i}$, i.e. $f(x) \in \mathrm{Z}(L_i)$. The centers $\mathrm{Z}(L_i) = \{0_{L_i}\}$ since L_i is a simple Lie algebra, for every $i = 1, \ldots, t$, and then $P_{ik} = 0 \in \mathrm{M}_{n_i \times n_k}(\mathbb{C})$. By similar computation we obtain $Q_{ik} = 0 \in \mathrm{M}_{n_i \times n_k}(\mathbb{C})$. Then P and Q are respectively the diagonal block matrices

$$\begin{pmatrix} P_{11} & 0 & \cdots & 0 \\ 0 & P_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & P_{tt} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Q_{11} & 0 & \cdots & 0 \\ 0 & Q_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & Q_{tt} \end{pmatrix}.$$

The linear maps φ_i and ψ_i, whose associated matrices are respectively P_{ii} and Q_{ii} with respect to the basis $\mathcal{B}_i$ of L_i, map every element $x_i \in L_i$ in L_i, for any $i = 1, \ldots, t$. Finally, for every $x = x_1 + \cdots + x_t, y = y_1 + \cdots + y_t \in L = L_1 \oplus \cdots \oplus L_t$ we have

$$\begin{aligned} B(x, y) &= B(x_1 + \cdots + x_t, y_1 + \cdots + y_t) \\ &= [\varphi_1(x_1), y_1]_{L_1} + \cdots + [\varphi_t(x_t), y_t]_{L_t} \\ &= [x_1, \psi_1(y_1)]_{L_1} + \cdots + [x_t, \psi_t(y_t)]_{L_t}. \end{aligned}$$

By Proposition 5.1.1 and since L_i is a simple Lie algebra, exist $\lambda_i, \mu_i \in \mathbb{C}$ such that $[\varphi_i(x_i), y_i]_{L_i} = \lambda_i [x_i, y_i]_{L_i}$ and $[x_i, \psi_i(y_i)]_{L_i} = \mu_i [x_i, y_i]_{L_i}$, for every $i = 1, \ldots, t$. We can conclude the proof by Corollary 5.1.2 and say that $\lambda_i = \mu_i$, for every $i = 1, \ldots, t$. □

In the last part of this subsection we want to show how to decompose the vector space BiDer(L) in a direct sum of its subspaces, when L is a complete Lie algebra. In other words, we will show that is always possible to decompose the vector space BiDer(L) of all biderivations of L into a direct sum of two simpler vector spaces. The space BiDer(L) is isomorphic to the vector space

$$\{\varphi \in \mathrm{End(L)} \mid \exists \psi \in \mathrm{End(L)} \ \text{such that} \ [\varphi(x), y] = [x, \psi(y)], \forall x, y \in L\}$$

that is isomorphic to

$$V = \{P \in \mathrm{M}_n(\mathbb{C}) \mid \exists Q \in \mathrm{M}_n(\mathbb{C}) \text{ such that } PA_i = A_iQ, \forall i = 1, \dots, n\},$$

where $\dim_{\mathbb{C}} L = n$ and $(A_1, \dots, A_n)$ are the structure matrices of L.

Theorem 5.1.5. *Let L be a complete Lie algebra over the complex field $\mathbb{C}$, with $\dim_{\mathbb{C}} L = n$, $(A_1, \dots, A_n)$ the structure matrices of L with respect to a fixed basis $\mathcal{B}$ of L and let V be the vector space defined above. Then $V = V^+ \oplus V^-$, where*

$$V^+ = \{P \in \mathrm{M}_n(\mathbb{C}) \mid (PA_i)' = (PA_i)\}$$
$$V^- = \{P \in \mathrm{M}_n(\mathbb{C}) \mid (PA_i)' = -(PA_i)\}$$

Proof. Remind that all matrices A_i are skew-symmetric. Then $V^+ \oplus V^- \subseteq V$ because, for all $P_+ \in V^+, P_- \in V^-$ and $A_i \in \{A_1, \dots, A_n\}$, we have

$$P_+A_i = (P_+A_i)' = A_i'P_+' = -A_iP_+' = A_i(-P_+'),$$

and this proves that $P_+ \in V$. On the other hand,

$$P_-A_i = -(P_-A_i)' = -A_i'(P_-)' = A_i(P_-'),$$

for every $A_i \in \{A_1, \dots, A_n\}$, and this proves that $P_- \in V$. Now, to prove that $V \subseteq V^+ \oplus V^-$ we consider $P \in V$ and $Q \in \mathrm{M}_n(C)$ such that $PA_i = A_iQ$. Then we have

$$\begin{aligned}
((P + Q')A_i)' &= (PA_i + Q'A_i)' \\
&= (A_iQ + Q'A_i)' \\
&= -Q'A_i - A_iQ \\
&= -PA_i - Q'A_i \\
&= -(P + Q')A_i
\end{aligned}$$

and

$$\begin{aligned}
((P - Q')A_i)' &= (PA_i - Q'A_i)' \\
&= (A_iQ - Q'A_i)' \\
&= -Q'A_i + A_iQ \\
&= (-Q'A_i + PA_i) \\
&= (P - Q')A_i,
\end{aligned}$$

for every $A_i \in \{A_1, \dots, A_n\}$. Then we showed that $P+Q' \in V^-$ and $P-Q' \in V^+$. Thus the matrix $P = \frac{1}{2}(P + Q') + \frac{1}{2}(P - Q')$ belongs to $V^+ \oplus V^-$ and this proves the second inclusion. □

By Theorem 5.1.4 it follows that the matrix P associated to the endomorphism φ of the semisimple Lie algebra L (that is equal to Q, the matrix associated to ψ) is direct sum of scalar matrices $\lambda_i I_{n_i}$, for $i = 1, \dots, t$, then P is a scalar matrix and $(PA_i)' = A_i'P' = -A_iP' = -A_iP$, for all structure matrices $A_i \in \{A_1, \dots, A_n\}$.

Thus the subspace V^+ is empty and if $L = L_1 \oplus \cdots \oplus L_t$, where L_i is a complex simple Lie algebra with $\dim_{\mathbb{C}} L_i = n_i$ for every $i = 1, \ldots, t$ and $n_1 + \cdots + n_t = n$, V^- is isomorphic to $\mathbb{C}^t$. We summarize these facts in the following proposition.

Proposition 5.1.6. *Let L be a semisimple Lie algebra over the complex field $\mathbb{C}$ like above, with $\dim_{\mathbb{C}} L = n$, $(A_1, \ldots, A_n)$ the structure matrices of L with respect to a fixed basis $\mathcal{B}$ of L and $V = V^+ \oplus V^-$ the vector space defined in Theorem 5.1.5. Then* $\mathrm{BiDer(L)} \cong V^- \cong \mathbb{C}^t$.

5.1.1 Symmetric and skew-symmetric biderivations of complete Lie algebras

There are several examples of papers in which biderivations (more precisely skew-symmetric biderivations) are determined in terms of linear commuting maps (see [16],[23],[24],[46], [78]). In this section we show how simple are symmetric and skew-symmetric biderivations of a complete Lie algebra.
A biderivation $B\colon L \times L \to L$ is called *symmetric* (resp. skew-symmetric) if $B(x, y) = B(y, x)$ (resp. $B(x, y) = -B(y, x)$), for all $x, y \in L$. If B is a symmetric biderivation the bilinear forms $\beta_1, \beta_2, \ldots, \beta_n$ associated with B are symmetric and then the matrices $B_1, B_2, \ldots, B_n$ are symmetric. Obviously, the same reasoning applies to a skew-symmetric biderivation. It is equivalent to say that φ maps symmetric (resp. skew-symmetric) biderivations into $n-$tuples of symmetric (resp. skew-symmetric) matrices. In general, every biderivation $B\colon L \times L \to L$ of a Lie algebra L can be written as $B = \frac{1}{2}B^+ + \frac{1}{2}B^-$, where B^+ and B^- are respectively the bilinear maps from $L \times L$ to L defined as

$$B^+\colon (x, y) \mapsto B(x, y) + B(y, x) \quad \text{and} \quad B^-\colon (x, y) \mapsto B(x, y) - B(y, x),$$

for all $x, y \in L$. We note that, since $\mathrm{BiDer(L)}$ is a vector space, B^+ and B^- are biderivations of L. With this assumptions we prove the following result.

Proposition 5.1.7. *Let B be a biderivation of a complete Lie algebra L, with*

$$B(x, y) = [\varphi(x), y] = [x, \psi(y)]$$

for some $\varphi, \psi \in \mathrm{End(L)}$ and for all $x, y \in L$.

- *If B is symmetric, then $\varphi = -\psi$.*
- *If B is skew-symmetric, then $\varphi = \psi$.*

Proof.

- Since B is a symmetric biderivation of L we have $B(x, y) = B(y, x)$, for every $x, y \in L$. Then $B(y, x) = [\varphi(y), x]$ and $B(x, y) = [\varphi(x), y] = [x, \psi(y)]) = -[\psi(y), x]$. By comparing these two expressions we obtain

$$[\varphi(y) + \psi(y), x] = 0$$

and this implies that $\varphi(y) + \psi(y) \in \mathrm{Z}(L)$. The Lie algebra L is centerless because it is complete and this allows us to conclude that $\varphi = -\psi$.

- If we start from the equality $B(y,x) = -B(x,y)$, with the same arguments and similar computations we obtain $\varphi = \psi$.

□

We recall here that a linear map $f : L \to L$ is called *commuting* if $[x, f(x)] = 0$, for every $x \in L$. If the characteristic $char(\mathbb{F}) \neq 2$, then a linear map $f\colon L \to L$ is commuting if and only if $[f(x), y] = [x, f(y)]$ for all $x, y \in L$. In a similar way f is called *skew-commuting* if and only if $[f(x), y] = -[x, f(y)]$ for all $x, y \in L$. To be more precise the definition of commuting linear maps can be given for a wider class of algebraic structures, for example for rings. Armed with these definitions and the last proposition, it is fairly straightforward to prove the following results in which are described symmetric and skew-symmetric biderivations of a complete Lie algebra L.

Corollary 5.1.8. *Let L be a complete Lie algebra. $B\colon L \to L$ is a symmetric biderivation of L if and only if there exists a unique skew-commuting linear map $\varphi \in \mathrm{End(L)}$ such that $B(x,y) = [\varphi(x), y]$, for any $x, y \in L$.*

Corollary 5.1.9. *Let L be a complete Lie algebra. $B\colon L \to L$ is a skew-symmetric biderivation of L if and only if there exists a unique commuting linear map $\varphi \in \mathrm{End(L)}$ such that $B(x,y) = [\varphi(x), y]$, for any $x, y \in L$.*

Appendix A

Lie algebras

A.1 Basic Definitions

Definition A.1.1. Let A be a vector space over the field $\mathbb{F}$. A is called an *algebra over* $\mathbb{F}$ if there is a bilinear operation $\cdot \colon A \times A \to A$ defined on it. A is called *associative* if $(xy)z = x(yz)$ for all $x, y, z \in A$. A is said to have a *unit* if there exists $1 \in A$ such that $x1 = 1x = x$ for all $x \in A$.

Definition A.1.2. Let $\mathbb{F}$ be a field. $\mathfrak{g}$ is a *Lie algebra* over $\mathbb{F}$ if $\mathfrak{g}$ is an $\mathbb{F}$-vector space equipped with a bilinear form, the *Lie bracket* defined as follows:

$$\begin{aligned} \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (x, y) &\mapsto [x, y] \end{aligned}$$

satisfying the following properties:

$$[x, x] = 0, \text{ for all } x \in \mathfrak{g}, \tag{A.1}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \text{ for all } x, y, z \in \mathfrak{g}. \tag{A.2}$$

The *Lie bracket* $[x, y]$ are often referred to as the *commutator* of x and y. The condition A.2 is called the *Jacobi identity*. Note that, since the Lie bracket $[-, -]$ define a bilinear form, it follows that

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x].$$

Therefore, condition A.1 implies

$$[x, y] = -[y, x], \text{ for all } x, y \in \mathfrak{g}. \tag{A.3}$$

For a field $\mathbb{F}$ with characteristic different from 2, conditions A.2 and A.3 are equivalent. Additionally, it is easy to observe that

- $[0, v] = 0$, for all $v \in \mathfrak{g}$;
- if $x, y \in \mathfrak{g}$ such that $[x, y] \neq 0$, then x and y are linearly independent.

Definition A.1.3. A Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ is *abelian* if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

Definition A.1.4. The *dimension* of a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ is defined as the dimension of $\mathfrak{g}$ as a vector space over $\mathbb{F}$, denoted as $\dim \mathfrak{g} := \dim_{\mathbb{F}} \mathfrak{g}$.

Here are some examples of Lie algebras that will be frequently mentioned:

1. On any vector space V, it is always possible to define a Lie bracket by setting $[x, y] = 0$ for all $x, y \in V$. Thus, V is equipped with the structure of a Lie algebra. In this case, it is an abelian Lie algebra. It is worth noting that a field $\mathbb{F}$ can be thought of as a Lie algebra of dimension 1.

2. Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Denote $\mathsf{gl}(V)$ as the set of linear maps from V to itself. From linear algebra, it is well known that $\mathsf{gl}(V)$ is a vector space over $\mathbb{F}$. Moreover, it forms a Lie algebra, referred to as the *general linear algebra*, by defining the following Lie bracket:

$$[x, y] := x \circ y - y \circ x \text{ for all } x, y \in \mathsf{gl}(V),$$

   where $\circ$ denotes the composition of linear maps.

3. Let $\mathsf{gl}(n, \mathbb{F})$ be the vector space of $n \times n$ matrices with entries in the field $\mathbb{F}$. Then we can define the following Lie bracket:

$$[x, y] := xy - yx,$$

   where xy is the standard matrix multiplication (row by column). It can be useful to observe how the Lie bracket acts on the canonical basis of $\mathsf{gl}(n, \mathbb{F})$. Let $\{e_{ij}\}_{i,j=1,\dots,n}$ be that basis, where e_{ij} denotes the $n \times n$ matrix with a 1 at position (i, j) and 0 elsewhere. Then we have

$$[e_{ij}, e_{kl}] = e_{ij}e_{kl} - e_{kl}e_{ij} = \delta_{jk}e_{il} - \delta_{il}e_{kj},$$

   where the last equality becomes clear when matrix multiplication is carried out, and δ represents the Kronecker delta, i.e., $\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$.

Definition A.1.5. Let $\mathfrak{g}$ be a Lie algebra over the field $\mathbb{F}$. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a subspace of $\mathfrak{g}$. $\mathfrak{h}$ is called a *subalgebra* of $\mathfrak{g}$ if $[x, y] \in \mathfrak{h}$ for every $x, y \in \mathfrak{h}$.

Here are some examples of subalgebras.

1. Let $\mathsf{sl}(n, \mathbb{F})$ denote the vector subspace of $\mathsf{gl}(n, \mathbb{F})$ consisting of matrices with zero trace. Remembering from linear algebra that $\mathrm{Tr}(xy) = \mathrm{Tr}(yx)$, then the matrix $xy - yx$ has zero trace. Thus, the same Lie bracket defined on $\mathsf{gl}(n, \mathbb{F})$ also define a Lie algebra on $\mathsf{sl}(n, \mathbb{F})$. This Lie algebra is known as the *special linear algebra*. So, by the above definition, $\mathsf{sl}(n, \mathbb{F})$ is a Lie subalgebra of $\mathsf{gl}(n, \mathbb{F})$.

2. Let $\mathsf{b}(n, \mathbb{F})$ be the vector subspace of $\mathsf{gl}(n, \mathbb{F})$ consisting of upper triangular matrices, i.e., matrices $x = (x_{ij})$ such that $x_{ij} = 0$ when $i > j$. This is a Lie algebra if we define the same bracket as in $\mathsf{gl}(n, \mathbb{F})$. Furthermore, since xy is still an upper triangular matrix, $\mathsf{b}(n, \mathbb{F})$ is a Lie subalgebra of $\mathsf{gl}(n, \mathbb{F})$.

3. The same considerations as above can be made for the vector subspace $\mathsf{n}(n, \mathbb{F})$ of $\mathsf{gl}(n, \mathbb{F})$ consisting of strictly upper triangular matrices ($x = (x_{ij})$ such that $x_{ij} = 0$ when $i \geq j$), which also turns out to be a Lie subalgebra of $\mathsf{gl}(n, \mathbb{F})$.

Definition A.1.6. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. Let $\mathfrak{i}$ be a subspace of $\mathfrak{g}$. If $[x, y] \in \mathfrak{i}$ for every $x \in \mathfrak{g}$ and $y \in \mathfrak{i}$, then $\mathfrak{i}$ is called an *ideal* of $\mathfrak{g}$.

Regarding the distinction between right and left ideals, here we do not need it because for Equation A.3, we have $[x, y] = -[y, x]$ for every $x, y \in \mathfrak{g}$.

Example A.1.6.1. Referring to the previous examples, it is easily noted that $\mathsf{sl}(n, \mathbb{F})$ is an ideal of $\mathsf{gl}(n, \mathbb{F})$ and $\mathsf{n}(n, \mathbb{F})$ is an ideal of $\mathsf{b}(n, \mathbb{F})$.

Remark A.1.1. An ideal is always a subalgebra; this follows trivially from the definitions. Conversely, however, is not always true. It has been shown that $\mathsf{b}(n, \mathbb{F})$ is a subalgebra of $\mathsf{gl}(n, \mathbb{F})$, but for $n \geq 2$, it is not its ideal. For example, if we consider $e_{11} \in \mathsf{b}(n, \mathbb{F})$ and $e_{21} \in \mathsf{gl}(n, \mathbb{F})$, then $[e_{11}, e_{21}] = -e_{21} \notin \mathsf{b}(n, \mathbb{F})$.

For every Lie algebra $\mathfrak{g}$, there are always two ideals, known as *trivial*, which are $\{0\}$ and $\mathfrak{g}$. A non-trivial ideal for non-abelian Lie algebras is defined as follows:

Definition A.1.7. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. The *center of* $\mathfrak{g}$ is defined as

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0, \text{ for every } y \in \mathfrak{g}\}.$$

As defined earlier, if $Z(\mathfrak{g}) = \mathfrak{g}$, then $\mathfrak{g}$ is an abelian Lie algebra.

Proposition A.1.8. $\mathrm{Z}(\mathfrak{g})$ *is an ideal of* $\mathfrak{g}$.

Proof. Let $x_1, x_2 \in \mathrm{Z}(\mathfrak{g})$ and $y \in \mathfrak{g}$. Then

$$[x_1 + x_2, y] = [x_1, y] + [x_2, y] = 0 + 0 = 0,$$

so $x + y \in \mathrm{Z}(\mathfrak{g})$. Additionally, let $\lambda \in \mathbb{F}$, $x \in \mathfrak{Z}(\mathfrak{g})$, and $y \in \mathfrak{g}$. Therefore,

$$[\lambda x, y] = \lambda [x, y] = 0,$$

so $\lambda x \in \mathrm{Z}(\mathfrak{g})$, and $\mathrm{Z}(\mathfrak{g})$ is a subspace of $\mathfrak{g}$. Furthermore, it is also an ideal. For every $x \in \mathfrak{g}$ and $y \in \mathbb{Z}(\mathfrak{g})$, $[x, y] \in \mathrm{Z}(\mathfrak{g})$, we have

$$[t, [x, y]] + [x, [y, t]] + [y, [t, x]] = 0$$

and this implies that

$$[t, [x, y]] = -[x, [y, t]] - [y, [t, x]] = -[x, 0] - 0 = -0 - 0 = 0,$$

for every $t \in \mathfrak{g}$. □

A.2 Homomorphisms of Lie Algebras

Definition A.2.1. Let $(\mathfrak{g}_1, [\cdot,\cdot]_{\mathfrak{g}_1 1})$ and $(\mathfrak{g}_2, [\cdot,\cdot]_{\mathfrak{g}_2})$ be two Lie algebras over the same field $\mathbb{F}$. Then $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a *homomorphism of Lie algebras* if ϕ is a linear map and if it satisfies

$$\phi([x,y]_{\mathfrak{g}_1}) = [\phi(x), \phi(y)]_{\mathfrak{g}_2} \text{ for all } x, y \in \mathfrak{g}_1.$$

Remark A.2.1. The Lie bracket $[x, y]$ are those defined on $\mathfrak{g}_1$, while those of $[\phi(x), \phi(y)]$ are those defined on $\mathfrak{g}_2$. Often, to simplify notation, subscripts indicating the Lie algebra to which reference is made will be omitted. When they are included, it will be to avoid ambiguity or errors.

For every Lie algebra $\mathfrak{g}$ over the field $\mathbb{F}$, there is a particular and extremely important homomorphism.

Definition A.2.2. The *adjoint homomorphism*, denoted by ad, acts as follows:

$$\begin{aligned} \mathrm{ad}\colon \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}) \\ x &\mapsto \mathrm{ad}(x), \end{aligned}$$

where $\mathrm{ad}(x)(y) := [x, y]$ for all $x, y \in \mathfrak{g}$.

From the bilinearity of Lie brackets, it follows that the adjoint homomorphism is linear. To demonstrate that the Lie brackets are *preserved*, it suffices to apply the Jacobi identity. Less obvious is the following.

Proposition A.2.3. *The kernel of* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *is the center of* $\mathfrak{g}$.

Proof. If $x \in \mathfrak{g}$, then

$$\mathrm{ad}(x) = 0 \Leftrightarrow \mathrm{ad}(x)(y) = 0, \forall y \in \mathfrak{g} \Leftrightarrow [x, y] = 0, \forall y \in \mathfrak{g} \Leftrightarrow x \in \mathrm{Z}(\mathfrak{g}).$$

□

Proposition A.2.4. *Let* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *be two Lie algebras over a field* $\mathbb{F}$*, and* $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ *be a homomorphism of Lie algebras. Then*

1. $\ker\phi$ *is an ideal of* $\mathfrak{g}_1$*;*

2. $\mathrm{Im}\,\phi$ *is a subalgebra of* $\mathfrak{g}_2$*.*

1. If $x \in \ker\phi$ and $y \in \mathfrak{g}_1$, then (recall that $\ker\phi$ is already a vector subspace of $\mathfrak{g}_1$):

$$\phi([x, y]) = [\phi(x), \phi(y)] = [0, \phi(y)] = 0 \Rightarrow [x, y] \in \ker\phi.$$

2. It must be shown that $[x', y'] \in \mathrm{Im}\,\phi$, for every $x', y' \in Im\phi$. Since $x', y' \in \mathrm{Im}\,\phi$, there exist $x, y \in \mathfrak{g}_1$ such that $x' = \phi(x)$ and $y' = \phi(y)$, thus

$$[x', y'] = [\phi(x), \phi(y)] = \phi([x, y]) \in \mathrm{Im}\,\phi.$$

Remark A.2.2. If V is a vector space over the field $\mathbb{F}$ with $\dim_{\mathbb{F}} V = n$, it is well known from linear algebra that $\mathrm{End}_{\mathbb{F}}(V) \cong \mathrm{M}_n(\mathbb{F})$ through the linear isomorphism $f \mapsto M_f$, where M_f is the matrix of the linear map f with respect to a fixed basis of V. It is not difficult to observe that this represents an isomorphism of Lie algebras, i.e., a linear isomorphism that preserves Lie brackets. Therefore, the identification made for vector spaces also holds for Lie algebras, namely in our language $\mathsf{gl}(V) \cong \mathsf{gl}(n, \mathbb{F})$.

Definition A.2.5. Let A be an algebra over the field $\mathbb{F}$. A *derivation* of A is a linear map $\delta : A \to A$ such that

$$\delta(ab) = a\delta(b) + \delta(a)b \quad \text{for all } a, b \in A.$$

If we denote $\mathrm{Der}(A)$ as the set of all derivations on A and equip it with the usual addition and scalar multiplication between linear maps, then $\mathrm{Der}(A)$ is a subspace of $\mathsf{gl}(A)$. In particular, it is observed that if δ_1 and δ_2 are two derivations of a Lie algebra A, then $[\delta_1, \delta_2]$ is still a derivation, so $\mathrm{Der}(A)$ is a subalgebra of $\mathsf{gl}(A)$.

Lemma A.2.6. *Let $\mathfrak{g}$ be a Lie algebra, and let $x \in \mathfrak{g}$. Then $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ is a derivation.*

Proof. Since the Jacobi identity holds, we have

$$\mathrm{ad}(x)\,[y, z] = [x, [y, z]] = [[x, y]\,, z] + [y, [x, z]] = [\mathrm{ad}(x)(y), z] + [y, \mathrm{ad}(x)(z)]\,,$$

for all $x, y, z \in \mathfrak{g}$. □

Definition A.2.7. Let $\mathfrak{g}$ be a Lie algebra, and let δ be a derivation on $\mathfrak{g}$. δ is called an *inner derivation* if $\delta = \mathrm{ad}(x)$ for some $x \in \mathfrak{g}$.

If we denote $\mathrm{IDer}\,\mathfrak{g}$ as the set of inner derivations on $\mathfrak{g}$, then $\mathrm{IDer}\,\mathfrak{g}$ is an ideal of $\mathrm{Der}\,\mathfrak{g}$.

A.2.1 Structure Constants

To understand how a bilinear form acts on any pair of vector spaces, it is sufficient to define its values on pairs of basis elements. Similarly, to understand how Lie brackets act on a given Lie algebra $\mathfrak{g}$, it is enough to define their action on the basis elements, essentially presenting the *multiplication table*. For example, let $\{e_1, \dots, e_n\}$ be a basis for $\mathfrak{g}$. Then $[e_i, e_j]$ is a linear combination of the basis elements as an element of $\mathfrak{g}$, that is,

$$[e_i, e_j] = \sum_{k=1}^{n} a_{ij}^k e_k.$$

The scalars $a_{ij}^k \in \mathbb{F}$ are naturally defined and are called the *structure constants* of $\mathfrak{g}$ with respect to the given basis. It is immediately apparent that the scalars a_{ij}^k depend on the chosen basis; different bases will define different structure constants. From the conditions A.1 and A.3 on Lie bracket, $[e_i, e_i] = 0$ for every

$i = 1, 2, \ldots, n$ and $[e_i, e_j] = -[e_j, e_i]$ for every $i, j = 1, 2, \ldots, n$. Therefore, it is sufficient to know the structure constants a_{ij}^k for every $1 \leq i < j \leq n$.

A.3 Sum of Ideals, Quotient Algebras, and Isomorphism Theorems

Given two ideals $\mathfrak{i}$ and $\mathfrak{j}$ of a Lie algebra $\mathfrak{g}$, it is possible to construct new ideals from $\mathfrak{i}$ and $\mathfrak{j}$. We define the sets

$$\mathfrak{i} + \mathfrak{j} := \{x + y \mid x \in \mathfrak{i}, y \in \mathfrak{j}\}$$

and

$$[\mathfrak{i}, \mathfrak{i}] := \langle [x, y] \mid x \in \mathfrak{i}, y \in \mathfrak{j} \rangle.$$

which turn out to be ideals of $\mathfrak{g}$. A particular case of this construction is when we consider $\mathfrak{i} = \mathfrak{j} = \mathfrak{g}$.

Definition A.3.1. Let $\mathfrak{g}$ be a Lie algebra over the field $\mathbb{F}$. The Lie algebra $[\mathfrak{g}, \mathfrak{g}]$, denoted as $\mathfrak{g}'$, is called the *derived algebra* of $\mathfrak{g}$.

If $\mathfrak{i}$ is an ideal of the Lie algebra $\mathfrak{g}$, then in particular, $\mathfrak{i}$ is a vector subspace of $\mathfrak{g}$, so we can consider the set $z + \mathfrak{i} = \{z + x | x \in \mathfrak{i}\}$, with $z \in \mathfrak{g}$, and we have the quotient vector space

$$\mathfrak{g}/\mathfrak{i} = \{z + \mathfrak{i} \mid z \in \mathfrak{g}\}$$ [1].

Definition A.3.2. In the above conditions, the vector space $\mathfrak{g}/\mathfrak{i}$ can be equipped with the structure of a Lie algebra, called the *quotient Lie algebra*, whose Lie brackets are defined as

$$[w + \mathfrak{i}, z + \mathfrak{i}] := [w, z] + \mathfrak{i} \quad \text{for all } w, z \in \mathfrak{g}.$$

The Lie brackets of $\mathfrak{g}/\mathfrak{i}$ are indeed well-defined, meaning they depend on the representatives w and z. If $w + \mathfrak{i} = w' + \mathfrak{i}$ and $z + \mathfrak{i} = z' + \mathfrak{i}$, then $w - w' \in \mathfrak{i}$ and $z - z' \in \mathfrak{i}$. By the bilinearity of Lie brackets on $\mathfrak{g}$, we have

$$\begin{aligned} [w', z'] &= [w + (w' - w), z + (z' - z)] \\ &= [w, z] + [w' - w, z] + [w, z' - z] + [w' - w, z' - z], \end{aligned}$$

where the last three terms belong to the ideal $\mathfrak{i}$. Therefore, $[w' + \mathfrak{i}, z' + \mathfrak{i}] = [w, z] + \mathfrak{i}$. Finally, it is observed that the axioms for the Lie brackets defined on the quotient Lie algebra $\mathfrak{g}/\mathfrak{i}$ follow from the axioms that hold for $\mathfrak{g}$. The following theorems present isomorphism results for Lie algebras.

Theorem A.3.3 (Isomorphism Theorems)**.**

[1] It is well-known that this set is a vector space once we define the operations *inherited* from $\mathfrak{g}$.

1. *Let* $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ *be a homomorphism of Lie algebras. Then* $\ker\phi$ *is an ideal of* $\mathfrak{g}_1$*,* $\operatorname{Im}\phi$ *is a subalgebra of* $\mathfrak{g}_2$*, and*
$$\mathfrak{g}_1/\ker\phi \cong \operatorname{Im}\phi.$$
2. *If* $\mathfrak{i}$ *and* $\mathfrak{j}$ *are two ideals of a Lie algebra, then* $(\mathfrak{i}+\mathfrak{j})/\mathfrak{j} \cong \mathfrak{i}/(\mathfrak{i}\cap\mathfrak{j})$.
3. *Suppose that* $\mathfrak{i}$ *and* $\mathfrak{j}$ *are ideals of the Lie algebra* $\mathfrak{g}$ *such that* $\mathfrak{i}\subseteq\mathfrak{j}$*. Then* $\mathfrak{j}/\mathfrak{i}$ *is an ideal of* $\mathfrak{g}/\mathfrak{i}$ *and*
$$(\mathfrak{g}/\mathfrak{i})/(\mathfrak{j}/\mathfrak{i}) \cong \mathfrak{g}/\mathfrak{j}.$$

In order, these are called the first, second, and third isomorphism theorem for Lie algebras.

Suppose that $\mathfrak{i}$ is an ideal of the Lie algebra $\mathfrak{g}$. Then there exists a bijective correspondence between ideals of $\mathfrak{g}$ containing $\mathfrak{i}$ and ideals of $\mathfrak{g}/\mathfrak{i}$. Specifically, this correspondence associates an ideal $\mathfrak{j}$ of $\mathfrak{g}$ containing $\mathfrak{i}$ with the ideal $\mathfrak{j}/\mathfrak{i}$ of $\mathfrak{g}/\mathfrak{i}$. Conversely, if $\mathfrak{k}$ is an ideal of $\mathfrak{g}/\mathfrak{i}$, then the set $\mathfrak{j} := \{z \in \mathfrak{g} | z+\mathfrak{i} \in \mathfrak{k}\}$ is an ideal of $\mathfrak{g}$ containing $\mathfrak{k}$. These two mappings are inverses of each other. The following theorem summarizes what has been said.

Theorem A.3.4 (Correspondence Theorem). *Let* $\mathfrak{i}$ *be an ideal of the Lie algebra* $\mathfrak{g}$*. There exists a bijective correspondence between ideals of* $\mathfrak{g}$ *containing* $\mathfrak{i}$ *and ideals of* $\mathfrak{g}/\mathfrak{i}$*.*
$$\{\textit{ideals of } \mathfrak{g} \textit{ containing } \mathfrak{i}\} \leftrightarrow \{\textit{ideals of } \mathfrak{g}/\mathfrak{i}\}$$

For completeness, we provide the following results related to quotient algebras and homomorphisms of Lie algebras.

Proposition A.3.5. $\mathfrak{g}/\mathfrak{i}$ *is abelian if and only if* $\mathfrak{g}' \leq \mathfrak{i}$*.*

Proof. If $\mathfrak{g}/\mathfrak{i}$ is abelian, then for every $x+\mathfrak{i}, y+\mathfrak{i}$, we have
$$[x+\mathfrak{i}, y+\mathfrak{i}] = 0_{\mathfrak{g}/\mathfrak{i}} \Rightarrow [x,y]+\mathfrak{i} = \mathfrak{i} \Rightarrow [x,y] \in \mathfrak{i} \Rightarrow \mathfrak{g}' \leq \mathfrak{i}.$$
Conversely, if $\mathfrak{g}' \leq \mathfrak{i}$, then for any $x', y' \in \mathfrak{g}/\mathfrak{i}$, we have
$$[x', y'] = [\phi(x), \phi(y)] = \phi([x,y]) \in \phi(\mathfrak{i}).$$
Hence, $\mathfrak{g}/\mathfrak{i}$ is abelian. □

Proposition A.3.6. *Let* $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ *be a surjective isomorphism of Lie algebras.*

1. $\phi(\mathfrak{g}_1') = \mathfrak{g}_2'$
2. *If* $x \in \mathfrak{g}_1$ *such that* $ad(x)$ *is diagonalizable, then* $\operatorname{ad}(\phi(x))$ *is diagonalizable.*

Proof.

1. To prove the thesis, we demonstrate the two inclusions. To show that $\phi(\mathfrak{g}_1') \leq \mathfrak{g}_2'$, we first consider $x, y \in \mathfrak{g}_1$. Then, $\phi([x,y]) = [\phi(x), \phi(y)] \in \mathfrak{g}_2'$

because ϕ is a Lie homomorphism. For a generic linear combination of n elements of $\mathfrak{g}_1'$, i.e., taking $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathfrak{g}_1$, we have

$$\phi(\lambda_1 [x_1, y_1] + \cdots + \lambda_n [x_n, y_n]) = \sum_{i=1}^{n} \lambda_i \phi([x_i, y_i]) \in \mathfrak{g}_2'$$

because, for every $i = 1, \ldots, n$, it was shown above that $\phi([x_i, y_i]) \in \mathfrak{g}_2'$. For the inverse inclusion, let $x', y' \in \mathfrak{g}_2$. Due to the surjectivity of ϕ, there exist $x, y \in \mathfrak{g}_1$ such that $x' = \phi(x)$ and $y' = \phi(y)$, so

$$[x', y'] = [\phi(x), \phi(y)] = \phi([x, y]) \in \phi(\mathfrak{g}_1').$$

2. If $x \in \mathfrak{g}_1$ is such that $\operatorname{ad}(x)$ is diagonalizable, then there exists a basis $\mathcal{B}$ of $\mathfrak{g}_1$ consisting of eigenvectors of $\operatorname{ad}(x)$, i.e., $\mathcal{B} = \{v_1, \ldots, v_n\}$, if $\dim \mathfrak{g}_1 = n$, with $\operatorname{ad}(x)(v_i) = \lambda_i v_i$. Then, $\mathfrak{g}_2 = \langle \phi(v_1), \ldots, \phi(v_n) \rangle$, and, by appropriate reordering, $\{\phi(v_1), \ldots, \phi(v_m)\}$ is a basis of $\mathfrak{g}_2$, where $\dim \mathfrak{g}_2 = m$. So, for $1 \leq i \leq m$, we have

$$\operatorname{ad}(\phi(x))(\phi(v_i)) = [\phi(x), \phi(v_i)] = \phi([x, v_i]) = \phi(\operatorname{ad}(x)(v_i)) = \phi(\lambda_i v_i) = \lambda_i \phi(v_i),$$

therefore, $\{\phi(v_1), \ldots, \phi(v_m)\}$ is a basis of $\mathfrak{g}_2$ consisting of eigenvectors of $\operatorname{ad}(\phi(x))$, and consequently, $\operatorname{ad}(\phi(x))$ is diagonalizable.

□

Remark A.3.1. Under the same assumptions as before, it cannot be concluded that $\phi(\mathrm{Z}(\mathfrak{g}_1)) = \mathrm{Z}(\mathfrak{g}_2)$, but in general, it can only be stated that $\phi(\mathrm{Z}(\mathfrak{g}_1)) \subset \mathrm{Z}(\mathfrak{g}_2)$. Indeed, if $x' \in \mathfrak{g}_2$ and $\phi(z) \in \phi(\mathrm{Z}(\mathfrak{g}_1))$ (i.e., $z \in \mathrm{Z}(\mathfrak{g}_1)$), then, due to the surjectivity of ϕ, $x' = \phi(x)$ for some $x \in \mathfrak{g}_1$, and

$$[x', \phi(z)] = [\phi(x), \phi(z)] = \phi([x, z]) = \phi(0_{\mathfrak{g}_1}) = 0_{\mathfrak{g}_2}.$$

To obtain the equality above, a stronger hypothesis is needed.

Proposition A.3.7. *Let $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ be an isomorphism of Lie algebras. Then $\phi(\mathrm{Z}(\mathfrak{g}_1)) = \mathrm{Z}(\mathfrak{g}_2)$.*

Proof. In the previous remark, it was shown that if ϕ is surjective, then $\phi(\mathrm{Z}(\mathfrak{g}_1)) \leq \mathrm{Z}(\mathfrak{g}_2)$. To establish the reverse inclusion, consider $z_2 \in \mathrm{Z}(\mathfrak{g}_2)$. For every $x_2 \in \mathrm{Z}(\mathfrak{g}_2)$, we have $[x_2, z_2] = 0$. Let $x_1 \in \mathrm{Z}(\mathfrak{g}_1) \Rightarrow [\phi(x_1), z_2] = 0$. Since ϕ is an isomorphism, there exists ϕ^{-1}, so

$$\Big[\phi(x_1), \phi(\phi^{-1}(z_2))\Big] = \phi(\Big[x_1, \phi^{-1}(z_2)\Big]) = 0.$$

Then $[x_1, \phi^{-1}(z_2)] = 0$ and this implies that $\phi^{-1}(z_2) \in \mathrm{Z}(\mathfrak{g}_1)$. Hence $z_2 \in \phi(\mathrm{Z}(\mathfrak{g}_1))$. □

Proposition A.3.8. *Let $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ be a homomorphism of Lie algebras. If $\mathfrak{i}$ is an ideal of $\mathfrak{g}_1$, then $\phi(\mathfrak{i})$ is an ideal of* $\operatorname{Im}\phi$.

Proof. If $\mathfrak{i}$ is an ideal of $\mathfrak{g}$, it is, in particular, a vector subspace of $\mathfrak{g}$, so $\phi(\mathfrak{i})$ is a vector subspace of $\mathfrak{g}_2$, and therefore, of $\operatorname{Im}\phi$, since $\phi(\mathfrak{i}) \leq \operatorname{Im}\phi$. Now, consider $x' \in \operatorname{Im}\phi$ and $y \in \phi(\mathfrak{i})$. Then, there exist $x \in \mathfrak{g}_1$ and $i \in \mathfrak{i}$ such that

$$x' = \phi(x) \text{ and } y = \phi(i),$$

so

$$[x', y] = [\phi(x), \phi(y)] = \phi([x, y]) \in \phi(\mathfrak{i}).$$

□

A.4 Direct Sum of Lie Algebras

In the following sections, we will frequently use the concept of a *direct sum of Lie algebras*. Therefore, it is necessary to provide a detailed explanation of this construction.

Definition A.4.1. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$, and let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be subalgebras of $\mathfrak{g}$. We will say that $\mathfrak{g}$ is the *direct sum of the subalgebras* $\mathfrak{g}_1$ and $\mathfrak{g}_2$, denoted as $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, if the following conditions hold:

1. $\mathfrak{g}$ is the direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ as vector spaces.
2. $[x_1, x_2] = 0$ for every $x_i \in \mathfrak{g}_i$, with $i = 1, 2$.

Indeed, if a Lie algebra $\mathfrak{g}$ is the direct sum of two of its subalgebras, they are shown to be more.

Proposition A.4.2. *If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, then $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are ideals of $\mathfrak{g}$.*

Proof. Let $y \in \mathfrak{g}$. Then, $y = y_1 + y_2$, with $y_1 \in \mathfrak{g}_1$ and $y_2 \in \mathfrak{g}_2$. For any $x_1 \in \mathfrak{g}_1$,

$$[y, x_1] = [y_1 + y_2, x_1] = [y_1, x_1] + [y_2, x_1] = [y_1, x_1] \in \mathfrak{g}_1.$$

Similarly, $\mathfrak{g}_2$ is an ideal of $\mathfrak{g}$. □

Example A.4.2.1. Let $\mathsf{gl}(2, \mathbb{F})$ be the Lie algebra of invertible 2×2 matrices with elements in the field $\mathbb{F}$, where $\operatorname{char}(\mathbb{F}) \neq 2$. We want to show that

$$\mathsf{gl}(2, \mathbb{F}) = \mathsf{sl}(2, \mathbb{F}) \oplus Z(\mathsf{gl}(2, \mathbb{F})).$$

First, we determine the center of this Lie algebra. Let $x \in Z(\mathsf{gl}(2, \mathbb{F}))$, then $[x, e_{ij}] = 0$ for every $i, j = 1, 2$. If $x = (x_{ij})$, then

$$[x, e_{11}] = xe_{11} - e_{11}x = \begin{pmatrix} x_{11} & 0 \\ x_{21} & 0 \end{pmatrix} - \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -x_{12} \\ x_{21} & 0 \end{pmatrix} = 0_{\mathsf{gl}(2,\mathbb{F})}$$

if and only if $x_{12} = x_{21} = 0$. Thus, $x = \operatorname{diag}\{x_{11}, x_{22}\}$. Continuing with e_{12}:

$$[x, e_{12}] = xe_{12} - e_{12}x = \begin{pmatrix} 0 & x_{11} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & x_{22} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x_{11} - x_{12} \\ 0 & 0 \end{pmatrix} = 0_{\mathsf{gl}(2,\mathbb{F})}$$

if and only if $x_{11} = x_{22}$, so x is a scalar matrix, more formally, $x = \lambda I_2$, where $\lambda \in \mathbb{F}$. It can be easily verified that for such a matrix, $[x, e_{21}] = [x, e_{22}] = 0$, so

$$\mathrm{Z}(\mathsf{gl}(2,\mathbb{F})) = \{\lambda I_2 \mid \lambda \in \mathbb{F}\}.$$

Before verifying the two conditions given in Definition A.4.1, we observe that the dimension of $\mathsf{gl}(2,\mathbb{F})$ is 4, while the dimensions of $\mathsf{sl}(2,\mathbb{F})$ and $\mathrm{Z}(\mathsf{gl}(2,\mathbb{F}))$ are 3 and 1, respectively.

$\mathsf{sl}(2,\mathbb{F})$ and $\mathrm{Z}(\mathsf{gl}(2,\mathbb{F}))$ have trivial intersection since the only scalar matrix with a trace zero is the zero matrix, which is in $\mathsf{gl}(2,\mathbb{F})$. Thus, $\mathsf{sl}(2,\mathbb{F}) \cap \mathrm{Z}(\mathsf{gl}(2,\mathbb{F})) = \{0_{\mathsf{gl}(2,\mathbb{F})}\}$.

To demonstrate that it is a direct sum of vector spaces, it remains to be shown that for every $x \in \mathsf{gl}(2,\mathbb{F})$, there exists $s \in \mathsf{sl}(2,\mathbb{F})$ and $z \in \mathrm{Z}(\mathsf{gl}(2,\mathbb{F}))$ such that $x = s + z$.

Let $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{gl}(2,\mathbb{F})$. First, we calculate the trace of x, $\mathrm{Tr}(x) = a + d$. We want to find a matrix with this trace which is traceless. This is achieved by subtracting the average of the diagonal entries of x from x. In other words,

$$\begin{aligned} s = x - \frac{Tr(x)}{2} I_2 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \frac{a+d}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a - \frac{a+d}{2} & b \\ c & d - \frac{a+d}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{a-d}{2} & b \\ c & \frac{d-a}{2} \end{pmatrix} \in \mathsf{sl}(2,\mathbb{F}). \end{aligned}$$

For the element z, we choose $z = \frac{a+d}{2} I_2$, which belongs to $\mathrm{Z}(\mathsf{gl}(2,\mathbb{F}))$. We have

$$\begin{aligned} x = s + z &= \begin{pmatrix} \frac{a-d}{2} & b \\ c & \frac{d-a}{2} \end{pmatrix} + \frac{a+d}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{a-d}{2} + \frac{a+d}{2} & b \\ c & \frac{d-a}{2} + \frac{a+d}{2} \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{aligned}$$

Thus, $\mathsf{gl}(2,\mathbb{F}) = \mathsf{sl}(2,\mathbb{F}) \oplus \mathrm{Z}(\mathsf{gl}(2,\mathbb{F}))$.

In this example, we showed that the Lie algebra $\mathsf{gl}(2,\mathbb{F})$ of invertible 2×2 matrices over a field $\mathbb{F}$ can be decomposed as the direct sum of $\mathsf{sl}(2,\mathbb{F})$ and the center $\mathrm{Z}(\mathsf{gl}(2,\mathbb{F}))$, where $\mathsf{sl}(2,\mathbb{F})$ is the special linear Lie algebra and $\mathrm{Z}(\mathsf{gl}(2,\mathbb{F}))$ is the center of $\mathsf{gl}(2,\mathbb{F})$.

Remark A.4.1. If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathfrak{i}$ is an ideal of $\mathfrak{g}_i$, $i = 1, 2$, then $\mathfrak{i}$ is an ideal of $\mathfrak{g}$. Indeed, suppose that $\mathfrak{i}$ is an ideal of $\mathfrak{g}_1$ (the same argument applies if $\mathfrak{i}$ is an ideal of $\mathfrak{g}_2$). Then, $\mathfrak{i}$ is a vector subspace of $\mathfrak{g}_1$, so $\mathfrak{i}$ is also a vector subspace of $\mathfrak{g}$. Furthermore, let $y \in \mathfrak{g}$ and $x \in \mathfrak{i}$, then $[y, x] \in \mathfrak{i}$ because $y = y_1 + y_2$, with

$y_1 \in \mathfrak{g}_1$ and $y_2 \in \mathfrak{g}_2$, and

$$[y, x] = [y_1 + y_2, x] = [y_1, x] + [y_2, x] = [y_1, x] + 0 = [y_1, x] \in \mathfrak{i}.$$

$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is supported in the set determined by the Cartesian product of the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, that is

$$\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2 = \{(x_1, x_2) | x_1 \in \mathfrak{g}_1, x_2 \in \mathfrak{g}_2\}.$$

For the underlying vector spaces the sum is defined componentwise, i.e.

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

and $0_{\mathfrak{g}} = (0_{\mathfrak{g}_1}, 0_{\mathfrak{g}_2})$. The sum defined in this way is associative in $\mathfrak{g}_1$ and $\mathfrak{g}_2$, so it will also be in $\mathfrak{g}$. The latter, with the sum defined above, is an abelian group. Defining scalar multiplication as

$$\lambda(x_1, x_2) = (\lambda x_1, \lambda x_2),$$

where $\lambda \in \mathbb{F}$ and $(x_1, x_2) \in \mathfrak{g}$, it follows that $(\mathfrak{g}, +, \cdot)$ is a vector space over the field $\mathbb{F}$. It is also worth noting that the dimension of $\mathfrak{g}$ over $\mathbb{F}$ is the sum of the dimensions of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. To see this, suppose that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ have dimensions n_1 and n_2, respectively. Therefore, there are two bases for these vector spaces, denoted respectively as

$$\mathcal{B}_1 = \left\{e_1^1, e_2^1, \ldots, e_{n_1}^1\right\}$$

and

$$\mathcal{B}_2 = \left\{e_1^2, e_2^2, \ldots, e_{n_2}^2\right\}.$$

With this in mind, it is easy to construct a basis $\mathcal{B}$ of $\mathfrak{g}$ using $\mathcal{B}_1$ and $\mathcal{B}_2$ as follows:

$$\mathcal{B} = \left\{(e_1^1, 0), \ldots, (e_{n_1}^1, 0), (0, e_1^2), \ldots, (0, e_{n_2}^2)\right\}$$

whose cardinality is $n_1 + n_2$. If, as assumed earlier, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ have the structure of Lie algebras, then it is possible to endow $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$ with such a structure by defining the following Lie brackets in this vector space:

$$\begin{aligned} [-,-] &: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \\ ((x_1, x_2), (y_1, y_2)) &\mapsto [(x_1, x_2), (y_1, y_2)] = ([x_1, y_1], [x_2, y_2]). \end{aligned}$$

Naturally, the Lie brackets in the first component of the image are those related to $\mathfrak{g}_1$, and those in the second component are those related to $\mathfrak{g}_2$. Due to the additivity of the Lie brackets in $\mathfrak{g}_1$ and $\mathfrak{g}_2$, it follows that the Lie brackets in $\mathfrak{g}$ are also additive. Let's see this only for the first component; a similar argument can be made for the second component:

$$\begin{aligned}[(x_1, x_2) + (x_1' + x_2'), (y_1, y_2)] &= [(x_1 + x_1', x_2 + x_2'), (y_1, y_2)] \\ &= ([x_1 + x_1', y_1], [x_2 + x_2', y_2]) \\ &= ([x_1, y_1] + [x_1', y_1], [x_2, y_2] + [x_2', y_2]) \\ &= ([x_1, y_1], [x_2, y_2]) + ([x_1', y_1], [x_2', y_2]) \\ &= [(x_1, x_2), (y_1, y_2)] + [(x_1', x_2'), (y_1, y_2)] .\end{aligned}$$

With similar steps, one can demonstrate the properties regarding linearity with respect to scalar multiplication. Finally, it is observed that antisymmetry and the Jacobi identity hold because they also hold for $\mathfrak{g}_1$ and $\mathfrak{g}_2$. In fact, if $(x_1, x_2) \in \mathfrak{g}$, then

$$[(x_1, x_2), (x_1, x_2)] = ([x_1, x_1], [x_2, x_2]) = (0_{\mathfrak{g}_1}, 0_{\mathfrak{g}_2}) = 0_{\mathfrak{g}},$$

and for generic $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathfrak{g}$, it follows that

$$\begin{aligned}&[(x_1, x_2), [(y_1, y_2), (z_1, z_2)]] + [(y_1, y_2), [(z_1, z_2), (x_1, x_2)]] \\ &+ [(z_1, z_2), [(x_1, x_2), (y_1, y_2)]] \\ &= [(x_1, x_2), ([y_1, z_1], [y_2, z_2])] + [(y_1, y_2), ([z_1, x_1], [z_2, x_2])] \\ &+ [(z_1, z_2), ([x_1, y_1], [x_2, y_2])] \\ &= 0_{\mathfrak{g}}.\end{aligned}$$

If you have two Lie algebras over the same field $\mathbb{F}$, namely $\mathfrak{g}_1$ and $\mathfrak{g}_2$, then it is possible to construct a third Lie algebra $\mathfrak{g}$ from $\mathfrak{g}_1$ and $\mathfrak{g}_2$, where $\mathfrak{g} = \widetilde{\mathfrak{g}_1} \oplus \widetilde{\mathfrak{g}_2}$ and $\widetilde{\mathfrak{g}}_i \cong \mathfrak{g}_i$, with $i = 1, 2$. Define

$$\widetilde{\mathfrak{g}_1} = \{(x_1, 0) \mid x_1 \in \mathfrak{g}_1\}$$

and

$$\widetilde{\mathfrak{g}_2} = \{(0, x_2) \mid x_2 \in \mathfrak{g}_2\} .$$

Clearly, $\widetilde{\mathfrak{g}_1}$ is a Lie subalgebra of $\mathfrak{g}$, as a linear combination of elements of $\widetilde{\mathfrak{g}_1}$ is still in $\widetilde{\mathfrak{g}_1}$. Additionally, by applying the Lie brackets of $\mathfrak{g}$ to any two elements of $\widetilde{\mathfrak{g}_1}$, you will obtain another element of $\widetilde{\mathfrak{g}_1}$. The same argument can be made for $\widetilde{\mathfrak{g}_2}$. Now, let's see how $\mathfrak{g}$ is a direct sum of Lie algebras:

1. As vector spaces, it is well known that given two vector spaces, you can construct their direct sum.

2. For any $(x_1, 0) \in \widetilde{\mathfrak{g}_1}$ and $(0, x_2) \in \widetilde{\mathfrak{g}_2}$, you have

$$[(x_1, 0), (0, x_2)] = ([x_1, 0], [0, x_2]) = (0_{\widetilde{\mathfrak{g}_1}}, 0_{\widetilde{\mathfrak{g}_2}}) = 0_{\mathfrak{g}}.$$

To conclude, it's enough to show that $\widetilde{\mathfrak{g}}_i = \mathfrak{g}_i$ for $i = 1, 2$. Take $i = 1$ and consider the map $\varphi_1 \colon \mathfrak{g}_1 \to \widetilde{\mathfrak{g}_1}$ defined as

$$x_1 \in \mathfrak{g}_1 \mapsto \varphi_1(x_1) = (x_1, 0) \in \widetilde{\mathfrak{g}_1}.$$

The map φ_1 is clearly well-defined and linear, as

$$\varphi_1(\alpha x_1 + \beta y_1) = (\alpha x_1 + \beta y_1, 0) = \alpha(x_1, 0) + \beta(y_1, 0) = \alpha\varphi_1(x_1) + \beta\varphi_1(y_1),$$

for all $\alpha, \beta \in \mathbb{F}$ and for all $x_1, y_1 \in \mathfrak{g}_1$. This map is surjective since $(x, 0) = \varphi_1(x)$ for all $x \in \widetilde{\mathfrak{g}_1}$ and injective because

$$x_1 \in \ker(\varphi_1) \Leftrightarrow \varphi_1(x_1) = 0_{\mathfrak{g}} = (0_{\widetilde{\mathfrak{g}_1}}, 0_{\widetilde{\mathfrak{g}_2}}) \Leftrightarrow x_1 = 0_{\mathfrak{g}_1}.$$

Therefore, φ_1 is a linear isomorphism. Similarly, it can be shown that $x_2 \in \mathfrak{g}_2 \mapsto \varphi_2(x_2) = (0, x_2) \in \widetilde{\mathfrak{g}_2}$ is a linear isomorphism. Consequently, $\mathfrak{g}_1 \cong \widetilde{\mathfrak{g}_1}$ and, by the same reasoning, $\mathfrak{g}_2 \cong \widetilde{\mathfrak{g}_2}$.

The following proposition clarifies a result that will be frequently used in the subsequent chapters without explaining the details every time.

Proposition A.4.3. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1$ be a Lie algebra. Then, $\mathfrak{g}/\mathfrak{g}_2 \cong \mathfrak{g}_1$ and $\mathfrak{g}/\mathfrak{g}_1 \cong \mathfrak{g}_2$.*

Proof. Consider the projections π_1 and π_2 defined as follows:

$$\begin{aligned} \pi_1 \colon \mathfrak{g} &\to \mathfrak{g}_1 \\ x = x_1 + x_2 &\mapsto x_1, \end{aligned}$$

$$\begin{aligned} \pi_2 \colon \mathfrak{g} &\to \mathfrak{g}_2 \\ x = x_1 + x_2 &\mapsto x_2. \end{aligned}$$

π_1 and π_2 are straightforwardly surjective linear mappings of vector spaces. They are also homomorphisms of Lie algebras because for any $x = x_1 + x_2, y = y_1 + y_2 \in \mathfrak{g}$, we have:

$$\begin{aligned} \pi_1([x, y]) &= \pi_1([x_1 + x_2, y_1 + y_2]) \\ &= \pi_1([x_1, y_1] + [x_2, y_2]) \\ &= [x_1, y_1] \\ &= [\pi_1(x_1 + x_2), \pi_1(y_1 + y_2)], \end{aligned}$$

$$\begin{aligned} \pi_2([x, y]) &= \pi_2([x_1 + x_2, y_1 + y_2]) \\ &= \pi_2([x_1, y_1] + [x_2, y_2]) \\ &= [x_2, y_2] \\ &= [\pi_2(x_1 + x_2), \pi_2(y_1 + y_2)]. \end{aligned}$$

The kernels of π_1 and π_2 are as follows:

$$\ker(\pi_1) = \ker(\pi_1) \cap \mathfrak{g}_2 = \mathfrak{g}_2 \cap \mathfrak{g}_2 = 0,$$

$$\ker(\pi_2) = \ker(\pi_2) \cap \mathfrak{g}_1 = \mathfrak{g}_1 \cap \mathfrak{g}_1 = 0.$$

Thus, by the first isomorphism theorem, π_1 and π_2 yield the isomorphisms $\mathfrak{g}/\mathfrak{g}_2 \cong \mathfrak{g}_1$ and $\mathfrak{g}/\mathfrak{g}_1 \cong \mathfrak{g}_2$. □

A.5 Lie Algebras of Dimension ≤ 3

In general, Lie algebras are difficult to classify. However, there is a complete classification for all Lie algebras of dimension at most 3 over an algebraically closed field.

Theorem A.5.1. *[36] Let $\mathfrak{g}$ be a Lie algebra over the field $\mathbb{F} = \mathbb{C}$. Then, we have:*

dim $\mathfrak{g}$	dim $\mathfrak{g}'$	Multiplication Table
1	0	Abelian
2	0	Abelian
2	1	$[x, y] = x$
3	0	Abelian
3	1	$\mathfrak{g}' \not\subseteq \mathrm{Z}(\mathfrak{g}) : [x, y] = x, [x, z] = [y, z] = 0$
3	1	$\mathfrak{g}' \subseteq \mathrm{Z}(\mathfrak{g}) : [x, y] = z, [x, z] = [y, z] = 0$
3	2	$[x, y] = y, [y, z] = 0, [x, z] = y + z$
3	2	$[x, y] = y, [y, z] = 0, [x, z] = \mu z, \mu \neq 0$
3	3	$\mathfrak{g} \cong \mathsf{sl}(2, \mathbb{F})$

Table A.1: Isomorphism classes of Lie Algebras of Dimension ≤ 3

The sixth Lie algebra in the list is known as the *Heisenberg algebra*. On the other hand, the penultimate Lie algebra is often denoted as $\mathfrak{g}_\mu$ because it depends on the parameter μ, and for it, the following holds.

Proposition A.5.2. $\mathfrak{g}_\mu \cong \mathfrak{g}_\nu \Leftrightarrow \nu = \mu$ *or* $\nu = \mu^{-1}$.

A.6 Solvable and Nilpotent Lie Algebras

Definition A.6.1. The *derived series* of a Lie algebra $\mathfrak{g}$ is defined recursively as follows:

$$\mathfrak{g}^{(1)} = \mathfrak{g}' \text{ and } \mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}] \text{ for } k \geq 2.$$

Definition A.6.2. A Lie algebra $\mathfrak{g}$ is called *solvable* if there exists $m \geq 1$ such that $\mathfrak{g}^{(m)} = 0$.

1. The Heisenberg algebra $\mathfrak{h}$ is solvable; indeed, $\mathfrak{h}^{(1)} = \mathfrak{h}' = \langle x \rangle$ and $\mathfrak{h}^{(2)} = [\mathfrak{h}^{(1)}, \mathfrak{h}^{(1)}] = [\mathfrak{h}', \mathfrak{h}'] = [x, x] = 0$ since $[x, x] = 0$.

2. The Lie algebra $\mathsf{sl}(2,\mathbb{C})$ is a non-solvable Lie algebra. To see this, consider a basis of $\mathsf{sl}(2,\mathbb{C})$ as $\{e_{12}, e_{21}, e_{11}-e_{22}\}$, then the multiplication table is:

$$\begin{aligned}
&[e_{12}, e_{21}] = e_{12}e_{21} - e_{21}e_{12} = e_{11} - e_{22},\\
&[e_{11}-e_{22}, e_{12}] = (e_{11}-e_{22})e_{12} - e_{12}(e_{11}-e_{22}) = e_{12} + e_{12} = 2e_{12},\\
&[e_{11}-e_{22}, e_{21}] = (e_{11}-e_{22})e_{21} - e_{21}(e_{11}-e_{22}) = -e_{21} - e_{21} = -2e_{21}.
\end{aligned}$$

Thus, we have $\mathsf{sl}(2,\mathbb{C})^{(1)} = \mathsf{sl}(2,\mathbb{C})' = \mathsf{sl}(2,\mathbb{C})$ and recursively $\mathsf{sl}(2,\mathbb{C})^{(m)} = [\mathsf{sl}(2,\mathbb{C})^{(m-1)}, \mathsf{sl}(2,\mathbb{C})] = \mathsf{sl}(2,\mathbb{C})$.

Lemma A.6.3. *If $\mathfrak{g}$ has ideals*

$$\mathfrak{g} = \mathfrak{i}_0 \supseteq \mathfrak{i}_1 \supseteq \cdots \supseteq \mathfrak{i}_{m-1} \supseteq \mathfrak{i}_m = 0$$

such that $\mathfrak{i}_{k-1}/\mathfrak{i}_k$ is abelian for $1 \leq k \leq m$, then $\mathfrak{g}$ is solvable.

As mentioned earlier, a homomorphism of Lie algebras preserves the derived series.

Lemma A.6.4. *Let $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ be a surjective homomorphism of Lie algebras. Then*

$$\phi(\mathfrak{g}_1^{(k)}) = \mathfrak{g}_2^{(k)}.$$

Lemma A.6.5. *Let $\mathfrak{g}$ be a Lie algebra.*

- *If $\mathfrak{g}$ is solvable, then every subalgebra and every homomorphic image of $\mathfrak{g}$ is solvable.*
- *Suppose $\mathfrak{g}$ has an ideal $\mathfrak{i}$ such that $\mathfrak{i}$ and $\mathfrak{g}/\mathfrak{i}$ are solvable. Then $\mathfrak{g}$ is solvable.*
- *If $\mathfrak{i}$ and $\mathfrak{j}$ are solvable ideals of $\mathfrak{g}$, then $\mathfrak{i}+\mathfrak{j}$ is a solvable ideal of $\mathfrak{g}$.*

Corollary A.6.6. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then, there exists a unique solvable ideal of $\mathfrak{g}$ that contains every solvable ideal of $\mathfrak{g}$.*

Definition A.6.7. Let $\mathfrak{g}$ be a Lie algebra. The maximal solvable ideal containing every solvable ideal of $\mathfrak{g}$ is called the *radical* of $\mathfrak{g}$, denoted by $\mathrm{rad}(\mathfrak{g})$.

Definition A.6.8. If $\mathfrak{g} \neq 0$, a Lie algebra $\mathfrak{g}$ is called *semisimple* if $\mathrm{rad}(\mathfrak{g}) = 0$.

Lemma A.6.9. *Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple.*

Definition A.6.10. Let $\mathfrak{g}$ be a Lie algebra. The *descending central series of* $\mathfrak{g}$ is defined recursively as follows:

$$\mathfrak{g}^1 = \mathfrak{g}' \text{ and } \mathfrak{g}^k = [\mathfrak{g}, \mathfrak{g}^{k-1}] \text{ for } k \geq 2.$$

Thus, $\mathfrak{g} \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \cdots$.

Remark A.6.1. $\mathfrak{g}^k$ is an ideal of $\mathfrak{g}$ since it is a product of ideals. In general, if $\mathfrak{i}$ and $\mathfrak{j}$ are ideals of $\mathfrak{g}$, then $[\mathfrak{i}, \mathfrak{j}]$ is also an ideal of $\mathfrak{g}$.

The series defined above is called *central* because $\mathfrak{g}^k/\mathfrak{g}^{k+1}$ is contained in the center of $\mathfrak{g}/\mathfrak{g}^{k+1}$. To see this, consider $x+\mathfrak{g}^{k+1} \in \mathfrak{g}^k/\mathfrak{g}^{k+1}$ and $y+\mathfrak{g}^{k+1} \in \mathfrak{g}/\mathfrak{g}^{k+1}$. We have

$$[x+\mathfrak{g}^{k+1}, y+\mathfrak{g}^{k+1}] = [x,y]+\mathfrak{g}^{k+1} = [x,y] \in [\mathfrak{g}^k, \mathfrak{g}^{k+1}] = \mathfrak{g}^{k+1},$$

since $[x,y] \in [\mathfrak{g}^k, \mathfrak{g}] = \mathfrak{g}^{k+1}$, which implies

$$x+\mathfrak{g}^{k+1} \in \mathrm{Z}(\mathfrak{g}/\mathfrak{g}^{k+1}) \Rightarrow \mathfrak{g}^k/\mathfrak{g}^{k+1} \subseteq \mathrm{Z}(\mathfrak{g}/\mathfrak{g}^{k+1}).$$

Definition A.6.11. A Lie algebra $\mathfrak{g}$ is called *nilpotent* if there exists $m \geq 1$ such that $\mathfrak{g}^m = 0$.

Definition A.6.12. If $\mathfrak{g}$ is a nilpotent Lie algebra, and m is the smallest index such that $\mathfrak{g}^m = 0$, then m is called the *nilpotency index*.

Here are some examples of nilpotent Lie algebras.

1. Every abelian Lie algebra is nilpotent, with a nilpotency index of 1.

2. The Lie algebra $\mathfrak{n}(n,\mathbb{F})$ of strictly upper triangular matrices is a nilpotent Lie algebra.

3. In A.5.1, an example of a nilpotent Lie algebra $\mathfrak{h}$ of dimension 3 called the *Heisenberg algebra* is given, with a basis $\{x,y,z\}$ such that $[x,y]=z$, $[x,z]=[y,z]=0$. Therefore,

   $$\mathfrak{h}^1 = \mathfrak{h}' = \langle z \rangle, \mathfrak{h}^2 = [\mathfrak{h}, \mathfrak{h}^1] = [\mathfrak{h}, \langle z \rangle] = 0.$$

   Here, the nilpotency index is 2.

4. Let $\mathfrak{g}$ be a Lie algebra of dimension n with a basis $\{e_1, e_2, \ldots, e_n\}$ and the following Lie brackets:

   $$[e_1, e_i] = e_{i+1} \text{ for } 2 \leq i \leq n-1.$$

   Consequently,

   $$\begin{aligned}
   &\mathfrak{g}^1 = \mathfrak{g}' = \langle e_3, \ldots, e_n \rangle, \\
   &\mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1] = [\mathfrak{g}, \langle e_3, \ldots, e_n \rangle] = \langle e_4, \ldots, e_n \rangle, \\
   &\mathfrak{g}^3 = [\mathfrak{g}, \mathfrak{g}^2] = \langle e_5, \ldots, e_n \rangle, \\
   &\vdots \\
   &\mathfrak{g}^{n-2} = \langle e_n \rangle, \\
   &\mathfrak{g}^{n-1} = 0.
   \end{aligned}$$

   The nilpotency index of this Lie algebra is $n-1$.

Remark A.6.2. Every nilpotent Lie algebra is also solvable. For each $k \geq 1$, we have $\mathfrak{g}^{(k)} \subseteq \mathfrak{g}^k$, and to see this, we can use induction on k. For $k=1$, $\mathfrak{g}^{(1)} = \mathfrak{g}^1$, which can serve as the base of induction. Assuming the induction hypothesis that $\mathfrak{g}^{(k-1)} \subseteq \mathfrak{g}^{k-1}$, we need to prove the thesis. By applying the definition of the derived series and using the induction hypothesis, we have:

$$\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}] \subseteq [\mathfrak{g}^{k-1}, \mathfrak{g}] \subseteq [\mathfrak{g}^{k}, \mathfrak{g}] = \mathfrak{g}^{k}.$$

Remark A.6.3. The converse of the above observation does not hold. Consider, for example, the Lie algebra $\mathfrak{g}$ presented in A.5.1 with dimension 2 and basis $\{x, y\}$, where $[x, y] = x$. Then,

$$\begin{aligned}
&\mathfrak{g}^{1} = \mathfrak{g}^{(1)} = \langle x \rangle, \\
&\mathfrak{g}^{2} = [\mathfrak{g}, \mathfrak{g}^{1}] = \langle [x, y] \rangle = \langle x \rangle; \\
&\vdots \\
&\mathfrak{g}^{k} = \langle x \rangle, \text{ for every } k \geq 1,
\end{aligned}$$

so the descending central series stabilizes, and $\mathfrak{g}$ is not nilpotent. However, it is solvable since

$$\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = [\langle x \rangle, \langle x \rangle] = 0.$$

Lemma A.6.13. *Let $\mathfrak{g}$ be a Lie algebra.*

1. *If $\mathfrak{g}$ is nilpotent, then every subalgebra of $\mathfrak{g}$ is nilpotent.*

2. *If $\mathfrak{g}/\mathrm{Z}(\mathfrak{g})$ is nilpotent, then $\mathfrak{g}$ is nilpotent.*

Remark A.6.4. If $\mathfrak{g}/\mathfrak{i}$ and $\mathfrak{i}$ are nilpotent, it does not necessarily mean that $\mathfrak{g}$ is nilpotent. To see this, consider an example. Let $\mathfrak{g}$ be a Lie algebra of dimension 2, which is non-abelian. Referring to Theorem A.5.1, we know that there exists a basis $\{x, y\}$ such that $[x, y] = x$. Then,

$$\begin{aligned}
&\mathfrak{g}^{1} = \mathfrak{g}^{(1)} = \langle x \rangle, \\
&\mathfrak{g}^{2} = [\mathfrak{g}, \mathfrak{g}^{1}] = \langle [x, y] \rangle = \langle x \rangle; \\
&\vdots \\
&\mathfrak{g}^{k} = \langle x \rangle, \text{ for every } k \geq 1,
\end{aligned}$$

so the descending central series stabilizes, and $\mathfrak{g}$ is not nilpotent.

A.7 Cartan's Criteria

In this section, we will present criteria for determining when a Lie algebra is *semisimple* or *solvable*. These criteria, known as *Cartan's criteria*, are very useful because it is often impractical to use the definitions to determine whether a Lie algebra is semisimple or solvable. Furthermore, we will assume that $\mathbb{F}$ is an algebraically closed field, so without loss of generality, we can take $\mathbb{F} = \mathbb{C}$, meaning we are dealing with *complex* Lie algebras.

Definition A.7.1. Let $\mathfrak{g}$ be a Lie algebra. It is called *simple* if its only ideals are the trivial ones, and $\mathfrak{g}$ is non-abelian.

Definition A.7.2. Let $\mathfrak{g}$ be a complex Lie algebra. The *Killing form* on $\mathfrak{g}$ is a symmetric bilinear form κ defined as:

$$\kappa(x, y) := \operatorname{Tr}(\operatorname{ad}(x) \circ \operatorname{ad}(y)) \qquad \text{for all } x, y \in \mathfrak{g}.$$

The Killing form is bilinear because ad is linear, the composition of linear maps is still a linear map, and Tr is linear. This bilinear form has a peculiar property, which we will call *associativity*.

Proposition A.7.3. *Let $\mathfrak{g}$ be a complex Lie algebra, and κ be the Killing form on it. Then, for all x, y, and $z \in \mathfrak{g}$, we have*

$$\kappa([x, y], z) = \kappa(x, [y, z]).$$

Proof. To simplify the notation, we will denote the composition of linear maps as multiplication. Remember that the *trace* function satisfies $\operatorname{Tr}(ab) = \operatorname{Tr}(ba)$ for all $a, b \in \mathrm{M}_n(\mathbb{C})$. Now, for all x, y, and $z \in \mathfrak{g}$:

$$\begin{aligned}
\kappa([x, y], z) &= \operatorname{Tr}(\operatorname{ad}([x, y]) \circ \operatorname{ad}(z)) \\
&= \operatorname{Tr}(\operatorname{ad}(x)\operatorname{ad}(y)\operatorname{ad}(z) - \operatorname{ad}(y)\operatorname{ad}(x)\operatorname{ad}(z)) \\
&= \operatorname{Tr}(\operatorname{ad}(x)\operatorname{ad}(y)\operatorname{ad}(z) - \operatorname{ad}(x)\operatorname{ad}(z)\operatorname{ad}(y)) \\
&= \operatorname{Tr}(\operatorname{ad}(x)([\operatorname{ad}(y), ad(z)])) \\
&= \operatorname{Tr}(\operatorname{ad}(x)\operatorname{ad}([y, z])) \\
&= \kappa(x, [y, z]).
\end{aligned}$$

This completes the proof. □

Theorem A.7.4 (First Cartan's Criterion)**.** *The complex Lie algebra $\mathfrak{g}$ is solvable if and only if $\kappa(x, y) = 0$ for all $x \in \mathfrak{g}$ and $y \in \mathfrak{g}'$.*

Example A.7.4.1. Let $\mathfrak{g}$ be the non-abelian Lie algebra of dimension 2 with a basis consisting of elements x and y such that $[x, y] = x$. In this basis, the linear maps $\operatorname{ad}(x)$ and $\operatorname{ad}(y)$ have matrix representations:

$$\operatorname{ad}(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \operatorname{ad}(y) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, $\kappa(x, x) = \kappa(x, y) = \kappa(y, x) = 0$, and $\kappa(y, y) = 1$. Thus, the matrix representing κ in the basis x and y is

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

So, for any $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathbb{C}$, we have

$$\kappa(\alpha x + \beta y, x) = \alpha\kappa(x, x) + \beta\kappa(y, x) = 0 + 0 = 0.$$

Recalling that for a symmetric bilinear form $B : V \times V \to \mathbb{F}$, where V is a vector space over the field $\mathbb{F}$, you can define, for any subset S of V, the *orthogonal complement to* S as

$$S^{\perp} = \{x \in C \mid B(x, s) = 0, \text{ for all } s \in S\}.$$

With that said, a symmetric bilinear form B is called *non-degenerate* if $V^{\perp} = 0$.

Theorem A.7.5 (Second Cartan's Criterion)**.** *The complex Lie algebra* $\mathfrak{g}$ *is semisimple if and only if the Killing form on* $\mathfrak{g}$ *is non-degenerate.*

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