REGULARITY AND DIRICHLET PROBLEM FOR DOUBLE PHASE ENERGY FUNCTIONALS OF DIFFERENT POWER GROWTH

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Abstract. We work with a double phase energy functional exhibiting log L-perturbed $p\&q$-growth. We look for regularity properties of such functional in the setting of Musielak-Orlicz-Sobolev space, by imposing suitable conditions on the data. We further obtain the existence and uniqueness results for the solution of perturbed Dirichlet double phase problems. We deal with both the cases of uncontrolled growth and controlled growth.

1. Introduction

Let e be the Euler constant and $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain in which its boundary $\partial\Omega$ is Lipschitz. The primary objective of this paper is to discuss the properties of the perturbed double phase differential operator

$$u \mapsto \Delta_{\mathcal{H}_L} u = \operatorname{div}\left(\frac{\mathcal{H}_L'(x, |\nabla u|)}{|\nabla u|}\nabla u\right) \quad \text{for all } x \in \Omega, \text{ all } u \in W^{1,\mathcal{H}_L}(\Omega), \tag{1}$$

where we consider the function

$$\mathcal{H}_L(x,t) = [t^p + b(x)t^q]\log(e+t), \text{ for all } x \in \Omega, \text{ all } t \geq 0, \tag{2}$$

with $1 < p < q$ and the logarithmic term acts as perturbation. For $r \in]1, +\infty[$ and $b(\cdot) \in L^\infty(\Omega) \setminus \{0\}$, $b(x) \geq 0$ for a.a. $x \in \Omega$, by $\Delta_r^{b(x)}$ we denote the weighted r-Laplace differential operator defined by

$$\Delta_r^{b(x)} u = \operatorname{div}\,(b(x)|\nabla u|^{r-2}\nabla u).$$

In the case $b \equiv 1$, then we retrieve the classical r-Laplace differential operator denoted by Δ_r. Since the differential operator in (1) originates by the sum of two such operators, then it is nonhomogeneous. The double phase differential operator $\Delta_p u + \Delta_q^{b(x)}$ corresponds to the following energy functional

$$u \to \int_\Omega [|\nabla u|^p + b(x)|\nabla u|^q]dx.$$

The integrand in this functional is the function

$$\mathcal{H}(x,t) = t^p + b(x)t^q \quad \text{for all } x \in \Omega, \text{ all } t \geq 0. \tag{3}$$

We do not impose that $b(\cdot)$ is bounded away from zero (that is, we do not assume that $\operatorname*{ess\,inf}_\Omega b > 0$). Therefore, $\mathcal{H}(x,\cdot)$ has unbalanced growth, that is we have the following inequality constraints

$$t^p \leq \mathcal{H}(x,t) \leq c_0[t^p + t^q] \quad \text{for a.a. } x \in \Omega, \text{ all } t \geq 0, \text{ some } c_0 > 0.$$

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Such energy functionals were first considered by Marcellini [16, 17] and Zhikov [26, 27, 28], in the context of problems of the calculus of variations and of nonlinear elasticity theory. Since the integrand $\mathcal{H}(x,\cdot)$ exhibits unbalanced growth, then the related problems are well-posed in the functional framework given by the so-called Musielak-Orlicz-Sobolev spaces (see Section 3), see Liu-Dai [15] for additional information. In the last years, Mingione and co-workers revitalized the interest in such problems, by obtaining significant interior regularity results of local minimizers for useful energy functionals. We mention the paper of Baroni et al. [5] and references cited therein. We also recall the paper of Ragusa & Tachikawa [22], where the local regularity theory of [5], was extended to anisotropic double phase problems (that is, the authors study the case of variable exponents $p\&q$). We note that a global regularity theory (that is, regularity up to the boundary of Ω) of double phase problems is yet missing and hence the analysis of double phase equations is more difficult. On the other hand, there is a recent literature focusing on the existence and multiplicity of (weak) solutions to certain classes of nonlinear elliptic equations driven by a double phase differential operator. We mention the papers of Albalawi et al. [1] (Dirichlet problems), Papageorgiou et al. [20, 21] (Robin problems), Gasiński & Winkert [12] and Vetro & Winkert [25] (elliptic problems with convection). Finally, we recall that double phase problems arise in many physical applications, for example we refer to the works of Bahrouni et al. [3, 4] (flow problems), Anthal et al. [2] and Sun et al. [23] (Choquard type logistic problems). Further extensions to more general nonlinear equations can be found in the work of Crespo-Blanco et al. [8], that is the starting point of our study here. Differently from [8], we introduce and incorporate in the operator the effects of logarithmic perturbation term, and to the best of our knowledge, this is the first manuscript dealing with the double phase function in the perturbed form (2) (that is, exhibiting the effects of both $L^p \log L$ growth and weighted $b(\cdot)L^q \log L$ growth). We underline that we are also able to prove the similar results of [8], but under different data. Hence this article provides content that do properly complement the above mentioned articles of related topic.

The manuscript is organized as follows. In Section 2, we introduce notation and collect the relevant facts about weak Φ-functions and N-functions. In Section 3, we discuss the properties of the new double phase differential operator in the context of Musielak-Orlicz spaces, focusing on the embeddings results. In Section 4, we establish the existence and uniqueness of weak solutions to certain Dirichlet double phase problems with both uncontrolled and controlled growth of the reaction term. The main features of these problems are the presence of a logarithmic perturbation in the principal operator and the reaction that may exhibit the effects of convection.

2. Weak Φ-functions and N-functions

A key notion in introducing and developing the theory of Musielak-Orlicz spaces, is the definition of weak Φ-function, see also Harjulehto & Hästö [13] and Harjulehto et al. [14] (here we follow the notation in [13, 14]). In the sequel, $B_R(x)$ will denote the open ball $B(x,R) = \{z \in \mathbb{R}^N : |x-z| < R\}$ centered at $x \in \mathbb{R}^N$ with radius $R > 0$. However, when there is not ambiguity in identifying the center of the ball, we will denote this ball only by B_R. Moreover, we recall that $u \colon \mathbb{R} \to \mathbb{R}$ is an almost increasing function, provided that we can find $L \geq 1$ satisfying $u(s) \leq Lu(t)$ for all $s \leq t$. In a similar fashion, we can give the notion of almost decreasing function.

Definition 1. We say that $\varphi : \Omega \times [0, +\infty[\to [0, +\infty]$ is a weak Φ-function, denoted by $\varphi \in \Phi_w(\Omega)$, if it satisfies the following assumptions:

(φ_1) for all $t \geq 0$, $x \mapsto \varphi(x, t)$ is a measurable function, and for all $x \in \Omega$, $t \mapsto \varphi(x, t)$ is an increasing function;
(φ_2) $\varphi(x, 0) = \lim_{t\to 0^+} \varphi(x, t) = 0$ and $\lim_{t\to+\infty} \varphi(x, t) = +\infty$ for all $x \in \Omega$;
(φ_3) the function $t \mapsto t^{-1}\varphi(x, t)$ is almost increasing for $t > 0$, some $L \geq 1$, all $x \in \Omega$;
(φ_4) the function $t \mapsto \varphi(x, t)$ is left-continuous for all $t > 0$, all $x \in \Omega$.

By $\varphi \in \Phi_c(\Omega)$, we mean a convex Φ-function, which is a weak Φ-function satisfying the additional condition:

(φ_5) $t \mapsto \varphi(x, t)$ is a convex function for all $x \in \Omega$.

We note that in [13] the authors removed assumption (φ_4) from Definition 1, and included it in the notion of convex Φ-function. Moreover, they pointed out the key role of (φ_3) in providing the space with an appropriate quasi-norm. Finally, a function φ satisfying only (φ_1), (φ_2) is known as Φ-prefunction (see Definition 2.1.3 of [13]). Next definition provides two properties that will be used in establishing certain embeddings results in Section 3, namely (aInc) and (aDec); essentially they give us certain growth features.

Definition 2. A weak Φ-function $\varphi \in \Phi_w(\Omega)$ is said to satisfy the property:

(aInc) if the function $t \mapsto \frac{\varphi(x,t)}{t^p}$ is almost increasing for some $p > 1$, a.a. $x \in \Omega$;
(aDec) if the function $t \mapsto \frac{\varphi(x,t)}{t^q}$ is almost decreasing for some $0 < q < +\infty$, a.a. $x \in \Omega$.

Let φ^{-1} denote the left-continuous inverse of a weak Φ-function φ, then we have

$$\varphi^{-1}(x, \tau) = \inf\{t \geq 0 : \varphi(x, t) \geq \tau\}.$$

Moreover, φ is said to be "doubling" if we can find a constant $c_\varphi \geq 1$ such that $\varphi(x, 2t) \leq c_\varphi \varphi(x, t)$ for all $x \in \Omega$, all $t \geq 0$. When φ is doubling, then we write $\varphi \in \Delta_2$ for short. In [13] (see Lemma 2.2.6, p. 22), the authors refer to (aDec) as a "quantitative" version of the doubling property. If $\varphi \in \Delta_2$, then we have the following

$$\varphi(x, t + s) \leq c_\varphi \varphi(x, t) + c_\varphi \varphi(x, s) \text{ for all } t, s \geq 0. \tag{4}$$

Another useful technical property is known as "∇_2-condition". Namely, $\varphi \in \Phi_w$ satisfies this condition ($\varphi \in \nabla_2$, for short) if we can find $c^\varphi > 1$ such that

$$\varphi(x, c^\varphi t) \geq 2c^\varphi \varphi(x, t) \quad \text{for all } x \in \Omega,\ t \geq 0.$$

This property can be linked to (aInc), see Corollary 2.4.11 of [13].

Remark 1. Referring to (2), in this paper we consider the weak Φ-function given by

$$\varphi(x, t) = \mathcal{H}_L(x, t) = [t^p + b(x)t^q] \log(e + t) \quad \text{for all } x \in \Omega, \text{ all } t \geq 0, \tag{5}$$

where $b(x) \geq 0$ for all $x \in \Omega$, $1 < p < q$. Then for all $\lambda > 1$, we get

$$\lambda^p \mathcal{H}_L(x, t) \leq \mathcal{H}_L(x, \lambda t) = (\lambda t)^p + b(x)(\lambda t)^q \log(e + \lambda t) \leq \lambda^{q+1}\mathcal{H}_L(x, t),$$

which says that $\mathcal{H}_L \in \Delta_2$ with constant $c_{\mathcal{H}_L} = c_{\mathcal{H}_L}(q)$ and that $\mathcal{H}_L \in \nabla_2$ with constant $c^{\mathcal{H}_L} = c^{\mathcal{H}_L}(p)$. We remark that $c_{\mathcal{H}_L}, c^{\mathcal{H}_L}$ do not depend on $b(\cdot)$.

To simplify the notation, in the sequel we will use to denote by $c > 0$ a constant whose value may change from line to line, and any relevant dependencies will be underlined by using round parentheses (as in the case of $c^{\mathcal{H}_L}$ above, where we point out the dependence by p). Moreover, some bound properties of the left-continuous inverse of a Φ-function can be noted in relation to a positive constant bounded from 1, as given in the first two items of the following definition. We recall that a classical problem of regularity theory is in establishing the density of smooth functions into the involved functional space (here, the Musielak-Orlicz spaces). Now, property $(A0)$ below, is a crucial condition in this direction.

Definition 3. A weak Φ-function $\varphi \in \Phi_w(\Omega)$ is said to satisfy the property:

$(A0)$ if we can find $0 < \beta \leq 1$ such that $\beta \leq \varphi^{-1}(x, 1) \leq \frac{1}{\beta}$ for a.a. $x \in \Omega$;

$(A1)$ if we can find $0 < \beta < 1$ such that $\beta\varphi^{-1}(x, t) \leq \varphi^{-1}(y, t)$ for all $t \in [1, 1/|B_R|]$, a.a. $x, y \in B_R \cap \Omega$, all balls B_R with $|B_R| \leq 1$;

$(A1')$ if we can find $0 < \beta < 1$ satisfying $\varphi(x, \beta t) \leq \varphi(y, t)$ for all $\varphi(y, t) \in [1, 1/|B_R|]$, a.a. $x, y \in B_R \cap \Omega$, all balls B_R with $|B_R| \leq 1$.

We refer again to [13] for more details, here we point out that $(A1)$ is a local property which plays a role in the comparison of values of $\varphi^{-1}(\cdot, t)$ at nearby points (i.e., points inside a suitable ball B_R).

Remark 2. We note that $(A0)$ gives us the existence of a constant $\beta \in]0, 1]$ satisfying $\varphi(x, \beta) \leq 1 \leq \varphi(x, 1/\beta)$ for a.a. $x \in \Omega$.

The strong relation between the properties $(A1)$ and $(A1')$ is established in [13], see Corollary 4.1.6. For readers convenience, we recall this result as follows.

Corollary 1. *If a weak Φ-function $\varphi \in \Phi_w(\Omega)$ satisfies $(A0)$, then φ satisfies $(A1)$ if and only if it satisfies $(A1')$.*

We will use Corollary 1 in the proof of next lemma. Indeed, we obtain the properties $(A0)$ and $(A1)$ for $\mathcal{H}_L$ defined by (5) (recall Remark 1) under some regularities as follows.

Lemma 1. *If $q/p \leq 1 + \alpha/N$ and $b(\cdot) \in C^{0,\alpha}(\overline{\Omega})$ is nonnegative with $0 < \alpha \leq 1$, then $\mathcal{H}_L$ defined by (5) satisfies both $(A0)$ and $(A1)$.*

Proof. We split the proof into two steps.

Step 1: $\mathcal{H}_L$ satisfies the property $(A0)$. If we look at Remark 2, we can argue the existence of some constant $0 < \beta \leq 1$ such that $\mathcal{H}_L(x, \beta) \leq 1 \leq \mathcal{H}_L(x, 1/\beta)$ for a.a. $x \in \Omega$. Indeed, since $1 < p < q$ (see (5)) we note that

$$\begin{aligned}\mathcal{H}_L(x, \beta) &= (\beta^p + b(x)\beta^q) \log(e + \beta) \\ &\leq 2\beta^p(1 + \|b\|_\infty).\end{aligned}$$

Now, if we solve the inequality

$$2\beta^p(1 + \|b\|_\infty) \leq 1,$$

then we get the interval range

$$0 < \beta \leq \left(\frac{1}{2(1 + \|b\|_\infty)}\right)^{1/p}.$$

So, if β falls within this interval, it follows that

$$1 \leq 2(1 + \|b\|_\infty) \leq \mathcal{H}_L(x, 1/\beta) \quad \text{for a.a. } x \in \Omega.$$

We conclude that the property $(A0)$ holds for all $\beta \leq \left(\frac{1}{2(1+\|b\|_\infty)}\right)^{1/p}$.

Step 2: $\mathcal{H}_L$ satisfies the property $(A1)$. We aim to obtain this result as a byproduct of Corollary 1. Thus, we show that $\mathcal{H}_L$ satisfies property $(A1')$. From Definition 2, we recall that $\mathcal{H}_L$ is said to satisfy the property $(A1')$ if we can find $0<\beta<1$ satisfying

$$\mathcal{H}_L(x,\beta t) \le \mathcal{H}_L(y,t) \quad \text{for all } \mathcal{H}_L(y,t) \in [1, 1/|B_R|], \text{ a.a. } x, y \in B_R \cap \Omega, \tag{6}$$

and every ball B_R with $|B_R| \le 1$. From the definition of $\mathcal{H}_L$ here, we obtain that

$$\frac{1}{|B_R|} \ge \mathcal{H}_L(y,t) = (t^p + b(y)t^q)\log(e+t) \ge t^p \log(e+t) \ge t^p,$$

and hence we get

$$t < \left(\frac{1}{|B_R|}\right)^{1/p}.$$

Since we have to establish (6) in the case $|B_R| = \omega(N)R^N \le 1$ (recall that $\omega(N) > 1$ is the Lebesgue measure of the ball $B_1(0)$), then we deduce that $R<1$ and we also have

$$t < \omega(N)^{-1/p}\, R^{-N/p}. \tag{7}$$

Letting c_b the Hölder constant of $b(\cdot)$, then for all $x, y \in B_R$, we obtain that

$$b(x) \le b(y) + |b(x) - b(y)| \le b(y) + 2^\alpha c_b R^\alpha \quad (\text{recall } R < 1). \tag{8}$$

Noting that $\beta < 1$ in (6), we get the following calculations

$$\begin{aligned}
\mathcal{H}_L(x,\beta t) &\le \beta^p (t^p + b(x)t^q)\log(e+t) \quad (\text{as } \log(e+\beta t) < \log(e+t))\\
&\le \beta^p(t^p + 2^\alpha c_b R^\alpha t^q)\log(e+t) + b(y)t^q \log(e+t) \quad (\text{by } (8))\\
&= \beta^p t^p (1 + 2^\alpha c_b R^\alpha t^{q-p})\log(e+t) + b(y)t^q\log(e+t)\\
&\le \beta^p t^p (1 + 2^\alpha c_b \omega(N)^{1-\frac{q}{p}}\, R^{\alpha+N-\frac{Nq}{p}})\log(e+t) + b(y)t^q\log(e+t) \quad (\text{by } (7))\\
&\le \mathcal{H}_L(y,t),
\end{aligned}$$

whenever $\beta^p(1 + 2^\alpha c_b\omega(N)^{1-\frac{q}{p}}) \le 1$, recall that $R^{\alpha+N-\frac{Nq}{p}} \le 1$ since in the statement of our lemma we assume the restriction $q/p \le 1 + \alpha/N$. We conclude that $(A1')$ holds true, and hence we can apply Corollary 1 to obtain $(A1)$. □

A special class of functions very useful in studying boundary value problems is the class of so-called N-functions and their generalized versions. We recall this concept in the following definition.

Definition 4. Let $\varphi : [0,+\infty[\to [0,+\infty[$. We say that

- φ is an N-function, if it is a continuous and convex function satisfying the following assumptions:

$$\lim_{t\to 0} t^{-1}\varphi(t) = 0 \quad \text{and} \quad \lim_{t\to+\infty} t^{-1}\varphi(t) = +\infty.$$

- φ is a generalized N-function, denoted by $\varphi \in N(\Omega)$, if it satisfies the following assumptions:
  (i) $\varphi(\cdot, t)$ is measurable for all $t \ge 0$;
  (ii) $\varphi(x, \cdot)$ is an N-function for all $x \in \Omega$.

Remark 3. Let $\mathcal{H}_L$ be given by (5) with $b(\cdot)$ measurable, then $\mathcal{H}_L \in N(\Omega)$.

We collect here some miscellaneous properties of generalized N-functions as follows. Given $\varphi, \psi \in N(\Omega)$, we say that φ is weaker than ψ, denoted by $\varphi \prec \psi$, if there exist two positive constants c_1, c_2 and a nonnegative function $h \in L^1(\Omega)$ such that

$$\varphi(x,t) \leq c_1\psi(x, c_2 t) + h(x) \quad \text{for all } x \in \Omega, \text{ all } t \geq 0.$$

Additionally, a function $\varphi \in N(\Omega)$ is locally integrable if $\varphi(\cdot, t) \in L^1(\Omega)$ for all $t > 0$. Moreover, a function $\varphi \in N(\Omega)$ admits right-derivative $\varphi'(x, \cdot)$ such that

$$\varphi(x,t) = \int_0^t \varphi'(x,s)ds \quad \text{for all } x \in \Omega, \text{ all } t \geq 0.$$

Also, $\varphi'(x, \cdot)$ is a non-decreasing and right-continuous function. Starting from the notion of complementary of a Φ-function $\varphi \in \Phi_w$ defined as

$$\varphi^*(x,t) = \sup_{s\geq 0}(st - \varphi(x,s)) \quad \text{for all } x \in \Omega, \tag{9}$$

we note that in the case $\varphi \in N(\Omega)$ the following hold: $\varphi^* \in N(\Omega)$ and $(\varphi^*)'(x,t) = (\varphi')^{-1}(x,t)$, where $(\varphi')^{-1}(x,t) = \sup\{s \in [0, +\infty[: \varphi'(x,s) \leq t\}$. Then we get

$$\varphi^*(x,t) = \int_0^t (\varphi')^{-1}(x,s)ds \quad \text{for all } x \in \Omega,\ t \geq 0,$$

see Theorem 2.6.8 of [9]. Also, we have $\varphi \in \nabla_2$ if and only if $\varphi^* \in \Delta_2$ with $2c^\varphi = c_{\varphi^*}$. Referring to (9), we obtain the Young inequality

$$ts \leq \varphi(x,t) + \varphi^*(x,s). \tag{10}$$

Some other estimates can be considered, assuming additional regularity on the function $\varphi \in \Phi_w(\Omega)$. Precisely, in the case that φ is doubling and convex, then for each $\varepsilon \in]0,1[$, we can find a positive constant $c(\varepsilon)$ depending only on c_φ, c^φ and ε such that for $s, t > 0$ and $x \in \Omega$ we have

$$ts \leq \varepsilon\varphi(x,t) + c(\varepsilon)\varphi^*(x,s).$$

On the other hand, in the case that φ is doubling and convex with derivative φ', then for all $t \geq 0$ and $x \in \Omega$, we get

$$c^{-1}\varphi(x,t) \leq \varphi'(x,t)t \leq c\varphi(x,t) \text{ and } \varphi^*(x, \varphi'(x,t)) \leq \widetilde{c}\,\varphi(x,t) \tag{11}$$

for some $c, \widetilde{c} > 1$ depending only on c_φ, c^φ. We refer the reader to [9] (see Lemma 2.6.6 and Lemma 2.6.11) for more details.

3. The function $\mathcal{H}_L$ in Musielak-Orlicz spaces

As we already noted in the Introduction, the right framework to deal with double phase equations is provided by the well-known Musielak-Orlicz-Sobolev spaces. For a comprehensive presentation of these spaces, we refer to the books of Diening et al. [9] and Harjulehto & Hästö [13] as well as the papers of Colasuonno & Squassina [7], Fan [10] and Liu & Dai [15].

We impose the following hypotheses on the weight $b(\cdot)$ and the exponents p, q:

$$0 \leq b(\cdot) \in L^1(\Omega), \quad 1 < p < N, \quad p < q. \tag{12}$$

Remark 4. The assumption $b(\cdot) \in L^1(\Omega)$ ensures that $\mathcal{H}_L$ defined by (5) is locally integrable for all $t > 0$.

If $\mathcal{M}$ denotes the space of all measurable functions $u : \Omega \to \mathbb{R}$, we identify two such functions differing only on a Lebesgue null subset of Ω. Then the Musielak-Orlicz space $L^{\mathcal{H}_L}(\Omega)$ is given as

$$L^{\mathcal{H}_L}(\Omega) = \{u \in \mathcal{M}(\Omega) : \rho_{\mathcal{H}_L}(u) < +\infty\},$$

where

$$\rho_{\mathcal{H}_L}(u) = \int_\Omega \mathcal{H}_L(x, |u|)\, dx.$$

We endow $L^{\mathcal{H}_L}(\Omega)$ with the Luxemburg norm $\|u\|_{\mathcal{H}_L}$ defined by

$$\|u\|_{\mathcal{H}_L} = \inf\left\{\lambda > 0 : \rho_{\mathcal{H}_L}\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

We note that $L^{\mathcal{H}_L}(\Omega)$ is complete with respect to this norm, that is, $(L^{\mathcal{H}_L}(\Omega), \|\cdot\|_{\mathcal{H}_L})$ is a Banach space (see Theorem 7.7 of [19]). From (10) we get the following Hölder inequality

$$\int_\Omega |u||v| dx \le 2\|u\|_{\mathcal{H}_L}\|v\|_{\mathcal{H}_L^*} \quad \text{for all } u \in L^{\mathcal{H}_L}(\Omega), \text{ all } v \in L^{\mathcal{H}_L^*}(\Omega).$$

Let $C_0^\infty(\Omega) = \{u \in C^\infty(\Omega) : \text{ supp } u \text{ is compact in } \Omega\}$. From Theorem 3.7.15 of [13], we get that $C_0^\infty(\Omega)$ is dense in $L^{\mathcal{H}_L}(\Omega)$ (recall that $\mathcal{H}_L$ satisfies both of $(A0)$ and (aDec)). Let ∇u denote the weak gradient of u, then we can define the Musielak-Orlicz-Sobolev spaces $W^{1,\mathcal{H}_L}(\Omega)$ and $W_0^{1,\mathcal{H}_L}(\Omega)$ as follow:

$$W^{1,\mathcal{H}_L}(\Omega) = \{u \in L^{\mathcal{H}_L}(\Omega) : |\nabla u| \in L^{\mathcal{H}_L}(\Omega)\}.$$

We equip this space with the norm

$$\|u\|_{1,\mathcal{H}_L} = \|u\|_{\mathcal{H}_L} + \|\nabla u\|_{\mathcal{H}_L},$$

where $\|\nabla u\|_{\mathcal{H}_L} = \||\nabla u|\|_{\mathcal{H}_L}$. Hence, we can also consider the space

$$W_0^{1,\mathcal{H}_L}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,\mathcal{H}_L}},$$

namely $W_0^{1,\mathcal{H}_L}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{H}_L}(\Omega)$. By Remark 4 we know that $\mathcal{H}_L \in N(\Omega)$ is locally integrable if (12) holds. Additionally, $\inf_{x\in\Omega} \mathcal{H}_L(x, 1) \ge 1$, and hence we deduce that $W^{1,\mathcal{H}_L}(\Omega)$ and $W_0^{1,\mathcal{H}_L}(\Omega)$ are separable Banach spaces (see Proposition 1.7 of [11]), when endowed with the norm $\|u\|_{1,\mathcal{H}_L}$ (see also Theorem 10.2 of [19]). On the other hand, the separability of the spaces is a consequence of the fact that $\mathcal{H}_L$ satisfies the property (aDec). Since $\mathcal{H}_L$ satisfies also the property (aInc) we deduce that $W^{1,\mathcal{H}_L}(\Omega)$ and $W_0^{1,\mathcal{H}_L}(\Omega)$ are uniformly convex and reflexive. For more details see Theorem 6.1.4 of [13] (and related comments on p. 124 about the use of Δ_2- and ∇_2-conditions instead of properties (aDec) and (aInc)).

From now on, the notation $X \hookrightarrow Y$ means that the space X is continuously embedded into the space Y, while $X \hookrightarrow\hookrightarrow Y$ means that X is compactly embedded into Y.

In our proofs, we will involve the notion of modular convergence, and hence we recall the following useful facts. For $\varphi \in \Phi_w(\Omega)$ and $u_n, u \in L^\varphi(\Omega)$, we say that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is modular convergent to u if $\rho_\varphi(\lambda(u_n - u)) \to 0$ as $n \to +\infty$ for some $\lambda > 0$.

By [13, Lemma 3.3.3], we get the following result.

Lemma 2. *The modular convergence and the norm convergence are equivant each other, in $L^{\mathcal{H}_L}(\Omega)$ and in $L^{\mathcal{H}_L^*}(\Omega)$.*

Moreover, we note a close relation between the modular function $\rho_{\mathcal{H}_L}(\cdot)$ and its norm $\|\cdot\|_{\mathcal{H}_L}$, as given in the following result.

Proposition 1. *If* (5) *and* (12) *hold, then we have the following:*

(i) *if* $u \in L^{\mathcal{H}_L}(\Omega) \setminus \{0\}$, *then* $\|u\|_{\mathcal{H}_L} = \lambda$ *if and only if* $\rho_{\mathcal{H}_L}\left(\frac{u}{\lambda}\right) = 1$;

(ii) $\|u\|_{\mathcal{H}_L} < 1$ *(resp.* $> 1, = 1$*) if and only if* $\rho_{\mathcal{H}_L}(u) < 1$ *(resp.* $> 1, = 1$*)*;

(iii) *If* $\|u\|_{\mathcal{H}_L} < 1$, *then* $\|u\|_{\mathcal{H}_L}^{q+1} \leq \rho_{\mathcal{H}_L}(u) \leq \|u\|_{\mathcal{H}_L}^{p}$;

(iv) *If* $\|u\|_{\mathcal{H}_L} > 1$, *then* $\|u\|_{\mathcal{H}_L}^{p} \leq \rho_{\mathcal{H}_L}(u) \leq \|u\|_{\mathcal{H}_L}^{q+1}$;

(v) $\|u\|_{\mathcal{H}_L} \to 0$ *if and only if* $\rho_{\mathcal{H}_L}(u) \to 0$;

(vi) $\|u\|_{\mathcal{H}_L} \to +\infty$ *if and only if* $\rho_{\mathcal{H}_L}(u) \to +\infty$;

(vii) $\|u\|_{\mathcal{H}_L} \to 1$ *if and only if* $\rho_{\mathcal{H}_L}(u) \to 1$;

(viii) *if* $u_n \to u$ *in* $L^{\mathcal{H}_L}(\Omega)$, *then* $\rho_{\mathcal{H}_L}(u_n) \to \rho_{\mathcal{H}_L}(u)$.

Proof. Letting $u \in L^{\mathcal{H}_L}(\Omega)$, then $\mathbb{R} \ni \lambda \mapsto \rho_{\mathcal{H}_L}(\lambda u)$ is a continuous, convex and even function. Moreover, this function is strictly increasing provided that $\lambda > 0$. Referring to the above mentioned definitions, we deduce that

$$\begin{aligned} & \|u\|_{\mathcal{H}_L} = \lambda \quad \Longleftrightarrow \quad \rho_{\mathcal{H}_L}\left(\frac{u}{\lambda}\right) = 1, \\ \Rightarrow \quad & \text{(i) holds true,} \\ \Rightarrow \quad & \text{(ii) holds true too.} \end{aligned}$$

To establish (iii), we observe that

$$\begin{aligned} & \xi^p \rho_{\mathcal{H}_L}(u) \leq \rho_{\mathcal{H}_L}(\xi u) \leq \xi^{q+1} \rho_{\mathcal{H}_L}(u) \quad \text{if } \xi > 1, \\ & \xi^{q+1} \rho_{\mathcal{H}_L}(u) \leq \rho_{\mathcal{H}_L}(\xi u) \leq \xi^{p} \rho_{\mathcal{H}_L}(u) \quad \text{if } 0 < \xi < 1. \end{aligned} \tag{13}$$

Now, if $\|u\|_{\mathcal{H}_L} = \lambda$ with $0 < \lambda < 1$, then we also have $\rho_{\mathcal{H}_L}\left(\frac{u}{\lambda}\right) = 1$ (by (i)). Given $\xi = \frac{1}{\lambda} > 1$, then the first inequality in (13) leads to

$$\frac{\rho_{\mathcal{H}_L}(u)}{\lambda^p} \leq \rho_{\mathcal{H}_L}\left(\frac{u}{\lambda}\right) = 1 \leq \frac{\rho_{\mathcal{H}_L}(u)}{\lambda^{q+1}},$$

and hence (iii) is proved. In a similar fashion, one can establish (iv), involving this time the second inequality in (13). Clearly (v) and (vi) can be obtained as byproduct of (iii) and (iv), respectively. On the other hand, (iii) together with (iv) give us (vii). To conclude the last one of items, we note that in the case that $u_n \to u$ in $L^{\mathcal{H}_L}(\Omega)$ as $n \to +\infty$, then by applying (v) we get the implications:

$$\begin{aligned} & \|u_n - u\|_{\mathcal{H}_L} \to 0 \quad \Rightarrow \rho_{\mathcal{H}}(u_n - u) \to 0 \quad \Rightarrow \rho_p(u_n - u) \to 0 \\ & \text{(recall that } \rho_p(u) = \textstyle\int_\Omega |u|^p \, dx), \\ \Rightarrow \quad & u_n \to u \text{ a.e. in } \Omega \text{ (by passing to a suitable subsequence if necessary).} \end{aligned}$$

Starting from

$$\mathcal{H}_L(x, |u_n|) \leq 2^{q+1} \left[\mathcal{H}_L(x, |u_n - u|) + \mathcal{H}_L(x, |u|)\right] \quad \text{(by (4))}$$

and since $\rho_{\mathcal{H}_L}(u_n - u) \to 0$ as $n \to +\infty$ (recall (v)), then the sequence $\{\mathcal{H}_L(x, |u_n|)\}_{n \in \mathbb{N}}$ is uniformly integrable. So, it furthermore converges a.e. to $\mathcal{H}(x, |u|)$ by the a.e. convergence of $\{u_n\}_{n \in \mathbb{N}}$ to u. Running through this subsequence, an application of Vitali's Theorem gives us that $\rho_{\mathcal{H}_L}(u_n) \to \rho_{\mathcal{H}_L}(u)$ as $n \to +\infty$. It follows that every subsequence $\{u_{n_k}\}_{k \in \mathbb{N}}$ of $\{u_n\}_{n \in \mathbb{N}}$, admits a subsequence (denoted also by $\{u_{n_k}\}_{k \in \mathbb{N}}$, for convenience) such that $\rho_{\mathcal{H}_L}(u_{n_k}) \to \rho_{\mathcal{H}_L}(u)$ as $n \to +\infty$. Hence, we can easily conclude that (viii) is established. □

Since the differential operator in (1) originates by the perturbed double phase function (5) which can be seen as the sum of two competing contributions, thus we consider the functions $\mathcal{H}_{p,L}$, $\mathcal{H}_{q,L}$, $\mathcal{H}_{q,L,b} : \Omega \times [0,+\infty[\to [0,+\infty[$ defined respectively by

$$\mathcal{H}_{p,L}(x,t) = t^p \log(e+t), \quad \mathcal{H}_{q,L}(x,t) = t^q \log(e+t), \quad \mathcal{H}_{q,L,b}(x,t) = b(x)t^q \log(e+t)$$

for all $x \in \overline{\Omega}$, all $t \geq 0$, where $b(\cdot) \in L^1(\overline{\Omega})$ is a given weight term.

Now, we observe that

$$\mathcal{H}_{p,L}(x,t) \leq \mathcal{H}_L(x,t)$$

and if $b(\cdot) \in L^\infty(\Omega)$,

$$\mathcal{H}_L(x,t) \leq (1 + \|b\|_\infty)\mathcal{H}_{q,L} + 2\,\chi_\Omega(x)$$

for all $x \in \overline{\Omega}$, all $t \geq 0$. Moreover, the inequality

$$\mathcal{H}_{q,L,b}(x,t) \leq \mathcal{H}_L(x,t)$$

holds for all $x \in \overline{\Omega}$, all $t \geq 0$. Therefore, we deduce that

$$\mathcal{H}_{p,L} \prec \mathcal{H}_L, \quad \mathcal{H}_{q,L,b} \prec \mathcal{H}_L \quad \text{and, if } b(\cdot) \in L^\infty(\Omega),\ \mathcal{H}_L \prec \mathcal{H}_{q,L}.$$

On this basis, we first consider the space

$$L^{\mathcal{H}_{p,L}}(\Omega) = \left\{ u \in \mathcal{M}(\Omega) : \rho_{\mathcal{H}_{p,L}}(u) < +\infty \right\},$$

equipped with the norm

$$\|u\|_{p,L} = \inf\left\{ \lambda > 0 : \rho_{\mathcal{H}_{p,L}}\left(\frac{u}{\lambda}\right) \leq 1 \right\},$$

then we introduce the space

$$L^{\mathcal{H}_{q,L,b}}(\Omega) = \left\{ u \in \mathcal{M}(\Omega) : \rho_{\mathcal{H}_{q,L,b}}(u) < +\infty \right\}$$

equipped with the semi-norm

$$\|u\|_{q,L,b} = \inf\left\{ \lambda > 0 : \rho_{\mathcal{H}_{q,L,b}}\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

In the above notation, we use the modular functions $\rho_{\mathcal{H}_{p,L}}(u) = \int_\Omega \mathcal{H}_{p,L}(x,|u|)\,dx$ and $\rho_{\mathcal{H}_{q,L,b}}(u) = \int_\Omega \mathcal{H}_{q,L,b}(x,|u|)\,dx$.

Theorem 8.5 of [19] says that given $\varphi, \psi \in N(\Omega)$ locally integrable with $\varphi \prec \psi$, then $L^\psi(\Omega) \hookrightarrow L^\varphi(\Omega)$. Consequently, by the definitions and properties of $\mathcal{H}_L$ and $\mathcal{H}_{p,L}$ (recall $\mathcal{H}_{p,L} \prec \mathcal{H}_L$) we deduce the embedding

$$L^{\mathcal{H}_L}(\Omega) \hookrightarrow L^{\mathcal{H}_{p,L}}(\Omega).$$

Next, we can deduce the following finer embeddings (see Proposition 2.15 of [7] for the case of generalized N-function $\mathcal{H}$ given by (3)), where the critical Sobolev exponents for p are defined by

$$p^* = \frac{Np}{N-p} \quad \text{and} \quad p_* = \frac{(N-1)p}{N-p}.$$

Proposition 2. *If* (5) *and* (12) *hold, then we have the following:*

(i) $L^{\mathcal{H}_L}(\Omega) \hookrightarrow L^{\mathcal{H}_{p,L}}(\Omega) \hookrightarrow L^r(\Omega)$ *for all* $1 \leq r \leq p$*;*
(ii) $W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow W^{1,\mathcal{H}_{p,L}}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ *for all* $1 \leq r \leq p$*;*
(iii) $W_0^{1,\mathcal{H}_L}(\Omega) \hookrightarrow W_0^{1,\mathcal{H}_{p,L}}(\Omega) \hookrightarrow W_0^{1,r}(\Omega)$ *for all* $1 \leq r \leq p$*;*
(iv) $W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$, $W_0^{1,\mathcal{H}_L}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ *for all* $1 \leq r < p^*$*;*
(v) $W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow\hookrightarrow L^r(\partial\Omega)$, $W_0^{1,\mathcal{H}_L}(\Omega) \hookrightarrow\hookrightarrow L^r(\partial\Omega)$ *for all* $1 \leq r < p_*$*;*
(vi) $L^{\mathcal{H}_L}(\Omega) \hookrightarrow L^{\mathcal{H}_{q,L,b}}(\Omega)$, *and if* $b(\cdot) \in L^\infty(\Omega)$ *then* $L^{\mathcal{H}_{q,L}}(\Omega) \hookrightarrow L^{\mathcal{H}_L}(\Omega)$*;*

(vii) $L^{\mathcal{H}_L}(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow W^{1,\mathcal{H}}(\Omega)$, $W_0^{1,\mathcal{H}_L}(\Omega) \hookrightarrow W_0^{1,\mathcal{H}}(\Omega)$.

On the space $W_0^{1,\mathcal{H}_L}(\Omega)$, the Poincaré inequality holds, that is, we can find $\widehat{c} > 0$ satisfying

$$\|u\|_{\mathcal{H}_L} \leq \widehat{c}\|\nabla u\|_{\mathcal{H}_L} \quad \text{for all } u \in W_0^{1,\mathcal{H}_L}(\Omega). \tag{14}$$

We note that (14) holds true by Theorem 6.2.8 of [13], as $\mathcal{H}_L$ satisfies both the properties $(A0)$ and $(A1)$. For readers' convenience we get this classical inequality as a part of the following proposition, where different from Theorems 6.2.8 and 6.3.7 of [13] we do not impose property $(A1)$ but only require that $b(\cdot) \in L^\infty(\Omega)$.

Proposition 3. *If* (5) *and* (12) *hold with $q < p^*$ and $b(\cdot) \in L^\infty(\Omega)$, then we have the following:*

(i) $W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow\hookrightarrow L^{\mathcal{H}_L}(\Omega)$;
(ii) *the Poincaré inequality* (14) *is true.*

Proof. First we obtain the embedding result (i) as a byproduct of Proposition 2. Precisely, since $q < p^*$ we can find $\alpha > 0$ and $t_0 > 0$ such that $q + \alpha < p^*$ and

$$\mathcal{H}_{q,L}(x,t) \leq t^{q+\alpha} + t_0^q \log(e + t_0)\chi_\Omega(x) \quad \text{for all } x \in \Omega, \text{ all } t \geq 0.$$

This permits to apply Proposition 2(iv), which says that

$$W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow\hookrightarrow L^{q+\alpha}(\Omega) \hookrightarrow L^{\mathcal{H}_{q,L}}(\Omega).$$

Consequently, by Proposition 2(vi) we deduce that (i) holds.

To prove the Poincaré inequality (14) we argue by contradiction, and suppose that we can find a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}_L}(\Omega)$ satisfying the inequality

$$\|u_n\|_{\mathcal{H}_L} > n\|\nabla u_n\|_{\mathcal{H}_L}.$$

Setting $v_n = \dfrac{u_n}{\|u_n\|_{\mathcal{H}_L}}$, then we deduce that

$$1 \geq \frac{1}{n} > \|\nabla v_n\|_{\mathcal{H}_L} \text{ and } \|v_n\|_{\mathcal{H}_L} = 1 \quad \text{for all } n \in \mathbb{N},$$

hence $\{v_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $W_0^{1,\mathcal{H}_L}(\Omega)$. Passing to a subsequence if necessary, we can find $v \in W_0^{1,\mathcal{H}_L}(\Omega)$ such that

$$v_n \xrightarrow{w} v \quad \text{in } W_0^{1,\mathcal{H}_L}(\Omega).$$

Since $w \mapsto \|\nabla w\|_{\mathcal{H}_L}$ is weakly lower semicontinuous on $W_0^{1,\mathcal{H}_L}(\Omega)$ (recall that it is convex and continuous), then we obtain that

$$\begin{aligned}
&\|\nabla v\|_{\mathcal{H}_L} \leq \liminf_{n\to+\infty} \|\nabla v_n\|_{\mathcal{H}_L} \leq \lim_{n\to+\infty} \frac{1}{n} = 0,\\
\Rightarrow\quad & v = c \in \mathbb{R} \text{ is a constant function,}\\
\Rightarrow\quad & v = 0 \text{ (as } v \in W_0^{1,p}(\Omega) \text{ by Proposition 2(iii)).} \qquad (15)
\end{aligned}$$

On the other hand, (i) implies that $v_n \to v$ in $L^{\mathcal{H}_L}(\Omega)$, and hence

$$\begin{aligned}
&\|v\|_{\mathcal{H}_L} = \lim_{n\to+\infty} \|v_n\|_{\mathcal{H}_L} = 1,\\
\Rightarrow\quad & v \neq 0,
\end{aligned}$$

a contradiction to (15). We conclude that (14) holds true. $\square$

Since (14) gives us an equivalence condition of norms, then we can endow $W_0^{1,\mathcal{H}_L}(\Omega)$ with the norm

$$\|u\| = \|\nabla u\|_{\mathcal{H}_L} \quad \text{for all } u \in W_0^{1,\mathcal{H}_L}(\Omega). \tag{16}$$

4. Dirichlet $\mathcal{H}_L$-Laplacian problems

In this section, we discuss the existence of weak solutions to certain Dirichlet double phase problems related to the perturbed differential operator given by (1). The monotonicity property of suitable operators is a cornerstore to build a comprehensive framework for the characterization of Euler energy functionals associated to nonlinear differential problems. Among the generalized versions of monotonicity, we will consider that of pseudomonotonicity. On the other hand, pseudomonotonicity is closely related to the so-called (S_+)-property. We recall that these notions play a crucial role in applying variational techniques to energy functionals. In particular, we will benefit from the last one in the proof of compactness of functionals. Additionally, we will use pseudomonotonicity in producing solutions of nonlinear operator equations according to the approaches developed by Minty (in Hilbert spaces) and Browder (reflexive Banach spaces). Finally, we mention coercivity as an useful tool of critical point theory. Here, the general framework to introduce these concepts is that of Banach spaces. Thus, we collect these properties in the following definition, where we involve the duality pairing $\langle \cdot , \cdot \rangle$ of Banach spaces.

Definition 5. Let X be a reflexive Banach space with dual space denoted by X^*. Then the operator $V\colon X \to X^*$ satisfies:

(i) (S_+)-property if $u_n \xrightarrow{w} u$ in X and $\limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \le 0$ imply
$$u_n \to u \text{ in } X;$$

(ii) pseudomonotonicity property if $u_n \xrightarrow{w} u$ in X and $\limsup_{n\to+\infty}\langle V(u_n), u_n - u\rangle \le 0$ imply
$$\liminf_{n\to+\infty}\langle V(u_n), u_n - v\rangle \ge \langle V(u), u - v\rangle \quad \text{for all } v \in X;$$

(iii) coercivity property if
$$\lim_{\|u\|_X\to+\infty} \frac{\langle V(u), u\rangle}{\|u\|_X} = +\infty.$$

Remark 5. If $V : X \to X^*$ is a bounded operator, then Definition 5(ii) (that is, the pseudomonotonicity property) is equivalent to the following property:

$$\begin{gathered} u_n \xrightarrow{w} u \text{ in } X \text{ and } \limsup_{n\to+\infty}\langle V(u_n), u_n - u\rangle \le 0 \\ \text{imply } V(u_n) \xrightarrow{w} V(u) \text{ and } \langle V(u_n), u_n\rangle \to \langle V(u), u\rangle \text{ as } n \to +\infty. \end{gathered}$$

We know that the pseudomonotonicity property of operators leads to remarkable surjectivity properties. This is the case of the following theorem, see, for example, Carl et al. [6] (Theorem 2.99, p. 40).

Theorem 1. *If X is a real reflexive Banach space, and we consider a pseudomonotone, bounded, and coercive operator $V\colon X \to X^*$, then the equation $V(u) = f$ with $f \in X^*$ admits a solution.*

In this section we first study the existence of weak solutions to Dirichlet problem of the form

$$-\Delta_{\mathcal{H}_L} u = f \text{ in } \Omega, \quad u\Big|_{\partial\Omega} = 0, \tag{17}$$

where $f \in L^{\mathcal{H}_L^*}(\Omega)$. This equation is governed by the following perturbed double phase differential operator (recall (2) and (5)):

$$\Delta_{\mathcal{H}_L} u = \operatorname{div}\left(\frac{\mathcal{H}_L'(x,|\nabla u|)}{|\nabla u|}\nabla u\right) \quad \text{for all } x \in \Omega, \text{ all } u \in W^{1,\mathcal{H}_L}(\Omega).$$

We impose the following hypotheses on the weight $b(\cdot)$ and the exponents p, q:

$$0 \le b(\cdot) \in L^\infty(\Omega) \setminus \{0\}, \quad 1 < p < N, \quad p < q < p^*. \tag{18}$$

This set of assumptions ensures that $\mathcal{H}_L$ defined by (5) satisfies (aInc) with respect to p and the almost increasing constant $L = 1$. Moreover, $\mathcal{H}_L$ satisfies (aDec) with respect to $q + 1$ and the almost decreasing constant $L = 1$. Consequently, we note that $W_0^{1,\mathcal{H}_L}(\Omega)$ is a separable, uniformly convex and reflexive Banach space. By Proposition 3(ii), the Poincaré inequality (14) holds for $\mathcal{H}_L$ considered herein. Now, we point out the precise definition of weak solution to problem (17), on the space $W_0^{1,\mathcal{H}_L}(\Omega)$ equipped with the norm $\|\cdot\|$ (recall (16)).

Definition 6. For a weak solution to problem (17), we mean a function $u \in W_0^{1,\mathcal{H}_L}(\Omega)$ satisfying the equality

$$\int_\Omega \frac{\mathcal{H}_L'(x,|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla v\, dx = \int_\Omega f v\, dx \quad \text{for all } v \in W_0^{1,\mathcal{H}_L}(\Omega). \tag{19}$$

Next, we consider the energy functional for problem (17), namely $J : W_0^{1,\mathcal{H}_L}(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \mathcal{F}(u) - \int_\Omega f u\, dx \quad \text{for all } u \in W_0^{1,\mathcal{H}_L}(\Omega),$$

where $\mathcal{F} : W_0^{1,\mathcal{H}_L}(\Omega) \to \mathbb{R}$ is given by $\mathcal{F}(u) = \int_\Omega \mathcal{H}_L(x,|\nabla u|)\, dx$ and is the integral functional related to the principal operator $\Delta_{\mathcal{H}_L}$. The study of differentiability properties of functionals on Banach spaces has continued on and off for over fifty years both in analysis and in geometry. Here we show that $\mathcal{F}(\cdot)$ has some regularities in the sense of $W_0^{1,\mathcal{H}_L}(\Omega)$ and $W_0^{1,\mathcal{H}_L}(\Omega)^*$, as established in the following lemma.

Lemma 3. *If (5) and (18) hold, then $\mathcal{F}(\cdot)$ is a C^1-functional and its Gateaux derivative $\mathcal{F}' : W_0^{1,\mathcal{H}_L}(\Omega) \to W_0^{1,\mathcal{H}_L}(\Omega)^*$ is given by*

$$\langle \mathcal{F}'(u), v\rangle = \int_\Omega \frac{\mathcal{H}_L'(x,|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla v\, dx.$$

Proof. It is immediate that for $u, v \in W_0^{1,\mathcal{H}_L}(\Omega)$ and $t \in \mathbb{R}$, we have the following

$$\begin{aligned}\frac{\mathcal{F}(u+tv) - \mathcal{F}(u)}{t} &= \int_\Omega \frac{\mathcal{H}_L(x,|\nabla u + t\nabla v|) - \mathcal{H}_L(x,|\nabla u|)}{t} dx \\ &= \int_\Omega \frac{1}{t}\int_{|\nabla u|}^{|\nabla u + t\nabla v|} \mathcal{H}_L'(x,s)\, ds dx.\end{aligned}$$

On the other hand, from $|\nabla u + t\nabla v| \to |\nabla u|$ in $L^{\mathcal{H}_L}(\Omega)$ as $t \to 0$, and hence in $L^1(\Omega)$, it follows that one can assume the convergence $|\nabla u + t\nabla v| \to |\nabla u|$ a.e. in Ω. From the monotonicity (increasing) of $\mathcal{H}'_L$, it follows that for t small enough we obtain the inequality

$$\left|\frac{1}{t}\int_{|\nabla u|}^{|\nabla u+t\nabla v|}\mathcal{H}'_L(x,s)\,ds\right|\leq \mathcal{H}'_L(x,|\nabla u|+|\nabla v|)|\nabla v|.$$

In our case (because of doubling and convex functions satisfy (11)), we note that

$$\mathcal{H}'_L(x,|\nabla w|)\in L^{\mathcal{H}^*_L}(\Omega)\quad\text{for all } w\in W_0^{1,\mathcal{H}_L}(\Omega).$$

By Hölder inequality, we deduce that $\mathcal{H}'_L(x,|\nabla u|+|\nabla v|)|\nabla v|\in L^1(\Omega)$. Therefore, by the theorem concerning dominated convergence, it follows that $\mathcal{F}'(\cdot)$ may be seen acting from $W_0^{1,\mathcal{H}_L}(\Omega)$ into $W_0^{1,\mathcal{H}_L}(\Omega)^*$ by

$$\begin{aligned}\langle\mathcal{F}'(u),v\rangle=&\lim_{t\to 0}\frac{\mathcal{F}(u+tv)-\mathcal{F}(u)}{t}\\ =&\frac{d}{dt}\mathcal{F}(u+tv)\Big|_{t=0}=\int_\Omega\frac{\mathcal{H}'_L(x,|\nabla u|)}{|\nabla u|}\nabla u\cdot\nabla v\,dx.\end{aligned}$$

Now, we give the proof that $\mathcal{F}'(\cdot)$ is continuous. Let $\{u_n\}_{n\in\mathbb{N}}\subseteq W_0^{1,\mathcal{H}_L}(\Omega)$ be a sequence such that $u_n\to u$ in $W_0^{1,\mathcal{H}_L}(\Omega)$ as $n\to+\infty$. For every $v\in W_0^{1,\mathcal{H}_L}(\Omega)$ with $\|v\|\leq 1$, we can apply the Hölder inequality to obtain the following estimate

$$\begin{aligned}|\langle\mathcal{F}'(u_n)-\mathcal{F}'(u),v\rangle|&=\left|\int_\Omega\left(\frac{\mathcal{H}'_L(x,|\nabla u_n|)}{|\nabla u_n|}\nabla u_n-\frac{\mathcal{H}'_L(x,|\nabla u|)}{|\nabla u|}\nabla u\right)\cdot\nabla v\,dx\right|\\ &\leq 2\left\|\frac{\mathcal{H}'_L(x,|\nabla u_n|)}{|\nabla u_n|}\nabla u_n-\frac{\mathcal{H}'_L(x,|\nabla u|)}{|\nabla u|}\nabla u\right\|_{\mathcal{H}^*_L}.\end{aligned}$$

Finally, we involve the modular $\rho_{\mathcal{H}^*_L}(\cdot)$ and derive the following modular convergence

$$\rho_{\mathcal{H}^*_L}\left(\frac{\mathcal{H}'_L(x,|\nabla u_n|)}{|\nabla u_n|}\nabla u_n-\frac{\mathcal{H}'_L(x,|\nabla u|)}{|\nabla u|}\nabla u\right)\to 0\quad\text{as } n\to+\infty,$$

which equivalently (see Lemma 2) leads to the norm convergence

$$\left\|\frac{\mathcal{H}'_L(x,|\nabla u_n|)}{|\nabla u_n|}\nabla u_n-\frac{\mathcal{H}'_L(x,|\nabla u|)}{|\nabla u|}\nabla u\right\|_{\mathcal{H}^*_L}\to 0.$$

Therefore, we deduce that

$$\sup_{v\in W_0^{1,\mathcal{H}_L}(\Omega),\,\|v\|\leq 1}|\langle\mathcal{F}'(u_n)-\mathcal{F}'(u),v\rangle|\to 0,$$

and so we conclude that $\mathcal{F}'(\cdot)$ is continuous. $\square$

On this basis, we define the operator $V: W_0^{1,\mathcal{H}_L}(\Omega)\to W_0^{1,\mathcal{H}_L}(\Omega)^*$ by

$$\langle V(u),v\rangle=\int_\Omega\frac{\mathcal{H}'_L(x,|\nabla u|)}{|\nabla u|}\nabla u\cdot\nabla v\,dx,\tag{20}$$

for all $u,v\in W_0^{1,\mathcal{H}_L}(\Omega)$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W_0^{1,\mathcal{H}_L}(\Omega)$ and its dual space $W_0^{1,\mathcal{H}_L}(\Omega)^*$. Now, we discuss some properties of $V(\cdot)$.

Theorem 2. *If* (5) *and* (18) *hold, then* $V: W_0^{1,\mathcal{H}_L}(\Omega)\to W_0^{1,\mathcal{H}_L}(\Omega)^*$ *defined by* (20) *is continuous, bounded, strictly monotone (hence, maximal monotone) and coercive.*

Proof. By the formula (20), $V(\cdot) = \mathcal{F}'(\cdot)$ being $\mathcal{F}$ a C^1-functional (recall Lemma 3), is continuous. Let us now prove that $V(\cdot)$ is bounded. Consider a bounded subset B of the Musielak-Orlicz-Sobolev space $W_0^{1,\mathcal{H}_L}(\Omega)$, that is, we assume that the following is the case

$$\|u\| \leq M \quad \text{for all } u \in B, \text{ some } M > 0.$$

Then, we have the computation

$$\begin{aligned}
\left\langle V(u), \frac{v}{\|v\|} \right\rangle &= \int_\Omega \frac{\mathcal{H}_L'(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \frac{\nabla v}{\|v\|}\, dx \\
&\leq \int_\Omega \mathcal{H}_L'(x, |\nabla u|) \frac{|\nabla v|}{\|v\|}\, dx \\
&\leq \int_\Omega \mathcal{H}_L^*(x, \mathcal{H}_L'(x, |\nabla u|))\, dx + \int_\Omega \mathcal{H}_L\left(x, \frac{|\nabla v|}{\|v\|}\right) dx \\
&\leq c \int_\Omega \mathcal{H}_L(x, |\nabla u|))\, dx + 1 \quad \text{(by (11) and Proposition 1)} \\
&\leq cM^r + 1 \quad \text{for all } u \in B,
\end{aligned}$$

where $r = 1$ if $M \leq 1$ or $r = q + 1$ if $M > 1$ (see (13)). Therefore, it is immediate that $V(\cdot)$ is bounded.

To establish the monotonicity of $V(\cdot)$, we observe the following

$$\begin{aligned}
\langle V(u) - V(v), u - v \rangle &= \int_\Omega \left(\frac{\mathcal{H}_L'(x, |\nabla u|)}{|\nabla u|} \nabla u - \frac{\mathcal{H}_L'(x, |\nabla v|)}{|\nabla v|} \nabla v) \right) \cdot (\nabla u - \nabla v)\, dx \\
&\geq \int_\Omega (\mathcal{H}_L'(x, |\nabla u|) - \mathcal{H}_L'(x, |\nabla v|)) \cdot (|\nabla u| - |\nabla v|)\, dx \\
&> 0 \quad \text{(by monotonicity (strictly increasing) of } \mathcal{H}_L'\text{)}.
\end{aligned}$$

In order to obtain that $V(\cdot)$ is coercive, we assume that $\|u\| \geq 1$ (without loss of generality). From this constraining assumption, we get

$$\begin{aligned}
\frac{\langle V(u), u \rangle}{\|u\|} &= \int_\Omega \frac{\mathcal{H}_L'(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \frac{\nabla u}{\|u\|}\, dx \\
&= \int_\Omega \mathcal{H}_L'(x, |\nabla u|) \frac{|\nabla u|}{\|u\|}\, dx \\
&\geq c \int_\Omega \mathcal{H}_L(x, |\nabla u|) \frac{1}{\|u\|}\, dx \quad \text{for some } c > 0 \text{ (see (11))} \\
&\geq c \int_\Omega \|u\|^{p-1} \mathcal{H}_L\left(x, \frac{|\nabla u|}{\|u\|}\right) dx \quad \text{(by (13))} \\
&\geq c\|u\|^{p-1} \to +\infty \quad \text{as } \|u\| \to +\infty,
\end{aligned}$$

hence Definition 5(iii) is fulfilled. $\square$

The next result establishes that our operator $V(\cdot)$ satisfies the (S_+)-property (see Definition 5(i)).

Theorem 3. *If* (5) *and* (18) *hold, then* $V : W_0^{1,\mathcal{H}_L}(\Omega) \to W_0^{1,\mathcal{H}_L}(\Omega)^*$ *defined by* (20) *satisfies the* (S_+)*-property.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}_L}(\Omega)$ be a sequence such that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,\mathcal{H}_L}(\Omega) \text{ and } \limsup_{n\to+\infty}\langle V(u_n), u_n - u\rangle \leq 0. \tag{21}$$

By the weak convergence of $\{u_n\}_{n\in\mathbb{N}}$ to u in $W_0^{1,\mathcal{H}_L}(\Omega)$, it follows that

$$\lim_{n\to+\infty}\langle V(u), u_n - u\rangle = 0.$$

This non-positivity condition along with (21) implies the following condition for the limit superior

$$\limsup_{n\to+\infty}\langle V(u_n) - V(u), u_n - u\rangle \leq 0.$$

In our case, $V(\cdot)$ is strictly monotone and hence we have

$$0 \leq \liminf_{n\to+\infty}\langle V(u_n) - V(u), u_n - u\rangle \leq \limsup_{n\to+\infty}\langle V(u_n) - V(u), u_n - u\rangle \leq 0.$$

To satisfy this chain of inequalities, we conclude that

$$\lim_{n\to+\infty}\langle V(u_n) - V(u), u_n - u\rangle = 0. \tag{22}$$

Since $\mathcal{H}_L$ defined by (5), along with its first and second derivatives, satisfies the constraining assumptions:

$$p(p-1)\mathcal{H}_L(x,t) \leq t^2\mathcal{H}_L''(x,t) \leq q(q+1)\mathcal{H}_L(x,t) \tag{23}$$

and

$$(p-1)\mathcal{H}_L'(x,t) \leq t\mathcal{H}_L''(x,t) \leq 2q\mathcal{H}_L'(x,t), \tag{24}$$

then according to (24), we apply Lemma 2.3 of [24] and deduce that

$$\begin{aligned}
&|\nabla u_n - \nabla u|^2\mathcal{H}_L''(x, |\nabla u_n| + |\nabla u|)\\
&\leq c\left(\frac{\mathcal{H}_L'(x,|\nabla u_n|)}{|\nabla u_n|}\nabla u_n - \frac{\mathcal{H}_L'(x,|\nabla u|)}{|\nabla u|}\nabla u\right)\cdot(\nabla u_n - \nabla u).
\end{aligned} \tag{25}$$

By (23) and (25), it follows that

$$\mathcal{H}_L(x, |\nabla u_n - \nabla u|) \leq c\left(\frac{\mathcal{H}_L'(x,|\nabla u_n|)}{|\nabla u_n|}\nabla u_n - \frac{\mathcal{H}_L'(x,|\nabla u|)}{|\nabla u|}\nabla u\right)\cdot(\nabla u_n - \nabla u).$$

Integrating both sides of this inequality over Ω we get

$$\begin{aligned}
&\int_\Omega \mathcal{H}_L(x, |\nabla u_n - \nabla u|)\,dx\\
&\leq c\int_\Omega\left(\frac{\mathcal{H}_L'(x,|\nabla u_n|)}{|\nabla u_n|}\nabla u_n - \frac{\mathcal{H}_L'(x,|\nabla u|)}{|\nabla u|}\nabla u\right)\cdot(\nabla u_n - \nabla u)\,dx\\
&= c\,\langle V(u_n) - V(u), u_n - u\rangle,
\end{aligned}$$

and hence passing to the limit as $n \to +\infty$ we conclude by (22) that

$$\lim_{n\to+\infty}\int_\Omega \mathcal{H}_L(x, |\nabla u_n - \nabla u|)\,dx = 0.$$

By Lemma 2 or Proposition 1 the sequence $\{\nabla u_n\}_{n\in\mathbb{N}}$ converges to ∇u in $L^{\mathcal{H}_L}(\Omega)$. This implies that the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges to u in $W_0^{1,\mathcal{H}_L}(\Omega)$. □

We recall that one of the relevant (maximal) regularity properties of a nonlinear differential operator is that the involved operator is a homeomorphism. Such result is useful to establish existence and uniqueness of solution to problem (17). Here, we show that $V(\cdot)$ is a homeomorphism of $W_0^{1,\mathcal{H}_L}(\Omega)$ to $W_0^{1,\mathcal{H}_L}(\Omega)^*$.

Theorem 4. *If* (5) *and* (18) *hold, then the operator* $V : W_0^{1,\mathcal{H}_L}(\Omega) \to W_0^{1,\mathcal{H}_L}(\Omega)^*$ *defined by* (20) *is a homeomorphism.*

Proof. Consider a sequence $\{w_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}_L}(\Omega)^*$ such that $w_n \to w$ in $W_0^{1,\mathcal{H}_L}(\Omega)^*$ and denote $u_n := V^{-1}(w_n)$ and $u := V^{-1}(w)$ (as the strict monotonicity of $V(\cdot)$ guarantees that $V(\cdot)$ is injective). By the strong convergence of $\{w_n\}_{n\in\mathbb{N}}$ and the boundedness of $V^{-1}(\cdot)$, it follows that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,\mathcal{H}_L}(\Omega)$. Therefore, we can find a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ satisfying the condition

$$u_{n_k} \xrightarrow{w} u_0 \quad \text{in } W_0^{1,\mathcal{H}_L}(\Omega),$$

where $u_0 \in W_0^{1,\mathcal{H}_L}(\Omega)$ is the weak limit. Letting $k \to +\infty$, we infer that

$$\begin{aligned}&\lim_{k\to+\infty}\langle V(u_{n_k}) - V(u_0), u_{n_k} - u_0\rangle\\ &= \lim_{k\to+\infty}\langle w_{n_k} - w, u_{n_k} - u_0\rangle + \lim_{k\to+\infty}\langle w - V(u_0), u_{n_k} - u_0\rangle = 0.\end{aligned}$$

By Theorem 3 the (S_+)-property of $V(\cdot)$ is established, and hence we deduce the strong convergence $u_{n_k} \to u_0$ in the solution space $W_0^{1,\mathcal{H}_L}(\Omega)$. By Theorem 2 we know that $V(\cdot)$ is also a continuous operator. It follows that

$$V(u_0) = \lim_{k\to+\infty} V(u_{n_k}) = \lim_{k\to+\infty} w_{n_k} = w = V(u).$$

Additionally, $V(\cdot)$ is injective, and hence we get that $u = u_0$. Thus, we can conclude that all subsequences of type $\{u_{n_k}\}_{k\in\mathbb{N}}$ which are weakly convergent have the same limit, and hence we can refer to the standard subsequence principle to obtain the convergence of the principal sequence $\{u_n\}_{n\in\mathbb{N}}$. □

As mentioned at the beginning of this section, an ingredient for the proof of our result is the following Minty-Browder theorem, which is linked to surjectivity property of the involved operator.

Theorem 5. *Assume that* $V : X \to X^*$ *is a bounded, continuous, coercive and monotone operator, where* X *is a real, reflexive Banach space. Then for each data* $f \in X^*$ *there is an element* $u \in X$ *satisfying* $V(u) = f$*, that is,* $V(X) = X^*$.

The previous results provide us with the tools to establish the existence and uniqueness of solution to problem (17).

Theorem 6. *If* $f \in L^{\mathcal{H}_L^*}(\Omega)$, (5) *and* (18) *hold, then problem* (17) *has a unique weak solution* $u \in W_0^{1,\mathcal{H}_L}(\Omega)$.

Proof. It is an easy consequence of Theorem 2. Indeed, by Theorem 2, we have the appropriate setting for considering $V(\cdot)$ defined by (20) in the sense of Theorem 5. Since $f \in L^{\mathcal{H}_L^*}(\Omega) \hookrightarrow W_0^{1,\mathcal{H}_L}(\Omega)^*$, then we can find a unique $u \in W_0^{1,\mathcal{H}_L}(\Omega)$ such that

$$\int_\Omega \frac{\mathcal{H}_L'(x,|\nabla u|)}{|\nabla u|}\nabla u\cdot\nabla v\,dx = \int_\Omega fv\,dx \quad \text{for all } v \in W_0^{1,\mathcal{H}_L}(\Omega),$$

that is, (19) holds and so $u \in W_0^{1,\mathcal{H}_L}(\Omega)$ is a unique weak solution to problem (17). □

Remark 6. Theorem 6 remains true if we assume that $f \in W_0^{1,\mathcal{H}_L}(\Omega)^*$ instead of $f \in L^{\mathcal{H}_L^*}(\Omega)$.

Finally, we explicitly consider the case where the right hand side of the equation (17) exhibits the effects of gradient dependency (convection). Now, the dependence on the gradient of f (see equation (26) below), implies that one can not solve directly the Dirichlet problem by using variational techniques. So, our approach here is topological based on Theorem 1. Namely, we consider the following problem:

$$-\Delta_{\mathcal{H}_L} u = f(x, u, \nabla u) \quad \text{in } \Omega, \quad u\Big|_{\partial\Omega} = 0, \tag{26}$$

and we give sufficient regularity and growth conditions on the data f ensuring the existence of some $u \in W_0^{1,\mathcal{H}_L}(\Omega)$ such that the equality

$$\int_\Omega \frac{\mathcal{H}_L'(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla v \, dx = \int_\Omega f(x, u, \nabla u) v \, dx \tag{27}$$

holds for all $v \in W_0^{1,\mathcal{H}_L}(\Omega)$. Such $u \in W_0^{1,\mathcal{H}_L}(\Omega)$ is known as weak solution to problem (17) (similarly to Definition 6). First of all, suitable conditions on f will give the well-posedness of the Nemytskii map corresponding to f, that is,

$$(N_f u)(x) = f(x, u(x), \nabla u(x)) \quad \text{for all } x \in \Omega, \tag{28}$$

then technical conditions must be formulated to control the growth of the energy functional associated to the Dirichlet problem. As usual in this sense, we assume that f is of Carathéodory, which ensures that $N_f u : \Omega \to \mathbb{R}$ is measurable for all $u \in W_0^{1,\mathcal{H}_L}(\Omega)$. The precise hypotheses on f are as follows:

$H(f)$ Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that $f(\cdot, 0, 0) \neq 0$, and the following hold:

(i) there exist $a_0 \in L^{\alpha'}(\Omega)$, $\beta \in [1, p^*[$, $\gamma \in [0, \frac{p}{(p^*)'}[$ ($(p^*)'$ is the conjugate exponent to p^* given by $\frac{1}{p^*} + \frac{1}{(p^*)'} = 1$), and $b_1, b_2 \geq 0$ such that

$$|f(x, y, z)| \leq a_0(x) + b_1 |y|^{\beta - 1} + b_2 |z|^\gamma \quad \text{for a.a. } x \in \Omega, \text{ all } y \in \mathbb{R}, \text{ all } z \in \mathbb{R}^N,$$

where $\alpha \in [1, p^*[$ with conjugate α' given by $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$;

(ii) there exist $a_1 \in L^1(\Omega)$ and $c_1, c_2 \geq 0$ such that

$$f(x, y, z) y \leq a_1(x) + c_1 |y|^p + c_2 |z|^p \quad \text{for a.a. } x \in \Omega, \text{ all } y \in \mathbb{R}, \text{ all } z \in \mathbb{R}^N.$$

Further, the nonnegative coefficients c_1, c_2 have to satisfy the inequality

$$c_1 \lambda_{1,p}^{-1} + c_2 < p \tag{29}$$

where $\lambda_{1,p}$ is the first eigenvalue of the Dirichlet eigenvalue problem for the p-Laplacian, that is,

$$0 < \lambda_{1,p} = \inf_{u \in W_0^{1,p}(\Omega),\, u \neq 0} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}. \tag{30}$$

(iii) there exist $a_2 \in L^1(\Omega)$ and $0 \leq s < p$ such that

$$f(x, y, z) y \leq a_2(x) + s \mathcal{H}(x, |z|) \quad \text{for a.a. } x \in \Omega, \text{ all } y \in \mathbb{R}, \text{ all } z \in \mathbb{R}^N.$$

Remark 7. We note that $H(f)$(i) together with the embeddings results in Proposition 2 ensure that (27) (that is, the definition of weak solution to (26)) is well-posed. Indeed, $p^* > \max\{\alpha, \frac{p^*}{p^*-\beta+1}, \frac{p}{p-\gamma}\}$ (by $H(f)$(i)), and we have the following estimate

$$\begin{aligned}\left|\int_\Omega f(x,u,\nabla u)v\,dx\right| &\le \int_\Omega |a_0|\,|v|\,dx + b_1\int_\Omega |u|^{\beta-1}\,|v|\,dx + b_2\int_\Omega |\nabla u|^\gamma\,|v|\,dx\\ &\le \|a_0\|_{\alpha'}\|v\|_\alpha + b_1\|u\|_{p^*}^{\beta-1}\|v\|_{\frac{p^*}{p^*-\beta+1}} + b_2\|\nabla u\|_p^\gamma\|v\|_{\frac{p}{p-\gamma}}\\ &\quad\text{(by Holder inequality)}\\ &\le c(\|a_0\|_{\alpha'} + b_1\|u\|_{p^*}^{\beta-1} + b_2\|\nabla u\|_p^\gamma)\|\nabla v\|_p\\ &\quad\text{(by Sobolev embeddings in Proposition 2)}\end{aligned}$$

for all $u, v \in W_0^{1,\mathcal{H}_L}(\Omega)$ (see Proposition 2). We also refer to Motreanu et al. [18] for additional information on the exponents in hypothesis $H(f)$(i) and the ones involved in other existing growth conditions. Finally, we underline that hypothesis $H(f)$(iii) is alternative to $H(f)$(ii) in establishing the existence result.

A simplest way to satisfy $H(f)$ occurs when we can find functions $\underline{a}, \overline{a} \in L^{\alpha'}(\Omega)$, with $\alpha \in [1, p^*[$ such that

$$0 < \underline{a}(x) \le f(x,y,z) \le \overline{a}(x)$$

for a.a. $x \in \Omega$, all $y \in \mathbb{R}$, all $z \in \mathbb{R}^N$.

Working with the Nemytskii operator originated by the Carathéodory function f in hypotheses $H(f)$, we establish the existence of weak solutions to (26) as follows. Here, by $\bar{\imath} : L^{r'}(\Omega) \to W_0^{1,\mathcal{H}_L}(\Omega)^*$ we mean the adjoint operator to $i : W_0^{1,\mathcal{H}_L}(\Omega) \to L^r(\Omega)$, for some $r \in [1, p^*[$.

Theorem 7. *If* (5), (18) *and* $H(f)$*(i),(ii) hold, then problem* (26) *has a nontrivial weak solution* $u \in W_0^{1,\mathcal{H}_L}(\Omega)$.

Proof. Let $\overline{N}_f : W_0^{1,\mathcal{H}_L}(\Omega) \to L^{r'}(\Omega)$ be the Nemytskii operator defined by

$$\overline{N}_f(u) = f(x,u,\nabla u) \quad \text{for all } u \in W_0^{1,\mathcal{H}_L}(\Omega) \quad \text{(see also (28))},$$

and recall that by Remark 7 (that is, combining $H(f)$(i) and Proposition 2) we know that $f(x,u,\nabla u) \in L^{r'}(\Omega)$ for all $u \in W_0^{1,\mathcal{H}_L}(\Omega)$ for some $r \in [1,p^*[$. It follows that this operator is well-defined, bounded and continuous.

Also, we introduce the operator $\mathcal{G} : W_0^{1,\mathcal{H}_L}(\Omega) \to W_0^{1,\mathcal{H}_L}(\Omega)^*$ given as

$$\mathcal{G}(u) = V(u) - N_f(u), \tag{31}$$

where $N_f = \bar{\imath} \circ \overline{N}_f$ and $V(\cdot)$ is defined by (20). So, $\mathcal{G}$ is a continuous and bounded operator by Theorem 2. Next step is to show that we can apply the surjectivity result in Theorem 1. Thus, we start verifying that our operator $\mathcal{G}(\cdot)$ is pseudomonotone (we refer to Remark 5). Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}_L}(\Omega)$ satisfying (21) with $\mathcal{G}$ in place of V, hence Proposition 2 (i.e., compact embedding result) gives us

$$u_n \to u \quad \text{in } L^r(\Omega) \quad \text{since } p^* > r \text{ and recall } p^* > \max\{\alpha, \tfrac{p^*}{p^*-\beta+1}, \tfrac{p}{p-\gamma}\}). \tag{32}$$

We can use the Hölder inequality, together with the weak convergence of $\{u_n\}_{n\in\mathbb{N}}$ in $W_0^{1,\mathcal{H}_L}(\Omega)$ (i.e., the boundedness in norm) to conclude the convergence

$$\left|\int_\Omega f(x,u_n,\nabla u_n)(u_n-u)\,dx\right| \le 2\|\overline{N}_f(u_n)\|_{r'}\|u-u_n\|_r \to 0 \quad \text{as } n \to +\infty, \tag{33}$$

since $\{\|\overline{N}_f(u_n)\|_{r'}\}_{n\in\mathbb{N}}$ is bounded (recall that $\overline{N}_f$ is bounded, see again the estimate in Remark 7) and $u_n \to u$ in $L^r(\Omega)$ (see (32)). By (31) and (33), we deduce easily that

$$0 \geq \limsup_{n\to+\infty} \langle \mathcal{G}(u), u_n - u\rangle = \limsup_{n\to+\infty} \langle V(u_n), u_n - u\rangle. \tag{34}$$

By Theorem 3, we know that $V(\cdot)$ defined by (20) possesses the (S_+)-property. So, (34) implies the strong convergence of $\{u_n\}_{n\in\mathbb{N}}$ to u in $W_0^{1,\mathcal{H}_L}(\Omega)$. We invoke the continuity of $\mathcal{G}(\cdot)$ to derive that $\mathcal{G}(\cdot)$ is a pseudomonotone operator.

It remains to verify the coercivity of $\mathcal{G}(\cdot)$. Referring to (30), we know that

$$\int_\Omega |u|^p\,dx \leq \lambda_{1,p}^{-1} \int_\Omega |\nabla u|^p\,dx \quad \text{for all } u \in W_0^{1,\mathcal{H}_L}(\Omega). \tag{35}$$

Since $u \in W_0^{1,p}(\Omega)$ (by $u \in W_0^{1,\mathcal{H}_L}(\Omega)$) and

$$t\mathcal{H}_L'(x,t) \geq p\mathcal{H}_L(x,t) \quad \text{for all } x\in\Omega, \text{ all } t\geq 0,$$

we can now use Proposition 1 to deduce the following estimate

$$\begin{aligned}
\langle \mathcal{G}(u,u)\rangle &= \int_\Omega \frac{\mathcal{H}_L'(x,|\nabla u|)}{|\nabla u|}\nabla u\cdot\nabla u\,dx - \int_\Omega f(x,u,\nabla u)u\,dx \\
&\geq p\int_\Omega \mathcal{H}_L(x,|\nabla u|)\,dx - \|a_1\|_1 - c_1\int_\Omega |u|^p\,dx - c_2\int_\Omega |\nabla u|^p\,dx \quad (\text{by } H(f)(\text{ii})) \\
&\geq p\int_\Omega \mathcal{H}_L(x,|\nabla u|)\,dx - \|a_1\|_1 - c_1\lambda_{1,p}^{-1}\int_\Omega |\nabla u|^p\,dx - c_2\int_\Omega |\nabla u|^p\,dx \quad (\text{by } (35)) \\
&\geq (p - c_1\lambda_{1,p}^{-1} - c_2)\int_\Omega \mathcal{H}_L(x,|\nabla u|)\,dx - \|a_1\|_1 \\
&\geq (p - c_1\lambda_{1,p}^{-1} - c_2)\|u\|^p - \|a_1\|_1 \quad \text{if } \|u\|>1.
\end{aligned}$$

The coefficient "$p - c_1\lambda_{1,p}^{-1} - c_2$" is positive by $H(f)$(ii), then we derive the coercivity of $\mathcal{G}(\cdot)$. We are in position to apply the surjectivity result (Theorem 1) to $\mathcal{G}$ to conclude that the following equation

$$\mathcal{G}(u) = 0$$

admits a solution $u \in W_0^{1,\mathcal{H}_L}(\Omega)$. This solution is a weak solution to problem (26), because of (31); further it is nontrivial as $f(\cdot,0,0)\neq 0$ by hypothesis. □

Example 1. Consider the function defined by

$$f(x,y,z) = \left(\frac{a(x)+|z|^{p-1}}{y^2+1} + |y|^{p-2}\right)y$$

for a.a. $x\in\Omega$, all $y\in\mathbb{R}$ and all $z\in\mathbb{R}^N$, with $a\in L^\infty(\Omega)\setminus\{0\}$.

It is left to the reader to easily verify that f fulfils hypotheses $H(f)$(i),(ii) (recall that $p-1 = \frac{p}{p'} < \frac{p}{(p^*)'}$, and set, for the sake of simplicity, $\alpha=\beta=p$ and $\gamma = p-1$).

We can establish the similar conclusion to Theorem 7, changing hypothesis $H(f)$(ii) by hypothesis $H(f)$(iii) as follows.

Theorem 8. *If* (5), (18) *and* $H(f)$*(i),(iii) hold, then problem* (26) *has a nontrivial weak solution* $u\in W_0^{1,\mathcal{H}_L}(\Omega)$.

Proof. It is clear that repeating the same arguments as in the proof of Theorem 7, we get that $\mathcal{G} : W_0^{1,\mathcal{H}_L}(\Omega) \to W_0^{1,\mathcal{H}_L}(\Omega)^*$ defined by (31) is a continuous, bounded and pseudomonotone operator (by $H(f)$(i)). We also recall that

$$t\mathcal{H}_L'(x,t) \geq p\mathcal{H}_L(x,t) \quad \text{for all } x \in \Omega, \text{ all } t \geq 0,$$

and note that $W_0^{1,\mathcal{H}_L}(\Omega) \subseteq W_0^{1,\mathcal{H}}(\Omega)$. Let us now prove that $\mathcal{G}(\cdot)$ is coercive. This time, by hypothesis $H(f)$(iii) along with Proposition 1 one has the estimate

$$\begin{aligned}
\langle \mathcal{G}(u,u)\rangle &= \int_\Omega \frac{\mathcal{H}_L'(x,|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla u \, dx - \int_\Omega f(x,u,\nabla u)u \, dx \\
&\geq p\int_\Omega \mathcal{H}_L(x,|\nabla u|)\, dx - \|\hat{a}_2\|_1 - s\int_\Omega \mathcal{H}(x,|\nabla u|)\, dx \\
&\geq (p-s)\int_\Omega \mathcal{H}_L(x,|\nabla u|)\, dx - \|a_2\|_1 \quad (\text{since } \rho_{\mathcal{H}}(\nabla u) \leq \rho_{\mathcal{H}_L}(\nabla u)) \\
&\geq (p-s)\|u\|^p - \|a_2\|_1 \quad \text{if } \|u\| > 1.
\end{aligned}$$

This is sufficient to conclude that $\mathcal{G}(\cdot)$ is a coercive operator, because of $s < p$. The rest of the proof remains the same proof as Theorem 7. Namely, from Theorem 1 we can find a function $u \in W_0^{1,\mathcal{H}_L}(\Omega)$ such that $\mathcal{G}(u) = 0$. This solution is a weak solution to problem (26), by (31); again it is nontrivial as $f(\cdot,0,0) \neq 0$. □

Example 2. Consider the function defined by

$$f(x,y,z) = \frac{y}{1+y^2}(1+b(x)|z|^\gamma)$$

for a.a. $x \in \Omega$ all $y \in \mathbb{R}$ and all $z \in \mathbb{R}^N$, with $\gamma \in [0, \frac{p}{(p^*)'}[$, $1 < p < q < p^*$, $b \in L^\infty(\Omega) \setminus \{0\}$, $b(x) \geq 0$ for a.a. $x \in \Omega$, according to (3) and (18).

Note that

$$\frac{|y|}{1+y^2}(1+b(x)|z|^\gamma) \leq \frac{1}{2}(1+\|b\|_\infty|z|^\gamma),$$

hence f satisfies $H(f)$(i) with $a_0 \equiv 1/2$, $b_1 = 0$ and $b_2 = \|b\|_\infty/2$. Further, f satisfies hypothesis $H(f)$(iii) with $s = 1$ and $a_2(x) = b(x) + 1$ for all $x \in \Omega$. On the other hand, f fails to satisfy $H(f)$(ii) if $\|b\|_\infty \geq p$ (recall that $c_2 < p$ in $H(f)$(ii)).

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