Positive solutions for parametric singular Dirichlet (p, q)-equations

Nikolaos S. Papageorgiou[a], Calogero Vetro[b], Youpei Zhang[c,*]

[a] *National Technical University, Department of Mathematics, Zografou Campus, 15780 Athens, Greece*
[b] *University of Palermo, Department of Mathematics and Computer Science, Via Archirafi 34, 90123 Palermo, Italy*
[c] *Central South University, School of Mathematics and Statistics, Changsha, Hunan 410083, People's Republic of China & Department of Mathematics, University of Craiova, Street A.I. Cuza no. 13, 200585 Craiova, Romania*

Abstract

We consider a nonlinear elliptic Dirichlet problem driven by the (p, q)-Laplacian and a reaction consisting of a parametric singular term plus a Carathéodory perturbation $f(z, x)$ which is $(p-1)$-linear as $x \to +\infty$. First we prove a bifurcation-type theorem describing in an exact way the changes in the set of positive solutions as the parameter $\lambda > 0$ moves. Subsequently, we focus on the solution multifunction and prove its continuity properties. Finally we prove the existence of a smallest (minimal) solution u^*_λ and investigate the monotonicity and continuity properties of the map $\lambda \to u^*_\lambda$.

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*Corresponding author

Email addresses: npapg@math.ntua.gr (Nikolaos S. Papageorgiou), calogero.vetro@unipa.it (Calogero Vetro), youpei.zhang@inf.ucv.ro zhangypzn@163.com (Youpei Zhang)

1. Introduction

In this paper we study the following parametric singular (p,q)-Dirichlet problem:

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) = \lambda u(z)^{-\eta} + f(z,u(z)) \quad \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0 \quad u > 0,\, \lambda > 0,\, 1 < q < p,\, 0 < \eta < 1. \end{cases} \tag{P_λ}$$

In this problem $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$. For every $r \in (1,+\infty)$, by Δ_r we denote the r-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u) \quad \text{for all } u \in W_0^{1,r}(\Omega).$$

So, in problem (P_λ) the differential operator is the sum of two such operators and therefore it is not homogeneous. Operators of this kind arise in many mathematical models of physical processes. We refer to the work of Bahrouni-Rădulescu-Repovš [2] and the references therein. In the reaction of (P_λ), we have a parametric singular term ($u \to \lambda u^{-\eta}$) and a Carathéodory perturbation $f(z,x)$ (that is, for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z,x)$ is continuous). We assume that this perturbation is $(p-1)$-linear as $x \to +\infty$. Our goal is to determine the precise dependence of the set of positive solutions on the parameter $\lambda > 0$. In this direction, we prove a bifurcation-type theorem describing the changes in the set of positive solutions of (P_λ) as the parameter λ moves in $\mathring{\mathbb{R}}_+ = (0,+\infty)$. Also, we study the properties of the solution multifunction and produce minimal positive solutions.

Recently Papageorgiou-Rădulescu-Repovš [18] examined problem (P_λ) assuming that the perturbation $f(z,\cdot)$ is $(p-1)$-superlinear but without satisfying the usual in such cases Ambrosetti-Rabinowitz condition. In fact the formulation of the problem in [18] is more general, since the differential operator is nonhomogeneous including as a special case the (p,q)-Laplacian and the singular term is more general having as a special case the function $x \to \lambda x^{-\eta}$. For the sake of simplicity in the presentation, we have decided to proceed with the (p,q)-Laplacian and the standard singularity $u \to \lambda u^{-\eta}$. The work here can also be extended to the more general framework in [18].

Bai-Motreanu-Zeng [3] considered parametric singular equations driven by the p-Laplacian and studied the continuity properties of the solution multifunction. Our results in Section 4 extend their work to (p,q)-equations. Finally we mention the nonsingular works on parametric $(p,2)$-equations of Papageorgiou-Rădulescu [12], Papageorgiou-Rădulescu-Repovš [14, 16, 17], Papageorgiou-Scapellato [19], Papageorgiou-Vetro-Vetro [21], Papageorgiou-Zhang [22].

2. Mathematical Background - Hypotheses

The main spaces in the analysis of problem (P_λ) are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\right\}$. By $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. On account of the Poincaré inequality, we have

$$\|u\| = \|\nabla u\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is ordered with positive (order) cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} < 0\right\}.$$

By $\frac{\partial u}{\partial n}$ we denote the normal derivative of $u(\cdot)$. We know that

$$\frac{\partial u}{\partial n}(z) = (\nabla u(z), n)_{\mathbb{R}^N} \quad \text{for all } z \in \partial\Omega, \text{ all } u \in W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$$

and $n(\cdot)$ is the outward unit normal on $\partial\Omega$.

For every $r \in (1,+\infty)$ by $A_r : W_0^{1,r}(\Omega) \to W_0^{1,r}(\Omega)^* = W^{-1,r'}(\Omega)$ $(\frac{1}{r} + \frac{1}{r'} = 1)$ we denote the nonlinear map defined by

$$\langle A_r(u), h\rangle = \int_\Omega |\nabla u|^{r-2}(\nabla u, \nabla h)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W_0^{1,r}(\Omega).$$

This map has the following well-known properties (see Gasiński-Papageorgiou [5], Problem 2.192, p. 279).

Proposition 1. *The map $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$ (that is, $u_n \xrightarrow{w} u$ in $W_0^{1,r}(\Omega)$ and $\limsup_{n\to+\infty} \langle A_r(u_n), u_n - u\rangle \leq 0$ imply $u_n \to u$ in $W_0^{1,r}(\Omega)$).*

We will need some facts about the spectrum of the Dirichlet r-Laplacian. So, we consider the following nonlinear eigenvalue problem

$$-\Delta_r u(z) = \widehat{\lambda}|u(z)|^{r-2}u(z) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{1}$$

We known (see Gasiński-Papageorgiou [4]) that this problem has a smallest eigenvalue $\widehat{\lambda}_1(r)$, which has the following properties:

- $0 < \widehat{\lambda}_1(r) = \inf \left[\dfrac{\|\nabla u\|_r^r}{\|u\|_r^r} : u \in W_0^{1,r}(\Omega), u \neq 0 \right];$ (2)
- $\widehat{\lambda}_1(r) > 0$ is isolated in the spectrum $\widehat{\sigma}(r)$ of (1), that is, there exists $\varepsilon > 0$ such that $(\widehat{\lambda}_1(r), \widehat{\lambda}_1(r) + \varepsilon) \cap \widehat{\sigma}(r) = \emptyset$;
- $\widehat{\lambda}_1(r) > 0$ is simple, that is, if $\widehat{u}, \widehat{v} \in W_0^{1,r}(\Omega)$ are eigenfunctions corresponding to $\widehat{\lambda}_1(r)$, then $\widehat{u} = \mu\widehat{v}$ for some $\mu \in \mathbb{R} \setminus \{0\}$.

So, by this last property, the eigenspace corresponding to $\widehat{\lambda}_1(r) > 0$ is one-dimensional. The infimum in (2) is realized on this eigenspace. Moreover, it is clear that the elements of this eigenspace have fixed sign. By $\widehat{u}_1(r)$ we denote the positive, L^r-normalized (that is, $\|\widehat{u}_1(r)\|_r = 1$) eigenfunction corresponding to $\widehat{\lambda}_1(r)$. The nonlinear regularity theory (see Lieberman [10]) and the nonlinear moaximum principle (see Pucci-Serrin [23]) imply that $\widehat{u}_1(r) \in \operatorname{int} C_+$. Employing the Ljusternik-Schnirelmann minimax scheme, we can produce a whole sequence of distinct eigenvalues $\{\widehat{\lambda}_k(r)\}_{k\geq 1}$ with $\widehat{\lambda}_k(r) \to +\infty$ as $k \to +\infty$. We do not know if these variational eigenvalues exhaust the spectrum $\widehat{\sigma}(r)$. This is the case if $r = 2$ (linear eigenvalue problem) or if $N = 1$ (scalar eigenvalue problem).

An easy consequence of the above properties of $\widehat{\lambda}_1(r) > 0$ is the following result (see Mugnai-Papageorgiou [11], Lemma 4.11).

Proposition 2. *If $\eta_0 \in L^\infty(\Omega)$, $\eta_0(z) \leq \widehat{\lambda}_1(r)$ for a.a. $z \in \Omega$ and $\eta_0 \not\equiv \widehat{\lambda}_1(r)$, then $\|\nabla u\|_r^r - \int_\Omega \eta_0(z)|u|^r dz \geq c_0\|u\|^r$ for some $c_0 > 0$, all $u \in W_0^{1,r}(\Omega)$.*

We mention that $\widehat{\lambda}_1(r)$ is the only eigenvalue with eigenfunctions of constant sign. All the other eigenvalues have eigenfunctions which are nodal (that is, sign changing).

We will also consider a weighted version of problem (2). So, let $m \in L^\infty(\Omega)$ such that $m(z) \geq 0$ for a.a. $z \in \Omega$, $m \not\equiv 0$ and consider the following nonlinear weighted eigenvalue problem:

$$-\Delta_r u(z) = \widetilde{\lambda} m(z)|u(z)|^{r-2}u(z) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{3}$$

Problem (3) has the same properties as problem (2). In particular there is a smallest eigenvalue $\widetilde{\lambda}_1(m,r) > 0$, which is isolated, simple and admits the following variational characterization

$$\widetilde{\lambda}_1(m,r) = \inf\left[\frac{\|\nabla u\|_r^r}{\int_\Omega m(z)|u|^r dz} : u \in W_0^{1,r}(\Omega), u \neq 0\right]. \tag{4}$$

The infimum in (4) is realized on the corresponding one-dimensional eigenspace. Using (4) one can show the following strict monotonicity property for the map $m \to \widetilde{\lambda}_1(m,r)$.

Proposition 3. *If $m_1, m_2 \in L^\infty(\Omega)$, $0 \leq m_1(z) \leq m_2(z)$ for a.a. $z \in \Omega$, $m_1 \not\equiv 0$, $m_1 \not\equiv m_2$, then $\widetilde{\lambda}_1(m_2,r) < \widetilde{\lambda}_1(m_1,r)$.*

For $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then given $u \in W_0^{1,p}(\Omega)$, we define $u^\pm(z) = u(z)^\pm$ for all $z \in \Omega$. We know that

$$u^\pm \in W_0^{1,p}(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^-.$$

Given $h_1, h_2 : \Omega \to \mathbb{R}$ measurable functions, we write $h_1 \prec h_2$ if for every $K \subseteq \Omega$ compact, we have

$$0 < c_K \leq h_2(z) - h_1(z) \quad \text{for a.a. } z \in K.$$

Evidently if $h_1, h_2 \in C(\Omega)$ and $h_1(z) < h_2(z)$ for all $z \in \Omega$, then $h_1 \prec h_2$.

A set $S \subseteq W_0^{1,p}(\Omega)$ is said to be "downward directed" if for every pair $(u_1, u_2) \in S \times S$, we can find $u \in S$ such that $u \leq u_1$, $u \leq u_2$.

Given $u, v \in W_0^{1,p}(\Omega)$ with $u \leq v$, we define

$$[u, v] = \left\{ h \in W_0^{1,p}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega \right\},$$
$$[u) = \left\{ h \in W_0^{1,p}(\Omega) : u(z) \leq h(z) \text{ for a.a. } z \in \Omega \right\}.$$

Let X be a Banach space and $\varphi \in C^1(X, \mathbb{R})$. We say that $\varphi(\cdot)$ satisfies the "C-condition" if the following property holds:

"Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to +\infty$, admits a strongly convergent subsequence".

We set $K_\varphi = \{u \in X : \varphi'(u) = 0\}$, the critical set of φ.

Our hypotheses on the perturbation $f(z, x)$ are the following:

$\underline{H}$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)$ such that

$$0 \leq f(z, x) \leq a_\rho(z) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \rho;$$

(ii) there exist functions $\eta, \widehat{\eta} \in L^\infty(\Omega)$ such that

$$\widehat{\lambda}_1(p) \leq \eta(z) \leq \widehat{\eta}(z) \quad \text{for a.a. } z \in \Omega,\ \eta \not\equiv \widehat{\lambda}_1(p),$$
$$\eta(z) \leq \liminf_{x\to+\infty} \frac{f(z, x)}{x^{p-1}} \leq \limsup_{x\to+\infty} \frac{f(z, x)}{x^{p-1}} \leq \widehat{\eta}(z) \quad \text{uniformly for a.a. } z \in \Omega;$$

(iii) there exists a function $\eta_0 \in L^\infty(\Omega)$ such that

$$\eta_0(z) \leq \widehat{\lambda}_1(q) \quad \text{for a.a. } z \in \Omega,\ \eta_0 \not\equiv \widehat{\lambda}_1(q),$$
$$\limsup_{x\to 0^+} \frac{f(z, x)}{x^{q-1}} \leq \eta_0(z) \quad \text{uniformly for a.a. } z \in \Omega;$$

(iv) for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$ the function $x \to f(z, x) + \widehat{\xi}_\rho x^{p-1}$ is nondecreasing on $[0, \rho]$.

Remark 1. Since our goal is to find positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality, we may assume that

$$f(z, x) = 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \leq 0. \tag{5}$$

Note that hypotheses $H\,(iii), (iv)$ imply that $f(z, 0) = 0$ for a.a. $z \in \Omega$. Hypothesis $H\,(iii)$ implies that $f(z, \cdot)$ is $(p-1)$-linear near $+\infty$ and also makes the energy functional of the problem noncoercive.

By a solution of problem (P_λ) we mean a function $u \in W_0^{1,p}(\Omega)$ such that $u^{-\eta}h \in L^1(\Omega)$ for all $h \in W_0^{1,p}(\Omega)$ and

$$\langle A_p(u), h\rangle + \langle A_q(u), h\rangle = \lambda \int_\Omega u^{-\eta} h dz + \int_\Omega f(z, u) h dz \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

The difficulty we encounter in dealing with problem (P_λ) is that the energy (Euler) functional $\varphi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ for the problem, defined by

$$\varphi_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \frac{\lambda}{1-\eta}\int_\Omega (u^+)^{1-\eta} dz - \int_\Omega F(z, u^+) dz$$

for all $u \in W_0^{1,p}(\Omega)$, with $F(z, x) = \int_0^x f(z, s) ds$, is not C^1 on account of the singular (third) term. Therefore the minimax techniques of critical point theory are not directly applicable to the functional $\varphi_\lambda(\cdot)$. We need to find ways to bypass the singularity and deal with C^1-functionals.

For this reason, first we consider the following purely singular auxiliary Dirichlet problem

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) = \lambda u(z)^{-\eta} \quad \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \quad u > 0, \ \lambda > 0, \ 1 < q < p, \ 0 < \eta < 1. \end{cases} \tag{Q_λ}$$

Consider the ordered Banach space $C_0(\overline{\Omega}) = \left\{u \in C(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\right\}$ with positive cone $K_+ = \left\{u \in C_0(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\right\}$. This cone has a nonempty interior given by

$$\operatorname{int} K_+ = \left\{u \in K_+ : c_u \widehat{d} \leq u \text{ for some } c_u > 0\right\},$$

where $\widehat{d}(z) = d(z, \partial\Omega)$ for all $z \in \overline{\Omega}$. Lemma 14.16, p. 335, of Gilbarg-Trudinger [6] implies that there exists $\delta > 0$ small such that $\widehat{d} \in C^2(\Omega_\delta)$ with $\Omega_\delta = \{z \in \overline{\Omega} : \widehat{d}(z) < \delta\}$. Therefore $\widehat{d} \in \operatorname{int} C_+$ and then using Proposition 4.1.22, p. 274, of Papageorgiou-Rădulescu-Repovš [15], we see that given $\underline{u} \in \operatorname{int} C_+$ we can find constants $0 < c_1 \leq c_2$ such that

$$\begin{aligned} & c_1 \widehat{d} \leq \underline{u} \leq c_2 \widehat{d}, \\ \Rightarrow \quad & \underline{u} \in \operatorname{int} K_+. \end{aligned} \tag{6}$$

Given $s > N$, we have $\widehat{u}_1(p)^{1/s} \in K_+$ and so on account of (6) we can find $c_3 > 0$ such that

$$\begin{aligned} & 0 \leq \widehat{u}_1(p)^{1/s} \leq c_3 \underline{u}, \\ \Rightarrow \quad & \underline{u}^{-\eta} \leq c_4 \widehat{u}_1(p)^{-\eta/s} \quad \text{for some } c_4 > 0. \end{aligned}$$

The Lemma in Lazer-McKenna [9] implies that

$$\underline{u}^{-\eta} \in L^s(\Omega) \quad (s > N). \tag{7}$$

For problem (Q_λ) we have the following result

Proposition 4. *For every $\lambda > 0$ problem (Q_λ) has a unique positive solution $\underline{u}_\lambda \in \operatorname{int} C_+$ and the map $\lambda \to \underline{u}_\lambda$ from $\mathring{\mathbb{R}}_+ = (0, +\infty)$ into $C_0^1(\overline{\Omega})$ is nondecreasing, that is, $0 < \mu < \lambda$ implies $\underline{u}_\mu \leq \underline{u}_\lambda$.*

Proof. The existence and uniqueness of the solution $\underline{u}_\lambda \in \operatorname{int} C_+$ follows from Proposition 10 of Papageorgiou-Rădulescu-Repovš [18]. We have

$$\underline{u}_\lambda^{-\eta} \in L^s(\Omega) \quad s > N \text{ (see (7))}. \tag{8}$$

Let $0 < \mu < \lambda$ and consider the Carathéodory function $\widehat{w}_\mu(z, x)$ defined by

$$\widehat{w}_\mu(z, x) = \begin{cases} \mu (x^+)^{-\eta} & \text{if } x \leq \underline{u}_\lambda(z), \\ \mu \underline{u}_\lambda(z)^{-\eta} & \text{if } \underline{u}_\lambda(z) < x, \end{cases} \quad \text{(see (8))}. \tag{9}$$

We consider the following Dirichlet problem

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) = \widehat{w}_\mu(z, u(z)) & \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \quad u > 0, \quad 1 < q < p. \end{cases}$$

Again using Proposition 10 of [18], we see that (9) has a solution $\widetilde{u}_\mu \in \operatorname{int} C_+$. We have

$$\langle A_p(\widetilde{u}_\mu), h\rangle + \langle A_q(\widetilde{u}_\mu), h\rangle = \int_\Omega \widehat{w}_\mu(z, \widetilde{u}_\mu) h dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{10}$$

In (10) we choose $h = (\widetilde{u}_\mu - \underline{u}_\lambda)^+ \in W_0^{1,p}(\Omega)$. Then we have

$$\begin{aligned}
&\langle A_p(\widetilde{u}_\mu), (\widetilde{u}_\mu - \underline{u}_\lambda)^+\rangle + \langle A_q(\widetilde{u}_\mu), (\widetilde{u}_\mu - \underline{u}_\lambda)^+\rangle \\
&= \int_\Omega \mu \underline{u}_\lambda^{-\eta} (\widetilde{u}_\mu - \underline{u}_\lambda)^+ dz \quad \text{(see (9))} \\
&\leq \int_\Omega \lambda \underline{u}_\lambda^{-\eta} (\widetilde{u}_\mu - \underline{u}_\lambda)^+ dz \quad (\text{since } \mu < \lambda) \\
&= \langle A_p(\underline{u}_\lambda), (\widetilde{u}_\mu - \underline{u}_\lambda)^+\rangle + \langle A_q(\underline{u}_\lambda), (\widetilde{u}_\mu - \underline{u}_\lambda)^+\rangle, \\
\Rightarrow \quad & \widetilde{u}_\mu \leq \underline{u}_\lambda \quad \text{(see Proposition 1)}, \\
\Rightarrow \quad & \widetilde{u}_\mu = \underline{u}_\mu \in \operatorname{int} C_+ \quad \text{(from the uniqueness of the solution of } (Q_\mu)), \\
\Rightarrow \quad & \underline{u}_\mu \leq \underline{u}_\lambda.
\end{aligned}$$

□

In the next section we will use this solution to isolate the singularity and use the minimax techniques of critical point theory.

3. Bifurcation-Type Theorem

In this section we prove a bifurcation-type theorem which describes in a precise way the changes in the set of positive solutions as the parameter $\lambda > 0$ moves.

We introduce the following two sets

$$\mathcal{L} = \{\lambda > 0 : \text{ problem } (P_\lambda) \text{ has a positive solution}\},$$
$$S_\lambda = \text{ set of positive solutions of } (P_\lambda).$$

Proposition 5. *If hypotheses H hold, then $\mathcal{L} \neq \emptyset$.*

Proof. For $\lambda > 0$, let $\underline{u}_\lambda \in \operatorname{int} C_+$ be the unique positive solution of (Q_λ) (see Proposition 4). We introduce the following truncation of the reaction in problem (P_λ)

$$g_\lambda(z,x) = \begin{cases} \lambda \underline{u}_\lambda(z)^{-\eta} + f(z,x^+) & \text{if } x \le \underline{u}_\lambda(z), \\ \lambda x^{-\eta} + f(z,x) & \text{if } \underline{u}_\lambda(z) < x, \end{cases} \tag{11}$$

(recall that $\underline{u}_\lambda(z)^{-\eta} \in L^s(\Omega)$, $s > N$, see (8)). This is a Carathéodory function. We set $G_\lambda(z,x) = \int_0^x g_\lambda(z,s)ds$ and consider the C^1-functional $\widehat{\varphi}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega G_\lambda(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

(see also Papageorgiou-Smyrlis [20], Proposition 3).

Let $r > p$. On account of hypotheses $H\,(i),(ii),(iii)$, given $\varepsilon > 0$, we can find $c_5 = c_5(\varepsilon, r) > 0$ such that

$$F(z,x) \le \frac{1}{p}[\eta_0(z) + \varepsilon]x^q + c_5 x^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0. \tag{12}$$

Using (11) and (12), we have

$$\begin{aligned} \widehat{\varphi}_\lambda(u) \ge\, & \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\left[\|\nabla u\|_q^q - \int_\Omega \eta_0(z)|u|^q dz - \varepsilon\|u\|_q^q\right] - c_6\|u\|^r \\ & - \lambda \int_{\{u \le \underline{u}_\lambda\}} \underline{u}_\lambda^{-\eta} u dz - \frac{\lambda}{1-\eta}\int_{\{\underline{u}_\lambda < u\}} \left[u^{1-\eta} - \underline{u}_\lambda^{1-\eta}\right] dz \\ & \text{for some } c_6 > 0, \text{ all } u \in W_0^{1,p}(\Omega). \end{aligned} \tag{13}$$

From Proposition 2, we have

$$\|\nabla u\|_q^q - \int_\Omega \eta_0(z)|u|^q dz \ge c_7\|\nabla u\|_q^q \quad \text{for some } c_7 > 0, \text{ all } u \in W_0^{1,p}(\Omega).$$

Therefore

$$\begin{aligned} \|\nabla u\|_q^q - \int_\Omega \eta_0(z)|u|^q dz - \varepsilon\|u\|_q^q &\ge \left[c_7 - \frac{\varepsilon}{\widehat{\lambda}_1(q)}\right]\|\nabla u\|_q^q \quad \text{(see (2))} \\ &\ge 0 \quad \text{choosing } \varepsilon \in (0, \widehat{\lambda}_1(q)c_7]. \end{aligned} \tag{14}$$

Also we have

$$
\begin{aligned}
&\lambda \int_{\{u \leq \underline{u}_\lambda\}} \underline{u}_\lambda^{-\eta} u dz + \frac{\lambda}{1-\eta} \int_{\{\underline{u}_\lambda < u\}} \left[u^{1-\eta} - \underline{u}_\lambda^{1-\eta} \right] dz \\
&\leq \lambda \int_{\{u \leq \underline{u}_\lambda\}} u^{1-\eta} u dz + \frac{\lambda}{1-\eta} \int_{\{\underline{u}_\lambda < u\}} u^{1-\eta} dz \\
&\leq \lambda c_8 \|u\|^{1-\eta} \quad \text{for some } c_8 > 0. \qquad (15)
\end{aligned}
$$

We return to (13) and use (14) and (15). We obtain

$$
\widehat{\varphi}_\lambda(u) \geq \frac{1}{p} \|u\|^p - c_6 \|u\|^r - \lambda c_8 \|u\|^{1-\eta}. \qquad (16)
$$

Choose $\rho \in (0,1)$ small such that

$$
\frac{1}{p} \rho^p - c_6 \rho^r \geq \mu_0 > 0 \quad (\text{recall } p < r). \qquad (17)
$$

We fix such a $\rho \in (0,1)$ and then choose $\lambda_0 > 0$ such that

$$
\lambda c_8 \rho^{1-\eta} < \frac{\mu_0}{2} \quad \text{for all } \lambda \in (0, \lambda_0]. \qquad (18)
$$

So, if in (16), we use (17), (18), then

$$
\widehat{\varphi}_\lambda(u) \geq \frac{\mu_0}{2} > 0 \quad \text{for all } u \in W_0^{1,p}(\Omega),\ \|u\| = \rho. \qquad (19)
$$

Let $B_\rho = \{u \in W_0^{1,p}(\Omega) : \|u\| < \rho\}$. Since $W_0^{1,p}(\Omega)$ is reflexive, by the Eberlein-Smulian theorem the set $\overline{B}_\rho$ is sequentially weakly compact. Moreover, using the Sobolev embedding theorem, we see that $\widehat{\varphi}_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. Therefore, by the Weierstrass-Tonelli theorem, there exists $u_\lambda \in \overline{B}_\rho$ such that

$$
\widehat{\varphi}_\lambda(u_\lambda) = \min[\widehat{\varphi}_\lambda(u) : u \in \overline{B}_\rho]. \qquad (20)
$$

Let $u \in \operatorname{int} C_+$ and choose $t \in (0,1)$ small such that

$$
tu \leq \underline{u}_\lambda \quad (\text{recall } \underline{u}_\lambda \in \operatorname{int} C_+). \qquad (21)
$$

Then from (21) and (11), we have

$$
\begin{aligned}
\widehat{\varphi}_\lambda(tu) &= \frac{t^p}{p} \|\nabla u\|_p^p + \frac{t^q}{q} \|\nabla u\|_q^q - \int_\Omega \left[\lambda \underline{u}_\lambda^{-\eta}(tu) + F(z, tu) \right] dz \\
&\leq t^q \left[\|\nabla u\|_p^p + \|\nabla u\|_q^q \right] - \lambda t \int_\Omega \underline{u}_\lambda^{-\eta} u dz \\
&\qquad\qquad (\text{since } t \in (0,1),\ 1 < q < p \text{ and } F \geq 0).
\end{aligned}
$$

Since $q > 1$, choosing $t \in (0,1)$ even smaller if necessary, we have

$$
\begin{aligned}
& \widehat{\varphi}_\lambda(tu) < 0 \text{ and } \|tu\| \leq \rho, \\
\Rightarrow \quad & \widehat{\varphi}_\lambda(u_\lambda) < 0 = \widehat{\varphi}_\lambda(0) \quad \text{(see (20))}, \\
\Rightarrow \quad & u_\lambda \neq 0.
\end{aligned}
$$

On account of (19) we have

$$
\|u_\lambda\| < \rho, \quad u_\lambda \neq 0. \tag{22}
$$

From (22) and (20), it follows that

$$
\begin{aligned}
& \widehat{\varphi}'_\lambda(u_\lambda) = 0, \\
\Rightarrow \quad & \langle A_p(u_\lambda), h\rangle + \langle A_q(u_\lambda), h\rangle = \int_\Omega g_\lambda(z, u_\lambda) h dz \quad \text{for all } h \in W_0^{1,p}(\Omega).
\end{aligned} \tag{23}
$$

In (23) first we choose $h = -u_\lambda^- \in W_0^{1,p}(\Omega)$. We obtain

$$
\begin{aligned}
& \|u_\lambda^-\|^p \leq 0, \\
\Rightarrow \quad & u_\lambda \geq 0, \quad u_\lambda \neq 0.
\end{aligned} \tag{24}
$$

Next in (23) we choose $h = (\underline{u}_\lambda - u_\lambda)^+ \in W_0^{1,p}(\Omega)$. We obtain

$$
\begin{aligned}
& \langle A_p(u_\lambda), (\underline{u}_\lambda - u_\lambda)^+\rangle + \langle A_q(u_\lambda), (\underline{u}_\lambda - u_\lambda)^+\rangle \\
& = \int_\Omega \left[\lambda \underline{u}_\lambda^{-\eta} + f(z, u_\lambda)\right] (\underline{u}_\lambda - u_\lambda)^+ dz \quad \text{(see (11) and (24))} \\
& \geq \int_\Omega \lambda \underline{u}_\lambda^{-\eta} (\underline{u}_\lambda - u_\lambda)^+ dz \quad (\text{since } f \geq 0) \\
& = \langle A_p(\underline{u}_\lambda), (\underline{u}_\lambda - u_\lambda)^+\rangle + \langle A_q(\underline{u}_\lambda), (\underline{u}_\lambda - u_\lambda)^+\rangle \\
\Rightarrow \quad & \underline{u}_\lambda \leq u_\lambda \quad \text{(see Proposition 1)}.
\end{aligned}
$$

Then from (11) and (23) it follows that $u_\lambda \in S_\lambda$ and so $(0, \lambda_0] \subseteq \mathcal{L} \neq \emptyset$. $\square$

Proposition 6. *If hypotheses H hold and $\lambda \in \mathcal{L}$, then $\underline{u}_\lambda \leq u$ for all $u \in S_\lambda$.*

Proof. Let $u \in S_\lambda \subseteq W_0^{1,p}(\Omega)$. We introduce the function $e_\lambda : \Omega \times \mathring{\mathbb{R}}_+ \to \mathbb{R}$ defined by

$$
e_\lambda(z, x) = \begin{cases} \lambda x^{-\eta} & \text{if } 0 < x \leq u(z), \\ \lambda u(z)^{-\eta} & \text{if } u(z) < x. \end{cases} \tag{25}
$$

This is a Carathéodory function on $\Omega \times \mathring{\mathbb{R}}_+$. We consider the following Dirichlet (p,q)-equation

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) = e_\lambda(z, u(z)) \quad \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \quad u > 0, \ \lambda > 0, \ 1 < q < p. \end{cases} \tag{E_λ}$$

Invoking Proposition 10 of Papageorgiou-Rădulescu-Repovš [18], we see that problem (E_λ) admits a solution $\widetilde{u}_\lambda \in W_0^{1,p}(\Omega)$, $\widetilde{u}_\lambda \geq 0$, $\widetilde{u}_\lambda \neq 0$. We have

$$\begin{aligned} & \langle A_p(\widetilde{u}_\lambda), (\widetilde{u}_\lambda - u)^+\rangle + \langle A_q(\widetilde{u}_\lambda), (\widetilde{u}_\lambda - u)^+\rangle \\ & = \int_\Omega \lambda u^{-\eta}(\widetilde{u}_\lambda - u)^+ dz \quad \text{(see (25))} \\ & \leq \int_\Omega \left[\lambda u^{-\eta} + f(z,u)\right](\widetilde{u}_\lambda - u)^+ dz \quad (\text{since } f \geq 0) \\ & = \langle A_p(u), (\widetilde{u}_\lambda - u)^+\rangle + \langle A_q(u), (\widetilde{u}_\lambda - u)^+\rangle \quad (\text{since } u \in S_\lambda), \\ \Rightarrow \quad & \widetilde{u}_\lambda \leq u. \end{aligned}$$

So, we have proved that

$$\widetilde{u}_\lambda \in [0, u], \quad \widetilde{u}_\lambda \neq 0. \tag{26}$$

From (26), (25) and Proposition 4 it follows that

$$\begin{aligned} & \widetilde{u}_\lambda = \underline{u}_\lambda \in \operatorname{int} C_+, \\ \Rightarrow \quad & \underline{u}_\lambda \leq u \quad \text{for all } u \in S_\lambda. \end{aligned}$$

□

Proposition 7. *If hypotheses H hold, then $S_\lambda \subseteq \operatorname{int} C_+$.*

Proof. Let $u \in S_\lambda$. From Proposition 6, we know that $\underline{u}_\lambda \leq u$, hence $u^{-\eta} \in L^s(\Omega)$, $s > N$ (see (8)). Then we have

$$-\Delta_p u(z) - \Delta_q u(z) = \lambda u(z)^{-\eta} + f(z, u(z)) \quad \text{for a.a. } z \in \Omega. \tag{27}$$

Consider the linear Dirichlet problem

$$-\Delta v(z) = \lambda u(z)^{-\eta} \quad \text{in } \Omega, \quad v\big|_{\partial\Omega} = 0. \tag{28}$$

Using Theorem 9.15, p. 241, of Gilbarg-Trudinger [6], we have that problem (28) admits a unique solution $v_\lambda \in W^{2,s}(\Omega)$. Since $s > N$, by the Sobolev embedding theorem we have that $v_\lambda \in C^{1,\alpha}(\overline{\Omega})$ with $\alpha = \frac{N}{s} \in (0,1)$. Let $w_\lambda = \nabla v_\lambda$. Then $w_\lambda \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N)$. Using $w_\lambda(\cdot)$, we rewrite (27) as follows

$$-\mathrm{div}\left(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u - w_\lambda\right) = 0 \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{29}$$

From (29) and Theorem 7.1, p. 286, of Ladyzhenskaya-Ural'tseva [8] we have that $u \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [10] implies that $u \in C_+ \setminus \{0\}$. Moreover, from (27) we see that

$$\begin{aligned} & \Delta_p u(z) + \Delta_q u(z) \leq 0 \quad \text{for a.a. } z \in \Omega, \\ \Rightarrow \quad & u \in \mathrm{int}\, C_+ \quad \text{(see Pucci-Serrin [23], pp. 111, 120).} \end{aligned}$$

□

Proposition 8. *If hypotheses H hold and $\lambda \in \mathcal{L}$, then $S_\lambda \subseteq C_0^1(\overline{\Omega})$ is compact.*

Proof. First we show that $S_\lambda \subseteq W_0^{1,p}(\Omega)$ is bounded.

We argue indirectly. So, suppose we can find $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$\|u_n\| \to +\infty \quad \text{as } n \to +\infty. \tag{30}$$

Let $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ as } n \to +\infty,\ y \geq 0. \tag{31}$$

For every $n \in \mathbb{N}$, we have

$$\begin{aligned} & \langle A_p(u_n), h\rangle + \langle A_q(u_n), h\rangle = \int_\Omega \left[\lambda u_n^{-\eta} + f(z, u_n)\right] h dz \ \text{ for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow\ & \langle A_p(y_n), h\rangle + \frac{1}{\|u_n\|^{p-q}}\langle A_q(y_n), h\rangle = \int_\Omega \left[\lambda \frac{u_n^{-\eta}}{\|u_n\|^{p-1}} + \frac{f(z,u_n)}{\|u_n\|^{p-1}}\right] h dz \quad (32) \\ & \text{for all } h \in W_0^{1,p}(\Omega). \end{aligned}$$

On account of hypotheses $H\,(i), (ii)$, we have

$$
\begin{aligned}
& 0 \le f(z,x) \le c_9[1+x^{p-1}] \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0, \text{ some } c_9 > 0, \\
\Rightarrow \quad & \left\{ \frac{f(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}} \right\} \subseteq L^{p'}(\Omega) \text{ is bounded.} \qquad (33)
\end{aligned}
$$

From (33) and hypothesis $H\,(ii)$, we infer that at least for a subsequence we have

$$
\frac{f(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}} \xrightarrow{w} \widetilde{\eta}(\cdot) y(\cdot)^{p-1} \quad \text{in } L^{p'}(\Omega) \text{ as } n \to +\infty, \qquad (34)
$$

with $\eta(z) \le \widetilde{\eta}(z) \le \widehat{\eta}(z)$ for a.a. $z \in \Omega$ (see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 16). In (32), we choose $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (30) (recall that $q < p$), (31) and (34). We obtain

$$
\begin{aligned}
& \lim_{n \to +\infty} \langle A_p(y_n), y_n - y \rangle = 0, \\
\Rightarrow \quad & y_n \to y \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 1), hence } \|y\| = 1,\ y \ge 0. \qquad (35)
\end{aligned}
$$

So, if in (32) we pass to the limit as $n \to +\infty$ and use (35), (30) and (34), then we have

$$
\begin{aligned}
& \langle A_p(y), h \rangle = \int_\Omega \widetilde{\eta}(z) y^{p-1} h dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \\
\Rightarrow \quad & -\Delta_p y(z) = \widetilde{\eta}(z) y(z)^{p-1} \quad \text{in } \Omega, \quad y\big|_{\partial\Omega} = 0.
\end{aligned}
$$

On account of Proposition 3, we have

$$
\begin{aligned}
& \widetilde{\lambda}_1(\widetilde{\eta}, p) < \widetilde{\lambda}_1(\widehat{\lambda}_1, p) = 1 \quad \text{(see (34) and hypothesis } H\,(iii)), \\
\Rightarrow \quad & y \ne 0, \quad \text{(see (35)) must be nodal.}
\end{aligned}
$$

This contradicts (35).

Therefore $S_\lambda \subseteq W_0^{1,p}(\Omega)$ is bounded. As before (see the proof of Proposition 7), using the nonlinear regularity theory (see [8] and [10]), we see that we can find $\beta \in (0,1)$ and $c_{10} > 0$ such that

$$
u \in C^{1,\beta}(\overline{\Omega}) \cap C_0^1(\overline{\Omega}) \text{ and } \|u\|_{C^{1,\beta}(\overline{\Omega})} \le c_{10}.
$$

The compact embedding of $C^{1,\beta}(\overline{\Omega}) \cap C_0^1(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ implies that

$$
S_\lambda \subseteq C_0^1(\overline{\Omega}) \text{ is relatively compact.}
$$

But clearly the set $S_\lambda \subseteq C_0^1(\overline{\Omega})$ is closed. Therefore we conclude that $S_\lambda \subseteq C_0^1(\overline{\Omega})$ is compact. $\square$

Next we show that $\mathcal{L}$ is connected (that is, $\mathcal{L}$ is an interval).

Proposition 9. *If hypotheses H hold, $\lambda \in \mathcal{L}$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}$.*

Proof. Since $\lambda \in \mathcal{L}$ we can find $u \in S_\lambda \subseteq \operatorname{int} C_+$ (see Proposition 7). We have $\underline{u}_\mu \leq \underline{u}_\lambda \leq u$ (see Propositions 4 and 6). Hence we can define the following truncation of the reaction in problem (P_μ)

$$k_\mu(z,x) = \begin{cases} \mu \underline{u}_\mu(z)^{-\eta} + f(z, \underline{u}_\mu(z)) & \text{if } x < \underline{u}_\mu(z), \\ \mu x^{-\eta} + f(z,x) & \text{if } \underline{u}_\mu(z) \leq x \leq u(z), \\ \mu u(z)^{-\eta} + f(z, u(z)) & \text{if } u(z) < x. \end{cases} \tag{36}$$

This is a Carathéodory function. We set $K_\mu(z,x) = \int_0^x k_\mu(z,s)ds$ and consider the functional $\sigma_\mu : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_\mu(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega K_\mu(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We have $\sigma_\mu \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ (see also Papageorgiou-Smyrlis [20], Proposition 3). From (36) it is clear that $\sigma_\mu(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. By the Weierstrass-Tonelli theorem, we can find $u_\mu \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} & \sigma_\mu(u_\mu) = \min\left[\sigma_\mu(u) : u \in W_0^{1,p}(\Omega)\right], \\ \Rightarrow \quad & \sigma_\mu'(u_\mu) = 0, \\ \Rightarrow \quad & \langle A_p(u_\mu), h\rangle + \langle A_q(u_\mu), h\rangle = \int_\Omega k_\mu(z,u_\mu)hdz \quad \text{for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{37}$$

In (37) first we choose $h = (\underline{u}_\mu - u_\mu)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned}
&\langle A_p(u_\mu), (\underline{u}_\mu - u_\mu)^+\rangle + \langle A_q(u_\mu), (\underline{u}_\mu - u_\mu)^+\rangle \\
&= \int_\Omega \left[\mu \underline{u}_\mu^{-\eta} + f(z, \underline{u}_\mu)\right] (\underline{u}_\mu - u_\mu)^+ dz \quad \text{(see (36))} \\
&\geq \int_\Omega \mu \underline{u}_\mu^{-\eta} (\underline{u}_\mu - u_\mu)^+ dz \quad (\text{since } f \geq 0) \\
&= \langle A_p(\underline{u}_\mu), (\underline{u}_\mu - u_\mu)^+\rangle + \langle A_q(\underline{u}_\mu), (\underline{u}_\mu - u_\mu)^+\rangle \quad \text{(see Proposition 4)},
\end{aligned}$$

$$\Rightarrow \quad \underline{u}_\mu \leq u_\mu \quad \text{(see Proposition 1)}.$$

Next in (37) we choose $h = (u_\mu - u)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned}
&\langle A_p(u_\mu), (u_\mu - u)^+\rangle + \langle A_q(u_\mu), (u_\mu - u)^+\rangle \\
&= \int_\Omega \left[\mu u^{-\eta} + f(z, u)\right] (u_\mu - u)^+ dz \quad \text{(see (36))} \\
&\leq \int_\Omega \left[\lambda u^{-\eta} + f(z, u)\right] (u_\mu - u)^+ dz \quad (\text{since } \mu < \lambda) \\
&= \langle A_p(u), (u_\mu - u)^+\rangle + \langle A_q(u), (u_\mu - u)^+\rangle \quad (\text{since } u \in S_\lambda),
\end{aligned}$$

$$\Rightarrow \quad u_\mu \leq u \quad \text{(see Proposition 1)}.$$

So, we have proved that

$$u_\mu \in [\underline{u}_\mu, u]. \tag{38}$$

From (38), (36) and (37) it follows that $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ and so $\mu \in \mathcal{L}$. $\square$

A byproduct of the above proof is the following corollary.

Corollary 1. *If hypotheses H hold, $\lambda \in \mathcal{L}$, $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}$ and we can find $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that $u_\mu \leq u_\lambda$.*

In fact we can improve this corollary as follows.

Proposition 10. *If hypotheses H hold, $\lambda \in \mathcal{L}$, $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}$ and we can find $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that $u_\lambda - u_\mu \in \operatorname{int} C_+$.*

Proof. From Corollary 1, we know that $\mu \in \mathcal{L}$ and that there exists $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that

$$u_\mu \leq u_\lambda. \tag{39}$$

Let $\rho = \|u_\lambda\|_\infty$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis $H\,(iv)$. We have

$$\begin{aligned}
&-\Delta_p u_\mu(z) - \Delta_q u_\mu(z) + \widehat{\xi}_\rho u_\mu(z)^{p-1} - \lambda u_\mu(z)^{-\eta} \\
&= [\mu - \lambda] u_\mu(z)^{-\eta} + f(z, u_\mu(z)) + \widehat{\xi}_\rho u_\mu(z)^{p-1} \\
&\leq f(z, u_\lambda(z)) + \widehat{\xi}_\rho u_\lambda(z)^{p-1} \quad \text{(see hypothesis } H\,(iv) \text{ and recall } \mu < \lambda) \\
&= -\Delta_p u_\lambda(z) - \Delta_q u_\lambda(z) + \widehat{\xi}_\rho u_\lambda(z)^{p-1} - \lambda u_\lambda(z)^{-\eta} \quad \text{for a.a. } z \in \Omega. \qquad (40)
\end{aligned}$$

Note that $0 \prec [\lambda - \mu] u_\mu^{-\eta}$. So (40) and Proposition 7 of Papageorgiou-Rădulescu-Repovš [18] imply that $u_\lambda - u_\mu \in \operatorname{int} C_+$. □

Let $\lambda^* = \sup \mathcal{L}$.

Proposition 11. *If hypotheses H hold, then $\lambda^* < +\infty$.*

Proof. On account of hypotheses $H\,(iii)$, we see that we can find $\widehat{\lambda} > 0$ such that

$$\widehat{\lambda} x^{-\eta} + f(z, x) \geq \frac{\widehat{\lambda}_1(p)}{2} x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x > 0. \qquad (41)$$

Let $\lambda > \widehat{\lambda}$ and suppose that $\lambda \in \mathcal{L}$. Then we can find $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ (see Proposition 7). Let $\Omega_0 \subset\subset \Omega$ with a C^2-boundary $\partial\Omega_0$ and let $m_0 = \min_{\overline{\Omega}_0} u_\lambda > 0$ (since $u_\lambda \in \operatorname{int} C_+$). For $\delta \in (0,1)$ we set $m_0^\delta = m_0 + \delta$, $\rho = \max\{\|u_\lambda\|_\infty, m_0^1\}$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis $H\,(iv)$. For $\delta \in (0, \min\{1, m_0\}]$, we have

$$\begin{aligned}
&-\Delta_p m_0^\delta - \Delta_q m_0^\delta + \widehat{\xi}_\rho (m_0^\delta)^{p-1} - \lambda (m_0^\delta)^{-\eta} \\
&\leq \left[\widehat{\xi}_\rho + \frac{\widehat{\lambda}_1(p)}{2}\right] m_0^{p-1} + \chi(\delta) - \frac{\lambda}{2} m_0^{-\eta} \quad \text{with } \chi(\delta) \to 0^+ \text{ as } \delta \to 0^+ \\
&\leq f(z, m_0) + \widehat{\xi}_\rho m_0^{p-1} + \chi(\delta) - \frac{\lambda}{2} m_0^{-\eta} \quad \text{(see (41) and recall } \widehat{\lambda} < \lambda) \\
&\leq f(z, u_\lambda(z)) + \widehat{\xi}_\rho u_\lambda(z)^{p-1} \quad \text{for } \delta > 0 \text{ small} \\
&= -\Delta_p u_\lambda(z) - \Delta_q u_\lambda(z) + \widehat{\xi}_\rho u_\lambda(z)^{p-1} - \lambda u_\lambda(z)^{-\eta} \quad \text{for a.a. } z \in \Omega_0. \qquad (42)
\end{aligned}$$

Note that for $\delta > 0$ small, we have

$$\widehat{\lambda} m_0^{-\eta} - \chi(\delta) \geq \widehat{\mu}_0 > 0.$$

Then from (42) and Proposition 6 of Papageorgiou-Rădulescu-Repovš [18], we have

$$m_0^\delta \le u_\lambda(z) \quad \text{for all } z \in \overline{\Omega}_0,\ \delta > 0 \text{ small},$$

a contradiction to the definition of m_0. Therefore $\lambda^* \le \widehat{\lambda} < +\infty$. $\square$

Proposition 12. *If hypotheses H hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) has at least two positive solutions $u_0, \widehat{u} \in \operatorname{int} C_+$, $u_0 \le \widehat{u}$, $u_0 \ne \widehat{u}$.*

Proof. Let $0 < \lambda < \vartheta < \lambda^*$. We have $\lambda, \vartheta \in \mathcal{L}$ and using Proposition 10 we can find $u_0 \in S_\lambda \subseteq \operatorname{int} C_+$ and $u_\vartheta \in S_\vartheta \subseteq \operatorname{int} C_+$ such that

$$u_\vartheta - u_0 \in \operatorname{int} C_+. \tag{43}$$

Also from Proposition 6 we know that $\underline{u}_\lambda \le u_0$ and so $u_0^{-\eta} \in L^s(\Omega)$, $s > N$ (see (8)). We introduce the following Carathéodory functions:

$$\beta_\lambda(z,x) = \begin{cases} \lambda u_0(z)^{-\eta} + f(z, u_0(z)) & \text{if } x \le u_0(z), \\ \lambda x^{-\eta} + f(z,x) & \text{if } u_0(z) < x\ , \end{cases} \tag{44}$$

$$\widehat{\beta}_\lambda(z,x) = \begin{cases} \beta_\lambda(z,x) & \text{if } x \le u_\vartheta(z), \\ \beta_\lambda(z, u_\vartheta(z)) & \text{if } u_\vartheta(z) < x\ . \end{cases} \tag{45}$$

We set $B_\lambda(z,x) = \int_0^x \beta_\lambda(z,s)ds$ and $\widehat{B}_\lambda(z,x) = \int_0^x \widehat{\beta}_\lambda(z,s)ds$ and consider the C^1-functionals $\gamma_\lambda, \widehat{\gamma}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\gamma_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega B_\lambda(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega),$$
$$\widehat{\gamma}_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega \widehat{B}_\lambda(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Using (44), (45) and the nonlinear regularity theory, we show that

$$K_{\gamma_\lambda} \subseteq [u_0) \cap \operatorname{int} C_+, \tag{46}$$

$$K_{\widehat{\gamma}_\lambda} \subseteq [u_0, u_\vartheta] \cap \operatorname{int} C_+. \tag{47}$$

From (44) and (46), we see that we may assume that

$$K_{\gamma_\lambda} \text{ is finite and } K_{\gamma_\lambda} \cap [u_0, u_\vartheta] = \{u_0\}. \tag{48}$$

Otherwise, we already have a second positive smooth solution, bigger than u_0 and so we are done. Also, it is clear from (44) and (45) that

$$\gamma_\lambda\Big|_{[0,u_\vartheta]} = \widehat{\gamma}_\lambda\Big|_{[0,u_\vartheta]} \quad \text{and} \quad \gamma_\lambda'\Big|_{[0,u_\vartheta]} = \widehat{\gamma}_\lambda'\Big|_{[0,u_\vartheta]}. \tag{49}$$

From (48) and (49) it follows that

$$K_{\widehat{\gamma}_\lambda} = \{u_0\}. \tag{50}$$

Evidently $\widehat{\gamma}_\lambda(\cdot)$ is coercive (see (45)) and sequentially weakly lower semicontinuous. So, we can find $\widetilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} & \widehat{\gamma}_\lambda(\widetilde{u}) = \min\left[\widehat{\gamma}_\lambda(u) : u \in W_0^{1,p}(\Omega)\right], \qquad (51)\\ \Rightarrow\quad & \widetilde{u} \in K_{\widehat{\gamma}_\lambda},\\ \Rightarrow\quad & \widetilde{u} = u_0 \in \operatorname{int} C_+ \quad \text{(see (50))}. \end{aligned}$$

From (51), (43) and (49) we infer that

$$\begin{aligned} & u_0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \gamma_\lambda(\cdot),\\ \Rightarrow\quad & u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \gamma_\lambda(\cdot) \qquad (52)\\ & \text{(see Papageorgiou-Rădulescu [13], Proposition 2.12).} \end{aligned}$$

Using (52), (48) and Theorem 5.7.6, p. 449, of Papageorgiou-Rădulescu-Repovš [15], we can find $\rho \in (0,1)$ small such that

$$\gamma_\lambda(u_0) < \inf\left[\gamma_\lambda(u) : \|u - u_0\| = \rho\right] = m_\lambda. \tag{53}$$

On account of hypothesis $H\,(ii)$, we have

$$\gamma_\lambda(t\widehat{u}_1(p)) \to -\infty \quad \text{as } t \to +\infty. \tag{54}$$

<u>Claim</u>: The functional $\gamma_\lambda(\cdot)$ satisfies the C-condition.

Consider a sequence $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ such that $\{\gamma_\lambda(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|u_n\|)\gamma_\lambda'(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \text{ as } n \to +\infty. \tag{55}$$

From (55) we have

$$\left|\langle A_p(u_n),h\rangle+\langle A_q(u_n),h\rangle-\int_\Omega \beta_\lambda(z,u_n)hdz\right|\leq\frac{\varepsilon_n\|h\|}{1+\|u_n\|} \tag{56}$$

$$\text{for all } h\in W_0^{1,p}(\Omega), \text{ with } \varepsilon_n\to 0^+.$$

In (56) first we choose $h=-u_n^-\in W_0^{1,p}(\Omega)$. Using (44) we see that

$$\begin{aligned}&\|u_n^-\|^p\leq c_{11}\quad\text{for some } c_{11}>0, \text{ all } n\in\mathbb{N},\\ \Rightarrow\quad &\{u_n^-\}_{n\geq 1}\subseteq W_0^{1,p}(\Omega)\text{ is bounded.}\end{aligned} \tag{57}$$

Next we show that $\{u_n^+\}_{n\geq 1}\subseteq W_0^{1,p}(\Omega)$ is bounded too. Arguing by contradiction, suppose that

$$\|u_n^+\|\to+\infty\quad\text{as } n\to+\infty. \tag{58}$$

We set $y_n=\frac{u_n^+}{\|u_n^+\|}$, $n\in\mathbb{N}$. Then $\|y_n\|=1$, $y_n\geq 0$ for all $n\in\mathbb{N}$. So, we may assume that

$$y_n\xrightarrow{w} y\text{ in } W_0^{1,p}(\Omega)\quad\text{and}\quad y_n\to y\text{ in } L^p(\Omega). \tag{59}$$

From (44) we see that

$$f(z,x)-c_{12}\leq\beta_\lambda(z,x)\leq\lambda u_0(z)^{-\eta}+f(z,x)+c_{12} \tag{60}$$

$$\text{for a.a. } z\in\Omega, \text{ all } x\geq 0, \text{ some } c_{12}>0,$$

$$\Rightarrow\quad\left\{\frac{\beta_\lambda(\cdot,u_n^+(\cdot))}{\|u_n^+\|^{p-1}}\right\}_{n\geq 1}\subseteq L^{p'}(\Omega)\text{ is bounded (see (8) and (58)).}$$

Then (60) and hypothesis $H\,(ii)$ imply that

$$\frac{\beta_\lambda(\cdot,u_n^+(\cdot))}{\|u_n^+\|^{p-1}}\xrightarrow{w}\widetilde{\eta}(\cdot)y(\cdot)^{p-1}\text{ in } L^{p'} \tag{61}$$

with $\eta(z)\leq\widetilde{\eta}(z)\leq\widehat{\eta}(z)$ for a.a. $z\in\Omega$ (see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 16).

From (56) and (57), we have

$$\left|\langle A_p(y_n),h\rangle+\frac{1}{\|u_n^+\|^{p-q}}\langle A_q(y_n),h\rangle-\int_\Omega\frac{\beta_\lambda(z,u_n^+)}{\|u_n^+\|^{p-1}}hdz\right|\leq\varepsilon_n' \tag{62}$$

$$\text{for all } h\in W_0^{1,p}(\Omega), \text{ with } \varepsilon_n'\to 0^+ \text{ as } n\to+\infty.$$

In (62) we choose $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (58), (59) and (61). We obtain

$$\begin{aligned} & \lim_{n\to+\infty} \langle A_p(y_n), y_n - y\rangle = 0, \\ \Rightarrow \quad & y_n \to y \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 1), hence } \|y\| = 1,\ y \geq 0. \end{aligned} \tag{63}$$

In (62) we pass to the limit as $n \to +\infty$ and use (58), (61) and (63). We obtain

$$\begin{aligned} & \langle A_p(y), h\rangle = \int_\Omega \widetilde{\eta}(z) y^{p-1} h dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow \quad & -\Delta_p y(z) = \widetilde{\eta}(z) y(z)^{p-1} \quad \text{in } \Omega, \quad y\big|_{\partial\Omega} = 0. \end{aligned} \tag{64}$$

On account of Proposition 3, we have

$$\widetilde{\lambda}_1(\widetilde{\eta}, p) < \widetilde{\lambda}_1(\widehat{\lambda}_1(p), p) = 1. \tag{65}$$

From (64), (65) and (63), we infer that y must be nodal, a contradiction. So, $\{u_n^+\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded, hence $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded (see (57)).

We may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^p(\Omega). \tag{66}$$

In (56) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (66) and (61). Then

$$\begin{aligned} & \lim_{n\to+\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A_q(u_n), u_n - u\rangle] = 0, \\ \Rightarrow \quad & \limsup_{n\to+\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A_q(u), u_n - u\rangle] \leq 0 \text{ (since } A_q(\cdot) \text{ is monotone)}, \\ \Rightarrow \quad & \limsup_{n\to+\infty} \langle A_p(u_n), u_n - u\rangle \leq 0, \\ \Rightarrow \quad & u_n \to u \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 1)}. \end{aligned}$$

Therefore $\gamma_\lambda(\cdot)$ satisfies the C-condition and this proves the Claim.

From (53), (54) and the Claim, we see that we can apply the Mountain Pass Theorem. Therefore we can find $\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$\widehat{u} \in K_{\gamma_\lambda} \subseteq [u_0) \cap \operatorname{int} C_+ \text{ (see (46)) and } m_\lambda \leq \gamma_\lambda(\widehat{u}). \tag{67}$$

From (67), (44) and (53) it follows that

$$\widehat{u} \in S_\lambda \subseteq \operatorname{int} C_+, \quad u_0 \leq \widehat{u}, \quad \widehat{u} \neq u_0.$$

□

Proposition 13. *If hypotheses H hold, then $\lambda^* \in \mathcal{L}$.*

Proof. Let $\lambda_n \in (0, \lambda^*)$, $n \in \mathbb{N}$, such that $\lambda_n \uparrow \lambda^*$ as $n \to +\infty$. Using the contradiction argument in the Claim in the proof of Proposition 12, we can find $u_n \in S_{\lambda_n} \subseteq \operatorname{int} C_+$, $n \geq 1$ with $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ bounded. Subsequently using Proposition 1, as in the proof of Proposition 12, we show that at least for a subsequence we have

$$u_n \to u_* \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to +\infty. \tag{68}$$

From Propositions 4 and 6, we have

$$\begin{aligned} & \underline{u}_{\lambda_1} \leq u_n \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow \quad & \underline{u}_{\lambda_1} \leq u_*. \end{aligned} \tag{69}$$

From (68) and (69), we conclude that

$$u_* \in S_{\lambda_*} \subseteq \operatorname{int} C_+ \text{ and so } \lambda^* \in \mathcal{L}.$$

□

We have proved that

$$\mathcal{L} = (0, \lambda^*].$$

Summarizing our findings in this section, we can state the following bifurcation-type theorem.

Theorem 1. *If hypotheses H hold, then there exists $\lambda^* > 0$ such that*

(a) *for every $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least two positive solutions u_0, $\widehat{u} \in \operatorname{int} C_+$, $u_0 \leq \widehat{u}$, $u_0 \neq \widehat{u}$;*

(b) *for $\lambda = \lambda^*$ problem (P_λ) has at least one positive solution $u_* \in \operatorname{int} C_+$;*

(c) *for every $\lambda > \lambda^*$ problem (P_λ) has no positive solutions.*

4. The Solution Multifunction

In this section we examine the continuity properties of the solution multifunction $\lambda \to S_\lambda$ from $\mathcal{L} = (0, \lambda^*]$ into $2^{C_0^1(\overline{\Omega})} \setminus \{\emptyset\}$.

First we need to recall some continuity notions for multifunctions. For details we refer to Hu-Papageorgiou [7].

Let X, Y be Hausdorff topological spaces and $S : X \to 2^Y \setminus \{\emptyset\}$ a multifunction. We introduce the following continuity concepts for $S(\cdot)$.

Definition 1. We say that $S(\cdot)$ is

(a) "upper semicontinuous" ("usc" for short) if for every $C \subseteq Y$ closed, $S^-(C) = \{x \in X : S(x) \cap C \neq \emptyset\}$ is closed in X;

(b) "lower semicontinuous" ("lsc" for short) if for every $C \subseteq Y$ closed, $S^+(C) = \{x \in X : S(x) \subseteq C\}$ is closed in X;

(c) "continuous" (or "Vietoris continuous") if it is both usc and lsc.

Remark 2. It is clear from Definition 1, that when $S(\cdot)$ is single-valued, then the notions of upper semicontinuity and lower semicontinuity coincide with the usual notion of continuity of a map between Hausdorff topological spaces. In general the two notions of upper and lower semicontinuity are distinct. Upper semicontinuity allows upward jumps (in the sense of inclusion), while lower semicontinuity allows downward jumps (in the sense of inclusion), see Hu-Papageorgiou [7], p. 38.

Next let (Y, d) be a metric space and $A, C \subseteq Y$. We set

$$h^*(A, C) = \sup[d(a, C) : a \in A] = \inf[\varepsilon > 0 : A \subseteq C_\varepsilon],$$

where $C_\varepsilon = \{y \in Y : d(y, C) < \varepsilon\}$ (the open ε-enlargment of C). The Hausdorff distance between A and C is defined by

$$h(A, C) = \max\{h^*(A, C), h^*(C, A)\} = \inf[\varepsilon > 0 : A \subseteq C_\varepsilon,\ C \subseteq A_\varepsilon].$$

Let $P_k(Y) = \{A \subseteq Y : A \neq \emptyset,\ A \text{ is compact}\}$. Then $(P_k(Y), h)$ is a complete metric space and it is separable (resp. Polish), if Y is separable (resp. Polish), see Hu-Papageorgiou [7], p. 15.

Let X be a Hausdorff topological space, (Y, d) a metric space and $S : X \to 2^Y \setminus \{\emptyset\}$ a multifunction.

Definition 2. We say that $S(\cdot)$ is

(a) "h-usc" if for all $x_0 \in X$, $x \to h^*(S(x), S(x_0))$ is continuous;

(b) "h-lsc" if for all $x_0 \in X$, $x \to h^*(S(x_0), S(x))$ is continuous;

(c) "h-continuous" if it is both h-usc and h-lsc.

Remark 3. Note that h-continuity of $S(\cdot)$ is continuity from X into the pseudometric space $(2^Y \setminus \{\emptyset\}, h)$. In general we have

$$\text{"usc} \Rightarrow h\text{-usc" and "}h\text{-lsc} \Rightarrow \text{lsc"}$$

with the converse implications failing in general. However, if $S(\cdot)$ is $P_k(Y)$-valued, then the notions are equivalent and so "$S(\cdot)$ is continuous if and only if $S(\cdot)$ is h-continuous" (see Hu-Papageorgiou [7], pp. 61-62).

According to Proposition 8, for the solution multifunction $\lambda \to S_\lambda$ from $\mathcal{L} = (0, \lambda^*]$ we know that it is $P_k(C_0^1(\overline{\Omega}))$-valued.

Proposition 14. *If hypotheses H hold, then the solution multifunction $\lambda \to S_\lambda$ is usc.*

Proof. Let $C \subseteq C_0^1(\overline{\Omega})$ be closed. According to Definition 1 (a), we need to show that $S^-(C) = \{\lambda \in \mathcal{L} : S_\lambda \cap C \neq \emptyset\}$ is closed in $\mathcal{L}$.

Let $\{\lambda_n\}_{n\geq 1} \subseteq S^-(C)$ and assume that $\lambda_n \to \lambda \in \mathcal{L}$. Let $u_n \in S_{\lambda_n} \subseteq \operatorname{int} C_+$, $u_n \in C$. Then as in the proof of Proposition 11 (see the Claim in that proof), we show that

$$\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{70}$$

Let $0 < \widetilde{\lambda} \leq \lambda_n$ for all $n \in \mathbb{N}$. From Propositions 4 and 6 we have

$$\underline{u}_{\widetilde{\lambda}} \leq u_n \quad \text{for all } n \in \mathbb{N}. \tag{71}$$

From (70) and the nonlinear regularity theory (see the proof of Proposition 7), we can find $\tau \in (0,1)$ and $c_{13} > 0$ such that

$$u_n \in C_0^{1,\tau}(\overline{\Omega}) = C^{1,\tau}(\overline{\Omega}) \cap C_0^1(\overline{\Omega}),\ \|u_n\|_{C^{1,\tau}(\overline{\Omega})} \leq c_{13} \quad \text{for all } n \in \mathbb{N}. \tag{72}$$

From (72) and the compact embedding of $C_0^{1,\tau}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$, we have that

$$\begin{aligned} & u_n \to u \text{ in } C_0^1(\overline{\Omega}),\ \underline{u}_{\tilde{\lambda}} \leq u \text{ (see (71)) and so } u \neq 0, \\ \Rightarrow\quad & u \in S_\lambda \subseteq \text{int } C_+, \quad u \in C, \\ \Rightarrow\quad & \lambda \in S^-(C) \text{ and so } S^-(C) \text{ is closed in } \mathcal{L}. \end{aligned}$$

This proves the upper semicontinuity of the solution multifunction $\lambda \to S_\lambda$. $\square$

By strengthening the conditions on the data of (P_λ), we can also show lower semicontinuity of the solution multifunction.

The new conditions on the perturbation $f(z,x)$ are the following:

$\underline{H'}$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function which is measurable in $z \in \Omega$ and such that

(i) $|f(z,x) - f(z,y)| \leq \widehat{k}(z)|x-y|$ for a.a. $z \in \Omega$, all $x, y \geq 0$ with $\widehat{k} \in L^\infty(\Omega)$;

hypotheses $H'\,(ii), (iii), (iv)$ are the same as the corresponding hypotheses $H\,(ii), (iii), (iv)$.

Remark 4. By taking $y = 0$ in hypothesis $H'\,(i)$ (recall $f(z,0) = 0$ for a.a. $z \in \Omega$), we have

$$0 \leq f(z,x) \leq \widehat{k}(z)\rho \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \rho,$$

and so hypothesis $H\,(i)$ holds.

Proposition 15. *If hypotheses H' hold, then the solution multifunction $\lambda \to S_\lambda$ is lsc.*

Proof. According to Proposition 2.6, p. 37, of Hu-Papageorgiou [7], it suffices to show that if $\lambda_n \to \lambda$ in $\mathcal{L}$, $\lambda \in \mathcal{L}$, then we have

$$S_\lambda \subseteq \liminf_{n \to +\infty} S_{\lambda_n} = \{u \in C_0^1(\overline{\Omega}) : u_n \in S_{\lambda_n},\ u_n \to u \text{ in } C_0^1(\overline{\Omega})\}.$$

Let $u \in S_\lambda \subseteq \operatorname{int} C_+$ and $0 < \widetilde{\lambda} \le \lambda_n$ for all $n \in \mathbb{N}$. Then $\widetilde{\lambda} \le \lambda$ and so we have

$$\begin{aligned} & \underline{u}_{\widetilde{\lambda}} \le u \quad \text{(see Propositions 4 and 6)}, \\ \Rightarrow \quad & u^{-\eta} \in L^s(\Omega) \quad s > N \text{ (see (8))}. \end{aligned}$$

We consider the following singular Dirichlet (p,q)-problem:

$$\begin{cases} -\Delta_p u_n^0(z) - \Delta_q u_n^0(z) = \lambda_n u(z)^{-\eta} + f(z, u(z)) \quad \text{in } \Omega, \\ u_n^0\big|_{\partial\Omega} = 0, \quad 1 < q < p. \end{cases} \tag{73}$$

We have $\lambda_n u(\cdot)^{-\eta} + f(\cdot, u(\cdot)) \in L^s(\Omega)$, $s > N$, and so problem (73) has a solution $u_n^0 \in \operatorname{int} C_+$ (nonlinear regularity, see [10], and the nonlinear maximum principle, see [23]) and this solution is unique on account of the strict monotonicity of the operator $u \to A_p(u) + A_q(u)$ (see Proposition 1). Moreover, since

$$0 \le \lambda_n u(z)^{-\eta} + f(z, u(z)) \le c_{14} \left[\underline{u}_{\widetilde{\lambda}}(z)^{-\eta} + 1 \right]$$

for a.a. $z \in \Omega$, all $n \in \mathbb{N}$, some $c_{14} > 0$ and $\underline{u}_{\widetilde{\lambda}}^{-\eta} \in L^s(\Omega)$, $s > N$, we have that

$$\{u_n^0\}_{n \ge 1} \subseteq C_0^1(\overline{\Omega}) \text{ is relatively compact}$$

(see the proof of Proposition 14). We may assume that

$$u_n^0 \to \widetilde{u} \text{ in } C_0^1(\overline{\Omega}) \text{ as } n \to +\infty, \;\; \underline{u}_{\widetilde{\lambda}} \le \widetilde{u}. \tag{74}$$

Passing to the limit as $n \to +\infty$ in (73) and using (74), we obtain

$$-\Delta_p \widetilde{u}(z) - \Delta_q \widetilde{u}(z) = \lambda u(z)^{-\eta} + f(z, u(z)) \quad \text{in } \Omega, \quad \widetilde{u}\big|_{\partial\Omega} = 0.$$

This problem has a unique solution $u \in S_\lambda$. Therefore

$$\widetilde{u} = u \in S_\lambda \subseteq \operatorname{int} C_+.$$

Hence for the original sequence we have

$$u_n^0 \to u \text{ in } C_0^1(\overline{\Omega}) \text{ as } n \to +\infty.$$

We consider the following singular Dirichlet (p,q)-problem:

$$\begin{cases} -\Delta_p u_n^1(z) - \Delta_q u_n^1(z) = \lambda_n u_n^0(z)^{-\eta} + f(z, u_n^0(z)) \quad \text{in } \Omega, \\ u_n^1\big|_{\partial\Omega} = 0, \quad 1 < q < p. \end{cases} \tag{75}$$

As above, we have that (75) has a unique solution $u_n^1 \in \operatorname{int} C_+$ and

$$u_n^1 \to u \text{ in } C_0^1(\overline{\Omega}) \text{ as } n \to +\infty.$$

We continue this way and generate a sequence $\{u_n^k\}_{k\geq 1} \subseteq \operatorname{int} C_+$ such that

$$\begin{cases} -\Delta_p u_n^k(z) - \Delta_q u_n^k(z) = \lambda_n u_n^{k-1}(z)^{-\eta} + f(z, u_n^{k-1}(z)) \quad \text{in } \Omega, \\ u_n^k\big|_{\partial\Omega} = 0, \quad \text{for all } n \in \mathbb{N}, \end{cases} \tag{76}$$

and for each $k \in \mathbb{N}_0$, we have

$$u_n^k \to u \text{ in } C_0^1(\overline{\Omega}) \text{ as } n \to +\infty. \tag{77}$$

We consider the sequence $\{u_n^k\}_{k\geq 0}$ and we show that it is bounded in $W_0^{1,p}(\Omega)$. Arguing by contradiction, suppose that

$$\|u_n^k\| \to +\infty \quad \text{as } k \to +\infty \ (n \in \mathbb{N}). \tag{78}$$

Let $y_k = \frac{u_n^k}{\|u_n^k\|}$, $k \in \mathbb{N}_0$. Then $\|y_k\| = 1$, $y_k \geq 0$ for all $k \in \mathbb{N}_0$. We may assume that

$$y_k \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_k \to y \text{ in } L^p(\Omega) \text{ as } k \to +\infty. \tag{79}$$

We have

$$\begin{aligned} & \langle A_p(u_n^k), h\rangle + \langle A_q(u_n^k), h\rangle = \int_\Omega \left[\lambda_n (u_n^{k-1})^{-\eta} + f(z, u_n^{k-1})\right] h dz \\ & \qquad\qquad\qquad\qquad \text{for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow \quad & \langle A_p(y_k), h\rangle + \frac{1}{\|u_n^k\|^{p-q}} \langle A_q(y_k), h\rangle \\ & = \int_\Omega \left[\lambda_n \frac{(u_n^{k-1})^{-\eta}}{\|u_n^k\|^{p-1}} + \frac{f(z, u_n^{k-1})}{\|u_n^k\|^{p-1}}\right] h dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{80}$$

Without any loss of generality we may assume that $\{\|u_n^k\|\}_{k\geq 0}$ is nondecreasing (see (78)). On account of hypotheses $H'\,(i),(ii)$, we have

$$\begin{aligned}
& 0 \leq f(z,u_n^{k-1}) \leq c_{15}\left[1+u_n^{k-1}(z)^{p-1}\right] \\
& \qquad\qquad \text{for a.a. } z\in\Omega, \text{ some } c_{15}>0, \\
\Rightarrow \quad & \frac{f(\cdot,u_n^{k-1}(\cdot))}{\|u_n^k\|^{p-1}} \leq c_{15}\left[\frac{1}{\|u_n^k\|^{p-1}}+\left(\frac{\|u_n^{k-1}\|}{\|u_n^k\|}\right)^{p-1} y_{k-1}(z)^{p-1}\right] \\
& \qquad\qquad \leq c_{15}\left[\frac{1}{\|u_n^k\|^{p-1}}+y_{k-1}(z)^{p-1}\right] \quad \text{for a.a. } z\in\Omega, \\
\Rightarrow \quad & \left\{\frac{f(\cdot,u_n^{k-1}(\cdot))}{\|u_n^k\|^{p}}\right\}_{k\geq 0} \subseteq L^{p'}(\Omega) \text{ is bounded.}
\end{aligned} \tag{81}$$

From hypothesis $H'\,(i)$ we have

$$\begin{aligned}
& \frac{f(z,u_n^k(z))}{\|u_n^k\|^{p-1}} - \frac{\widehat{k}(z)|u_n^k(z)-u_n^{k-1}(z)|}{\|u_n^k\|^{p-1}} \leq \frac{f(z,u_n^{k-1}(z))}{\|u_n^k\|^{p-1}} \\
& \leq \frac{f(z,u_n^k(z))}{\|u_n^k\|^{p-1}} + \frac{\widehat{k}(z)|u_n^k(z)-u_n^{k-1}(z)|}{\|u_n^k\|^{p-1}} \quad \text{for a.a. } z\in\Omega.
\end{aligned} \tag{82}$$

From (81), (82) and using hypothesis $H'\,(ii)$, we have that

$$\frac{f(\cdot,u_n^{k-1}(\cdot))}{\|u_n^k\|^{p-1}} \xrightarrow{w} \widetilde{\eta}(\cdot)y(\cdot)^{p-1} \text{ in } L^{p'} \tag{83}$$

with $\eta(z) \leq \widetilde{\eta}(z) \leq \widehat{\eta}(z)$ for a.a. $z\in\Omega$ (see [1]). In (80) first we choose $h=y_k-y\in W_0^{1,p}(\Omega)$, pass to the limit as $k\to+\infty$ and use (78), (81). We obtain

$$\begin{aligned}
& \lim_{n\to+\infty}\langle A_p(y_k),y_k-y\rangle = 0, \\
\Rightarrow \quad & y_k\to y \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 1), hence } \|y\|=1,\ y\geq 0.
\end{aligned} \tag{84}$$

Then if in (80) we pass to the limit as $k\to+\infty$ and use (84), (79) and (83), we obtain

$$\begin{aligned}
& \langle A_p(y),h\rangle = \int_\Omega \widetilde{\eta}(z)y^{p-1}h\,dz \quad \text{for all } h\in W_0^{1,p}(\Omega), \\
\Rightarrow \quad & -\Delta_p y(z) = \widetilde{\eta}(z)y(z)^{p-1} \quad \text{in } \Omega, \quad y\big|_{\partial\Omega}=0.
\end{aligned}$$

As before (see, for example, the Claim in the proof of Proposition 12), using (83) and Proposition 3, we infer that y must be nodal, a contradiction to (84).

Therefore we have that

$$\{u_n^k\}_{k\geq 0} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

From this, via the nonlinear regularity theory (see [8] and [10]), we obtain that

$$\{u_n^k\}_{k\geq 0} \subseteq C_0^1(\overline{\Omega}) \text{ is relatively compact.}$$

So, we may assume that

$$u_n^k \to u_n \text{ in } C_0^1(\overline{\Omega}) \quad \text{as } k \to +\infty.$$

From (76) we see that

$$\begin{aligned}
& -\Delta_p u_n(z) - \Delta_q u_n(z) = \lambda_n u_n(z)^{-\eta} + f(z, u_n(z)) \text{ in } \Omega, \quad u_n\big|_{\partial\Omega} = 0, \\
\Rightarrow \quad & u_n \in S_{\lambda_n} \subseteq \operatorname{int} C_+ \quad \text{for all } n \in \mathbb{N}.
\end{aligned}$$

From (77) and the double limit lemma (see Hu-Papageorgiou [7], Proposition A.1.107, p. 901), we have

$$\begin{aligned}
& u_n \to u \text{ in } C_0^1(\overline{\Omega}) \text{ as } n \to +\infty, \\
\Rightarrow \quad & u \in \liminf_{n\to+\infty} S_{\lambda_n}, \\
\Rightarrow \quad & \lambda \to S_\lambda \text{ is lsc on } \mathcal{L} = (0, \lambda^*].
\end{aligned}$$

□

Then from Propositions 14 and 15 it follows that under hypotheses H', the solution multifunction is Vietoris continuous. Since the multifunction is $P_k(C_0^1(\overline{\Omega}))$-valued, it is also h-continuous.

Summarizing our findings for the solution multifunction $\lambda \to S_\lambda$ on $\mathcal{L} = (0, \lambda^*]$, we can state the following theorem.

Theorem 2. *If hypotheses H' hold, then the solution multifunction $\lambda \to S_\lambda$ from $\mathcal{L} = (0, \lambda^*]$ into $P_k(C_0^1(\overline{\Omega}))$ is Vietoris continuous and h-continuous.*

5. Minimal Positive Solutions

In this section we show that for every $\lambda \in \mathcal{L} = (0, \lambda^*]$ problem (P_λ) has a smallest positive solution $u_\lambda^* \in \operatorname{int} C_+$ (that is, $u_\lambda^* \leq u$ for all $u \in S_\lambda$) and we examine the monotonicity and continuity properties of the map $\lambda \to u_\lambda^*$.

Proposition 16. *If hypotheses H hold and $\lambda \in \mathcal{L} = (0, \lambda^*]$, then problem (P_λ) admits a smallest positive solution $u_\lambda^* \in \operatorname{int} C_+$.*

Proof. From Proposition 18 of Papageorgiou-Rădulescu-Repovš [18], we know that S_λ is downward directed. Then using Lemma 3.10, p. 178, of Hu-Papageorgiu [7], we can find $\{u_n\}_{n\geq 1} \subseteq S_\lambda$ decreasing such that

$$\underline{u}_\lambda \leq u_n \leq u_1 \text{ for all } n \in \mathbb{N}, \quad \inf_{n\geq 1} u_n = \inf S_\lambda. \tag{85}$$

We have

$$\langle A_p(u_n), h\rangle + \langle A_q(u_n), h\rangle = \int_\Omega [\lambda u_n^{-\eta} + f(z, u_n)] h dz \text{ for all } h \in W_0^{1,p}(\Omega). \tag{86}$$

If in (86) we choose $h = u_n \in W_0^{1,p}(\Omega)$ and use (85), we obtain that

$$\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

We may assume that

$$u_n \xrightarrow{w} u_\lambda^* \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u_\lambda^* \text{ in } L^p(\Omega). \tag{87}$$

From (87) as in the proof of Proposition 12 (see the part of the proof after (66)), we obtain that

$$u_n \to u_\lambda^* \text{ in } W_0^{1,p}(\Omega). \tag{88}$$

Passing to the limit as $n \to +\infty$ in (86) and using (88) and (85), we have

$$\langle A_p(u_\lambda^*), h\rangle + \langle A_q(u_\lambda^*), h\rangle = \int_\Omega [\lambda (u_\lambda^*)^{-\eta} + f(z, u_\lambda^*)] h dz \text{ for all } h \in W_0^{1,p}(\Omega),$$

$$\underline{u}_\lambda \leq u_\lambda^*.$$

It follows that

$$u_\lambda^* \in S_\lambda \subseteq \operatorname{int} C_+ \quad \text{and} \quad u_\lambda^* = \inf S_\lambda.$$

$\square$

Next we determine the monotonicity and continuity properties of the map $\lambda \to u_\lambda^*$ from $\mathcal{L}$ into $C_0^1(\overline{\Omega})$.

Proposition 17. *If hypotheses H hold, then the minimal positive solution map $\lambda \to u_\lambda^*$ from $\mathcal{L} = (0, \lambda^*]$ into $C_0^1(\overline{\Omega})$ has the following properties:*

(a) it is strictly increasing, that is,

$$0 < \mu < \lambda \leq \lambda^* \Rightarrow u_\lambda^* - u_\mu^* \in \operatorname{int} C_+;$$

(b) it is left continuous.

Proof. (a) According to Proposition 10, we can find $u \in S_\mu \subseteq \operatorname{int} C_+$ such that

$$\begin{aligned} & u_\lambda^* - u \in \operatorname{int} C_+, \\ \Rightarrow \quad & u_\lambda^* - u_\mu^* \in \operatorname{int} C_+ \quad \text{(since } u_\mu^* \leq u\text{)}, \\ \Rightarrow \quad & \lambda \to u_\lambda^* \text{ is strictly increasing.} \end{aligned}$$

(b) Let $\{\lambda_n\}_{n\geq 1} \subseteq \mathcal{L} = (0, \lambda^*]$ and assume that $\lambda_n \to \lambda^-$. Let $u_n^* = u_{\lambda_n}^* \in S_{\lambda_n} \subseteq \operatorname{int} C_+$ be the minimal positive solution of (P_{λ_n}) produced in Proposition 16. We have

$$\begin{aligned} & \underline{u}_{\lambda_1} \leq u_n^* \leq u_{\lambda^*}^* \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow \quad & \{u_n^*\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \end{aligned}$$

From this as before (see, for example, the last part of the proof of Proposition 8), from the nonlinear regularity theory of Lieberman [10] we have that

$$\{u_n^*\}_{n\geq 1} \subseteq C_0^1(\overline{\Omega}) \text{ is relatively compact.}$$

Since $\{u_n^*\}_{n\geq 1}$ is increasing (see part (a)), we have

$$u_n^* \to \widetilde{u}_\lambda^* \text{ in } C_0^1(\overline{\Omega}) \text{ as } n \to +\infty. \tag{89}$$

We claim that $\widetilde{u}_\lambda^* = u_\lambda^*$. If this is not true, we can find $z_0 \in \Omega$ such that

$$\begin{aligned} & u_\lambda^*(z_0) < \widetilde{u}_\lambda^*(z_0), \\ \Rightarrow \quad & u_\lambda^*(z_0) < u_n^*(z_0) \quad \text{for all } n \geq n_0 \text{ (see (89))}, \end{aligned}$$

which contradicts part (a). Therefore $\widetilde{u}_\lambda^* = u_\lambda^*$ and this proves that the map $\lambda \to u_\lambda^*$ is left continuous.

$\square$

We can state the following theorem.

Theorem 3. *If hypotheses H hold and $\lambda \in \mathcal{L} = (0, \lambda^*]$, then problem (P_λ) has a smallest positive solution $u_\lambda^* \in \operatorname{int} C_+$ and the minimal positive solution map $\lambda \to u_\lambda^*$ from $\mathcal{L}$ into $C_0^1(\overline{\Omega})$ is strictly increasing and left continuous.*

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