NONLINEAR CONCAVE-CONVEX PROBLEMS WITH INDEFINITE WEIGHT

NIKOLAOS S. PAPAGEORGIOU, CALOGERO VETRO, FRANCESCA VETRO

Abstract. We consider a parametric nonlinear Robin problem driven by the p-Laplacian and with a reaction having the competing effects of two terms. One is a parametric $(p-1)$-sublinear term (concave nonlinearity) and the other is a $(p-1)$-superlinear term (convex nonlinearity). We assume that the weight of the concave term is indefinite (that is, sign changing). Using the Nehari method, we show that for all small values of the parameter $\lambda > 0$, the problem has at least two positive solutions and also we provide information about their regularity.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear parametric Robin problem

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u + \xi(z)u(z)^{p-1} = \lambda a(z)u(z)^{\tau-1} + \vartheta(z)u(z)^{\eta-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega,\ 1 < \tau < p < \eta < p^*,\ \lambda > 0,\ u > 0. \end{cases}$$

In this problem Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad \text{for all } u \in W^{1,p}(\Omega),\ p < N.$$

The potential term $\xi \in L^s(\Omega)$, $s > N/p$ (so, it may be unbounded) and $\xi \geq 0$. In the reaction (right hand side of (P_λ)), we have the combined effects of two different nonlinearities. The first is a parametric concave (that is, $(p-1)$-sublinear) term (since $\tau < p$), while the second term is a convex (that is, $(p-1)$-superlinear) perturbation (since $p < \eta$). The distinguishing feature in this reaction is that the weight $a(\cdot)$ is sign-changing and unbounded. In the boundary condition, $\dfrac{\partial u}{\partial n_p}$ denotes the conormal derivative of u corresponding to the p-Laplace differential operator. This conormal derivative is interpreted using the nonlinear Green's identity (see Papageorgiou-Rădulescu-Repovš [13], p. 35) and we have

$$\frac{\partial u}{\partial n_p} = |\nabla u|^{p-2}\frac{\partial u}{\partial n} \quad \text{for all } u \in C^1(\overline{\Omega}),$$

Key words and phrases. Nehari manifold, local minimizers, Lagrange multiplier, positive solutions, nonlinear regularity.

2010 Mathematics Subject Classification: 35J65, 35J50, 35J92.

Corresponding author: Calogero Vetro (calogero.vetro@unipa.it).

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. For the boundary coefficient $\beta(\cdot)$ we assume that $\beta \in L^\infty(\partial\Omega)$, $\beta \geq 0$.

Using the Nehari method, we show that for all $\lambda > 0$ small problem (P_λ) has at least two positive solutions and we also determine the regularity properties of these solutions.

The study of problems with combined nonlinearities was initiated with the seminal paper of Ambrosetti-Brezis-Cerami [1], where $p = 2$ (semilinear equation) with Dirichlet boundary condition and $a \equiv \vartheta \equiv 1$. They prove a bifurcation-type result, producing a critical parameter value $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least two positive solutions, for $\lambda = \lambda^*$ problem (P_λ) has at least one positive solution, and for $\lambda > \lambda^*$ there are no positive solutions. Their work was extended to p-Laplacian equations by Garcia Azorero-Manfredi-Peral Alonso [5] and Guo-Zhang [7]. Further generalizations can be found in the works of Papageorgiou-Rădulescu-Repovš [12] (semilinear Robin problems) and Papageorgiou-Winkert [14] (nonlinear Dirichlet problems). Problems with indefinite weights were studied primarily in the context of semilinear Dirichlet equations. We mention the important works of Brown [2], Brown-Wu [3], Brown-Zhang [4], Papageorgiou-Rădulescu [10], Wu [18]. For nonlinear problems driven by the Dirichlet p-Laplacian, there is the recent work of Silva-Macedo [17], where $a \equiv 1$, $\vartheta \in L^\infty(\Omega)$ is sign-changing.

2. Preliminaries - Hypotheses

Throughout this work we assume that $p < N$. Then the critical Sobolev exponent is $p^* = \dfrac{Np}{N-p}$. Also if $q \in [1, +\infty)$, then $q' \in (1, +\infty]$ is defined by $\dfrac{1}{q} + \dfrac{1}{q'} = 1$ (conjugate exponent).

We will be working on the Sobolev space $W^{1,p}(\Omega)$. By $\|\cdot\|$ we denote the norm of $W^{1,p}(\Omega)$ defined by

$$\|u\| = \left[\|u\|_p^p + \|\nabla u\|_p^p\right]^{1/p} \quad \text{for all } u \in W^{1,p}(\Omega).$$

Also, we will use the space $C^1(\overline{\Omega})$ which is an ordered Banach space with positive (order) cone given by $C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by $\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}$.

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define in the usual way the boundary Lebesgue spaces $L^q(\partial\Omega)$, $1 \leq q \leq +\infty$. We know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u\Big|_{\partial\Omega} \quad \text{for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map extends the notion of boundary values to all Sobolev functions. We know that

$$\operatorname{im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega) \quad \text{and} \quad \ker\gamma_0 = W_0^{1,p}(\Omega).$$

Moreover, the trace map is compact.

In the sequel for the sake of notational simplicity, we drop the use of the trace map $\gamma_0(\cdot)$. All the restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

Our hypotheses on the data of problem (P_λ) are the following:

H_1: $\xi \in L^s(\Omega)$ with $s > N/p$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$, $\beta \in L^\infty(\partial\Omega)$, $\beta(z) \geq 0$ for σ-a.a. $z \in \partial\Omega$ and $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

H_2: $a \in L^s(\Omega)$ with $s > N/p$, and $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \geq 0$ for a.a. $z \in \Omega$.

Remark 1. So, the weight of the concave term is indefinite.

In what follows $\gamma_p : W^{1,p}(\Omega) \to \mathbb{R}$ is the C^1-functional defined by

$$\gamma_p(u) = \|\nabla u\|_p^p + \int_\Omega \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

For $\lambda > 0$, let $\varphi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{p}\gamma_p(u) - \frac{\lambda}{\tau}\int_\Omega a(z)|u|^\tau dz - \frac{1}{\eta}\int_\Omega \vartheta(z)|u|^\eta dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

We have $\varphi_\lambda \in C^1(W^{1,p}(\Omega))$.

We introduce the functional $k_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$k_\lambda(u) = \langle \varphi'_\lambda(u), u\rangle = \gamma_p(u) - \lambda\int_\Omega a(z)|u|^\tau dz - \int_\Omega \vartheta(z)|u|^\eta dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Evidently $k_\lambda \in C^1(W^{1,p}(\Omega))$. We introduce the Nehari manifold for the functional $\varphi_\lambda(\cdot)$ defined by

$$N_\lambda = \left\{u \in W^{1,p}(\Omega) : \langle \varphi'_\lambda(u), u\rangle = k_\lambda(u) = 0,\ u \neq 0\right\}.$$

We decompose N_λ into three disjoint parts

$$\begin{aligned} N_\lambda^+ &= \left\{u \in N_\lambda : \langle k'_\lambda(u), u\rangle > 0\right\}, \\ N_\lambda^0 &= \left\{u \in N_\lambda : \langle k'_\lambda(u), u\rangle = 0\right\}, \\ N_\lambda^- &= \left\{u \in N_\lambda : \langle k'_\lambda(u), u\rangle < 0\right\}. \end{aligned}$$

The Nehari manifold N_λ is much smaller than $W^{1,p}(\Omega)$ and contains the critical points of $\varphi_\lambda(\cdot)$. We expect $\varphi_\lambda\big|_{N_\lambda}$ to have properties which fail globally.

3. Multiple positive solutions

Hypotheses H_1 together with Lemma 4.11 of Mugnai-Papageorgiou [9] and Proposition 2.4 of Gasiński-Papageorgiou [6], imply that

$$\gamma_p(u) \geq c_0\|u\|^p \quad \text{for all } u \in W^{1,p}(\Omega), \text{ some } c_0 > 0. \tag{1}$$

Proposition 1. *If hypotheses H_1, H_2 hold and $\lambda > 0$, then $\varphi_\lambda\big|_{N_\lambda}$ is coercive.*

Proof. From the definition of the Nehari manifold, we have

$$u \in N_\lambda \text{ if and only if } \gamma_p(u) = \int_\Omega [\lambda a(z)|u|^\tau + \vartheta(z)|u|^\eta] dz,\ u \neq 0. \tag{2}$$

Then for $u \in N_\lambda$, we have

$$\begin{aligned}
\varphi_\lambda(u) &= \frac{1}{p}\gamma_p(u) - \frac{1}{\tau}\left[\gamma_p(u) - \int_\Omega \vartheta(z)|u|^\eta dz\right] - \frac{1}{\eta}\int_\Omega \vartheta(z)|u|^\eta dz \quad \text{(see (2))} \\
&= \left[\frac{1}{p} - \frac{1}{\tau}\right]\gamma_p(u) + \left[\frac{1}{\tau} - \frac{1}{\eta}\right]\int_\Omega \vartheta(z)|u|^\eta dz \\
&= \left[\frac{1}{p} - \frac{1}{\tau}\right]\gamma_p(u) + \left[\frac{1}{\tau} - \frac{1}{\eta}\right]\gamma_p(u) - \lambda\left[\frac{1}{\tau} - \frac{1}{\eta}\right]\int_\Omega a(z)|u|^\tau dz \quad \text{(see (2))} \\
&= \left[\frac{1}{p} - \frac{1}{\eta}\right]\gamma_p(u) - \lambda\left[\frac{1}{\tau} - \frac{1}{\eta}\right]\int_\Omega a(z)|u|^\tau dz \\
&\geq c_1\|u\|^p - \lambda c_2\int_\Omega a(z)|u|^\tau dz \quad \text{for some } c_1 > 0, \text{ with } c_2 = \frac{1}{\tau} - \frac{1}{\eta} > 0 \text{ (see (1))}. \qquad (3)
\end{aligned}$$

By hypothesis H_1 we have

$$\begin{aligned}
s > \frac{N}{p} &\Rightarrow s' < \left(\frac{N}{p}\right)' = \frac{N}{N-p}, \\
&\Rightarrow \quad \tau s' < ps' = \frac{Np}{N-p} = p^* \quad (\text{recall } \tau < p).
\end{aligned}$$

Therefore $|u|^\tau \in L^{s'}(\Omega)$ (by the Sobolev embedding theorem) and using Hölder's inequality, we have

$$\left|\int_\Omega a(z)|u|^\tau dz\right| \leq \|a\|_s \|u\|_{\tau s'}^\tau. \tag{4}$$

Since $\tau s' < p^*$ and $\tau < \tau s'$, using the Sobolev embedding theorem we have

$$W^{1,p}(\Omega) \hookrightarrow L^{\tau s'}(\Omega) \hookrightarrow L^\tau(\Omega)$$

with all embeddings being continuous and the first being also compact. So by Ehrling's inequality (see Papageorgiou-Winkert [15], Lemma 4.2.48, p. 317), we can find $c_3 > 0$ such that

$$\|u\|_{\tau s'}^\tau \leq c_3\left[\|u\|^\tau + \|u\|_\tau^\tau\right]. \tag{5}$$

We use (5) in (4) and obtain

$$\left|\int_\Omega a(z)|u|^\tau dz\right| \leq c_4\|u\|^\tau \quad \text{for some } c_4 > 0, \text{ all } u \in W^{1,p}(\Omega). \tag{6}$$

We return to (3) and use (6). We obtain

$$\varphi_\lambda(u) \geq \left[c_1\|u\|^{p-\tau} - \lambda c_5\right]\|u\|^\tau \quad \text{for some } c_5 > 0. \tag{7}$$

Since $\tau < p$, from (7) we conclude that $\varphi_\lambda\big|_{N_\lambda}$ is coercive. ∎

Next we show that for all $\lambda > 0$ small, we have $N^0_\lambda = \emptyset$.

Proposition 2. *If hypotheses H_1, H_2 hold, then there exists $\lambda^* > 0$ such that $N^0_\lambda = \emptyset$ for all $\lambda \in (0, \lambda^*)$.*

Proof. We argue by contradiction. So, suppose that for all $\lambda > 0$, $N^0_\lambda \neq \emptyset$. Therefore, we can find $u \in N_\lambda$ such that

$$
\begin{aligned}
& \langle k'_\lambda(u), u\rangle = 0, \\
\Rightarrow \quad & p\langle \gamma'_p(u), u\rangle = \lambda\tau \int_\Omega a(z)|u|^\tau dz + \eta \int_\Omega \vartheta(z)|u|^\eta dz, \\
\Rightarrow \quad & p\gamma_p(u) = \lambda\tau \int_\Omega a(z)|u|^\tau dz + \eta \int_\Omega \vartheta(z)|u|^\eta dz. \qquad (8)
\end{aligned}
$$

Since $u \in N_\lambda$ we have

$$
\begin{aligned}
& \tau\gamma_p(u) = \lambda\tau \int_\Omega a(z)|u|^\tau dz + \tau \int_\Omega \vartheta(z)|u|^\eta dz \quad \text{(see (2))}, \\
\Rightarrow \quad & \tau\gamma_p(u) - \tau \int_\Omega \vartheta(z)|u|^\eta dz = \lambda\tau \int_\Omega a(z)|u|^\tau dz. \qquad (9)
\end{aligned}
$$

We use (9) in (8) and obtain

$$
\begin{aligned}
& [p-\tau]\gamma_p(u) - [\eta-\tau] \int_\Omega \vartheta(z)|u|^\eta dz = 0, \\
\Rightarrow \quad & c_6\|u\|^p - c_7\|u\|^\eta \le 0 \quad \text{for some } c_6, c_7 > 0 \\
& \text{(see (1), hypothesis } H_1 \text{ and recall that } \tau < p < \eta) \\
\Rightarrow \quad & \left[\frac{c_6}{c_7}\right]^{\frac{1}{\eta-p}} \le \|u\|. \qquad (10)
\end{aligned}
$$

On the other hand from (2), we have

$$
\eta\gamma_p(u) - \lambda\eta \int_\Omega a(z)|u|^\tau dz = \eta \int_\Omega \vartheta(z)|u|^\eta dz. \qquad (11)
$$

We use (11) in (8) and it follows that

$$
\begin{aligned}
& [\eta - p]\gamma_p(u) = \lambda[\eta-\tau] \int_\Omega a(z)|u|^\tau dz, \\
\Rightarrow \quad & \|u\|^p \le \lambda c_8 \|u\|^\tau \quad \text{for some } c_8 > 0 \text{ (see (2), (4) and recall that } \tau < p < \eta), \\
\Rightarrow \quad & \|u\| \le [\lambda c_8]^{\frac{1}{p-\tau}}.
\end{aligned}
$$

We let $\lambda \to 0^+$ and we have a contradiction to (10). This means that we can find $\lambda^* > 0$ such that $N^0_\lambda = \emptyset$ for all $\lambda \in (0, \lambda^*)$. ∎

Proposition 3. *If hypotheses H_1, H_2 hold and $\lambda \in (0, \lambda^*)$, then $\varphi_\lambda\big|_{N^+_\lambda} < 0$.*

Proof. Let $u \in N_\lambda^+$. From the definition of N_λ^+, we have

$$0 < \langle k_\lambda'(u), u\rangle = p\gamma_p(u) - \lambda\tau \int_\Omega a(z)|u|^\tau dz - \eta \int_\Omega \vartheta(z)|u|^\eta dz. \tag{12}$$

Since $u \in N_\lambda$, from (2) we have

$$-\tau\gamma_p(u) + \lambda\tau \int_\Omega a(z)|u|^\tau dz + \tau \int_\Omega \vartheta(z)|u|^\eta dz = 0. \tag{13}$$

We add (12) and (13) and obtain

$$[\eta - \tau] \int_\Omega \vartheta(z)|u|^\tau dz < [p-\tau]\gamma_p(u),$$

$$\Rightarrow \quad \int_\Omega \vartheta(z)|u|^\tau dz < \left[\frac{p-\tau}{\eta-\tau}\right]\gamma_p(u). \tag{14}$$

Then we have

$$\begin{aligned}
\varphi_\lambda(u) &= \frac{1}{p}\gamma_p(u) - \frac{\lambda}{\tau}\int_\Omega a(z)|u|^\tau dz - \frac{1}{\eta}\int_\Omega \vartheta(z)|u|^\eta dz \\
&= \frac{1}{p}\gamma_p(u) - \frac{1}{\tau}\left[\gamma_p(u) - \int_\Omega \vartheta(z)|u|^\eta dz\right] - \frac{1}{\eta}\int_\Omega \vartheta(z)|u|^\eta dz \quad (\text{since } u \in N_\lambda, \text{ see } (2)) \\
&= \left[\frac{1}{p} - \frac{1}{\tau}\right]\gamma_p(u) + \left[\frac{1}{\tau} - \frac{1}{\eta}\right]\int_\Omega \vartheta(z)|u|^\eta dz \\
&< \left[\frac{1}{p} - \frac{1}{\tau}\right]\gamma_p(u) + \left[\frac{1}{\tau} - \frac{1}{\eta}\right]\left[\frac{p-\tau}{\eta-\tau}\right]\gamma_p(u) \quad (\text{see } (14)) \\
&= \left[\frac{\tau-p}{\tau p}\right]\gamma_p(u) + \left[\frac{p-\tau}{\tau\eta}\right]\gamma_p(u) \\
&< 0 \quad (\text{since } \tau < p < \eta).
\end{aligned}$$

$\square$

Let $m_\lambda^+ = \inf_{N_\lambda^+} \varphi_\lambda$. From Proposition 3 we have $m_\lambda^+ < 0$.

Proposition 4. *If hypotheses H_1, H_2 hold, then there exists $\widehat{\lambda}^* \in (0, \lambda^*]$ such that for all $\lambda \in (0, \widehat{\lambda}^*)$ we can find $u_\lambda^* \in N_\lambda^+$ such that $\varphi_\lambda(u_\lambda^*) = m_\lambda^+$.*

Proof. Let $\{u_n\}_{n\geq 1} \subseteq N_\lambda^+$ be a minimizing sequence, that is

$$\varphi_\lambda(u_n) \downarrow m_\lambda^+ \quad \text{as } n \to +\infty.$$

From Proposition 3 we have

$$\varphi_\lambda(u_n) < 0 \quad \text{for all } n \in \mathbb{N}. \tag{15}$$

Also, since $\{u_n\}_{n\geq 1} \subseteq N_\lambda^+ \subseteq N_\lambda$, using Proposition 1, we have that

$$\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

By passing to a suitable subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u_\lambda^* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_\lambda^* \text{ in } L^\eta(\Omega) \text{ and in } L^p(\partial\Omega). \tag{16}$$

Since $\varphi_\lambda(\cdot)$ is sequentially weakly lower semicontinuous, from (16) and since $\tau s' < p$, we have

$$\begin{aligned} & \varphi_\lambda(u_\lambda^*) < 0 = \varphi_\lambda(0) \quad (\text{recall } m_\lambda^+ < 0), \\ \Rightarrow \quad & u_\lambda^* \neq 0. \end{aligned}$$

From (15) we have

$$\begin{aligned} & \frac{1}{p}\gamma_p(u_n) - \frac{\lambda}{\tau}\int_\Omega a(z)|u_n|^\tau dz - \frac{1}{\eta}\int_\Omega \vartheta(z)|u_n|^\eta dz < 0 \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow \quad & \left[\frac{1}{p} - \frac{1}{\eta}\right]\gamma_p(u_n) - \lambda\left[\frac{1}{\tau} - \frac{1}{\eta}\right]\int_\Omega a(z)|u_n|^\tau dz < 0 \quad \text{for all } n \in \mathbb{N}, \\ & \qquad\qquad (\text{recall } u_n \in N_\lambda \text{ and see (2)}) \\ \Rightarrow \quad & \left[\frac{1}{p} - \frac{1}{\eta}\right]\gamma_p(u_\lambda^*) - \lambda\left[\frac{1}{\tau} - \frac{1}{\eta}\right]\int_\Omega a(z)|u_\lambda^*|^\tau dz \leq 0 \quad (\text{see (16) and recall that } ts' < p) \\ \Rightarrow \quad & \left[\frac{1}{p} - \frac{1}{\eta}\right]\gamma_p(u_\lambda^*) \leq \lambda\left[\frac{1}{\tau} - \frac{1}{\eta}\right]\int_\Omega a(z)|u_\lambda^*|^\tau dz. \end{aligned}$$

Since $\tau < p < \eta$ and $u_\lambda^* \neq 0$, using (1) we infer that

$$0 < \int_\Omega a(z)|u_\lambda^*|^\tau dz. \tag{17}$$

Consider the fibering map

$$\mu_{\lambda,u_\lambda^*}(t) = \varphi_\lambda(tu_\lambda^*) \quad \text{for all } t > 0.$$

Also, let

$$w_{u_\lambda^*}(t) = t^{p-\tau}\gamma_p(u_\lambda^*) - t^{\eta-\tau}\int_\Omega \vartheta(z)|u_\lambda^*|^\eta dz \quad \text{for all } t > 0.$$

Since $\tau < p < \eta$, we see that there exists a unique $t_0 > 0$ such that

$$\begin{aligned} & w_{u_\lambda^*}(t_0) = \max\{w_{u_\lambda^*}(t) : t > 0\}, \qquad (18) \\ \Rightarrow \quad & w'_{u_\lambda^*}(t_0) = 0, \\ \Rightarrow \quad & (p-\tau)\gamma_p(u_\lambda^*) = (\eta-\tau)t_0^{\eta-p}\int_\Omega \vartheta(z)|u_\lambda^*|^\eta dz, \\ \Rightarrow \quad & t_0 = \left[\frac{(p-\tau)\gamma_p(u_\lambda^*)}{(\eta-\tau)\int_\Omega \vartheta(z)|u_\lambda^*|^\eta dz}\right]^{\frac{1}{\eta-p}}, \\ \Rightarrow \quad & w_{u_\lambda^*}(t_0) = \left[\left(\frac{p-\tau}{\eta-\tau}\right)^{\frac{p-\tau}{\eta-p}} - \left(\frac{p-\tau}{\eta-\tau}\right)^{\frac{\eta-\tau}{\eta-p}}\right]\frac{\gamma_p(u_\lambda^*)^{\frac{\eta-\tau}{\eta-p}}}{\left[\int_\Omega \vartheta(z)|u|^\eta dz\right]^{\frac{p-\tau}{\eta-p}}} > 0. \qquad (19) \end{aligned}$$

Note that for $t > 0$, we have

$$tu_\lambda^* \in N_\lambda \text{ if and only if } w_{u_\lambda^*}(t) = \lambda\int_\Omega a(z)|u_\lambda^*|^\tau dz \quad (\text{see (2)}). \tag{20}$$

From (20) and (18), we see that

$$\text{if } \lambda \int_\Omega a(z)|u_\lambda^*|^\tau dz > w_{u_\lambda^*}(t_0) > 0 \quad \text{(see (19)), then for all } t > 0,\ tu_\lambda^* \notin N_\lambda.$$

So, let $\widehat{\lambda}^* \in (0, \lambda^*]$ be small so that

$$0 < \lambda \int_\Omega a(z)|u_\lambda^*|^\tau dz \leq w_{u_\lambda^*}(t_0) \quad \text{ for all } \lambda \in (0, \widehat{\lambda}^*] \text{ (see (19))}.$$

Since $\tau < p < \eta$, we can find $0 < t_1 < t_0 < t_2$ such that

$$w'_{u_\lambda^*}(t_2) < 0 < w'_{u_\lambda^*}(t_1), \tag{21}$$

$$\mu'_{\lambda, u_\lambda^*}(t_1) = 0 = \mu'_{\lambda, u_\lambda^*}(t_2) \quad \text{(see Brown-Wu [3], p. 4)}. \tag{22}$$

Then we have

$$\begin{aligned}
\mu''_{u_\lambda^*}(t_1) &= (p-1)t_1^{p-2}\gamma_p(u_\lambda^*) - \lambda(t-1)t_1^{\tau-2}\int_\Omega a(z)|u_\lambda^*|^\tau dz - (\eta-1)t_1^{\eta-2}\int_\Omega \vartheta(z)|u_\lambda^*|^\eta dz \\
&= (p-1)t_1^{p-2}\gamma_p(u_\lambda^*) - (\tau-1)t_1^{p-2}\gamma_p(u_\lambda^*) \\
&\quad - (\eta-1)t_1^{\eta-2}\int_\Omega \vartheta(z)|u_\lambda^*|^\eta dz + (\tau-1)t_1^{\eta-2}\int_\Omega \vartheta(z)|u_\lambda^*|^\eta dz \quad \text{(see (22))} \\
&= t_1^{\tau-1}\left[(p-\tau)t_1^{p-\tau-1}\gamma_p(u_\lambda^*) - (\eta-\tau)t_1^{\eta-\tau-1}\int_\Omega \vartheta(z)|u_\lambda^*|^\eta dz\right] \\
&= t_1^{\tau-1}w'_{u_\lambda^*}(t_1) > 0 \quad \text{(see (21))}. \qquad (23)
\end{aligned}$$

Similarly we show that

$$\mu''_{u_\lambda^*}(t_2) = t_2^{\tau-1}w'_{u_\lambda^*}(t_2) < 0. \tag{24}$$

From (23) and (24) it follows that

$$t_1 u_\lambda^* \in N_\lambda^+ \text{ and } t_2 u_\lambda^* \in N_\lambda^-.$$

Claim: $u_n \to u_\lambda^*$ in $W^{1,p}(\Omega)$ as $n \to +\infty$.

If the Claim is not true, then on account of (16) and the sequential weak lower semicontinuity of $\gamma_p(\cdot)$, we have

$$\gamma_p(u_\lambda^*) < \liminf_{n\to+\infty} \gamma_p(u_n). \tag{25}$$

Then we have

$$\begin{aligned}
&\liminf_{n\to+\infty} \mu'_{\lambda, u_n}(t_1) \\
&= \liminf_{n\to+\infty}\left[t_1^{p-1}\gamma_p(u_n) - \lambda t_1^{\tau-1}\int_\Omega a(z)|u_n|^\tau dz - t_1^{\eta-1}\int_\Omega \vartheta(z)|u_n|^\eta dz\right] \\
&> t_1^{p-1}\gamma_p(u_\lambda^*) - \lambda t_1^{\tau-1}\int_\Omega a(z)|u_\lambda^*|^\tau dz - t_1^{\eta-1}\int_\Omega \vartheta(z)|u_\lambda^*|^\eta dz \quad \text{(see (25) and (16))} \\
&= \mu'_{\lambda, u_\lambda^*}(t_1) = 0 \quad \text{(see (22))},
\end{aligned}$$

$$\Rightarrow \quad \mu'_{\lambda,u_n}(t_1) > 0 \quad \text{for all } n \geq n_0. \tag{26}$$

Since $u_n \in N_\lambda^+ \subseteq N_\lambda$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} & \mu'_{\lambda,u_n}(t) < 0 \quad \text{for all } t \in (0,1) \text{ and } \mu'_{\lambda,u_n}(1) = 0 \quad \text{for all } n \in \mathbb{N} \\ & \text{(see also Silva-Macedo [17], pp. 1897-1898)}, \\ \Rightarrow \quad & t_1 > 1. \end{aligned} \tag{27}$$

The fibering function $\mu_{\lambda,u_\lambda^*}(\cdot)$ is decreasing on $(0, t_2)$. So, we have

$$\begin{aligned} \varphi_\lambda(t_1 u_\lambda^*) &\leq \varphi_\lambda(u_\lambda^*) \quad \text{(see (27))} \\ &\leq \liminf_{n\to+\infty} \varphi_\lambda(u_n) \quad \text{(see (25), (16))} \\ &= m_\lambda^+. \end{aligned} \tag{28}$$

But recall that $t_1 u_\lambda^* \in N_\lambda^+$. So, we have

$$m_\lambda^+ \leq \varphi_\lambda(t_1 u_\lambda^*). \tag{29}$$

Comparing (28) and (29), we have a contradiction. This proves the Claim.
On account of the Claim, we have

$$\begin{aligned} & \varphi_\lambda(u_n) \to \varphi_\lambda(u_\lambda^*), \\ \Rightarrow \quad & \varphi_\lambda(u_\lambda^*) = m_\lambda^+. \end{aligned}$$

Since $u_n \in N_\lambda^+$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} & \langle k'_\lambda(u_n), u_n \rangle > 0 \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow \quad & \langle k'_\lambda(u_\lambda^*), u_\lambda^* \rangle \geq 0 \quad \text{(see the Claim)}. \end{aligned}$$

Recall that $\widehat{\lambda}^* \leq \lambda^*$ and so by Proposition 2 equality is not possible. Therefore

$$\begin{aligned} & \langle k'_\lambda(u_\lambda^*), u_\lambda^* \rangle > 0, \\ \Rightarrow \quad & u_\lambda^* \in N_\lambda^+. \end{aligned}$$

□

Next we consider the following minimization problem

$$m_\lambda^- = \inf_{N_\lambda^-} \varphi_\lambda. \tag{30}$$

Proposition 5. *If hypotheses H_1, H_2 hold, then there exists $\widehat{\lambda}_0^* \in (0, \widehat{\lambda}^*]$ such that $\varphi_\lambda\big|_{N_\lambda^-} > 0$ for all $\lambda \in (0, \lambda_0^*)$.*

Proof. Let $u \in N_\lambda^-$ and consider the function

$$e_u(t) = \frac{t^p}{p}\gamma_p(u) - \frac{t^\eta}{\eta}\int_\Omega \vartheta(z)|u|^\eta dz \quad \text{for all } t > 0.$$

Since $p < \eta$, we see that $e_u(\cdot)$ has a unique maximum $\widehat{t} > 0$ determined by the equation

$$\begin{aligned} & e_u'(\widehat{t}) = 0, \\ \Rightarrow\quad & \widehat{t}^{p-1}\gamma_p(u) = \widehat{t}^{\eta-1}\int_\Omega \vartheta(z)|u|^\eta dz, \\ \Rightarrow\quad & \widehat{t} = \left[\frac{\gamma_p(u)}{\int_\Omega \vartheta(z)|u|^\eta dz}\right]^{\frac{1}{\eta-p}}. \end{aligned}$$

Then we have

$$e_u(\widehat{t}) = \frac{1}{p}\left[\frac{\gamma_p(u)}{\int_\Omega \vartheta(z)|u|^\eta dz}\right]^{\frac{p}{\eta-p}}\gamma_p(u) - \frac{1}{\eta}\left[\frac{\gamma_p(u)}{\int_\Omega \vartheta(z)|u|^\eta dz}\right]^{\frac{\eta}{\eta-p}}\int_\Omega \vartheta(z)|u|^\eta dz.$$

Note that $\dfrac{p}{\eta-p} + 1 = \dfrac{\eta}{\eta-p}$. Hence we have

$$\begin{aligned} e_u(\widehat{t}) &= \left[\frac{1}{p} - \frac{1}{\eta}\right]\frac{\gamma_p(u)^{\frac{\eta}{\eta-p}}}{\left[\int_\Omega \vartheta(z)|u|^\eta dz\right]^{\frac{p}{\eta-p}}} \\ &= \left[\frac{1}{p} - \frac{1}{\eta}\right]\left[\frac{\gamma_p(u)^\eta}{\left(\int_\Omega \vartheta(z)|u|^\eta dz\right)^p}\right]^{\frac{1}{\eta-p}}. \end{aligned} \tag{31}$$

Note that

$$\begin{aligned} \left[\int_\Omega \vartheta(z)|u|^\eta dz\right]^{\frac{1}{\eta}} &\leq c_9\|u\|_\eta \quad \text{for some } c_9 > 0 \text{ (see hypotheses } H_2) \\ &\leq c_{10}\|u\| \quad \text{for some } c_{10} > 0 \text{ (by the Sobolev embedding theorem)} \\ &\leq c_{11}\gamma_p(u)^{\frac{1}{p}} \quad \text{for some } c_{11} > 0 \text{ (see (1))}, \\ \Rightarrow\quad \left[\int_\Omega \vartheta(z)|u|^\eta dz\right]^{\frac{p}{\eta}} &\leq c_{11}^p\gamma_p(u), \\ \Rightarrow\quad \left[\int_\Omega \vartheta(z)|u|^\eta dz\right]^{p} &\leq c_{12}\gamma_p(u)^\eta \quad \text{with } c_{12} = c_{11}^{p\eta} > 0. \end{aligned} \tag{32}$$

Returning to (31) and using (32), we obtain

$$e_u(\widehat{t}) \geq \left[\frac{1}{p} - \frac{1}{\eta}\right]\left[\frac{\gamma_p(u)^\eta}{c_{12}\gamma_p(u)^\eta}\right]^{\frac{1}{\eta-p}} = c_{13} > 0 \tag{33}$$

(the constant $c_{13} > 0$ is independent of $u \in N_\lambda^-$).

Then we have

$$\begin{aligned} & \frac{\widehat{t}^\tau}{\tau}\int_\Omega a(z)|u|^\tau dz \\ & \leq \frac{\widehat{t}^\tau}{\tau}c_{14}\|u\|^\tau \quad \text{for some } c_{14} > 0 \text{ (see (6))} \end{aligned}$$

$$\leq \frac{1}{\tau} c_{15} \left[\frac{\gamma_p(u)}{\int_\Omega \vartheta(z)|u|^\eta dz} \right]^{\frac{\tau}{\eta-p}} \gamma_p(u)^{\frac{\tau}{p}} \quad \text{for some } c_{15} > 0$$

$$\text{(recall the value of } \widehat{t} > 0 \text{ and use (1))}$$

$$\leq \frac{1}{\tau} c_{15} \left[\frac{\gamma_p(u)^\eta}{\left(\int_\Omega \vartheta(z)|u|^\eta dz\right)^p} \right]^{\frac{\tau}{p(\eta-p)}}$$

$$\leq c_{16} e_u(\widehat{t})^{\frac{\tau}{p}} \quad \text{for some } c_{16} > 0 \text{ (see (33)).} \tag{34}$$

It follows that the fibering function $\mu_{\lambda,u}(\cdot)$ satisfies

$$\begin{aligned} \mu_{\lambda,u}(\widehat{t}) &= \frac{\widehat{t}^p}{p}\gamma_p(u) - \frac{\widehat{t}^\eta}{\eta}\int_\Omega \vartheta(z)|u|^\eta dz - \lambda \frac{\widehat{t}^\tau}{\tau}\int_\Omega a(z)|u|^\tau dz \\ &\geq e_u(\widehat{t}) - \lambda c_{16} e_u(\widehat{t})^{\frac{\tau}{p}} \quad \text{(see (34))} \\ &= \left[e_u(\widehat{t})^{\frac{p-\tau}{p}} - \lambda c_{16} \right] e_u(\widehat{t})^{\frac{\tau}{p}} \\ &\geq \left[c_{13}^{\frac{p-\tau}{p}} - \lambda c_{16} \right] e_u(\widehat{t})^{\frac{\tau}{p}} \quad \text{(see (33)).} \end{aligned}$$

So, we can find $\widehat{\lambda}_0^* \in (0, \widehat{\lambda}^*]$ small such that

$$\mu_{\lambda,u}(\widehat{t}) > 0 \quad \text{for all } \lambda \in (0, \widehat{\lambda}_0^*).$$

Then we have

$$\begin{aligned} & \varphi_\lambda(u) = \mu_{\lambda,u}(1) \geq \mu_{\lambda,u}(\widehat{t}) > 0 \quad \text{(since } u \in N_\lambda^-), \\ \Rightarrow \quad & \varphi_\lambda\big|_{N_\lambda^-} > 0. \end{aligned}$$

□

Proposition 6. *If hypotheses H_1, H_2 hold and $\lambda \in (0, \widehat{\lambda}_0^*)$, then there exists $v_\lambda^* \in N_\lambda^-$ such that $m_\lambda^- = \varphi_\lambda(v_\lambda^*) > 0$.*

Proof. The proof is similar to that of Proposition 4.

Consider a minimizing sequence $\{v_n\}_{n\geq 1} \subseteq N_\lambda^-$, that is

$$\varphi_\lambda(v_n) \downarrow m_\lambda^- \quad \text{as } n \to +\infty.$$

From Proposition 1, we have that

$$\{v_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

Hence, by passing to a subsequence if necessary, we may assume that

$$v_n \xrightarrow{w} v_\lambda^* \text{ in } W^{1,p}(\Omega) \text{ and } v_n \to v_\lambda^* \text{ in } L^\eta(\Omega) \text{ and in } L^p(\partial\Omega). \tag{35}$$

From the proof of Proposition 4, we know that there exists $t_2 > t_0$ such that

$$t_2 v_\lambda^* \in N_\lambda^- \text{ and } \varphi_\lambda(t_2 v_\lambda^*) \geq \varphi_\lambda(t_0 v_\lambda^*). \tag{36}$$

<u>Claim:</u> $v_n \to v_\lambda^*$ in $W^{1,p}(\Omega)$ as $n \to +\infty$.

Arguing by contradiction, suppose that the Claim is not true. Then on account of (35) we have

$$\gamma_p(v_\lambda^*) < \liminf_{n\to+\infty} \gamma_p(v_n). \tag{37}$$

Since $v_n \in N_\lambda^-$, we have

$$\begin{aligned}
&\varphi_\lambda(tv_n) \le \varphi_\lambda(v_n) \quad \text{for all } t>0, \text{ all } n\in\mathbb{N} \text{ (see [3], p. 6)} \qquad (38)\\
\Rightarrow\quad &\varphi_\lambda(t_2 v_\lambda^*) < \liminf_{n\to+\infty}\left[\frac{t_2^p}{p}\gamma_p(u_n) - \frac{\lambda t_2^\tau}{\tau}\int_\Omega a(z)|v_n|^\tau dz - \frac{t_2^\eta}{\eta}\int_\Omega \vartheta(z)|v_n|^\eta dz\right]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(see (37) and (35))}\\
&= \liminf_{n\to+\infty}\varphi_\lambda(t_2 v_n)\\
&\le \liminf_{n\to+\infty}\varphi_\lambda(v_n) = m_\lambda^-,
\end{aligned}$$

a contradiction since $t_2 v_\lambda^* \in N_\lambda^-$ (see (36)). So, the Claim is true and we have

$$v_n \to v_\lambda^* \text{ in } W^{1,p}(\Omega).$$

From this fact as in the proof of Proposition 4, we obtain

$$\varphi_\lambda(v_\lambda^*) = m_\lambda^- \text{ and } v_\lambda^* \in N_\lambda^-.$$

□

Proposition 7. *If hypotheses H_1, H_2 hold, $u \in N_\lambda$ is a local minimizer of $\varphi_\lambda\big|_{N_\lambda}$ and $u \notin N_\lambda^0$, then $u \in K_{\varphi_\lambda}$.*

Proof. Recall that $k_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ is the C^1-functional defined by

$$k_\lambda(u) = \gamma_p(u) - \lambda\int_\Omega a(z)|u|^\tau dz - \int_\Omega \vartheta(z)|u|^\eta dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

We consider the following constrained minimization problem

$$\inf\left[\varphi_\lambda(u) : k_\lambda(u) = 0,\ u \ne 0\right]. \tag{39}$$

On account of the hypothesis and (2), $u \in N_\lambda$ is a local minimizer of (39). Using the Lagrange multiplier rule (see Papageorgiou-Rădulescu-Repovš [14], p. 422), we can find $s \ge 0$ such that

$$\begin{aligned}
&\varphi_\lambda'(u) + s k_\lambda'(u) = 0,\\
\Rightarrow\quad & s\langle k_\lambda'(u), u\rangle = 0 \quad (\text{since } u \in N_\lambda). \qquad (40)
\end{aligned}$$

But by hypothesis $u \notin N_\lambda^0$. So, from (40) it follows that $s = 0$. Hence

$$\begin{aligned}
&\varphi_\lambda'(u) = 0,\\
\Rightarrow\quad & u \in K_{\varphi_\lambda}.
\end{aligned}$$

□

Now we are ready for the multiplicity theorem for problem (P_λ).

Theorem 1. *If hypotheses H_1, H_2 hold, then there exists $\widehat{\lambda}_0^* > 0$ such that for all $\lambda \in (0, \widehat{\lambda}_0^*)$ problem (P_λ) has at least two positive solutions $u_\lambda^*, v_\lambda^* \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$ $(0 < \alpha < 1)$ such that $u_\lambda^*(z) > 0$, $v_\lambda^*(z) > 0$ for all $z \in \Omega$, $\varphi_\lambda(u_\lambda^*) < 0 < \varphi_\lambda(v_\lambda^*)$.*

Proof. From Propositions 2, 4 and 6, we know that there exists $\widehat{\lambda}_0^* > 0$ such that for all $\lambda \in (0, \widehat{\lambda}_0^*)$ we can find $u_\lambda^* \in N_\lambda^+$ and $v_\lambda^* \in N_\lambda^-$ such that

$$u_\lambda^* \text{ and } v_\lambda^* \text{ are local minimizers of } \varphi_\lambda\big|_{N_\lambda}, \tag{41}$$

$$N_\lambda^0 = \emptyset. \tag{42}$$

From (41), (42) and Proposition 7 it follows that

$$u_\lambda^*, v_\lambda^* \in K_{\varphi_\lambda}.$$

Also from Propositions 4 and 6, we have

$$\varphi_\lambda(u_\lambda^*) < 0 < \varphi_\lambda(v_\lambda^*).$$

We have that u_λ^*, v_λ^* are weak solutions of (P_λ). From Proposition 2.10 of Papageorgiou-Rădulescu [11], we have that $u_\lambda^*, v_\lambda^* \in L^\infty(\Omega)$. Since $\varphi_\lambda(u) = \varphi_\lambda(|u|)$ for all $u \in W^{1,p}(\Omega)$, we may assume that $u_\lambda^*, v_\lambda^* \geq 0$.

Then using Theorem 7.2.1 (Harnack inequality) and Theorem 7.3.1 (local regularity) of Pucci-Serrin [16], we have that

$$\begin{aligned} &u_\lambda^*, v_\lambda^* \in C^{0,\alpha}(\Omega) \quad \text{for some } \alpha \in (0,1), \\ &u_\lambda^*(z) > 0, v_\lambda^*(z) > 0 \quad \text{for all } z \in \Omega. \end{aligned}$$

□

If we strengthen the conditions on the data of (P_λ), we can improve the regularity of the two solutions.

H_1': $\xi \in L^\infty(\Omega)$, $\beta \in C^{0,\alpha}(\partial\Omega)$ $(0 < \alpha < 1)$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$, $\beta(z) \geq 0$ for all $z \in \partial\Omega$ and at least one of them is nonzero.

H_2': $a \in L^\infty(\Omega)$, $\vartheta \in L^\infty(\Omega)$ and $\vartheta(z) \geq 0$ for a.a. $z \in \Omega$.

Proposition 8. *If hypotheses H_1', H_2' hold and $\lambda \in (0, \widehat{\lambda}_0^*)$, then $u_\lambda^*, v_\lambda^* \in \operatorname{int} C_+$.*

Proof. From Theorem 1 we have two nontrivial solutions

$$u_\lambda^*, v_\lambda^* \in W^{1,p}(\Omega) \cap L^\infty(\Omega).$$

Invoking Theorem 2 of Lieberman [8], we have $u_\lambda^*, v_\lambda^* \in C_+$. In fact the nonlinear maximum principle of Pucci-Serrin [16] (p. 120) implies that $u_\lambda^*, v_\lambda^* \in \operatorname{int} C_+$.

□

References

[1] A. Ambrosetti, H. Brezis, G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal., **122** (1994), 519–543.

[2] K.J. Brown, *The Nehari manifold for a semilinear elliptic equation involving a sublinear term*, Calc. Var., **22** (2005), 483–494.

[3] K.J. Brown, T.-F. Wu, *A fibering map approach to a semilinear elliptic boundary value problem*, Electron. J. Differential Equations, **2007**:69 (2007), 1–9.
[4] K.J. Brown, Y. Zhang, *The Nehari manifold for a semilinear elliptic problem with a sign changing weight function*, J. Differential Equations, **193** (2003), 481–499.
[5] J. Garcia Azero, J. Manfredi, I. Peral Alonso, *Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations*, Commun. Contemp. Math., **2** (2000), 385–404.
[6] L. Gasiński, N.S. Papageorgiou, *Positive solutions for the Robin p-Laplacian problem with competing nonlinearities*, Adv. Calc. Var., **12** (2019), 31–56.
[7] Z. Guo, Z. Zhang, *$W^{1,p}$ versus C^1 local minimizers and multiplicity results for quasilinear elliptic equations*, J. Math. Anal. Appl., **286** (2003), 32–50.
[8] G.M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal., **12** (1988), 1203–1219.
[9] D. Mugnai and N.S. Papageorgiou, *Resonant nonlinear Neumann problems with indefinite weight*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), **XI** (2012), 729–788.
[10] N.S. Papageorgiou, V.D. Rădulescu, *Multiple solutions for asymptotically linear equations with sign-changing weight*, Kyoto J. Math., **55** (2015), 593–606.
[11] N.S. Papageorgiou, V.D. Rădulescu, *Nonlinear nonhomogeneous Robin problems with superlinear reaction term*, Adv. Nonlinear Stud., **16** (2016), 737–764.
[12] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, *Robin problems with indefinite linear part and competition phenomena*, Comm. Pure Appl. Anal., **16** (2017), 1293–1314.
[13] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, *Nonlinear Analysis - Theory and Methods*, Springer, Switzerland, (2019).
[14] N.S. Papageorgiou, P. Winkert, *Positive solutions for nonlinear nonhomogeneous Dirichlet problems with concave-convex nonlinearites*, Positivity, **20** (2016), 945–979.
[15] N.S. Papageorgiou, P. Winkert, *Applied Nonlinear Functional Analysis*, Walter de Gruyter GmbH & Co., Berlin (2018).
[16] P. Pucci, J. Serrin, *The Maximum Principle*, Birkhäuser, Basel (2007).
[17] K. Silva, A. Macedo, *Local minimizers over the Nehari manifold for a class of concave-convex problems with sign changing nonlinearity*, J. Differential Equations, **265** (2018), 1894–1921.
[18] T.-F. Wu, *On semilinear elliptic equations involving concave-convex nonlinearities and sign changing weight function*, J. Math. Anal. Appl., **318** (2006), 253–270.

(N.S. Papageorgiou) National Technical University, Department of Mathematics, Zografou campus, Athens 15780, Greece
E-mail address: npapg@math.ntua.gr

(C. Vetro) University of Palermo, Department of Mathematics and Computer Science, Via Archirafi 34, 90123, Palermo, Italy
E-mail address: calogero.vetro@unipa.it

(F. Vetro) 90123, Palermo, Italy
E-mail address: francescavetro80@gmail.com