Journal of Hydroinformatics

 Journal of Hydroinformatics Vol 26 No 7, 1558 doi: 10.2166/hydro.2024.280

A new approach for explicit approximation of the Colebrook–White formula for pipe flows

Giovanni B. Ferreri

Università degli Studi di Palermo, Dipartimento di Ingegneria, Viale delle Scienze Ed. 8, Palermo IT-90128, Italy
E-mail: giovannibattista.ferreri@unipa.it

GBF, 0000-0002-2468-4270

ABSTRACT

A novel approach is presented for explicit assessment of the friction factor of the Darcy–Weisbach formula for a pipe turbulent flow, a topic being especially useful in practical applications requiring a large number of pipes has to be calculated a great many times in a short time. In such applications, in fact, an explicit formula shortening computation time, with respect to a trial-and-error solution of the Colebrook–White formula, is very advisable. To this aim, in the paper analytical simplicity is pursued besides accuracy, the latter being kept high enough for practical purposes. Unlike previous studies, the ratio between the actual friction factor and that relating to a fully turbulent flow is analysed as a function of the relative roughness and the relative Reynolds number (i.e., the ratio between the actual Reynolds number and the Reynolds number separating the transition regime from the fully turbulent regime). By processing a dataset obtained by systematically solving the C-W formula over suited ranges of the Reynolds number and the relative roughness, two expressions are obtained: a simpler first-step accuracy expression giving generally acceptable accuracy for most engineering practical purposes; and a second-step accuracy expression allowing adequately high accuracy for all situations.

Key words: Colebrook–White formula, explicit solution, friction factor, friction head-loss, pipe liquid flow

HIGHLIGHTS

- A novel approach is presented for an explicit approximation of the Colebrook–White (C-W) formula.
- Two expressions giving acceptable accuracy for most engineering practical purposes are proposed.
- Approximated explicit solutions of the C-W formula are very useful to noticeably shorten computation time in several practical applications requiring a large number of pipes has to be calculated a great many times in a short time.

LIST OF SYMBOLS

a	ratio between the actual friction factor and that relating to a fully turbulent regime
a^{I}	first-step a value assessed by the approximate expression obtained by the writer
b, c	parameters of the approximate expression for a
D	pipe diameter
g	gravity acceleration
h_f	friction head-loss
J	friction head-loss per unit length of pipe
k	sand grain size in Nikuradse's experiments
L	pipe length
Q	flow rate
Re	Reynolds number
Re'	Reynolds number value at which smooth regime ends in Nikuradse's experiments
Re''	Reynolds number value separating transition regime from the fully turbulent regime
Re^*	friction Reynolds number
V	mean flow velocity
V^*	friction velocity
ε	absolute pipe roughness

λ friction factor
λ_l, λ_0, λ_∞ λ values relating, respectively, to laminar, smooth and fully turbulent regimes
λ^I, λ^{II} λ values assessed by, respectively, first-step (a^I) and second-step accuracy (a^I plus C-W formula)
$\Delta\lambda$ percentage relative error between assessed and actual λ value
ν kinematic viscosity

1. INTRODUCTION

The law of flow resistance in circular pipes is fundamental for all calculations relating to pipe flows, such as, for instance, waterworks, water distribution networks, fire-fighting systems, oil pipelines and industrial plants in general. This law expresses friction head-loss as a function of the physical and geometrical characteristics of the flow and pipe. For a laminar flow, the law is given by the empirical Hagen–Poiseuille formula, which can be theoretically obtained as an application of the Navier–Stokes equations, whereas for a turbulent flow a few theoretical treatments exist, which, however, require all the same the support of experiments to suitably model the flow–wall interaction and to assess coefficients and exponents. In practical cases, a pipe turbulent flow relates to all civil applications and most industrial applications, whereas only some industrial applications involve a pipe laminar flow. Several formulas, empirical or semi-empirical, have been given in the last two centuries for friction head-loss in turbulent flows, each relating to the flow regime examined by the respective authors (namely, smooth, transitional or fully turbulent). The best-known formulas are those of Chézy, Darcy–Weisbach, Kutter, Manning and Hazen–Williams. Furthermore, several exponential pipe-friction formulas have been given by pipe industries derived from experiments on their pipes. However, since a few decades back the Darcy–Weisbach formula has been adopted worldwide as a *general* formula (i.e., for all flow regimes in pipes), estimating just the *friction factor* by a suitable formula relating to the specific flow regime (usually, for turbulent flows, the two Prandtl–von Kármán formulas for smooth pipe and fully turbulent regimes, and the Colebrook–White formula for the transitional flow regime). In practice, for estimation of the friction factor, the sole Colebrook–White (C-W) formula is usually adopted over the whole turbulent regime, both smooth and transitional and fully turbulent, in virtue of its asymptotic trends for the lower and higher Reynolds numbers.

Actually, further investigations on the transitional regime in the presence of a wall surface honed and polished down to an extremely low relative roughness of about 10^{-6} (Shockling *et al.* 2006) showed, unlike the C-W formula, an inflectional trend of friction factor vs. Reynolds number, analogous to that of Nikuradse's experiments with artificial sand grain roughness. Other experiments on commercial steel pipe having a honed surface too (Langelandsvik *et al.* 2008) confirmed a descending trend like the C-W formula but exhibiting, with respect to the C-W curve relating to the same relative roughness, delayed (i.e., higher Reynolds number) and more abrupt detaching of the experimental points from the smooth regime curve. In both cases having an artificially equalized roughness, therefore, the C-W formula proves to overestimate the friction factor, up to almost 10% in the former case and a few percent in the latter case. Such errors are noticeable from the point of view of an exact description of the physical phenomenon, but for practical purposes they could be accepted when falling within uncertainty ranges affecting the data and running conditions of the examined pipe or plant. In addition, it is very useful to adopt a sole formula for the whole turbulent regime (both smooth and transitional and fully turbulent) without selecting a specific one depending on both the Reynolds number and the relative roughness and the pipe-wall finish. All this explains practical interest in the C-W formula, up to now the only accepted and the most used for friction factor assessment over the whole turbulent regime.

Unfortunately, in most practical applications, the friction factor is not given explicitly by the C-W formula, and a trial-and-error procedure is needed for its determination. Actually, in many cases, this is not a problem as nowadays every computer or simple calculator has an equation solver, but in cases where a big number of pipes has to be calculated a great many times in a short time, as for example in long-term simulation of long oil pipelines (Clamond 2009) or in water-supply network optimization problems for sizing up pipes or management of the networks themselves (e.g., Cunha & Sousa 1999; Marques *et al.* 2015b), an explicit formula giving a good approximation of the friction factor would prove to be very useful to noticeably shorten computation time. This would even allow management decisions in real time.

Since the publication of the C-W formula (Colebrook 1939) and the subsequent Moody Chart (Moody 1944), a number of researchers have proposed more or less simple and more or less approximate formulas for explicit estimation of the friction factor for given Reynolds number and relative roughness, which is the most usual problem to solve in pipe calculations, as will be specified in the next section. Reviews of the best-known formulas were made, among others, by Genić *et al.* (2011), Brkić (2011a) and Winning & Coole (2013). Some formulas result from adjustment of one or both terms forming the argument of the logarithm (e.g., Churchill 1973; Swamee & Jain 1976; Bonvissuto 1977; Haaland 1983), others from successive approximations of the friction factor inside the argument of the logarithm (e.g., Chen 1979; Zigrang & Sylvester 1982; Praks & Brkić 2018), others from suited mathematical manipulation and subsequent approximation of the formula terms (e.g., Barr 1981; Sonnad & Goudar 2007; Giustolisi *et al.* 2011; Vatankhah 2018), others from series expansion truncated after a few terms (e.g., Romeo *et al.* 2002; Rollmann & Spindler 2015; Hafsi 2021; Lamri & Easa 2022), others from use of the Lambert function W (e.g., Keady 1998; Sonnad & Goudar 2004; Brkić 2011b), and others more from best-fitting among several basic equations to a big dataset obtained by systematically solving the C-W formula (e.g., Azizi *et al.* 2019). Recently, Avci & Karagoz (2019) have proposed a sole formula that correctly combines the effects of viscosity and roughness, valid over both the laminar regime and the whole turbulent regime. Such a formula is especially useful in applications of industrial type where, differently from civil cases, both laminar and turbulent flow may be met. Generally, simpler formulas give worse approximations while very good approximations relate to really elaborate formulas, so that none of the known formulas prove to be fully satisfactory. Moreover, approximation usually varies appreciably with relative roughness and Reynolds number. Therefore, further solutions are still proposed in the technical literature, which by itself shows interest in the topic.

The present paper deals with a novel approach to assess the friction factor by approximate explicit formulas. Unlike many past studies, mainly targeted at increasing the accuracy of the approximate friction factor but paying a price in terms of simplicity and handiness of the related formulas as well as of computation time, in the present paper, simplicity is firstly considered besides accuracy which, however, is kept high enough for practical purposes, as shown in the next sections. To the aim, a big dataset was generated by systematically solving the C-W formula over suited ranges of the Reynolds number and the relative roughness and then processed suitably for best-fitting of the considered mathematical expressions. The final issues are two explicit expressions for the friction factor: a simpler first-step accuracy expression, having not so intricate a mathematical structure as others of the technical literature that gives generally acceptable accuracy for most engineering practical purposes; and a less simple second-step accuracy expression always giving fully adequate accuracy for all practical purposes.

2. DETERMINATION OF THE FRICTION FACTOR FOR SOLUTION OF PRACTICAL PROBLEMS

The head-loss in a circular pipe due to wall friction, h_f, is usually calculated by the Darcy–Weisbach formula:

$$h_f = \frac{\lambda}{D}\frac{V^2}{2g}L \rightarrow J = \frac{h_f}{L} = \frac{\lambda}{D}\frac{V^2}{2g} = \frac{8}{g\pi^2}\lambda\frac{Q^2}{D^5} \tag{1}$$

where λ is the friction factor, D is the pipe diameter, V is the flow velocity, g is gravity acceleration, L is the pipe length, J is the unit head-loss (i.e., the friction head-loss per unit length of pipe), and Q is the flow rate. Suitable evaluation of the friction factor λ, based on the specific values of the relative roughness ε/D and of the Reynolds number

$$Re = \frac{V\,D}{\nu} = \frac{4}{\pi}\frac{Q}{\nu D} \tag{2}$$

where ε is the pipe roughness and ν is the kinematic viscosity, allows the formula to be adopted for all flow regimes.

It is well known from the experiments by Nikuradse (1933) and Colebrook & White (1937), relating to pipes having, respectively, artificial sand roughness and natural roughness, that in a *laminar regime* (i.e., $Re < 2{,}000$), the friction factor, here indicated as λ_l, is a function of the Reynolds number only:

$$\lambda_l = \frac{64}{Re} \tag{3}$$

As a turbulent regime is steady (i.e., $Re > 4{,}000$), if the roughness protrusions are so small that they are completely contained within the laminar sub-layer, even a rough pipe practically behaves in the same way as a zero-roughness pipe and a *smooth regime* occurs. The friction factor, here indicated as λ_0, is given by the Prandtl–von Kármán formula (Schlichting 1955):

$$\frac{1}{\sqrt{\lambda_0}} = -2\log\frac{2.51}{Re\sqrt{\lambda_0}} \tag{4}$$

whose coefficients were obtained by processing data up to $Re \sim 3.2 \cdot 10^6$. Actually, experiments carried out by Zagarola & Smits (1998) up to $Re \sim 35 \cdot 10^6$ led the authors themselves and later McKeon *et al.* (2005) to suggest slight adjustments of the coefficients −2 and 2.51 of Equation (4), for better approximation of the whole data.

A smooth regime occurs until *Re* reaches the value *Re*′, a decreasing function of the relative roughness, for which the higher roughness protrusions come out of the laminar sub-layer starting to affect turbulence and resistance to flow. For artificial sand roughness, Nikuradse (1933) found this situation to occur as:

$$Re^* = \frac{V^*\, k}{\nu} \approx 5 \rightarrow Re' \approx \frac{5\,\sqrt{8}}{\sqrt{\lambda_0}\,\frac{k}{D}} \tag{5}$$

where Re^* and $V^* = V\sqrt{(\lambda/8)}$ are, respectively, the friction Reynolds number and the friction velocity, and k is the sand grain size. Actually, for natural roughness, characterized by variously mixed protuberances, an expression for predicting *Re*′ has not been given so far, as the rough regime actually starts as the *highest* roughness protuberances shed eddies (Colebrook & White 1937), the highest protuberances being unknown for any given *equivalent* roughness ε.

For the higher Reynolds numbers, i.e., $Re > Re''$, the latter being the value (a function of ε/D only) for which the roughness protrusions emerge almost completely from the laminar sub-layer and the resistance to flow is practically due only to drag on them, a *fully turbulent regime* occurs. The friction factor, which does not depend on *Re* anymore and is indicated here as λ_∞, is given by the Prandtl–von Kármán formula (Schlichting 1955):

$$\frac{1}{\sqrt{\lambda_\infty}} = -2\,\log\left(\frac{1}{3.71}\frac{\varepsilon}{D}\right) \tag{6}$$

As for Re'', the experiments by Nikuradse (1933) led the author to assess a fully turbulent regime started when Re^* was about 60, but for natural roughness a value of 70 is usually considered more suitable; thus:

$$Re^* = \frac{V^*\varepsilon}{\nu} \approx 70 \rightarrow Re'' \approx \frac{70\,\sqrt{8}}{\sqrt{\lambda_\infty}\,\frac{\varepsilon}{D}} \tag{7}$$

where the Nikuradse sand roughness k is replaced by the natural equivalent roughness ε.

In the *transition regime* in a rough pipe, i.e., $Re' < Re < Re''$, the experiments of both Nikuradse (1933) and Colebrook & White (1937) showed the friction factor to depend on both the Reynolds number and the relative roughness, but a theoretical-type expression, like Equations (4) and (6), has never been found. However, based on the structural analogies between Equations (4) and (6), and the mildly descending trend of experimental data from the smooth regime curve to the fully turbulent regime horizontal line, Colebrook (1939) proposed the following formula:

$$\frac{1}{\sqrt{\lambda}} = -2\,\log\left(\frac{2.51}{Re\,\sqrt{\lambda}} + \frac{1}{3.71}\frac{\varepsilon}{D}\right) \tag{8}$$

often referred to as Colebrook–White's (C-W), which matched rather well experimental data provided by Colebrook and White themselves (1937) and by other researchers considered by the author. Actually, because of its structure, for lower *Re* values (i.e., $Re \rightarrow 4{,}000$; in practice $Re < Re'$, the latter not being specified for natural roughness), the C-W formula gives λ values close to the corresponding λ_0 as well as for higher *Re* values (in practice, $Re > Re''$) close to the corresponding λ_∞. Therefore, for practical purposes, this sole formula is usually accepted over the whole turbulent regime (i.e., $Re > 4{,}000$). This is advantageous because it avoids having to determine Re'' (Re' is not specified) and to place the actual *Re* with respect to it for selecting the suited formula for λ. Rouse (1943) carried out further validation of the C-W formula by matching the fitting of the latter to the measurements of other researchers too. In particular, Rouse pointed out that the deviations of the experimental points from Equation (8) were not much greater than the experimental scatter of the individual measurements in any one series, thus attesting the practical acceptability of the C-W formula. The latter, therefore, was assumed as *a benchmark* for the description of resistance to flow in pipes. The Moody chart (Moody 1944) represents Equation (8) for several relative roughnesses ε/D ranging between 0.000001 and 0.05, covering almost all practical situations.

Practical problems concerning a single pipe, for given ν, ε and L, consist in determining one of the three variables h_f (or J), Q and D, the other two being known. Available equations are (1), (2) and (8). A first-type problem is the determination of Q for given J and D (and then ε/D), which is the only problem one can solve directly, i.e., without implementation of a trial-and-error method. In fact, although neither Re (Equation (2)) nor λ (Equation (1)) are known, the product

$$Re^2\lambda = \frac{2gJD^3}{\nu^2} \tag{9}$$

is known, which allows Equation (8) to become explicit with respect to λ. A second-type problem is the determination of J for given Q and D. In this case, Re is known from Equation (2), but the implicit structure of Equation (8) requires a trial-and-error method for calculation of λ. A third-type problem, pipe sizing, is the determination of D for given Q and J. Neither Re (Equation (2)) nor λ (Equation (1)) nor ε/D are known, and implementation of a trial-and-error method is necessary. However, in practical cases, commercial diameters have to be adopted, with the further restriction that flow velocities have to fall, usually, in the range 0.5–2 m/s. Therefore, pipe sizing is often carried out by the following procedure: (i) choosing a commercial diameter allowing a flow velocity falling in the range above, (ii) determining the relating head-loss (solution of a second-type problem) and (iii) checking the latter is less than the available head-drop. In short, practical pipe sizing turns on the solution of a *second-type problem* under constraints concerning (*a*) the use of commercial diameters, (*b*) allowed flow velocity and (*c*) available head-drop. Therefore, any approximated explicit solution of Equation (8) has to be targeted at the solution of a second-type problem: determination of J for given Q, D and ε, i.e., determination of λ for given values of Re and ε/D.

3. A NEW APPROACH FOR ESTIMATION OF THE FRICTION FACTOR

The new approach stems from the observation of the Moody chart, in which each curve relating to a relative roughness ε/D exhibits a descending trend asymptotic to a specific straight line having equation $\lambda = \lambda_\infty$, where λ_∞ is given by Equation (6). This suggests that λ can be expressed as a function of λ_∞ by the relationship

$$\lambda = a\,\lambda_\infty \tag{10}$$

where, for the fixed ε/D, the coefficient $a > 1$ given by

$$a = \frac{\lambda}{\lambda_\infty} \tag{11}$$

is a decreasing function of Re tending to the horizontal asymptote $a = 1$. Determination of a relationship $a = f\,(Re, \varepsilon/D)$, from which λ by Equation (10), is the core of the novel approach and the goal of the present paper.

Introducing Equation (10) in Equation (8) the latter becomes:

$$\frac{1}{\sqrt{a\,\lambda_\infty}} = -2\log\left(\frac{2.51}{Re\sqrt{a\,\lambda_\infty}} + \frac{1}{3.71}\frac{\varepsilon}{D}\right) = -2\log\left[\frac{1}{3.71}\frac{\varepsilon}{D}\left(\frac{2.51\cdot 3.71}{Re\sqrt{a\,\lambda_\infty}\frac{\varepsilon}{D}} + 1\right)\right]$$

which, taking Equation (7) into account, gives:

$$\frac{1}{\sqrt{a\,\lambda_\infty}} = -2\log\left(\frac{1}{3.71}\frac{\varepsilon}{D}\right) - 2\log\left(\frac{2.51\cdot 3.71}{70\sqrt{8}\sqrt{a}\frac{Re}{Re''}} + 1\right)$$

and finally, considering Equation (6):

$$\frac{1}{\sqrt{a}} = 1 - 2\sqrt{\lambda_\infty}\log\left(\frac{2.51\cdot 3.71}{\sqrt{a}\,70\sqrt{8}\frac{Re}{Re''}} + 1\right) \tag{12}$$

According to Equation (12), a is a function of ε/D (through λ_∞) and the *relative Reynolds number*, Re/Re'', referred to the specific Re'' value, which in turn is a function of ε/D only (Eq. (7)).

Actually, solving Equation (12) for a presents the same difficulty as Equation (8) for λ. However, the representation of a vs. log (Re/Re'') (Figure 1) shows a family of mildly descending curves, each relating to an ε/D value, all being asymptotic, as expected, to the horizontal line $a = 1$. In the figure, 16 ε/D values are considered ranging between 0.0001 and 0.05, and only the stretch of each curve relating to the range $4{,}000 \leq Re \leq Re''$ is represented, as for $Re > Re''$ the curve asymptotic trend guarantees a sufficient approximation of any well-fitting curve that might be proposed for the range $4{,}000 \leq Re \leq Re''$. Each curve is labelled by the relating ε/D value placed close to the point having the lowest Re/Re'' value. The figure shows that as ε/D increases the relating curve moves upwards and shortens noticeably. The curves become very close to one another as $Re/Re'' \rightarrow 1$ whereas they stray more and more from one another as Re/Re'' decreases. Moreover, for given ε/D, the maximum a value (relating to $Re = 4{,}000$) reaches a few units, as well as the lower ε/D the higher the maximum a value. The next step consisted in searching for a simple mathematical expression well-fitting the curves of Figure 1, by processing a dataset obtained by systematically solving Equation (12), as specified below, which will be referred to as Dataset 1. The analysis concerned only the data relating to the ε/D values of Figure 1 (i.e., $\varepsilon/D \geq 0.0001$), because the data relating to lower ε/D proved to bear noticeably worse fit to the simple mathematical expressions tested. However, as will be shown in Section 4, the results can be accepted even in most practical situations relating to $\varepsilon/D < 0.0001$.

Dataset 1 was obtained as follows. For each relative roughness $(\varepsilon/D)_i$ ($i = 1, 2, \ldots, 16$), $\lambda_{\infty,i}$ was first determined by Equation (6), from which the related Re_i'' by Equation (7). Then, the range [log 4000, log Re_i''] was divided into 100 equal parts, obtaining 101 equidistant values of log Re, from which 101 Re values. Finally, Equation (12) was solved for a for each Re/Re_i'') value, by a trial-and-error method with an accuracy in the difference between left and right members set at 10^{-13}.

To detect an explicit expression well-fitting the curves of Figure 1, a few simple relationships relating to descending asymptotic curves to the line $a = 1$ were tested. To this aim, considering in turn the i-th relative roughness $(\varepsilon/D)_i$ ($i = 1, 2, \ldots, 16$), each relationship was fitted to the 101 points of Dataset 1 relating to the considered $(\varepsilon/D)_i$ by the least squares method, and then it was compared with the corresponding original curve of Figure 1. The comparison was carried out through the percentage relative error, taken in absolute value, given by

$$|\Delta a^{\mathrm{I}}| = \left|\frac{a^{\mathrm{I}} - a}{a}\right| \cdot 100 = \left|\frac{\lambda^{\mathrm{I}} - \lambda}{\lambda}\right| \cdot 100 = |\Delta\lambda^{\mathrm{I}}| \tag{13}$$

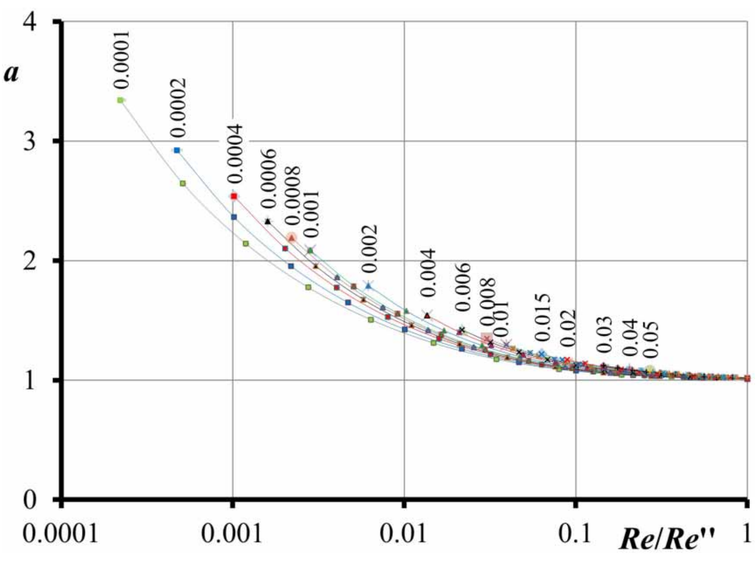


Figure 1 | Mildly descending trend of the curves a vs. log (Re/Re'') for the 16 ε/D values considered.

were a and λ are the actual values (i.e., the values given by the solution of Equations (12) and (8), respectively, a^{I} is the a value estimated by the fitted relationship, and λ^{I} the related λ value given by

$$\lambda^{\mathrm{I}} = a^{\mathrm{I}}\lambda_\infty \tag{14}$$

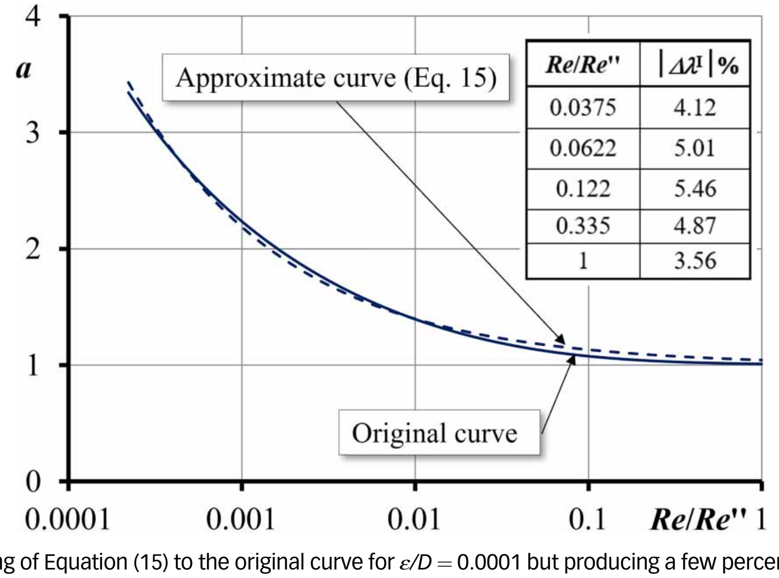
Therefore, for each relationship tested, a set of the errors over total Dataset 1 (i.e., concerning all the 16 $(\varepsilon/D)_i$ values) was obtained, which was then analysed to detect the relationship accuracy. In the end, weighing both accuracy and simplicity of the tested relationships, the following one was chosen:

$$a^{\mathrm{I}} = 1 + b\, c^{\log(Re/Re'')} \tag{15}$$

in which b and c are two coefficients depending on ε/D only. The coefficients b_i and c_i relating to each $(\varepsilon/D)_i$ ($i = 1, 2, \ldots, 16$) were determined, as already specified above, by fitting Equation (15) to the points of Dataset 1 relating to $(\varepsilon/D)_i$, by the least squares method. The whole coefficients b_i and c_i ($i = 1, 2, \ldots, 16$) constituted further Dataset 2.

The semi-logarithmic diagram in Figure 2 shows, as an example, the fitting of Equation (15) to the actual curve for $\varepsilon/D = 0.0001$. Although a general good fit ($R^2 = 0.996$), for the higher Re/Re'' values the error $|\Delta\lambda^{\mathrm{I}}|$ reaches a few percent (box in Figure 2), because of the slightly different curvature of the actual and approximate curves in that zone. Actually, such higher errors only concern the lower relative roughnesses ε/D and sensibly decrease down to tenths or even hundredths percent for higher ε/D values, for which the curve shortening and the absence of the left curve stretch with the higher curvature allow Equation (15) to better fit the actual curves. The mean and standard deviation of $|\Delta\lambda^{\mathrm{I}}|$ over the whole 16 relative roughnesses were equal, respectively, to 0.897 and 1.17%, an approximation that, in general, proves to be acceptable for practical engineering purposes, especially in water systems. However, this result was sensibly improved by assuming λ^{I} (Equation (14)) as a first-step approximation, and then using it as a starting point for the well-known iterative solution of a fixed-point problem (e.g., Traub 1964; Phillips & Taylor 1973), applied to the implicit C-W formula. According to this method, after an implicit equation $f(x) = 0$, having $x = \alpha$ among its zeros, is manipulated to one of the possible forms $x = \Phi(x)$, the value α can be found by the algorithm $x_{i+1} = \Phi(x_i)$ ($i = 0, 1, \ldots, n$; $n \to \infty$), provided a suitable convergence condition to be verified (as it occurs for the C-W equation). The higher n is the closer x_{i+1} to α. Therefore, introducing λ^{I} (Equation (14)) in the right-hand side of Equation (8), the latter gave

$$\frac{1}{\sqrt{\lambda^{\mathrm{II}}}} = -2\log\left(\frac{2.51}{Re\sqrt{\lambda^{\mathrm{I}}}} + \frac{1}{3.71}\frac{\varepsilon}{D}\right) = -2\log\left(\frac{2.51}{Re\sqrt{a^{\mathrm{I}}\lambda_\infty}} + \frac{1}{3.71}\frac{\varepsilon}{D}\right) \tag{16}$$



Re/Re''	|Δλ^I|%
0.0375	4.12
0.0622	5.01
0.122	5.46
0.335	4.87
1	3.56

Figure 2 | General good fitting of Equation (15) to the original curve for $\varepsilon/D = 0.0001$ but producing a few percent errors for the higher Re/Re'' values (box inside the figure).

where the left-hand side yielded a second-step λ^{II} value which was much closer than λ^{I} to the actual λ. The method of solution of a fixed-point problem for approaching the actual λ value was used, with specific adjustments by the respective authors, for the formulas indicated in Section 1 as being obtained from successive approximations of the friction factor inside the argument of the logarithm. Actually, the second-step λ^{II} value of Equation (16) proved already to be close enough to the actual one for every practical engineering purpose, because the mean and standard deviation of $|\Delta\lambda^{II}|$ (the latter being calculated likewise Equation (13)) over the whole 16 relative roughnesses fell, respectively, to 0.049 and 0.076%, with a maximum value of 0.452%. Therefore, Equation (16) combined with Equation (14) is a further explicit expression, more accurate than Equation (14) only, for assessment of the friction factor λ.

Based on these first results, Dataset 2 (composed of the values b_i and c_i obtained for the 16 relative roughnesses) was then processed to express the parameters b and c of Equation (15) by suitable functions of ε/D combining accuracy and, primarily, simplicity. Several simple relationships (such as, linear, exponential, logarithmic, etc., also considering ε/D raised to a power or added to a constant to be in turn determined) were assumed both for b and c, and each was fitted to the related points of Dataset 2 by the least squares method. Then, all the combinations of these relationships for b and c were used in Equation (15). For each combination, the related errors $|\Delta\lambda^{I}|$ and $|\Delta\lambda^{II}|$ were calculated over the total Dataset 1 and were then analysed. Actually, the deviations $|\Delta\lambda^{II}|$ proved to be acceptable for all the combinations, and therefore the latter ones were compared in terms of the errors $|\Delta\lambda^{I}|$ only. The most satisfactory results (weighing both accuracy and simplicity) were achieved by adopting the following pair of relationships for b and c:

$$b = 0.0066\left(\frac{\varepsilon}{D}\right)^{-0.203} \tag{17}$$

$$c = -0.0985\,\log\left(4.60\,\frac{\varepsilon}{D}\right) \tag{18}$$

which led to:

$$a^{I} = 1 + 0.0066\left(\frac{\varepsilon}{D}\right)^{-0.203}\left\{\log\left[\frac{0.860}{\left(\frac{\varepsilon}{D}\right)^{0.0985}}\right]\right\}^{\log(Re/Re'')} \tag{19}$$

For total Dataset 1, Equation (19) produces mean, standard deviation and maximum value of $|\Delta\lambda^{I}|$ equal, respectively, to 1.21, 0.998 and 5.18%, whereas for $|\Delta\lambda^{II}|$ equal, respectively, to 0.060, 0.081 and 0.79%. However, the actual occurrence of practical situations affected by the higher deviations has to be thoroughly checked. Error analysis and repercussions on practical calculations will be closely examined in the next section.

However, before broaching this important issue, two numerical examples, preparatory to the analysis itself, are given here, in order to observe practical consequences on pipe calculations when using the proposed approximated formulas. Let's consider an iron pipe having a diameter $D = 600$ mm and a roughness $\varepsilon = 0.6$ mm ($\varepsilon/D = 0.001$) that conveys, in a first case, water ($\nu = 1.14 \cdot 10^{-6}$ m^2/s) with a flow velocity $V = 1.60$ m/s. By Equation (2) we obtain $Re = 842{,}105$, by Equation (6) $\lambda_\infty = 0.01962$ and by Equation (7) $Re'' = 1{,}413{,}400$. By solving the C-W formula (Equation (8)) $\lambda = 0.01999$ is obtained and then by Equation (1) $J = 0.004347$ (i.e., the energy line falls by 4.347 m/km of pipe), whereas by Equation (11) $a = 1.0186$. Note that, in these examples, more digits than usual are used to stress the little differences between the actual and approximated quantities. If the approximated formulas are used, $a^{I} = 1.0373$ is given by Equation (19), $\lambda^{I} = 0.02036$ by Equation (14) and $J^{I} = 0.004427$ by Equation (1), whereas $\lambda^{II} = 0.01998$ by Equation (16) and $J^{II} = 0.004346$ by Equation (1). It is easily checked that $\Delta\lambda^{I} = 1.84\%$ whereas $\Delta\lambda^{II} = -0.016\%$ as well as that the same percent errors affect the respective friction slopes J^{I} and J^{II}. As for the practical consequences, assuming the friction slope J^{I} instead of J implies a difference in the friction head-loss of 8.0 cm/km of pipe, whereas assuming the friction slope J^{II} implies a difference in the friction head-loss less than 1 mm/km of pipe. In a second case, it is assumed the same pipe conveys an oil having $\nu = 30 \cdot 10^{-6}$ m^2/s with the same flow velocity. The parameters λ_∞ and Re'' are the same as in the former case. With these data, following the same procedure as above, it is obtained: $Re = 32{,}000$, $\lambda = 0.02569$, $J = 0.005586$, $a = 1.3090$, $a^{I} = 1.3010$, $\lambda^{I} = 0.02553$, $J^{I} = 0.005552$, $\lambda^{II} = 0.02570$, $J^{II} = 0.005589$, $\Delta\lambda^{I} = -0.609\%$ and $\Delta\lambda^{II} = 0.054\%$. In this case, therefore, assuming the friction slope J^{I} instead of J implies a difference in the friction head-loss of only 3.4 cm/km of pipe, whereas assuming the friction slope J^{II} a difference in the head-loss of 3.0 mm/km of pipe.

4. ERROR ANALYSIS AND PRACTICAL DEDUCTIONS

If the C-W formula is assumed as the benchmark for the calculation of circular pipes conveying liquid flows, the deviation $\Delta\lambda$ of any approximate λ value from that given by the solution of Equation (8) has to be weighed up taking into account its actual *practical consequences* on results of pipe-flow calculation, e.g., in terms of differing pipe-friction head-losses, nodal pressure heads, pipe-flow rates, service quality of pipeline system, etc. For a solution of a *second-type problem*, as considered here, the same percent error in λ assessment affects friction head-loss in the relating pipe, but, as discussed for instance by Lira (2013), the use of an approximate expression of λ is not the only source of errors in pipe-flow calculation, because there occur several implied and unavoidable uncertainties affecting important factors, such as actual pipe roughness, uncounted local head-losses, flow-rate ranging during pipe operation, etc. In addition, distribution networks are affected by constant changes in supplied urban areas and people, which require regular adaptation of the water-supply infrastructure to meet new circumstances. All this involves that, in practice, design and operation decisions have to be taken under uncertainties (Marques *et al.* 2015a). Therefore, a tolerance in pipe-flow calculation is effectively obligatory, and then analysis and weighing up of the errors $|\Delta\lambda|$ has to be carried out taking the presence of all these sources of errors into account.

Figure 3 shows the cumulative percent curves of the relative errors $|\Delta\lambda^{\mathrm{I}}|$ and $|\Delta\lambda^{\mathrm{II}}|$. As anticipated in the end part of Section 3, the errors $|\Delta\lambda^{\mathrm{II}}|$ are always negligible, for engineering purposes, as they remain noticeably below 1% (exactly, below 0.79%), and most of them (95%) even below 0.2%. Therefore, λ^{II} (Equation (16) combined with Equation (19)) can always be assumed (i.e., for every *Re* and ε/D) as a good approximation of the 'exact' C-W λ value. The $|\Delta\lambda^{\mathrm{I}}|$ errors are markedly higher than the $|\Delta\lambda^{\mathrm{II}}|$ ones, as they reach a few units, with a maximum error of about 5.2%. However, only in about 7% of cases does $|\Delta\lambda^{\mathrm{I}}|$ exceed 3% and only in about 0.7% of cases does it exceed 5%. Indeed, from an engineering point of view, even a 5–6% error is a good accuracy threshold, taking into account the many implied and unavoidable uncertainties in both data and plant operation mentioned above. Therefore, in most practical situations, where special accuracy is not needed, also λ^{I} (Equation (14) combined with Equation (19)) can be adopted as a good solution.

In this perspective, further analysis was carried out to detect just the cases (i.e., ranges of *Re* and ε/D) to which the higher $|\Delta\lambda^{\mathrm{I}}|$ values relate. Figure 4 shows the $|\Delta\lambda^{\mathrm{I}}|$ values, marked by the different ε/D values, vs. the Reynolds number *Re* (continuous thin lines). For a wider analysis, the figure also reports the curves relating to the ε/D values 0.00001 and 0.00005 not considered in the data processing above (dotted thin lines). The solid line represents the boundary from a transition regime to a fully turbulent regime (indeed, the line enters the point cloud down to $Re''_{\varepsilon/D=0.05} = 14800$, although it is not distinguishable anymore because of the point density). When crossing this boundary, all the error curves considerably decrease due to the asymptotic trend of Equation (19) to the line $a = 1$ as Re/Re'' increases.

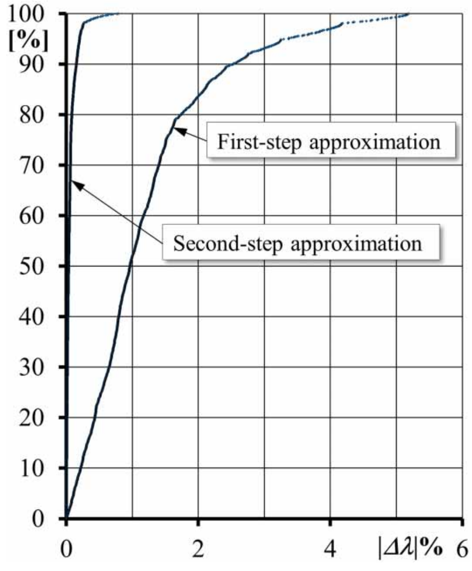


Figure 3 | Cumulative percent curves of the relative errors $|\Delta\lambda^{\mathrm{I}}|$ (first-step approximation) and $|\Delta\lambda^{\mathrm{II}}|$ (second-step approximation).

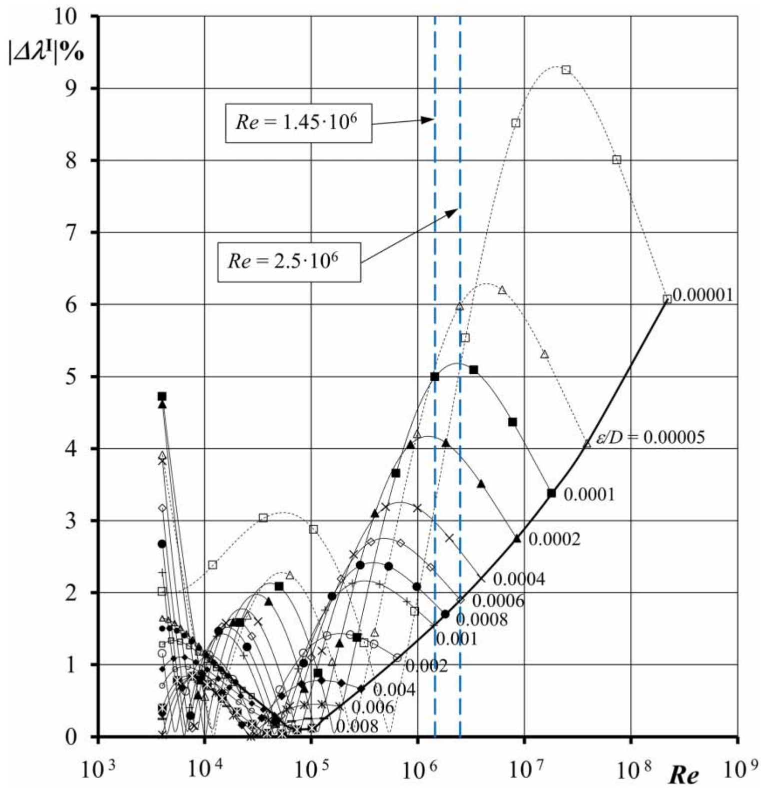


Figure 4 | Variation of the relative errors $|\Delta\lambda^{\mathrm{I}}|$ vs. *Re* for each ε/D (labelled next to the right-side point). The curves relating to $\varepsilon/D \geq 0.01$ throng in the left-down area are not labelled. The curves relating to $\varepsilon/D = 0.00001$ and 0.00005 (dotted thin lines) are also reported for a wider analysis of the results.

The figure allows one to recognize that, including in this analysis all the curves, $|\Delta\lambda^{\mathrm{I}}|$ higher than 5% can only occur for the three curves with $\varepsilon/D \leq 0.0001$ but only if $Re > \sim 1.45 \cdot 10^6$. However, if we accept an error threshold of 6%, the curves giving higher error are only the two with $\varepsilon/D < 0.0001$ and the limit *Re* value increases up to $\sim 2.5 \cdot 10^6$ (approximately a double value). As ε/D increases, the related maximum $|\Delta\lambda^{\mathrm{I}}|$ decreases down to less than 1% for higher ε/D values, being always less than 3.25% for $\varepsilon/D \geq 0.0004$. As for the condition $Re < \sim 1.45 \cdot 10^6$ in order to obtain $|\Delta\lambda^{\mathrm{I}}| < 5\%$ even for $\varepsilon/D \leq 0.0001$, it has to be noted that, in practical terms, this condition allows, for instance, a water pipe diameter up to about 1,650 mm, assuming a water kinematic viscosity $\nu = 1.14 \cdot 10^{-6}$ m^2/s and a flow velocity $V = 1$ m/s (as usual in water pipelines). The theoretical allowed maximum pipe diameter increases proportionally to kinematic viscosity for a pipeline conveying a more viscous liquid, such as an oil up to tens of times more viscous than water. On the other hand, the other condition $Re < \sim 2.5 \cdot 10^6$ ($|\Delta\lambda^{\mathrm{I}}| < 6\%$ for every ε/D) gives proportionally larger allowed values of *D* and *V* (e.g., water pipe diameters up to 2,850 mm with $V = 1$ m/s). In short, the condition $Re < \sim 2.5 \cdot 10^6$ (or $Re < \sim 1.45 \cdot 10^6$ accepting a 5% maximum error) includes most practical situations of turbulent liquid flows. Specifically for higher errors (i.e., those occurring for the lower ε/D and the higher *Re* values), it can further be observed that, for the involved ε/D and *Re* values, the 'exact' C-W λ approaches 0.01 (or less) and, with usual kinetic heads $V^2/2g = 5$–20 cm and the high *D* values causing those high *Re* values, practical values of friction head-loss usually amount to a few decimetres only per kilometre of pipeline. Therefore, an error of even a few percent in λ assessment turns into a difference in friction head-loss of a few centimetres only per kilometre and, after all, in a difference in the total head-loss being much less than the head-drop usually available (see the numerical examples at the end of Section 3).

Finally, the two expressions proposed (namely, Equations (14) and (16), where a^{I} is given by Equation (19)) were compared with seven of the many approximated formulas of the technical literature, summarized in Table 1. The latter were chosen taking into account both accuracy and formula handiness (i.e., in accordance with the guidelines of the present paper, too complicated formulas were not considered in the comparison). Figure 5 reports, for each formula, the cumulative percent

Table 1 | Formulas of the technical literature considered for a comparison with Equations (14) and (16); formula No. 7 is the simplest one proposed by the authors (first-step approximation), in accordance with the guidelines of the present paper

No.	Author(s)	Formula
1	Chen (1979)	$\frac{1}{\sqrt{\lambda}} = -2\,\log\left\{\frac{1}{3.7065}\frac{\varepsilon}{D} - \frac{5.0452}{Re}\log\left[\frac{1}{2.8257}\left(\frac{\varepsilon}{D}\right)^{1.1098} + \left(\frac{7.149}{Re}\right)^{0.8981}\right]\right\}$
2	Barr (1981)	$\frac{1}{\sqrt{\lambda}} = -2\,\log\left\{\frac{1}{3.7}\frac{\varepsilon}{D} - \frac{4.518\log\left(\frac{Re}{7}\right)}{Re\left[1+\frac{Re^{0.52}}{29}\left(\frac{\varepsilon}{D}\right)^{0.7}\right]}\right\}$
3	Zigrang & Sylvester (1982) – 1	$\frac{1}{\sqrt{\lambda}} = -2\,\log\left\{\frac{1}{3.7}\frac{\varepsilon}{D} - \frac{5.02}{Re}\log\left[\frac{1}{3.7}\frac{\varepsilon}{D} - \frac{5.02}{Re}\log\left(\frac{1}{3.7}\frac{\varepsilon}{D} + \frac{13}{Re}\right)\right]\right\}$
4	Zigrang & Sylvester (1982) – 2	$\frac{1}{\sqrt{\lambda}} = -2\,\log\left[\frac{1}{3.7}\frac{\varepsilon}{D} - \frac{5.02}{Re}\log\left(\frac{1}{3.7}\frac{\varepsilon}{D} + \frac{13}{Re}\right)\right]$
5	Haaland (1983)	$\frac{1}{\sqrt{\lambda}} = -1.8\,\log\left[\frac{1}{3.7}\left(\frac{\varepsilon}{D}\right)^{1.11} + \frac{6.9}{Re}\right]$
6	Brkič (2011b)	$\frac{1}{\sqrt{\lambda}} = -2\,\log\left\{\frac{1}{3.71}\frac{\varepsilon}{D} + \frac{2.18}{Re}\,\ln\left[\frac{Re}{1.816\,\ln\left[\frac{1.1\,Re}{\ln\left(1+1.1\,Re\right)}\right]}\right]\right\}$
7	Praks & Brkič (2018)	$\frac{1}{\sqrt{\lambda}} = -2\,\log\left[\frac{1}{3.7}\frac{\varepsilon}{D} - \frac{2.51}{Re}\left(8 - \frac{2\,A}{2 - A\,B}\right)\right]$ $A = 8 + 2\log\left(\frac{1}{3.7}\frac{\varepsilon}{D} + \frac{16}{Re}\right)$ $\quad B = \frac{-74,914,381.46}{\left(74,205.5 + 1,000\frac{\varepsilon}{D}Re\right)^2}$

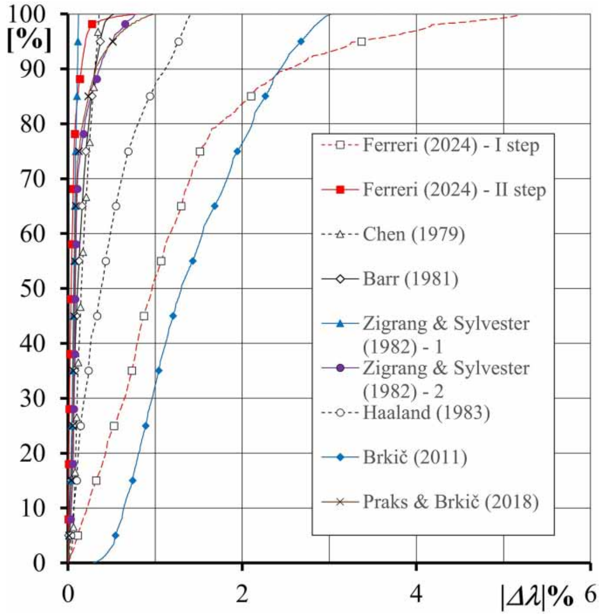


Figure 5 | Comparison of Equation (14) (λ^{I}) and Equation (16) (λ^{II}) with the formulas of the technical literature summarized in Table 1, by the respective cumulative percent curves of the relative errors $|\Delta\lambda|$ over total Dataset 1.

curve of the relative errors $|\Delta\lambda|$ over total Dataset 1 ($\varepsilon/D \leq 0.0001$; $4000 \leq Re \leq Re''$), calculated likewise Equation (13). The figure shows that, except for Equation (14) and formulas No. 5 and 6 (Table 1), the other six exhibit close high accuracy, and the related curves slightly stray from one another only for the higher errors (being always less than 1%) concerning about the upper 15% of cases. However, a closer examination allows one to recognize that Equation (16) and formula No. 3 are slightly more accurate than the others, and in particular the former is the most accurate in about 85% of cases. As for the three less accurate equations (but with maximum errors within about 5%), stated in the discussion above that an accuracy of a few percent is suitable for most practical pipe calculations, formula No. 5 gives rather close accuracy to the former six, whereas Equation (14) is more accurate than formula No. 6 in about 90% of cases, the latter concerning the most usual ranges of *Re* and ε/D (see Figure 4). On the whole (handiness and accuracy), formula No. 5 may be the best choice. As a general remark, examination of Figure 5 along with Table 1 allows one to recognize that the higher complexity of a formula does not imply necessarily higher accuracy. However, the effort of researchers to find more and more accurate explicit formulas is useful to better focus theoretical, numerical, and practical aspects (as, for instance, those examined in the present section) of resistance to flow as well as the role of the involved variables.

In conclusion, both the two expressions proposed (namely, Equations (14) and (16), where a^{I} is given by Equation (19)) answer the demand for an explicit equation for friction factor assessment giving adequate accuracy for most engineering practical purposes, without having too intricate a mathematical structure. Actually, several expressions available in the technical literature give better accuracy, especially with respect to Equation (14) combined with Equation (19), accuracy that sometimes can reach tenths percent, hundredths percent or even less, but in spite of a really intricate structure that may even thwart computation time-saving with respect to solving by a trial-and-error method the C-W formula itself. Therefore, the choice of one or another approximate expression of the technical literature, including the present two, has to take simplicity into account and not only high accuracy exceeding that requested in many practical cases. This is particularly true in water-supply systems, because of all the considerations discussed above (data uncertainties and practical consequences of assessment errors).

5. CONCLUSIONS

A novel approach has been developed for explicit assessment of the friction factor λ in the case, indicated here as a second-type operational problem, in which the friction head-loss has to be determined, the Reynolds number and the relative roughness being known. Based on the analogous descending trend of each curve of the Moody chart towards the respective horizontal asymptote relating to a fully turbulent regime, whose friction factor has been indicated here as λ_∞, the ratio $a = \lambda/\lambda_\infty$ was studied as a function of the relative roughness and the *relative* Reynolds number (the latter being the ratio between the actual Reynolds number and the Reynolds number separating transition regime from fully turbulent regime).

A dataset of points was obtained by systematically solving the C-W formula, and then used for testing the fitting of a few relationships having a suited asymptotic trend. An exponential-type relationship was selected among those tested, and its two parameters were then expressed as functions of the relative roughness only. Despite general good fitting, the relative errors in the friction factor given by the fitted curves, indicated as λ^{I}, proved not to be always very low. However, these errors were mostly within a range generally acceptable for many engineering practical purposes, especially in water-supply systems where implied uncertainties, in both data and system operation, recommend some ‘largeness’ in calculations. By contrast, for cases where high accuracy is requested in the determination of friction head-loss, a much better-approximated friction factor value, indicated as λ^{II}, was obtained using λ^{I} as a first-step approximation to be introduced in the C-W formula. This second-step λ^{II} value exhibits very low relative error, less than 0.79%, for every Reynolds number and relative roughness, as well as it proves to be even slightly more accurate than several of the most known formulas of the technical literature, in most practical cases.

In short, the new approach led to two possible solutions for the explicit assessment of the friction factor: the simpler one having lower accuracy but being adequate for most engineering practical purposes, the other entailing little more calculation but having fully adequate accuracy for all practical cases.

ACKNOWLEDGEMENTS

The writer wishes to thank his Master, emeritus Professor Guglielmo Benfratello (1927–2017) who, in his last appearances at the University of Palermo a few years before his demise, shared with the writer some notes of his own about an original

reformulation of the law of resistance to flow in circular pipes, and asked him to be helped in developing his idea. This yielded a paper on the proceedings of an Italian national conference (Benfratello *et al.* 2012). Starting from that idea, after Master's demise the writer made further thorough reformulations published in the present paper. The latter is a posthumous homage of the writer to his old Master.

DATA AVAILABILITY STATEMENT

All relevant data are included in the paper or its Supplementary Information.

CONFLICT OF INTEREST

The authors declare there is no conflict.

REFERENCES

Avci, A. & Karagoz, I. 2019 A new explicit friction factor formula for laminar, transition and turbulent flows in smooth and rough pipes. *European Journal of Mechanics/B Fluids* **78**, 182–187.

Azizi, N., Homayoon, R. & Hojjati, M. R. 2019 Predicting the Colebrook-White friction factor in the pipe flow by new explicit correlations. *Journal of Fluids Engineering, Transactions of the ASME* **141** (5), art. No. 051201.

Barr, D. I. H. 1981 Solutions of the Colebrook-White function for resistance to uniform turbulent flow. *Proceedings of the Institution of Civil Engineers Part 2: Research and Theory* **71** (2), 529–535.

Benfratello, G., Lo Re, C. & Ferreri, G. B. 2012 Applicazione alla verifica delle reti di condotte in una rivisitazione del capitolo del moto uniforme turbolento nelle correnti liquide (Implementation to distribution network calculation of a reassessed head on turbulent uniform liquid flows). Proc. of XXXIII Convegno Nazionale di Idraulica e Costruzioni Idrauliche (National Congress of Hydraulics and Hydraulic Constructions), Brescia (IT), September 10th–15th, on CD Rom, pp. 1–11.

Bonvissuto, G. 1977 Direct analytical solution for turbulent pipe-flow problems. *Meccanica* **12** (1), 42–44. https://doi.org/10.1007/BF02172206.

Brkić, D. 2011a Review of explicit approximations to the Colebrook relation for flow friction. *Journal of Petroleum Science and Engineering* **77**, 34–48.

Brkić, D. 2011b W solutions of the CW equation for flow friction. *Applied Mathematics Letters* **24**, 1379–1383.

Chen, N. H. 1979 An explicit equation for friction factor in pipes. *Industrial and Engineering Chemistry Fundamentals* **18** (3), 296–297.

Churchill, S. W. 1973 Empirical expressions for the shear stress in turbulent flow in commercial pipe. *AIChE Journal* **19**, 375–376.

Clamond, D. 2009 Efficient resolution of the Colebrook equation. *Industrial and Engineering Chemistry Research* **48**, 3665–3671.

Colebrook, C. F. 1939 Turbulent flow in pipes, with particular reference to the transition region between the smooth and the rough pipe laws. *Journal of the Institution of Civil Engineers* **11** (4), 133–156.

Colebrook, C. F. & White, C. 1937 Experiments with fluid friction in roughened pipes. *Proceedings of the Royal Society. Lond. Series A Mathematical, Physical Sciences* **161**, 367–381.

Cunha, M. C. & Sousa, J. 1999 Water distribution network design optimization: Simulated annealing approach. *Journal of Water Resources Planning and Management* **125** (4), 215–221.

Genić, S., Arandjelović, I., Kolendić, P., Jarić, M., Budimir, N. & Genić, V. 2011 A review of explicit approximations of Colebrook's equation. *FME Transactions* **39** (2), 67–71.

Giustolisi, O., Berardi, L. & Walski, M. 2011 Some explicit formulations of Colebrrok-White friction factor considering accuracy vs. computational speed. *Journal of Hydroinformatics* **13** (3), 401–418.

Haaland, S. E. 1983 Simple and explicit formulas for the friction factor in turbulent pipe flow. *Journal of Fluids Engineering* **105**, 89–90.

Hafsi, Z. 2021 Accurate explicit analytical solution for Colebrook-White equation. *Mechanics Research Communications* **111**, art. No. 103646, 1–8.

Keady, G. 1998 Colebrook-White formula for pipe flows. *Journal of Hydraulic Engineering, ASCE* **124** (1), 96–97.

Lamri, A. A. & Easa, S. M. 2022 Computationally efficient and accurate solution for Colebrook equation based on Lagrange theorem. *Journal of Fluids Engineering, Transactions of the ASME* **144** (1). art. No. 014504.

Langelandsvik, L. I., Kunkel, G. J. & Smits, A. J. 2008 Flow in commercial steel pipe. *Journal of Fluid Mechanics* **595**, 323–339.

Lira, I. 2013 On the uncertainties stemming from use of the Colebrook-White equation. *Industrial & Engineering Chemistry Research* **52** (22), 7550–7555.

Marques, J., Cunha, M. & Savić, D. 2015a Using real options in the optimal design of water distribution networks. *Journal of Water Resources Planning and Management, ASCE* **141** (2). art. No. 04014052.

Marques, J., Cunha, M. & Savić, D. 2015b Multi-objective optimization of water distribution systems based on a real options approach. *Environmental Modelling and Software* **63** (1), 1–13.

McKeon, B. J., Zagarola, M. V. & Smits, A. J. 2005 A new friction factor relationship for fully developed pipe flow. *Journal of Fluid Mechanics* **538**, 429–443.

Moody, L. F. 1944 Friction factors for pipe flow. *Transactions of the ASME* **66** (8), 671–684.

Nikuradse, J. 1933 Stromungsgesetze in Rauhen Rohren (Laws of flow in rough pipes). *VDI-Forschungsheft*, 361, B, 4.
Phillips, G. M. & Taylor, P. J. 1973 *Theory and Applications of Numerical Analysis*. Academic Press Inc., London.
Praks, P. & Brkić, D. 2018 Advanced iterative procedures for solving the implicit Colebrook equation for fluid flow friction. *Advances in Civil Engineering* **2018**, art. No. 5451034, 1–18.
Rollmann, P. & Spindler, K. 2015 Explicit representation of the implicit Colebrook-White equation. *Case Studies in Thermal Engineering* **5**, 41–47.
Romeo, E., Royo, C. & Monzón, A. 2002 Improved explicit equations for estimation of the friction factor in rough and smooth pipes. *Chemical Engineering Journal* **86**, 369–374.
Rouse, H. 1943 Evaluation of boundary roughness. In *Proc. of the 2nd Hydraulic Conference*, Iowa City (USA), Paper No. 27.
Schlichting, H. 1955 *Boundary Layer Theory*. Pergamon Press Ltd, London.
Shockling, M. A., Allen, J. J. & Smits, A. J. 2006 Roughness effects in turbulent pipe flow. *Journal of Fluid Mechanics* **564**, 267–285.
Sonnad, J. R. & Goudar, C. T. 2004 Constrains for using the Lambert W function based Colebrook-White equation. *Journal of Hydraulic Engineering, ASCE* **130** (9), 929–931.
Sonnad, J. R. & Goudar, C. T. 2007 Explicit reformulation of the Colebrook-White equation for turbulent flow friction factor calculation. *Industrial and Engineering Chemistry Research* **46**, 2593–2600.
Swamee, P. K. & Jain, A. K. 1976 Explicit equations for pipe-flow problems. *Journal of Hydraulic Div., ASCE* **102** (5), 657–664.
Traub, J. F. 1964 *Iterative Methods for the Solution of Equations*. Prentice-Hall, Inc., Englewood Cliffs, NJ.
Vatankhah, A. R. 2018 Approximate analytical solutions for the Colebrook equation. *Journal of Hydraulic Engineering, ASCE* **144** (5). Art. No. 06018007.
Winning, H. K. & Coole, T. 2013 Explicit friction factor accuracy and computational efficiency for turbulent flow in pipes. *Flow Turbulence Combustion* **90**, 1–27.
Zagarola, M. V. & Smits, A. J. 1998 Mean-flow scaling of turbulent pipe flow. *Journal of Fluid Mechanics* **373**, 33–79.
Zigrang, D. J. & Sylvester, N. D. 1982 Explicit approximations to the solution of Colebrook's friction factor equation. *AIChE Journal* **28**, 514–515.

First received 28 November 2023; accepted in revised form 19 June 2024. Available online 3 July 2024