A visual representation of the Steiner triple systems of order 13

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Abstract

Steiner triple systems (STSs) are a basic topic in combinatorics. In an STS the elements can be collected in threes in such a way that any pair of elements is contained in a unique triple. The two smallest nontrivial STSs, with 7 and 9 elements, arise in the context of finite geometry and nonsingular cubic curves, and have well-known pictorial representations. On the contrary, an STS with 13 elements does not have an intrinsic geometric nature, nor a natural pictorial illustration. In this paper we present a visual representation of the two non-isomorphic Steiner triple systems of order 13 by means of a regular hexagram. The thirteen points of each system are the vertices of the twelve equilateral triangles inscribed in the hexagram. In the case of the non-cyclic system, our representation allows one to visualize in a simple, elegant and highly symmetric way the twenty-six triples, the six automorphisms and their orbits, the eight quadrilaterals, the ten mitres, the thirteen grids, the four 3-colouring patterns, the block-colouring and some distinguished ovals. Our construction is based on a very simple idea (seeing the blocks as much as possible as equilateral triangles), which can be further extended to get new representations of the STSs of order 7 and 9, and of one of the STSs of order 15.

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1 Introduction

Steiner triple systems are one of the oldest and most studied classes of combinatorial objects, sinking their roots in the pioneering studies at the dawn of abstract algebra, finite geometry, projective geometry, and topology [8]. They date back to Plücker's study of algebraic curves in 1835, and to the publication in 1850 of Kirkman's fifteen schoolgirl problem [23], one of the most far-reaching and popular problems in combinatorics. Since then, they have gained more and more popularity among design-theoretists, universal-algebraists, geometers, statisticians, computer scientists, map-colourers, recreational mathematicians, and others.

Definition 1. (see [2], [6], [8]) A *Steiner triple system* of order v (STS(v), for brevity) is a pair $(\mathcal{V}, \mathcal{B})$, where $\mathcal{V}$ is a finite set with v elements, called *points*, and $\mathcal{B}$ is a collection of 3-element subsets of $\mathcal{V}$, called *blocks*, or *triples* (formerly *triads*), or *lines*, with the property that any two distinct points in $\mathcal{V}$ belong to precisely one triple in $\mathcal{B}$. An *isomorphism* from an STS $(\mathcal{V}_1, \mathcal{B}_1)$ to another STS $(\mathcal{V}_2, \mathcal{B}_2)$ is a one-to-one map π from $\mathcal{V}_1$ onto $\mathcal{V}_2$ that preserves triples (that is, $\{x, y, z\} \in \mathcal{B}_1$ if and only if $\{\pi(x), \pi(y), \pi(z)\} \in \mathcal{B}_2$). An *automorphism* of an STS $(\mathcal{V}, \mathcal{B})$ is an isomorphism of the STS with itself. An STS(v) is *cyclic* if it has an automorphism with a single orbit of length v.

A necessary and sufficient condition for the existence of at least one STS(v) is that $v \equiv 1$ or $3 \pmod 6$ (equivalently, v is odd and $v(v-1)$ is a multiple of 6) [22]. Such values of v are called *admissible*. An STS(v) contains precisely $\frac{1}{3}\binom{v}{2}$ blocks, and, moreover, it may be regarded as a *triangle decomposition* of the complete graph K_v: if the points of the STS are identified with the v vertices of K_v, then every triple of the STS corresponds to a triangle of edges of the graph, and the block-set of the STS can be seen as a partition of the edges of K_v into triangles. However, the resulting picture may not be particularly revealing nor aesthetically appealing. When representing an STS as a planar or spatial object, the main goal is, on the one hand, to merge the two structures so that the symmetries of the "geometric" object reflect some (or all) symmetries and automorphisms of the STS, and, on the other hand, to capture and convey some of the abstract beauty of the STS [32]. Also, representing a block design graphically can make some of its properties obvious that would otherwise require tedious reasoning [25].

In spite of its purely combinatorial definition, a Steiner triple system can be thought of as a finite geometry of points and lines, where each line contains three points and, as with planar geometry, there exists a unique line through any two distinct points. In the spirit of this premise, some of the terminology concerning Steiner triple systems is formulated with a geometric language. This is the reason why, in particular, a block of the system is often called a line (see Definition 1), although an STS, in general, does not necessarily arise in the context of finite geometry or algebraic geometry. By the same token, the notions of *oval* and *(complete) quadrilateral* in classical projective planes led to the following definitions.

Definition 2. (see [42, 12]) An *oval* in an STS(v) is a set of $r = \frac{v-1}{2}$ points, no three of which are collinear (that is, belonging to the same triple). A line is called *secant*, *tangent* or *exterior* (or *passant*) if it intersects the oval in $2, 1$, or 0 points, respectively. A point off the oval is called an *interior point* (respectively, an *exterior point*) if it lies on no tangent line (respectively, on at least one tangent line).

Definition 3. (see, e.g., [8, p. 147]) A *quadrilateral* (or *Pasch configuration*, or *fragment*) in a Steiner triple system is a configuration consisting of four lines, no three of which through one point, and of the six intersection points, one for each pair of lines.

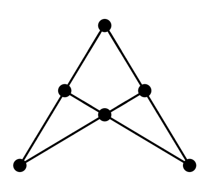

Figure 1: A quadrilateral (Pasch configuration).

More generally, a *configuration* $\mathcal{C}$ in a Steiner triple system $(\mathcal{V}, \mathcal{B})$ is a subset of the block-set $\mathcal{B}$ (see, for instance, [8, Chapter 13], [19], and [10]). The *points* of $\mathcal{C}$ are the elements of the union of all the blocks in $\mathcal{C}$. There are configurations that must occur in every nontrivial STS, while others may be avoided altogether. Configurations are an important invariant for the classification of Steiner triple systems, since the number of occurrences of a given configuration must be the same for any two isomorphic STSs. In this respect, besides the quadrilateral, another significant configuration is the so-called *mitre* (see Definition 4, where we also recall the definition of the *grid* configuration).

Definition 4. (see, e.g., [8, pp. 211, 245], [5, Figure 1]) A *mitre* in a Steiner triple system (sometimes called D_1-configuration) is a configuration of seven points and five lines, three of which through one point (called the *center* of the mitre), the other two lines forming a partition of the remaining six points. A *grid* (sometimes called E_5-configuration) is a configuration of nine points and six lines, which can be partitioned into two sets of three mutually parallel lines.

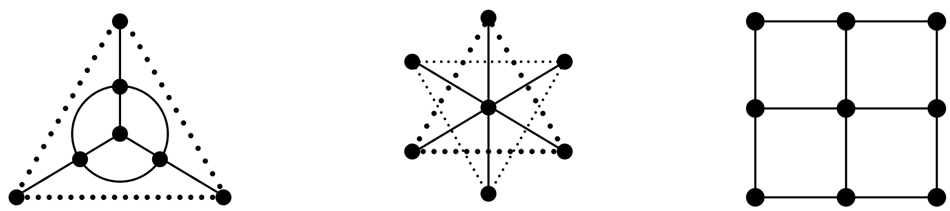

Figure 2: A mitre (left and center) and a grid (right).

A configuration is *full* if every point belongs to at least two lines. The quadrilateral is the only full configuration with four lines, and the mitre is the only full configuration with five lines. In an STS(v), the number of occurrences of any configuration with at most five lines depends only on v and on the numbers of quadrilaterals and mitres [10]. The grid and the so-called *prism* (or *double triangle*, or E_4-configuration) are the only full configurations with six lines and with the property that every point belongs to an even number of lines.

For any admissible v other than 7 and 13, there exists at least one STS(v) containing no quadrilaterals [20], and for any admissible v other than 9, there exists at least one STS(v) containing no mitres [40], whereas it is much harder to find STSs containing no quadrilaterals and no mitres. In this regard, there exists only one Steiner triple system of order $7 \leq v \leq 19$ that contains no quadrilaterals and no mitres, although there exist $11,084,874,829$ non-isomorphic STS(19)s (the only exception has order $v = 19$ and is denoted by A4) [5]. Finding STSs without grids is even harder, since each STS of order $9 \leq v \leq 19$ contains some grids. As far as we know, the existence of an STS(v), $v \geq 9$, with no grids is still an open problem.

Some STSs present a somehow intrinsic geometric structure, as they arise within the frame of finite geometry. Besides the trivial STS(3), this is the case for the two smallest STS(v)s, that is, for $v = 7$ and $v = 9$ (which are unique up to isomorphism), and for the *projective* STS of order 15 (see the following Example 7). The system of order 9 appears also in the context of plane algebraic curves, since every nonsingular cubic has nine points of inflection, and every pair of inflection points defines a three-point line. These three STS(v)s have always been considered the quintessential Steiner triple systems, and have been thouroughly investigated also in terms of their visual representations, as we recall in the following examples. The case $v = 13$, instead, has been somehow neglected as the ugly duckling of the small STSs, and we shall try to make it blossom into a swan.

Example 5. (The STS of order 7) The standard model of the (unique) STS(7) is the system of points and lines of the projective plane of order 2 (commonly called *Fano plane*), with seven points and seven lines. The STS(7) contains seven quadrilaterals (each obtained by removing one point and the three lines through the point), no mitres and no grids. The Fano plane is one of the most famous combinatorial structures, and is, by far, the most popular combinatorial design and the most common finite geometry, actually representing the very emblem of the whole theory. Its standard picture, which has become even more familiar than the Fano plane itself, appears on the front cover of one of the most commonly used textbooks in this area [6], and is the logo of the British Combinatorial Committee. The standard picture of the Fano plane is given on the left in Figure 3, where an alternative model is given on the right.

Figure 3: The Fano plane. Standard model (left) and simplicial model (right).

The model on the left shows that in the Fano plane there are seven points, represented by the seven dots, and seven three-point lines, represented by the three sides, the three altitudes, and the circle. In the model on the right the seven points of the STS are the seven simplicial elements of a triangle (2-simplex), that is, the three vertices, the three edges, and the whole triangle. In this model a triple appears either as two vertices and the edge between them, or as a vertex, the opposite edge, and the face (the whole triangle), or as the three edges of the triangle [14]. Another nice picture is given in [33, §1.4]. For all three models, the six symmetries of the regular triangle that underlies the diagram correspond to just as many automorphisms of the Fano plane. One more triangle-based picture will be introduced in the present paper.

In [34, Figure 4] one finds another remarkable planar and highly symmetric model of the Fano plane, which highlights an automorphism of order 7 and depicts the *homogeneous* nature of the geometry, in that none of the points or of the lines is distinguished among the others. The same can be said about the picture on the right in [21, Figure 3], where, however, each element of the Fano plane is represented by two distinct points. The homogeneous nature of the system can also be depicted by embedding the complete graph K_7 in a torus (see Figure 1 in [18], where the embedding visualizes at the same time two orthogonal Fano planes).

Figure 3 in [34] (see also Example 3 in [32]) shows a spatial model of the Fano plane, where the seven points are the centers of the six edges of a tetrahedron plus the center of the tetrahedron, and the lines are defined in such a way that the 24 symmetries of the tetrahedron translate into automorphisms of the Fano plane, and correspond precisely to the point stabilizer of the center point in the full automorphism group. Finally, Figure 7 in [25] represents the seven points of the Fano plane as the six vertices and the center point of a octahedron, showing the action of the alternating group A_4 (see also Figure 1 in [31]).

Example 6. (The STS of order 9) Although not as ubiquitous as the traditional picture of the Fano plane, there exists also a standard picture of the (unique) Steiner triple system of order 9, which can be found in many books, articles and conference posters on incidence geometry, finite geometries, block designs and recreational mathematics (see Figure 4). The lines of the system are the twelve lines of the affine plane of order 3, that is, the twelve affine lines of the two-dimensional vector space $\mathbb{F}_3 \times \mathbb{F}_3$ over the ternary field $\mathbb{F}_3 = \{0, 1, 2\}$, where each element (x, y) of the vector space is represented by the corresponding point in a cartesian plane (see, e.g., [39, Figure 5]).

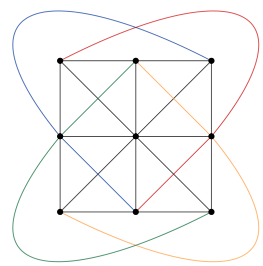

Figure 4: The standard picture of the Steiner triple system of order 9.

Equivalently, the twelve lines of the affine plane of order 3 can be seen as the rows, the columns, the forward diagonals, and the back diagonals of a 3×3 matrix (see Figure 5).

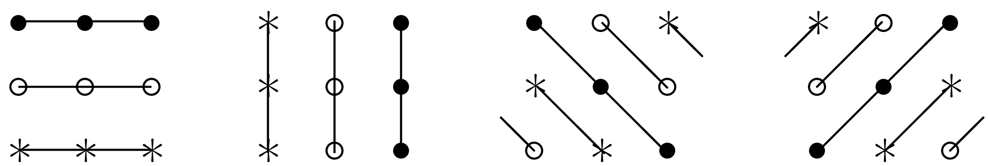

Figure 5: The four parallel classes of the affine plane of order 3.

Four further highly symmetric pictures are described in [33], among which a nice *generator-only* model on the regular octagon, which allows one to visualize an automorphism of order 8. An alternative, triangle-based, picture will be introduced in the present paper, as an extension of the main geometric idea of the article.

The STS(9) contains 36 mitres (four for each point), 6 grids (one for each of the $\binom{4}{2}$ pairs of parallel classes of lines), and no quadrilaterals. Each mitre is obtained by choosing its center x and one of the four lines through x, say $\{x, y, z\}$, and by removing from the STS(9) the points y, z and the seven lines through either y or z.

Example 7. (The *projective* STS of order 15) Just as the 2-simplex gives a model for the STS with $7 = 2^3 - 1$ elements, the fifteen simplicial elements of a tetrahedron (3-simplex) can be combined to give a model for one of the eighty non-isomorphic STSs with $15 = 2^4 - 1$ elements, precisely the geometry of points and lines of the three-dimensional projective space $\mathrm{PG}(3, 2)$ over the field with two elements [14]. Other symmetric planar or spatial models of $\mathrm{PG}(3, 2)$, which reflect automorphisms of order 3, 4, or 7, are given in [33] and [31]. More generally, a planar or spatial model for each of the seven non-isomorphic solutions of Kirkman's fifteen schoolgirl problem is given in [31].

The projective STS(15) contains 105 quadrilaterals, 280 grids and no mitres.

Unlike the STSs of order 7, 9, and 15 in the previous examples, a Steiner triple system of order 13 does not arise as a set of points and lines in a projective or affine geometry, nor can be embedded in a finite Desarguesian projective plane [26, §2.1], hence it does not have an "intrinsic" geometric nature, nor a "natural" visual representation. The goal of this paper is to present a simple and pleasing picture for each of the two non-isomorphic STS(13)s; in particular, for one of the two systems, the symmetries of the picture and the automorphisms of the STS merge perfectly. Our representation, moreover, allows one to visualize in a highly symmetric way some of the relevant sub-configurations of the system.

2 The two Steiner triple systems of order 13

According to Cole [9], the first arrangements of an STS(13) were given by Kirkman [24, D., p. 39] (1853), followed by those by Reiss [35] (1859), Netto [29] (1892), and de Vries [38] (1894). In 1897 Zulauf [44] reduced the known systems to two, those by Kirkman, Reiss, and de Vries being equivalent and with a non-transitive automorphism group of order 6, and that of Netto having a transitive automorphism group of order 39 (Netto actually gave a direct construction of an STS(v) for any prime order v). Finally, De Pasquale [11] (1899) and Brunel [4] (1901) showed that there exist, up to isomorphism, only two Steiner triple systems of order 13 (cf., e.g., [26, §2]). An alternative proof was given by Cole [9] (1913) by means of the notion of "interlacing", which, in modern terms, corresponds to that of Pasch configuration (four blocks $\{x, a, b\}$, $\{x, c, d\}$, $\{y, a, c\}$, $\{y, b, d\}$, are seen as an interlacing of x and y).

It is worth mentioning that algebraic representations of the STS(13)s were recently given in [30] and in [13, Proposition 4]. In the former case, an incidence structure isomorphic to a Steiner triple system of order 13 is constructed by defining a set $\mathcal{B}$ of twenty-six vectors in the 13-dimensional vector space V over the finite field $\mathbb{F}_5$, with the property that there exist precisely thirteen 6-subsets of $\mathcal{B}$ whose elements sum up to zero in V, whereas in the latter case the STS(13)s are described as sections of $A(13)/A(12)$ in the alternating group $A(13)$.

In any STS(13) there are thirteen points and twenty-six blocks, and each point belongs to precisely six blocks. One of the two systems has an automorphism of order 13, hence it is usually referred to as the *cyclic* STS(13) (see [27, system No. 1], [6, Table II.1.27, n. 2] and [8, Table 5.7, n. 2])), whereas the other system has a fixed point under all automorphisms and is often called the *non-cyclic* STS(13) (see [27, system No. 2], [6, Table II.1.27, n. 1] and [8, Table 5.7, n. 1])). The two systems contain, respectively, thirteen and eight quadrilaterals (see, e.g., [1, p. 41], [27], and [43]). The non-cyclic STS(13) contains ten mitres, the cyclic system contains no mitres, and both systems contain 13 grids.

In the cyclic case, the point-set of the STS(13) can be identified with the cyclic group $\mathbb{Z}/13\mathbb{Z}$, and the triples are the twenty-six 3-sets that are cyclically generated (mod 13) by the base blocks $\{1, 3, 9\}$, $\{2, 5, 6\}$ under the transformation $x \mapsto x + 1$ (see, e.g., [8, Theorem 2.11]; in modern terms, the two base blocks form a $(13, 3, 1)$-*cyclic difference family*). In this case, the group of automorphisms of the STS is the (transitive) Frobenius group of all affine transformations of the form $x \mapsto ax + b$, with $x, b \in \mathbb{Z}/13\mathbb{Z}$ and $a \in \{1, 3, 9\}$, which is, up to isomorphism, the unique non-abelian group of order 39. The group is generated by the transformations $x \mapsto x + 1$ and $x \mapsto 3x$. The thirteen quadrilaterals contained in the system are cyclically generated (mod 13) by the four blocks $\{1, 3, 9\}$, $\{1, 7, 12\}$, $\{2, 3, 12\}$, $\{2, 7, 9\}$, whereas the thirteen grids are cyclically generated (mod 13) by the six blocks $\{1, 3, 9\}$, $\{4, 6, 12\}$, $\{5, 7, 13\}$, $\{1, 4, 5\}$, $\{3, 6, 7\}$, $\{9, 12, 13\}$.

As to the non-cyclic STS(13), the blocks are the same as above, with the only exception of the four blocks $\{2, 4, 10\}$, $\{2, 5, 6\}$, $\{4, 6, 12\}$, $\{5, 10, 12\}$, which form a Pasch configuration and are replaced by the Pasch configuration $\{2, 4, 6\}$, $\{2, 5, 10\}$, $\{4, 10, 12\}$, $\{5, 6, 12\}$. Equivalently, each block in the second quadrilateral is obtained as a complement $\{2, 4, 5, 6, 10, 12\} \setminus \{x, y, z\}$, where $\{x, y, z\}$ is a block in the first quadrilateral. This operation is called a *Pasch switch* (see, e.g., [5, p. 7]). Given a Pasch configuration in a Steiner triple system, the corresponding Pasch switch always transforms the STS into another STS, which differs from the previous one by only four blocks.

The automorphism group of the non-cyclic STS(13) is isomorphic to the symmetric group S_3, the unique non-abelian group of order 6 (see, e.g., [8]). Note that $x \mapsto 3x$ is again an (order-3) automorphism of the system (if the point-set is $\mathbb{Z}/13\mathbb{Z}$ and the blocks are defined as above). Moreover, the orbits under the automorphism group have length $1, 3, 3, 6$ (in particular, there exists a fixed point). The eight quadrilaterals contained in the system, together with the ten mitres and the thirteen grids, will be described "geometrically" in the following part of the paper.

In the case of the cyclic STS(13), the "natural" picture to draw is a regular 13-gon, whose vertices represent the points of the STS, together with two triangles representing two base blocks, which, by rotation, generate all the 26 blocks of the system (see, for instance, [33, p. 24] and [37, Figure 7.7]). Such a picture not only describes the blocks of the system, but also visually illustrates an automorphism of order 13 (the $2\pi/13$ rotation). Such a picture, however, is not suitable to illustrate the automorphisms of order 3. A picture of the non-cyclic STS(13) is given again in [33, p. 24], by modifying only four blocks of the illustration for the cyclic system, and leaving the remaining twenty-two blocks unchanged. The resulting picture, however, only describes the blocks of the STS, without being related in any way to the automorphisms of the system (which no longer has an automorphism of order 13).

A highly symmetric picture of the non-cyclic STS(13) is given in [3, Figure 3] (see also Figure 6 below), where one can visualize an automorphism of order 3, the fixed point, the orbit of length 6 and the two orbits of length 3. This picture, however, is somehow not fully satisfactory, as it makes use of double points (each of the points labelled as $7, 8, 9$ appears twice).

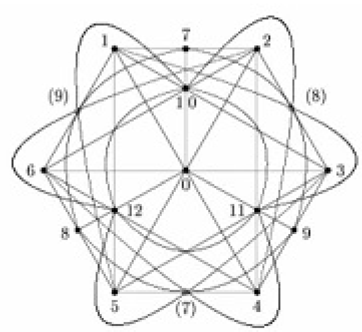

Figure 6: A picture of the non-cyclic STS(13) with double points.

We will now present a new picture of the non-cyclic STS(13), which enjoys the same properties as Figure 3 in [3], and mirrors the blocks, the sub-configurations and the symmetries of the system, but does not make use of double points. The basic idea of our geometric construction is to take as a starting block of an STS(13) the set of the three vertices A, B, C of an equilateral triangle (see Figure 7). If one now considers a new point D, then, being ABC a block, the triangle with vertices B, C, D cannot in turn represent a block, by definition of Steiner triple system. But if one considers two new points E and F, then the two triangles with vertices B, D, E and C, D, F can represent two new blocks of the system (see Figure 7, where the dotted triangles are those associated with blocks of the system).

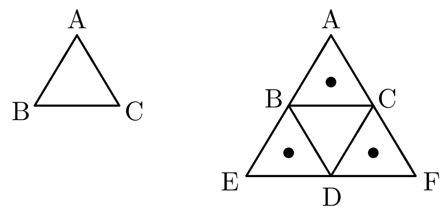

Figure 7: The first three blocks of the STS(13).

By iterating the same kind of argument, one finally gets twelve equilateral triangles, which form a regular hexagram and whose thirteen vertices $A, B, \ldots, M$ can be taken as the points of an STS(13). This produces also the first six blocks of the STS (see Figure 8, where again the dotted triangles are those associated with blocks of the system).

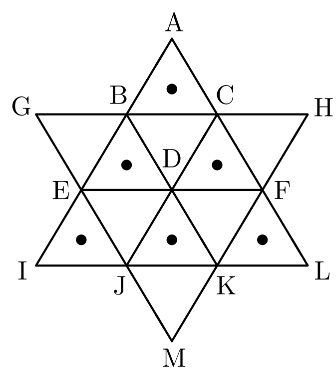

Figure 8: The hexagram and the first six blocks of the STS(13).

Another set of six blocks is obtained by rotating the base block AGK around the central point D by multiples of 60 degrees. Next, six further blocks are obtained by rotating the base blocks ADM and AEF around D by multiples of 120 degrees. Finally, AIL is also taken as a block, which amounts to a total number of nineteen blocks, as follows.

ABC	BDE	CDF	DJK	EIJ	FKL	
AGK	AHJ	BLM	CIM	EHL	FGI	
ADM	DGL	DHI				(1)
AEF	BIK	CJL				
AIL						

These nineteen blocks will be the "core" of the twenty-two common blocks of the two non-isomorphic Steiner triple systems of order 13. The main geometric property of this set of blocks is that it is invariant under the six symmetries of the hexagram that are induced by the symmetries of the equilateral triangle with vertices A, I, L, that is, the identity, the two rotations around D by 120 degrees, and the three axial symmetries with respect to the line segments AM, GL, and HI.

We now complete the nineteen blocks in (1) to the twenty-six blocks of a Steiner triple system of order 13. One of the seven blocks that are missing in the list (1) is the block through the points G and H. From the analysis of the blocks in (1) that contain either G or H it follows that there are only three possibilities: GHM, BGH, and CGH.

Case 1. The block through G and H is GHM. This choice is the only one that is consistent with the invariance under the vertical axial symmetry. Moreover, the choice of the block GHM uniquely determines the remaining six blocks, which are precisely BFH, BGJ, FJM, CEG, CHK, and EKM. Note that these six blocks are precisely those that are generated by the first one, BFH, under the six symmetries of the hexagram described above.

Case 2. The block through G and H is not GHM. Then, up to a vertical axial symmetry with respect to the line segment AM, one may assume that CGH is a block. Note that, unlike for the previous system, this determines the first symmetry breaking in the picture (on the other hand, the vertical axial symmetry is an order-2 permutation of the vertices, hence it cannot belong to the automorphism group of the system, which happens to have order 39). Also, the choice of the block CGH uniquely determines the remaining six blocks, which are precisely BFH, BGJ, FJM, CEK, EGM, and HKM, the first three of which also appear in the previous system.

Our geometric construction ultimately produces two Steiner triple systems of order 13, whose blocks are collected in the two following tables.

ABC	BDE	CDF	DJK	EIJ	FKL	AGK	AHJ	BLM	CIM	EHL
FGI	ADM	DGL	DHI	AEF	BIK	CJL	AIL	BFH	BGJ	FJM
GHM	CEG	CHK	EKM							

Table 1: The STS(13) n. 1 (non-cyclic).

ABC	BDE	CDF	DJK	EIJ	FKL	AGK	AHJ	BLM	CIM	EHL
FGI	ADM	DGL	DHI	AEF	BIK	CJL	AIL	BFH	BGJ	FJM
CGH	CEK	EGM	HKM							

Table 2: The STS(13) n. 2 (cyclic).

Note that the twenty-two blocks in the first two rows of the two tables are precisely the common blocks of the two systems, whereas the remaining sets of four blocks are both Pasch configurations and are obtained from one another by switching the points E and H (or, equivalently, by means of a Pasch switch).

If one sets A=0, B=3, C=4, D=5, E=1, F=2, G=7 (respectively, G=8), H=9, I=c, J=a, K=8 (respectively, K=7), L=b, M=6, then system n. 1 in Table 1 (respectively, system n. 2 in Table 2) becomes precisely the STS(13) denoted by n. 1 (respectively, n. 2) in [6, Table II.1.27] and [8, Table 5.7].

System n. 1 in Table 1 is the non-cyclic STS(13). Its automorphism group consists of the identity and of the permutations

$$(\mathrm{ALI})(\mathrm{GHM})(\mathrm{ECK})(\mathrm{BFJ})(\mathrm{D})$$

$$(\mathrm{AIL})(\mathrm{GMH})(\mathrm{EKC})(\mathrm{BJF})(\mathrm{D})$$

$$(\mathrm{EF})(\mathrm{BC})(\mathrm{GH})(\mathrm{JK})(\mathrm{IL})(\mathrm{D})(\mathrm{A})(\mathrm{M})$$

$$(\mathrm{AL})(\mathrm{EJ})(\mathrm{CF})(\mathrm{BK})(\mathrm{GM})(\mathrm{D})(\mathrm{H})(\mathrm{I})$$

$$(\mathrm{AI})(\mathrm{BE})(\mathrm{FK})(\mathrm{CJ})(\mathrm{HM})(\mathrm{D})(\mathrm{G})(\mathrm{L})$$

(cf. [27] and [43]), which are precisely the six symmetries of the hexagram in Figure 8 that are induced by the symmetries of the equilateral triangle with vertices A, I, L that we described earlier. The orbits of the thirteen points under the automorphism group are $\{\mathrm{D}\}$, $\{\mathrm{A, I, L}\}$, $\{\mathrm{G, H, M}\}$, and $\{\mathrm{B, C, F, K, J, E}\}$, which are precisely the center point, the two sets of vertices of the two main equilateral triangles, and the vertices of the inner hexagon, respectively. In Figure 9 we describe only the base blocks (representing each triple by three marks of the same kind), whose orbits under the automorphisms above determine all the twenty-six blocks of the system.

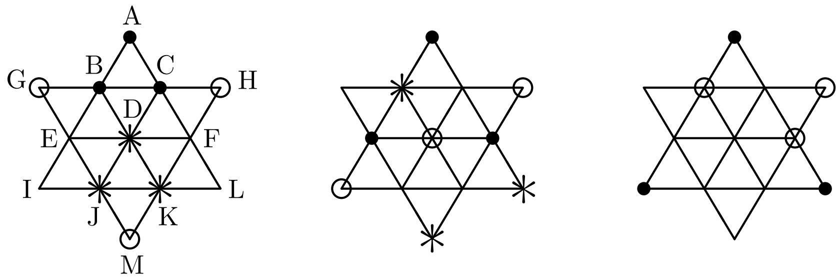

Figure 9: The base blocks of the non-cyclic STS(13).

Moreover, the (non-cyclic) system n. 1 contains precisely eight quadrilaterals, which can be easily visualized in the hexagram. The first one consists of the blocks ABC, BDE, CDF, and AEF, which correspond, respectively, to the three dotted triangles in Figure 7 and to the triangle AEF in the same figure. Two more quadrilaterals are obtained by applying to the previous one the two order-3 automorphisms of the system (that is, the two 120-degree rotations around the center of the hexagram). Another quadrilateral consists of the blocks ABC, AIL, BLM, and CIM (see the four triples in Figure 10), and is invariant under the axial symmetry with respect to the vertical line segment AM. Again, two more quadrilaterals are generated by the previous one under the two 120-degree rotations around the center of the hexagram.

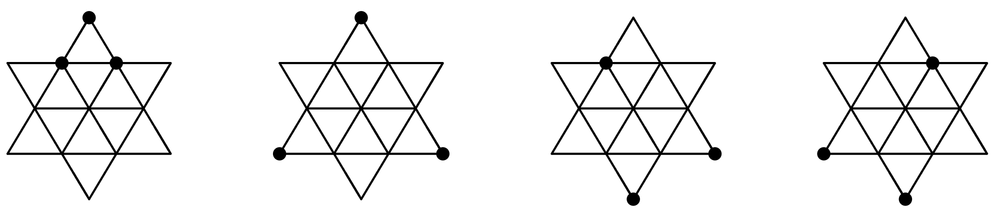

Figure 10: A quadrilateral in the non-cyclic STS(13) with vertical symmetry.

Finally, the four blocks GHM, CEG, CHK, and EKM, which are the only blocks not belonging to system n. 2 in Table 2, form a quadrilateral (see the four triples in Figure 11), which is invariant under the two 120-degree rotations, and which generates a further quadrilateral under the axial symmetry with respect to the vertical line segment AM.

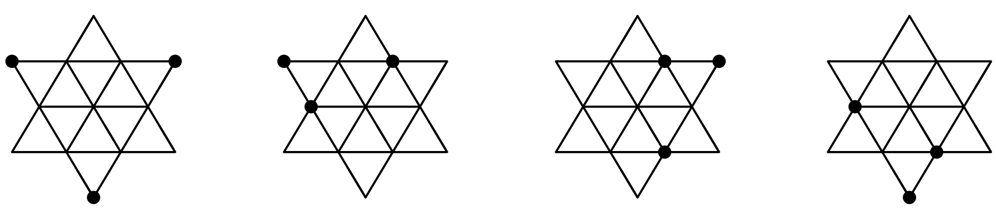

Figure 11: A rotation-invariant quadrilateral in the non-cyclic STS(13).

System n. 2 in Table 2 is the cyclic STS(13). Its automorphism group has order 39 and is generated by the order-13 permutation

$$\varphi = (\mathrm{A\ K\ I\ D\ G\ B\ H\ L\ J\ F\ E\ C\ M}) \tag{2}$$

and by the order-3 permutation ψ = (ALI)(GHM)(ECK)(BFJ)(D), which is precisely the clockwise 120-degree rotation around the center of the hexagram (and which is also an automorphism of system n. 1). The automorphism group acts on the 26 blocks of the system with two orbits of length 13, which are generated under φ by the only two blocks that are invariant under ψ (AIL and CEK). The four blocks CEK, CGH, EGM, and HKM, which are the only blocks not belonging to system n. 1 in Table 1, form a quadrilateral (see the four triples in Figure 12), which is invariant under the rotation ψ and which cyclically generates, under the permutation φ, all the thirteen quadrilaterals contained in the system.

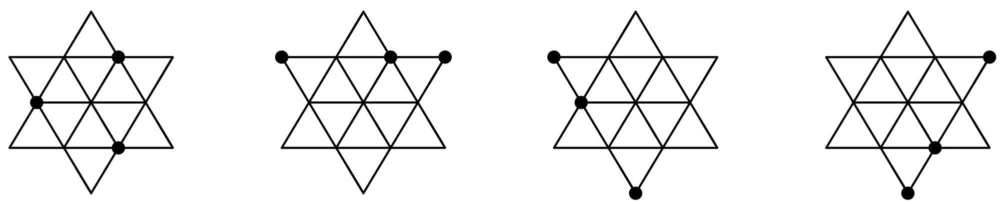

Figure 12: The base (and rotation-invariant) quadrilateral in the cyclic STS(13).

As we mentioned before, the cyclic STS(13) contains no mitres (see Definition 4), whereas the non-cyclic system contains 10 mitres, which, moreover, can be partitioned into four orbits of length 1, 3, 3, 3, under the action of the automorphism group. The fixed mitre is {ADM, DHI, DGL, AIL, GHM}, and is (necessarily) centered at the center D of the hexagram, D being the only fixed point under the automorphism group. The other three base mitres, one for each orbit, are {ADM, AGK, AHJ, DJK, GHM}, {ABC, AGK, AHJ, BGJ, CHK}, and {ADM, EKM, FJM, AEF, DJK}, which are centered at A, A, and M, respectively. Note that all the centers of the base mitres are on the vertical line ADM, and that each base mitre is invariant under the vertical axial symmetry. The four base mitres are represented in the same order as above in Figure 13, where we adopt the convention that the center of the mitre is always labelled as Ω, the two parallel (that is, disjoint) lines are marked with dotted line segments, and each of the three lines ΩXY through Ω is represented by labelling X and Y with two marks of the same kind. Note how the first mitre in Figure 13 looks precisely like the abstract model of mitre in the middle of Figure 2.

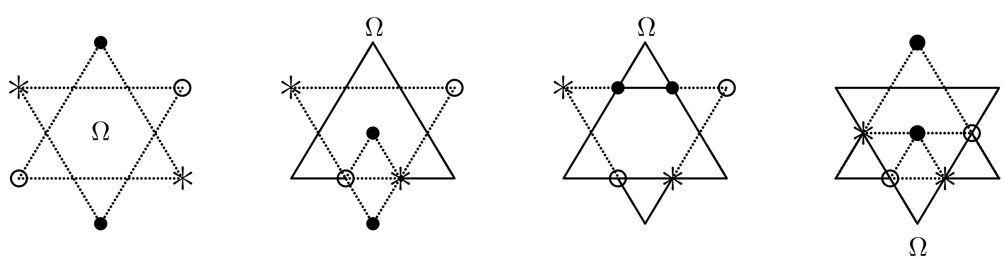

Figure 13: The four base mitres in the non-cyclic STS(13).

We also noted that both STS(13)s contain exactly thirteen grids (see Definition 4). This may appear somehow surprising at first thought, since for each system the family of grids is invariant under the automorphism group, which has order 39 for the cyclic system and order 6 for the non-cyclic sistem. The 26 blocks of the cyclic system, as we mentioned earlier, form two orbits of length 13 under the automorphism group, and each grid has three parallel lines in one orbit and three parallel lines in the other orbit, so that the 13 grids are cyclically generated by a single base grid under the action of the order-13 automorphism φ defined above in (2). Since our hexagram model is not suitable to visualize such an automorphism, the grids of the cyclic system do not show a particularly symmetric feature.

In the non-cyclic system there exist, instead, four base grids, whose orbits under the automorphism group S_3 have orders $1, 3, 3, 6$. The first three base grids are symmetric with respect to the vertical axial symmetry, whereas the base grid in the last orbit is necessarily not symmetric (else the length of its orbit would be at most 3). In Figures 14 and 15 one can visualize the four base grids, each of which is represented by two distinct hexagrams, one for each parallel class of lines, where, as usual, each line is represented by three marks of the same kind.

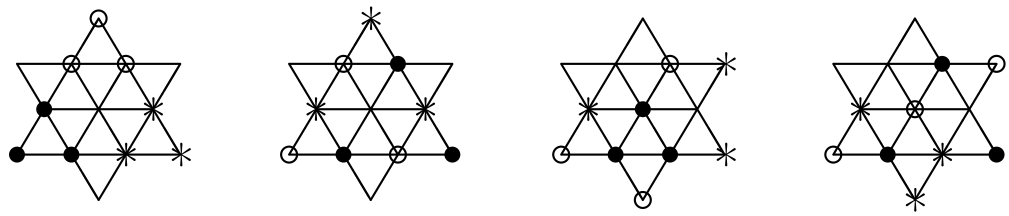

Figure 14: The base grids of order 1 (left) and 6 (right) in the non-cyclic STS(13).

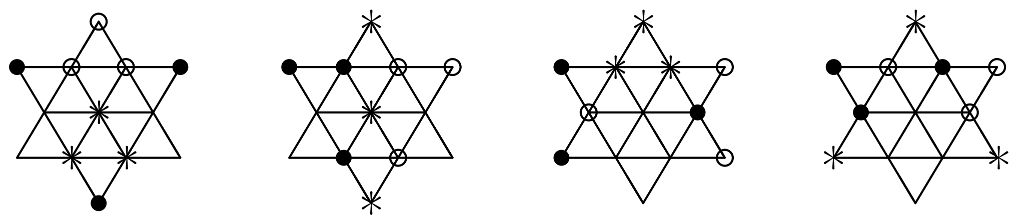

Figure 15: The two base grids of order 3 in the non-cyclic STS(13).

Remark 8. If $\mathcal{S} = (\mathcal{V}, \mathcal{B})$ is any of the two STS(13)s, the family $\mathcal{G}$ of the thirteen grids of $\mathcal{S}$ shows a quite interesting property. As it can be checked by inspection, any two grids have exactly one block $B \in \mathcal{B}$ in common, whereas each point $\mathrm{P} \in \mathcal{V}$ appears in precisely nine grids, and each of the twenty-six blocks in $\mathcal{B}$ appears in precisely three grids. This shows that $\mathcal{G}$ can be seen as the point-set of a Steiner triple system $\overline{\mathcal{S}} = (\mathcal{G}, \overline{\mathcal{B}})$ of order 13, whose blocks consist of the triples $\overline{B} = \{G_1(B), G_2(B), G_3(B)\}$, as B ranges in $\mathcal{B}$, where $G_1(B), G_2(B), G_3(B)$ are the only three grids containing B. For instance, for the non-cyclic system, the base grid of order 1 in Figure 14, together with the two base grids of order 3 in Figure 15, are the only three grids containing the block $\mathrm{ABC} \in \mathcal{B}$, hence they form a block in $\overline{\mathcal{B}}$, which we denote by $\overline{\mathrm{ABC}}$. Also, as we noted before, $\mathcal{S}$ and $\overline{\mathcal{S}}$ have the same automorphism group, hence they are in either case two isomorphic Steiner triple systems of order 13.

Now one can iterate the same construction, and consider the STS(13) $\overline{\overline{\mathcal{S}}}$, whose point-set consists of "grids of grids". It can be checked that in each of the thirteen grids of grids the nine "points" are the nine grids in $\mathcal{G}$ with a point P in common, for some $\mathrm{P} \in \mathcal{V}$, and the six "lines" are precisely the six blocks $\overline{\mathrm{PXY}}$ in $\overline{\mathcal{B}}$, as PXY ranges over the six triples in $\mathcal{B}$ containing P. For each $\mathrm{P} \in \mathcal{V}$, we denote by GG_{P} the grid of grids associated with P. In Table 3 we show the grid of grids GG_{D} associated with the central point D of the hexagram. Finally, the mapping $\mathrm{P} \to GG_{\mathrm{P}}$ defines a "natural" isomorphism between $\mathcal{S}$ and $\overline{\overline{\mathcal{S}}}$. We leave it as an exercise to the reader to verify the details of the isomorphism.

D	J	K	D	J	K	D	J	K
A	B	C	H	L	E	G	F	I
M	G	H	I	C	M	L	M	B
D	B	E	D	B	E	D	B	E
A	K	G	H	J	A	G	H	M
M	I	C	I	G	F	L	F	K
D	F	C	D	F	C	D	F	C
A	H	J	H	M	G	G	A	K
M	B	L	I	J	E	L	E	H

Table 3: The "grid of grids" GG_{D} in the non-cyclic STS(13).

A fundamental concept in the theory of block designs (and in graph theory) is the notion of *colouring* (see, for instance, [7, 15, 16, 36], and Chapters 18 and 19 in [8]). In general, a "colouring" of a block design is the assignment of a colour to each point of the design, together with a rule that specifies how many colours can appear in a block. In the special case of Steiner triple systems, the most widespread rule in the literature requires that no block can be monochromatic. A somehow complementary alternative is the rule that all blocks be monochromatic, as long as two intersecting blocks never have the same colour (in analogy with the more famous map-colouring problem); equivalently, two distinct blocks of the same colour are necessarily disjoint. In either case, one looks for the smallest number of colours for which the rule is respected.

Definition 9. (see, for instance, [8, §18.1]) Let $\mathcal{S} = (\mathcal{V}, \mathcal{B})$ be a Steiner triple system. A (*weak*) *colouring* of $\mathcal{S}$ is a mapping $\Phi : \mathcal{V} \to C$ (the set of *colours*), such that $|\Phi(B)| > 1$ for all $B \in \mathcal{B}$. For each $c \in C$, the set $\Phi^{-1}(\{c\}) = \{x \in \mathcal{V} : \Phi(x) = c\}$ is a *colour class*. The (weak) *chromatic number* of $\mathcal{S}$, $\chi(\mathcal{S})$, is the smallest value of k for which $\mathcal{S}$ admits a (weak) colouring with k colours, and $\mathcal{S}$ is said to be χ-*chromatic*. A *colouring pattern* of $\mathcal{S}$ is a χ-tuple $(c_1, c_2, \ldots, c_\chi)$, with $c_1 \geq c_2 \geq \ldots \geq c_\chi$, where $c_1, c_2, \ldots, c_\chi$ are the cardinalities of the colour classes of a (weak) colouring of $\mathcal{S}$ with χ colours.

Definition 10. (see, for instance, [8, §19.4]) Let $\mathcal{S} = (\mathcal{V}, \mathcal{B})$ be a Steiner triple system. A *colouring of triples* or *block-colouring* of $\mathcal{S}$ is a mapping $\Psi : \mathcal{B} \to C$ (the set of *colours*), such that if $\Psi(B) = \Psi(B')$, for $B, B' \in \mathcal{B}$, $B \neq B'$, then $B \cap B' = \emptyset$. For each $c \in C$, the set $\Psi^{-1}(\{c\}) = \{B \in \mathcal{B} : \Psi(B) = c\}$ is a *colour class*. The *chromatic index* of $\mathcal{S}$ is the smallest k for which $\mathcal{S}$ admits a block-colouring with k colours.

All Steiner triple systems of orders 7, 9, 13 and 15 are 3-chromatic [27]. The possible colouring patterns of the two STS(13)s are $(6, 5, 2), (6, 4, 3), (5, 5, 3), (5, 4, 4)$, and all are attainable for both systems [15] . In Figure 16 we represent four 3-colourings of the non-cyclic STS(13), one for each colouring pattern, in the same order as above. The leftmost picture and the rightmost picture represent also 3-colourings of the cyclic STS(13). It is possible to find mappings Φ that are 3-colourings of both systems also for the two remaining patterns [15], but their visual representations on the hexagram would not be particularly symmetric.

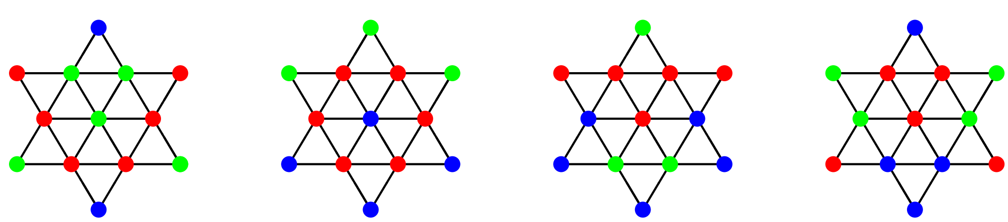

Figure 16: Four 3-colouring patterns in the non-cyclic STS(13).

It is also well-known that the chromatic index of any of the two Steiner triple systems of order 13 is equal to 8 [27]. In Figures 17 and 18 we represent a block-colouring with eight colours of the non-cyclic STS(13). We visualize each colour class on a separate hexagram, by representing only the (mutually disjoint) blocks that are associated with that colour. For instance, the mostleft picture in Figure 17 indicates that the blocks ABC, EIJ, FKL, and GHM are given the same colour. Note how, as usual, some of the pictures are invariant under rotation and/or axial symmetry, whereas other pictures can be obtained from one another by means of a 120-degree rotation. Also, the two blocks marked in violet are the only ones that are invariant under the 120-degree rotations.

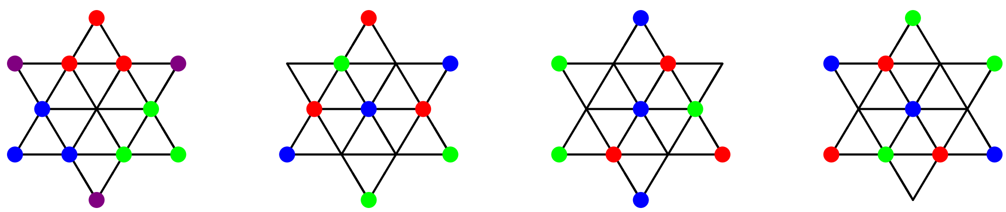

Figure 17: Block-colouring in the non-cyclic STS(13): colour classes I-IV.

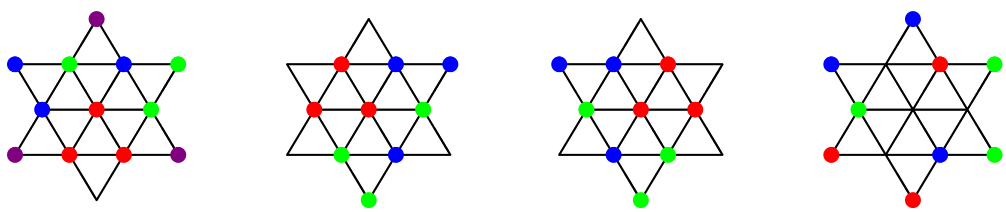

Figure 18: Block-colouring in the non-cyclic STS(13): colour classes V-VIII.

Similarly, in Figures 19 and 20 we represent a block-colouring with eight colours of the cyclic STS(13). Although the two systems differ by only four blocks, all eight colour classes of the block-colouring of the non-cyclic system must necessarily be modified in order to get a block-colouring of the cyclic system. Again, some of the pictures are invariant under rotation and/or axial symmetry, whereas other pictures can be obtained from one another by means of a 120-degree rotation. Also, the two blocks marked in violet are the only ones that are invariant under the 120-degree rotations.

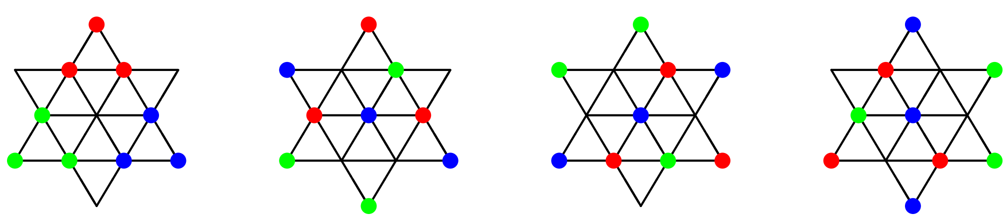

Figure 19: Block-colouring in the cyclic STS(13): colour classes I-IV.

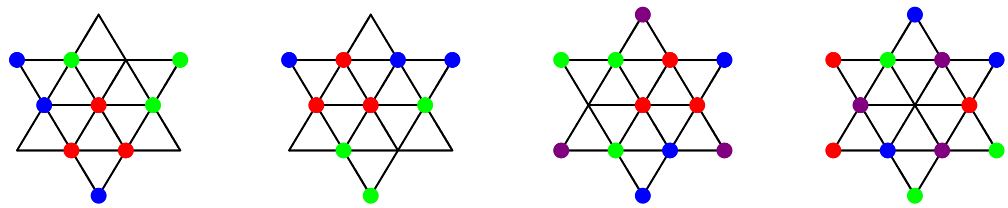

Figure 20: Block-colouring in the cyclic STS(13): colour classes V-VIII.

An oval in an STS(13) is a 6-set of points that does not contain any line (see Definition 2). Ovals in the two non-isomorphic STS(13)s are thoroughly investigated in [41] (where an alternative oval-based geometric construction is given). A simple counting argument shows that there exist only two types of ovals. In the first kind (called Ovale 1 in [41]) there exists a unique interior point and six exterior points, each of which lies in two secant, two tangent and two exterior lines. In the second kind (called Ovale 2 in [41]), there exist precisely two interior points, one exterior point lying in four tangent lines, one secant line and one exterior line, and four exterior points each lying in two secant, two tangent and two exterior lines. In either case, by Definitions 1 and 2, each point of the oval lies necessarily on five secant lines and on a unique tangent line (which leads to a total number of five exterior lines).

The "nicest" and most symmetric oval to visualize in the (non-cyclic) system n. 1 is $\{B, C, F, K, J, E\}$, that is, the set of vertices of the inner hexagon in the hexagram (picture on the left in Figure 21). Moreover, in this case, the nature of the seven points off the oval is consistent with what visually appears in the picture, since the center point of the hexagram, "inside" the oval (marked with a little circle in the picture on the left), is precisely the unique interior point, whereas the six remaining points (the outer points of the "star") are all exterior points. This is, however, just a fortunate case and not a general rule. Indeed, in the picture on the right in Figure 21, the two points that appear "inside" the oval (J and K in Figure 8) are actually exterior points, whereas the unique interior point happens to be the top point of the hexagram (marked with a little circle in the picture), which is actually the farthest point from the oval.

Figure 21: Two ovals of the first type in the non-cyclic STS(13).

In Figure 22 we show two ovals of the second type in the non-cyclic STS(13). In either case, the two interior points are marked with two little circles, whereas the point marked with an asterisk is the unique exterior point lying in four tangent lines, one secant line and one exterior line.

Figure 22: Two ovals of the second type in the non-cyclic STS(13).

Finally, in Figure 23 we describe two ovals in the (cyclic) system n. 2 (of the first type on the left, of the second type on the right). Again, each interior point is marked with a little circle, whereas the point marked with an asterisk is the unique exterior point lying in four tangent lines, one secant line and one exterior line.

Figure 23: Two ovals (first and second type) in the cyclic STS(13).

The oval on the right in Figure 23 is remarkable in that it is symmetric with respect to the vertical axis through the center point, although the STS itself is not. Also, the five exterior lines are precisely the four blocks of the quadrilateral on the right in Figure 7, plus the block of the three points on the vertical axis.

3 Triangle-based models of STSs of order 7, 9, 15

We conclude this paper by presenting new pictures of the Fano plane (STS(7)), of the affine plane of order 3 (STS(9)), and of one of the eighty non-isomorphic STS(15)s. Interestingly enough, perhaps even surprisingly, the two former pictures both readily arise as derived pictures of the previous models for the Steiner triple systems of order 13 in a simple and "natural" way, and with the same initial geometric idea.

Let us consider the six points A, B, C, D, E, F in Figure 7, and let us take the blocks in Table 1 above (or, indifferently, in Table 2) that can be formed with these points. This gives us the blocks ABC, BDE, CDF, AEF, which already appeared in the list (1), and which define a quadrilateral in both systems. The first three blocks are precisely the three dotted blocks in the picture on the right in Figure 7, whereas the forth block represents the vertices of the triangle AEF in the same picture. One can extend these four blocks uniquely to the block-set of a Fano plane by taking a seventh point, say, X, and three more blocks ADX, BFX, CEX. If we represent the point X as the center of the triangle AEF in Figure 7, then the resulting picture is given in Figure 24, where we describe only the three base blocks that generate all the blocks of the system under the two 120-degree rotations around the center point. Note that if in this picture we perform a Pasch switch with respect to the quadrilateral obtained by removing the center point and the three blocks through it, and we finally add again the three blocks through the center point, then we get precisely the classical representation of the Fano plane (on the left in Figure 3).

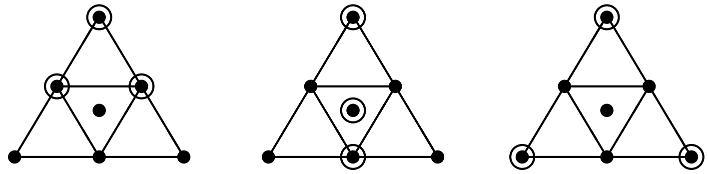

Figure 24: The base blocks of the Fano plane.

Similarly, in order to get a picture of the affine plane of order 3, let us consider the nine points A, B, C, E, F, I, J, K, L in Figure 8 (that is, the only points of the hexagram on the sides of the triangle AIL), and let us take the blocks in Table 1 above (or, indifferently, in Table 2) that can be formed with these points. This gives us the seven blocks ABC, AEF, AIL, BIK, CJL, EIJ, FKL, which already appeared in the list (1). One can extend these seven blocks uniquely to the block-set of a Steiner triple system of order 9 by adding the five blocks AJK, BEL, CFI, BFJ, CEK. The resulting picture is given in Figure 25, where we partition the twelve blocks in the four parallel classes of the affine plane of order 3. The six symmetries of the regular triangle that underlies the diagram correspond to just as many automorphisms of the Steiner triple system.

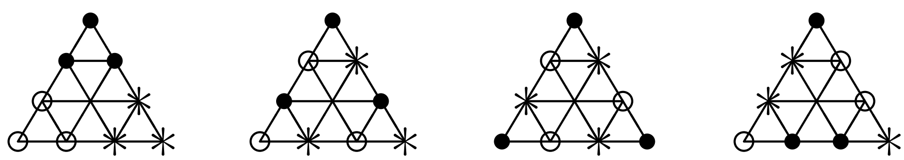

Figure 25: The four parallel classes of the affine plane of order 3.

Finally, we extend again the main geometric idea of this paper to get a triangle-based visual representation of one of the eighty non isomorphic Steiner triple systems of order 15. We start from the hexagram in Figure 8, we remove the points G, H, M and the three line segments connecting them, and we add seven more equilateral triangles below the base IL. The fifteen points of the STS are precisely the vertices of the sixteen small equilateral triangles in Figure 26 (note that the labelling of some of the vertices has been changed).

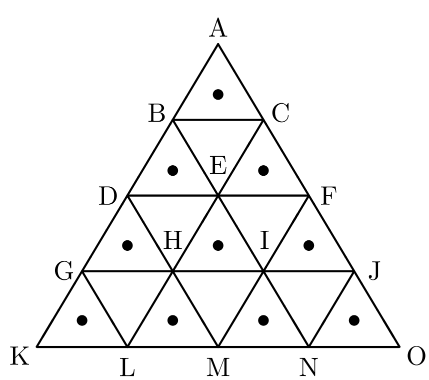

Figure 26: The points of the STS(15) #6.

The Steiner triple system is defined in a quite "natural" way as follows. The first ten blocks are obtained by taking as a block the set of the three vertices of any of the ten dotted triangles in Figure 26. The next ten blocks are the sets of vertices of all the equilateral triangles properly containing some dotted triangle, that is, the six medium-small triangles of the same size as ADF, the three medium-large triangles of the same size as AGJ, and the large triangle AKO. This choice of the first twenty blocks determines the remaining fifteen blocks uniquely. More precisely, the next nine blocks are generated by the base blocks AEM and AHN under the six symmetries of the equilateral triangle AKO, that is, the identity, the two rotations around the center point by 120 degrees, and the three axial symmetries with respect to the line segments AM, DO, and FK. The six final blocks are generated by the vertex-set DJL of the base isosceles triangle DJL under the same six symmetries of the underlying equilateral triangle AKO. By construction, such six symmetries are all automorphisms of the Steiner triple system.

It can be checked by inspection that the STS contains precisely three Fano planes. The first one consists of the blocks AEM, AHN, AIL, ELN, EHI, HLM, IMN, whereas the other two are generated by the first one under the two 120-degree rotations around the center of the triangle. Note that the base Fano plane is invariant under the axial symmetry with respect to the line segment AM. Also, it can be found by exhaustion that, in addition to the 7×3 quadrilaterals contained in the three Fano planes, there exist 16 quadrilaterals not contained in any Fano plane, for a total number of 37 quadrilaterals. It follows from [6, Table 1.29, p. 32] that the STS is necessarily (isomorphic to) the STS(15) #6. Alternatively, if one sets A=4, B=10, C=15, D=11, E=1, F=14, G=9, H=2, I=3, J=13, K=12, L=7, M=5, N=6, O=8, then the STS defined above becomes precisely the STS(15) listed as No. 6 in [27].

Following [27], the full automorphism group of the system is the order-24 group generated by either of the two 120-degree rotations around the center of the triangle and by the order-4 permutation (B J G C)(D K O F)(A M)(H I)(E)(L)(N). Note that our triangular model is particularly suitable to visualize the two rotations, but not the order-4 automorphism.

Finally, it is worth noting that ADF, AKO, DKM, FMO is a quadrilateral of the system with the property that the STS(15) obtained by means of the corresponding Pasch switch is precisely the resolvable STS(15) #7, hence the above model, with the four new blocks AFO, ADK, KMO, DFM, allows one to visualize the solutions 7a and 7b of the fifteen schoolgirl problem (see the discussion in [31, §3, Example 3]).

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