PHILOSOPHICAL TRANSACTIONS A

royalsocietypublishing.org/journal/rsta

Research

Check for updates

Cite this article: Bologna E, Di Paola M, Dayal K, Deseri L, Zingales M. 2020 Fractional-order nonlinear hereditariness of tendons and ligaments of the human knee. *Phil. Trans. R. Soc. A* **378**: 20190294.
http://dx.doi.org/10.1098/rsta.2019.0294

Accepted: 31 March 2020

One contribution of 15 to a theme issue 'Advanced materials modelling via fractional calculus: challenges and perspectives'.

Subject Areas:
biomedical engineering, mechanical engineering, mathematical modelling, biomechanics, mechanics, integral equations

Keywords:
nonlinear hereditariness, quasi-linear viscoelasticity, single-integral, tendons and ligaments

Author for correspondence:
M. Zingales
e-mail: massimiliano.zingales@unipa.it

THE ROYAL SOCIETY PUBLISHING

Fractional-order nonlinear hereditariness of tendons and ligaments of the human knee

E. Bologna[1,2], M. Di Paola[1], K. Dayal[3,4,5], L. Deseri[6,7,8,9] and M. Zingales[1,2]

[1]Dipartimento di Ingegneria, Viale delle Scienze ed.8, 90128 Palermo, Italy
[2]Bio/NanoMechanics for Medical Sciences Laboratory, Viale delle Scienze ed.8, 90128 Palermo, Italy
[3]Department of Civil and Environmental Engineering, Carnegie Mellon University Pittsburgh, Pittsburgh, PA 15213, USA
[4]Center for Nonlinear Analysis, Carnegie Mellon University Pittsburgh, Pittsburgh, PA 15213, USA
[5]Department of Materials Science and Engineering, Carnegie Mellon University Pittsburgh, Pittsburgh, PA 15213, USA
[6]Department of Mechanical Engineering, University of Pittsburgh, Benedum Hall, Pittsburgh, PA, USA
[7]Dipartimento Civile, ambientale e meccanica, Università degli Studi di Trento, Via Mesiano, 77 - 38123 Trento, Italy
[8]Department of Mechanical Engineering, Department of Civil & Env. Engineering, Carnegie Mellon University, Pittsburgh, PA 15213-3890, USA
[9]The Methodist Hospital Research Institute, Department of Nanomedicine, 6565 Fannin St., MS B-490, Houston, TX 77030, USA

EB, 0000-0002-6209-1599; MDP, 0000-0002-9564-1446; KD, 0000-0002-0516-3066; MZ, 0000-0001-9093-9529

In this paper the authors introduce a nonlinear model of fractional-order hereditariness used to capture experimental data obtained on human tendons of the knee. Creep and relaxation data on fibrous tissues have been obtained and fitted with logarithmic relations that correspond to power-laws with nonlinear dependence of the coefficients. The use of a proper nonlinear transform allows one to use Boltzmann superposition in the transformed variables yielding a fractional-order model for the nonlinear material hereditariness. The fundamental relations among the nonlinear creep and relaxation

royalsocietypublishing.org/journal/rsta Phil. Trans. R. Soc. A 378: 20190294

functions have been established, and the results from the equivalence relations have been contrasted with measures obtained from the experimental data. Numerical experiments introducing polynomial and harmonic stress and strain histories have been reported to assess the provided equivalence relations.

This article is part of the theme issue 'Advanced materials modelling via fractional calculus: challenges and perspectives'.

1. Introduction

Material hereditariness has been a challenging topic in the field of engineering and physical sciences since the very first observations on the long-term behaviour of polymers, rubbers and concrete at the end of the nineteenth century. Indeed experimental campaigns conducted under long-standing loads (creep tests) and long-standing displacements (relaxation tests) show that, besides an initial elastic displacement (strain in creep) or load (stress in relaxation), a time evolution of the initially measured strain/stress is clearly observed.

Mathematical modelling of materials time-dependence in terms of creep and relaxation has been intensively investigated since the beginning of the twentieth century [1–4].

In such studies, the use of the Boltzmann superposition assumption yields constitutive equations involving convolution integrals among a kernel function that is material dependent and the material state variables [5,6]. The observation involving effect superposition corresponds to a linear behaviour of the material such that linear hereditariness of engineering materials such as wood, glass and bitumen has been extensively studied to account for long-standing loads [1,7]. In this regard, the use of power-law kernels to describe experimental material functions led several authors to introduce the fractional-order differintegral operators to capture material behaviour in linear conditions [8].

Fractional-order constitutive equations have also been extended to three-axial conditions [9] and rheological equivalence has also been introduced [10–13]. The wide success of fractional-order calculus to deal with linear hereditariness is, however, not justified in the wider field of nonlinear material hereditariness. Indeed, it has been well known since the very first long-standing experiments on rubbers and polymers [1,2] that carbon-based materials undergo large strains and a significative nonlinear behaviour is observed from experimental tests [14,15]. This latter observation has also been more evident in presence of fibrous biological tissues where no linear conditions may be observed even for a small level of stress/strain [16–20].

In the framework of nonlinear material hereditariness, fundamental contributions trace back to studies from the middle of the last century that are based on the principle of fading memory [21] that allows, after some manipulations, the constitutive equations to be expressed as a sum of multiple integrals involving several material functions [22]. Such an approach was simplified in the mid 1990s transforming multiple integrals into multiple convolution integrals with material functions independent of the material state variables [23]. Neglecting multiple integrals with respect to the single integral term leads to the widely used quasi-linear viscoelasticity (QLV) [16,24]. A comprehensive review of linear and nonlinear material hereditariness was reported in some studies at the beginning of the present century [25–27].

QLV has not, however, been used in the context of fractional-order calculus to take full advantage of fractional-order formalism. Indeed, the formulation of QLV is usually presented in terms of material relaxation that is not readily obtained from experimental data for the inertia of loading equipment. Creep functions are, instead, more easily obtained but no closed-form relations among creep and relaxations for the QLV have been reported so far in scientific literature to the best of the authors knowledge. Moreover, no mechanical justification for the use of QLV models has ever been introduced, providing a severe limitation in the use of the quasi-linear formulations in biomechanics and material engineering.

In this paper, the authors aim to introduce the use of fractional-order calculus to capture the experimental behaviour of biological fibrous tissues of the human knee in the presence of material nonlinear behaviour. A complete physical and mathematical framework for the relations among creep and relaxation functions in the presence of material nonlinearities, contrasted with experimental data from a large experimental campaign have been provided by means of fractional-operators formalism.

The paper is organized as follows. In §2, some preliminary remarks about linear hereditariness and fractional-order calculus is introduced. In §3, the main results of an extensive experimental campaign conducted on fibrous biological tissues of the human knee in terms of creep and relaxation tests. In §4, the proposed formulation for the nonlinear constitutive equations based on the results of the experimental campaign is introduced and discussed in the context of fractional calculus.

2. Remarks on fractional-order linear hereditariness

Material hereditariness is experienced each time a long-standing controlled load (controlled displacement) experimental test shows time evolution of the measured displacements (measured load). In this section, we assume a one-dimensional load–displacement relation and we switch to engineering measure of stress, namely $\sigma(t)$, and of strain, namely $\varepsilon(t)$, without loss of generality. Under these circumstances, the constitutive behaviour involves material function for strain evolution, namely $\phi_c(\sigma, t)$, that provides the strain evolution under constant stress as well as a different material function yielding the stress decay under constant strain, namely $\phi_r(\varepsilon, t)$. In passing, we observe that the material functions ϕ_c and ϕ_r depend, in general, on the applied stress and strain, respectively.

In the framework of linear hereditariness, the creep function satisfies the linearity conditions, namely

$$\phi_c\,(\lambda\sigma, t) = \lambda\phi_c\,(\sigma, t) \quad \forall\,\lambda \in R; \quad \phi_c\,(\sigma_1+\sigma_2, t) = \phi_c\,(\sigma_1, t) + \phi_c\,(\sigma_2, t). \tag{2.1}$$

A similar consideration holds true for the relaxation function that satisfies the linearity conditions in (3.1) as

$$\phi_r\,(\lambda\varepsilon, t) = \lambda\phi_r\,(\varepsilon, t) \quad \forall\,\lambda \in R; \quad \phi_r\,(\varepsilon_1+\varepsilon_2, t) = \phi_r\,(\varepsilon_1, t) + \phi_r\,(\varepsilon_2, t)\,. \tag{2.2}$$

The linearity assumptions for the creep and relaxation functions allow one to introduce material hereditariness for unitary value of applied stress and strain, namely $\sigma = 1$, $\varepsilon = 1$, resulting in stress- and strain-independent material hereditary functions as $\phi_c(\sigma, t) = \sigma J(t)$, and $\phi_r(\varepsilon, t) = \varepsilon G(t)$, that is creep and relaxation are linear functions of σ and ε, respectively. The function $J(t)$ and $G)t)$ are creep and relaxation functions of linear viscoelasticity. Time-varying functions $[G(t)] = F/L^2$ and $[J(t)] = L^2/F$ are well-known relaxation and creep functions, respectively.

In the following, linearity conditions will be extensively used to introduce the linear mathematical description of material hereditariness as well as to provide a rheological description of the experimental linear behaviour observed for several conventional materials.

(a) Mathematical modelling of linear hereditariness

Knowledge of the material functions $J(t)$ and $G(t)$, creep and relaxation functions, respectively, allows for the use of the Boltzmann superposition principle [3], yielding the stress and the strain at a generic time instant t due to an arbitrary stress $\sigma(\tau)$ or strain $\varepsilon(\tau)$ history as

$$\varepsilon\,(t) = \int_0^t J\,(t-\tau)\,\dot{\sigma}\,(\tau)\,\mathrm{d}\tau \tag{2.3a}$$

and

$$\sigma\,(t) = \int_0^t G\,(t-\tau)\,\dot{\varepsilon}\,(\tau)\,\mathrm{d}\tau. \tag{2.3b}$$

The equalities in equation (2.3*a*,*b*) are well-known integral constitutive relations in one-dimensional linear hereditariness and it is well known, after some straightforward manipulations,

royalsocietypublishing.org/journal/rsta Phil. Trans. R. Soc. A 378: 20190294

that creep and relaxation functions must satisfy the fundamental equation of linear hereditariness $\hat{G}(s)\hat{J}(s) = 1/s^2$, with $[s] = 1/T$ the Laplace parameter and $[\hat{\bullet}]$ denoting Laplace transform.

The specific functional class of creep and relaxation functions reported in equation (2.3*a*,*b*) may be guessed from experimental data collected in the course of experimental campaigns and they are very often expressed as single or linear combinations of exponential functions by means of the Prony representation theorem [28] as

$$J(t) = \sum_{l=1}^{M} J_I \left(1 - \exp\left(-\frac{t}{\tau_l^{(c)}}\right)\right) \tag{2.4a}$$

and

$$G(t) = \sum_{l=1}^{N} G_I \exp\left(-\frac{t}{\tau_I^{(r)}}\right), \tag{2.4b}$$

where the coefficients of the expansions have physical measures $[G_I] = F/L^2$ and $[J_I] = L^2/F$ and the material characteristic times in creep and relaxation, namely, $\tau_I^{(c)}$ and $\tau_I^{(r)}$, are additional material parameters that may be estimated by best fitting procedures together with the expansion coefficients. The integer numbers in the expansions, namely M and N, are, respectively, the order of the Prony series used for creep and relaxation, respectively.

The expressions for creep and relaxation functions reported in equation (2.4) cannot, however, satisfy the fundamental relation of linear hereditariness, and, henceforth, they must be used separately in stress- and strain-based constitutive relations reported in equation (2.3). Some attempts to introduce analogous formulations jointly in creep and relaxation led to unphysical negative values of the material relaxation times in the Prony expansion [28].

The lack of mathematical consistency in the use of Prony series for creep and relaxation, respectively, in conjunction with the Nutting experiments obtained by best fitting on experimental data performed on rubbers, concrete and ceramics. It has pushed several authors to select the creep and relaxation function in the class of power laws as

$$J(t) = \frac{J_\beta}{\Gamma(\beta)} t^\beta = \frac{1}{G_\beta \Gamma(\beta)} t^\beta = \frac{1}{G_0 \Gamma(\beta)} \left(\frac{t}{\tau_0}\right)^\beta \tag{2.5a}$$

and

$$G(t) = \frac{G_\beta}{\Gamma(1-\beta)} t^{-\beta} = \frac{G_0}{\Gamma(1-\beta)} \left(\frac{t}{\tau_0}\right)^{-\beta}, \tag{2.5b}$$

where the anomalous terms $[G_\beta] = FT^\beta/L^2 \geq 0$, $[J_\beta] = L^2/FT^\beta \geq 0$ are material-dependent coefficients, the exponent $0 \leq \beta \leq 1$ is a material-dependent decaying order and $\Gamma(\bullet)$ is the Euler–Gamma function. The physical dimensions of the material coefficients allows the creep and relaxation functions to be represented introducing a two-term factorization $G_\beta = G_0 \tau_0^\beta$ with τ_0 a material-dependent characteristic time and G_0 the conventional elastic modulus of the material observed in a monotone tensile test.

Straightforward manipulations show that equation (2.5) satisfies the fundamental relations of linear hereditariness and substitution into equation (2.2), the stress–strain constitutive equations of linear hereditariness read

$$\sigma(t) = \frac{G_0 (\tau_0)^\beta}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} \dot{\varepsilon}(\tau)\, \mathrm{d}\tau = G_0 (\tau_0)^\beta \left(D_0^\beta \varepsilon\right)(t) \tag{2.6a}$$

and

$$\varepsilon(t) = \frac{1}{G_0 (\tau_0)^\beta \Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \sigma(\tau)\, \mathrm{d}\tau = \frac{1}{G_0 (\tau_0)^\beta} \left(I_0^\beta \sigma\right)(t). \tag{2.6b}$$

Notations introduced in the last equality at the right-hand side of equation (2.6), namely $(D_0^\beta f)(t)$ and $(I_0^\beta f)(t)$, denote, respectively, the Caputo fractional derivative and the Riemann–Liouville fractional integral of order β of the generic function $f(t)$, respectively. Details about fractional-order operators are out of the scope of the paper, and the reader may refer to more complete readings about the topic as in [5,29].

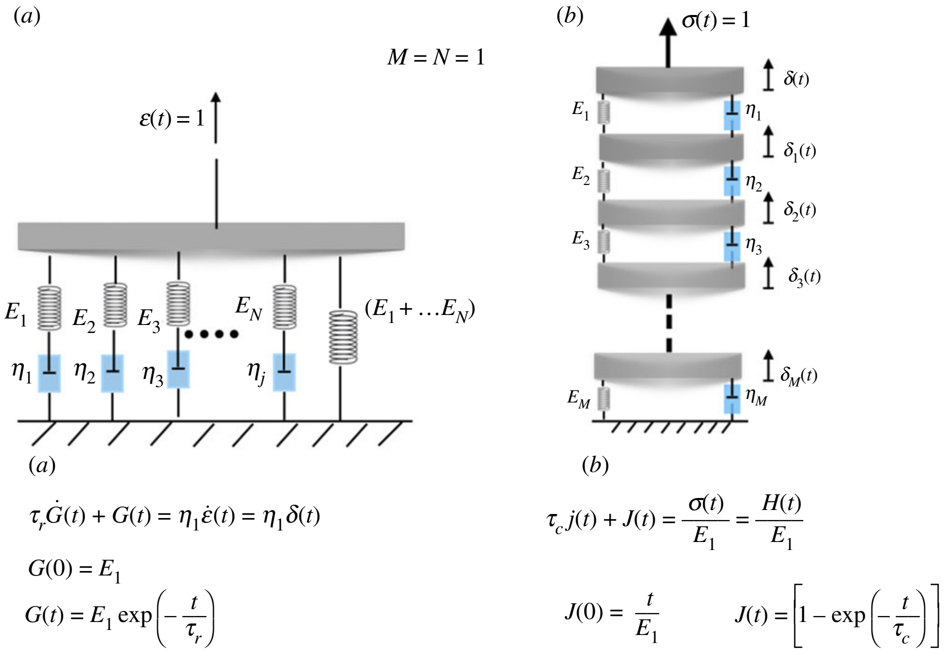

Figure 1. (*a*) Rheological model for exponential relaxation; (*b*) rheological model for exponential creep. (Online version in colour.)

In the following, for simplicity's sake, we refer to non-dimensional stress $s(t) = \sigma(t)/G_0$ yielding the constitutive equations for fractional-order nonlinear hereditariness as

$$s\,(t) = (\tau_0)^{\beta}\left(D_0^{\beta}\varepsilon\right)(t) \tag{2.7a}$$

and

$$\varepsilon\,(t) = (\tau_0)^{-\beta}\left(I_0^{\beta}s\right)(t)\,. \tag{2.7b}$$

Fractional-order operators used in the constitutive relations in equation (2.6) have been shown to be very useful to describe the mechanical behaviour of several engineering materials such as concrete [10,30], composites, polymers and rubbers [31–33] under some restrictions.

Fractional-order calculus has also been applied in other fields of applied mechanics, such as heat transfer modelling [34], diffusive flow [35,36], wave propagation [37], non-local elasticity [38,39]. For a comprehensive review, readers may refer to [10]. Some stability mechanics problems involving non-conventional description of material external restraints have also been represented by means of fractional calculus [40].

(b) The rheological models of linear material hereditariness

The functional stress–strain relations reported in equation (2.6) possess an equivalent differential formulation in terms of elastic (Hookean) and viscous (Newtonian) elements.

In more detail, the differential formulation, named rheological representation of the Prony series expansion of the creep function $J(t)$, in equation (2.4a) is provided in figure 1*a*. Similarly, the mechanical arrangement springs and dashpots reported in figure 1*b* corresponds to the rheological representation of the relaxation function $G(t)$ reported in equation (2.4*b*). In passing, we observe that as far as $N = M = 1$, the well-known Maxwell elements representing relaxation and Kelvin–Voigt element for creep are obtained. Direct comparisons of figure 1*a*,*b* show that

royalsocietypublishing.org/journal/rsta *Phil. Trans. R. Soc. A* **378**: 20190294

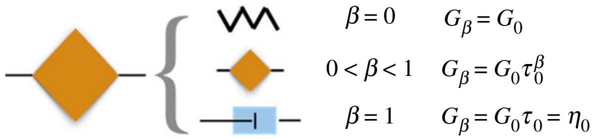

Figure 2. Springpot representation. (Online version in colour.)

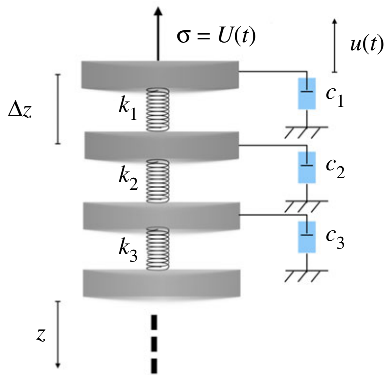

Figure 3. Mechanical model of springpot. (Online version in colour.)

the mechanics beyond the creep and relaxation functions described by Prony series expansion is quite different, as shown by the series and parallel arrangements of springs and dashpots that correspond to the prescribed analytical expression in equation (3.3). Such a consideration is a direct consequence of the lack of mathematical consistency of creep and relaxation functions expressed in terms of Prony series expansions.

A different scenario appears as we consider the rheological model corresponding to power-law functional classes that has been dubbed *springpot* [41] and it is modelled as a two-parameter element, namely the characteristic time τ_0 and the order of power-law β as depicted in figure 2. It has been recently proved [10–12] that the springpot element may be represented as a functionally graded sequence of springs and dashpots with different arrangement (figure 1*b*). The order of decay of the sequence is related to the order β of the power-law of the springpot. In more detail, values of $0 \leq \beta \leq \frac{1}{2}$ are obtained with decaying values of the elastic and viscosity moduli, namely with $\alpha \geq 0$, and they correspond to elastoviscous (EV) materials. Instead the coefficients of springs and dashpots increase with $\alpha \leq 0$, then values of $\frac{1}{2} \leq \beta \leq 1$ are obtained corresponding to the so-called viscoelastic (VE) materials. In figure 3, the representation of the mechanical model for the springpot is depicted in discretized form. It is an infinite column of spring and dashpot modelled as $c(z_j) = c_0 z_j^{-\alpha}/\Gamma(1-\alpha)$ and $k(z_j) = k_0 z_j^{-\alpha}/\Gamma(1-\alpha)$. Where $[k_0] = FL^{\alpha+1}$ and $[c_0] = FTL^{\alpha+1}$, $\alpha \in [-1, 1]$, $z_j = (j-1)\Delta z$ and $\alpha = 2\beta - 1$. As $\Delta z \to 0$ the discretized form of the springpot reverts into its continuous form ruled by a differential equation

$$\frac{\partial}{\partial z}\left[k_a\,(z)\,\frac{\partial u}{\partial z}\right] = c\,(z)\,\frac{\partial u}{\partial t}, \tag{2.8}$$

and by simple manipulations the creep function exactly given as in equation (2.5*a*) provides $c_0/k_0 = 1/G_0\Gamma(\beta)$, where $G_0 = \Gamma(2\beta)k_0 2^{2\beta}$.

Table 1. Average values for hamstring and patellar tendons obtained from best fitting of experimental campaigns for creep test.

creep							
hamstring				patellar			
σ	β_c	τ_c	α_c	σ	β_c	τ_c	α_c
2.61	0.1266	3.976	1.255	2.21	0.0917	4.938	0.635
5.65	0.0832	11.405	1.811	4.55	0.0902	2.742	0.717
8.61	0.0744	8.701	1.649	5.37	0.0885	4.000	0.789
9.98	0.0735	4.052	1.591	6.88	0.011	5.799	0.672
16.26	0.067	11.29	1.568	7.57	0.058	5.22	0.729

3. Experimental campaign on tendons hereditariness: nonlinear relaxation and creep

In this section, the results of an experimental campaign conducted on tendons of human knee is outlined. The experimental campaign involved 30 samples of human patellar and hamstring tendons, subjected to simple uniaxial tensile. Details about the used protocol as well as about the findings are provided in the next subsections.

(a) Material and methods

The experimental campaign involved two kinds of human tissue, namely patellar (P) and hamstring (H) tendons. Human tissues were obtained from a tissue bank (Lifelegacy Foundation, AZ, USA) with the requirement that each ensemble of P and H were obtained from the same human knee to avoid donor variability. Biological specimens were stored at 80°C and thawed in a 37°C water bath for 15 min prior to testing [42], then prepared for the test and finally each specimen was cut approximately at the same length before clamping for the uniaxial test. A commercial electromechanic system (Electroforce, Bose 3330) was used to test both tendon groups. We have used a specific protocol for the repeatability of the experimental campaign. Initially, the samples were preconditioned by cycling between 20 and 100 N, for 20 cycles at 0.25 Hz, to remove any crimping in the tendon fibrils [43]; after preconditioning, we performed a relaxation test with prescribed values of the strain level in the range 1–5% [43]. We conducted the relaxation tests applying a linear ramp of displacement with speed 250 mm s^{-1} and after the hold was fixed for 100 s after achieving the preselected value of strain; at the end of the relaxation test the sample was rested for 15 min in order to achieve the same length of the initial specimen measured at the end of the first phase. In the last phase, the creep test was obtained by applying the same initial stress reached at the end of the relaxation test with a linear load ramp of 315 N s^{-1} and holding the load 100 s. During the test, the sample was continuously moistened with saline solution.

(b) Data analysis

The experimental data in terms of the axial engineering strain $\varepsilon(\sigma, t)$ have been averaged for each level of applied stress. The averaged creep functions, namely $\langle\varepsilon^{(P)} > (\sigma_i, t)\rangle$ and $\langle\varepsilon^{(H)}(\sigma_i, t)\rangle$, are reported in figure 1. A more detailed representation of the averaged creep functions may be observed in a $\log[\langle\varepsilon\rangle] - \log[t]$ plot reported in figures 1 and 2 for the patellar and hamstring tendons, respectively.

royalsocietypublishing.org/journal/rsta Phil. Trans. R. Soc. A 378: 20190294

Data analysis reported in tables 1 and 2 for the log–log plots reveals that a good candidate to fit averaged values of creep functions $\mu_\varepsilon^{(P)}(\bar{\sigma}_i, t)$ and $\mu_\varepsilon^{(H)}(\bar{\sigma}_i, t)$ is the linear model with equation

$$\log\left[\langle \varepsilon^{(j)}(\sigma, t)\rangle\right] = \beta_j \log\left(\frac{t}{\Gamma(1-\beta_j)\,\tau_c^{(j)}}\right)^{\frac{1}{\beta_j}} + \alpha_j \log\left(\frac{\sigma_j}{G_0}\right), \tag{3.1}$$

where $j = P, H$ denotes the specific tissue considered, $\tau_c^{(j)}$ and $\bar{\sigma}_j$ are, respectively, a characteristic time and the non-dimensional stress $\bar{\sigma}_j = \sigma_j/E$, where E is the tangent elastic modulus obtained at the origin of a monotone test.

Straightforward manipulation of equation (3.1) yields the relation for the average of the strain omitting j-dependence

$$\langle \varepsilon(\sigma, t)\rangle = \left(\frac{\sigma}{G_0}\right)^{\alpha}\left(\frac{t}{\Gamma(1-\beta_c)^{\frac{1}{\beta}}\,\tau_c}\right)^{\beta} = \frac{\|s\|^{\alpha_c} sign(\sigma)}{\Gamma(1-\beta_c)}\left(\frac{t}{\tau_c}\right)^{\beta_c}, \tag{3.2}$$

with $0 \le \beta_c \le 1$, $0 \le \alpha_c \le 1$ two material parameters, $[\tau_c] = [T]$ is an additional material constant representing the characteristic time of the material observed in a creep test and $sign(\bullet)$ is the signum function.

It may be observed that values of α_c, β_c and τ_c are represented in figures 6 and 7 for the considered tissues. Inspection of equation (3.2) reveals that the creep function coalesces with the original formulation of Nutting obtained by experimental data conducted for rubbers, concrete, steel, but not for biological tissues as in fact $\langle \varepsilon(\sigma, t)\rangle$ for an assigned value of σ is a creep function, that is $\langle \varepsilon(\sigma, t)\rangle = \phi_c(s, t) = s^{\alpha_c} = J(t)$ for $s > 0$. Solid lines in figures 4 and 5 represent fits of the data with equation (3.1) and excellent agreement among curves and data may be observed as expected for power-law representation of ligament and tendon hereditariness [4]. The Nutting Law given in the form $|s|^{\alpha_c} sgn(\sigma) J(t)$ has been obtained by considering a creep test. The equation, since equation (3.2) is nonlinear, may be obtained from the corresponding relation for the relaxation test as happens in the linear case. In order to archive this result, we proceed with the relation test on the specimen for patellar and hamstring tendons. Previous considerations about the averaged values of the creep test results may be reported for the relaxation averaged data in figures 10 and 11 and for the log–log plots reported in figures 8 and 9 for the patellar and hamstring tendons, respectively. Solid lines in figures 8 and 9 represent linear fitting with equations (omitting j-dependence)

$$\log\left[\frac{\langle \sigma(\varepsilon, t)\rangle}{G_0}\right] = \log\left[\langle s(t)\rangle\right] = -\beta_r \log\left[\frac{\Gamma(\beta_r)^{\frac{1}{\beta_r}}\, t}{\tau_r}\right] + \alpha_r \log[\varepsilon], \tag{3.3}$$

which corresponds after straightforward manipulations to the stress average relaxation expressed as

$$\langle s(t)\rangle = \frac{\varepsilon^{\alpha_r}}{\Gamma(\beta_r)}\left(\frac{t}{\tau_r}\right)^{-\beta_r} = \frac{\|\varepsilon\|^{\alpha_r} sign(\varepsilon)}{\Gamma(\beta_r)}\left(\frac{t}{\tau_r}\right)^{-\beta_r}, \tag{3.4}$$

with α_r, β_r relaxation material parameters and $[\tau_r] = [T]$ the characteristic time of the tissue obtained in a relaxation test (figures 10 and 11).

Observation of equations (3.2) and (3.4) shows that both creep and relaxation functions of the fibrous tissue are nonlinear functions of the stress and the strain respectively. Under the assumption that $\alpha = \gamma = 1$, a linear dependence is experienced so that creep and relaxation may be expressed as

$$\langle \varepsilon(t)\rangle = \frac{s}{\Gamma(1-\beta_c)}\left(\frac{t}{\tau_c}\right)^{\beta_c} = sJ(t) \tag{3.5a}$$

and

$$\langle s(t)\rangle = \frac{\varepsilon}{\Gamma(\beta_r)}\left(\frac{t}{\tau_r}\right)^{-\beta_r} = \varepsilon G(t), \tag{3.5b}$$

with $J(t)$ and $G(t)$ the well-known creep and relaxation functions of linear hereditariness as $\tau_r = \tau_c = \tau_0$ and $\beta_r = \beta_c = \beta$.

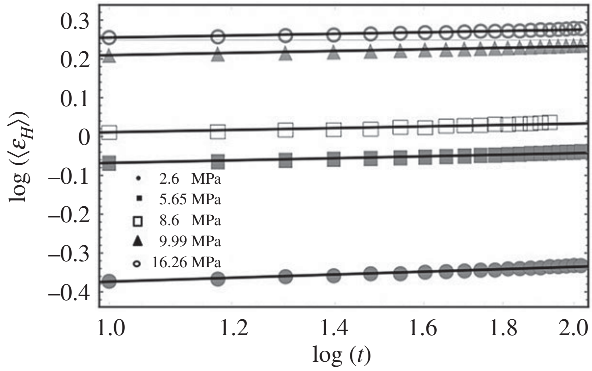

Figure 4. Log–log plots averaged creep functions hamstring ligaments.

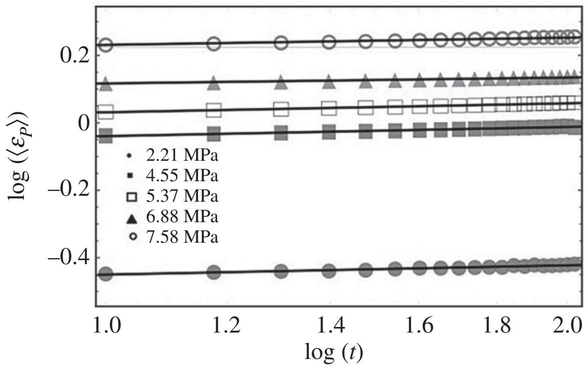

Figure 5. Log–log plots averaged creep functions patellar tendons.

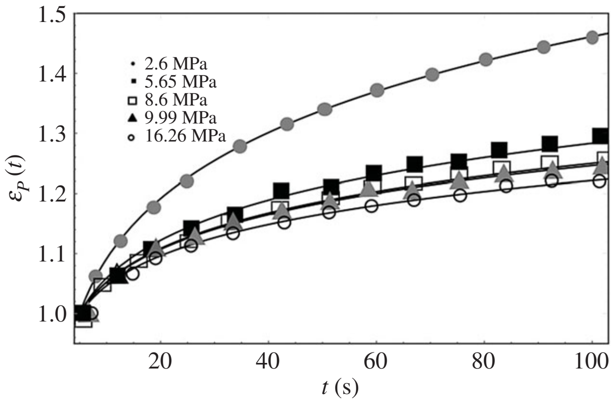

Figure 6. Averaged creep functions hamstring ligaments.

(c) Relations among creep and relaxation parameters

The nonlinear dependence of the strain and the stress observed in the experimental campaign was extensively investigated in several papers on ligament and tendon hereditariness [14,17,42]. However, despite the large efforts in the description of material parameters observed in relaxation

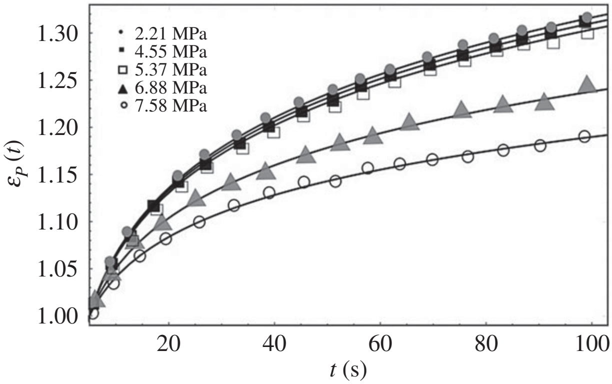

Figure 7. Averaged creep functions patellar tendons.

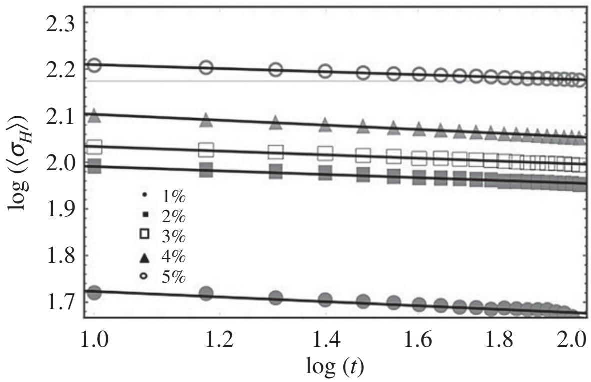

Figure 8. Log–log plots averaged relaxation functions hamstring ligaments.

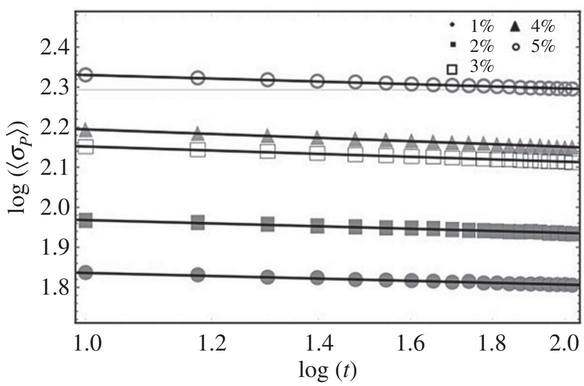

Figure 9. Log–log plots averaged relaxation functions patellar tendons.

tests no relations among α_c, β_c, τ_c for creep tests and α_r, β_r, τ_r for relaxation could be observed as reported by several authors. This latter comment is discussed in detail in this section, obtaining the fundamental conditions that must be fulfilled for fractional-order modelling of nonlinear hereditariness of tendons of the knee.

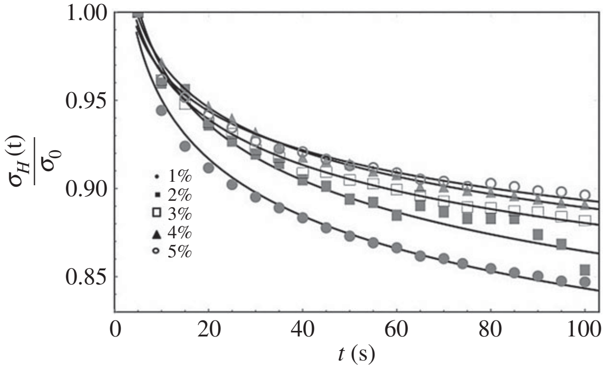

Figure 10. Averaged relaxation functions hamstring ligaments.

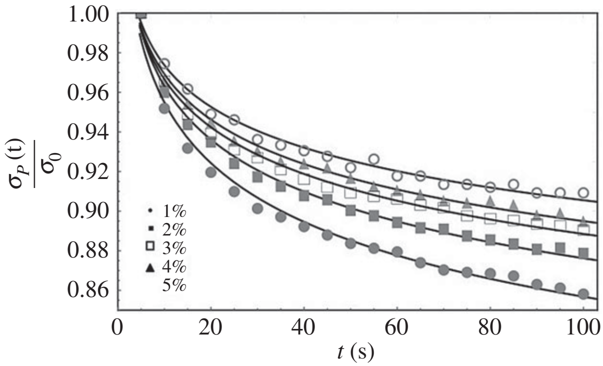

Figure 11. Averaged relaxation functions patellar tendons.

Table 2. Average values for hamstring and patellar tendons obtained from experimental campaigns for relaxation test.

relaxation							
hamstring				patellar			
ε	β_r	τ_r	α_r	ε	β_r	τ_r	α
1%	0.1589	5.191	0.796	1%	0.1444	5.39	1.574
2%	0.1507	5.211	0.552	2%	0.1258	5.2311	1.394
3%	0.1227	4.686	0.606	3%	0.1122	4.971	1.268
4%	0.117	4.503	0.628	4%	0.1060	4.601	1.489
5%	0.1051	4.532	0.637	5%	0.098	4.98	1.69

With this aim, let us evaluate the strain $\varepsilon(t)$ at time instant $t = \tau_c$ yielding a one-to-one relation among the applied, non-dimensional, stress s and the measured strain $\varepsilon(\tau_c)$, omitting arguments

$$s = \varepsilon^{1/\alpha_c} \left(\Gamma\left(\beta_c + 1\right)\right)^{1/\alpha_c}, \tag{3.6}$$

that after substitution in equation (3.6) yields the equality

$$\varepsilon^{1/\alpha_c}(\Gamma\,(\beta_c+1))^{1/\alpha_c}=\frac{\varepsilon^{\alpha_r}}{\Gamma\,(\beta_r)}\left(\frac{\tau_c}{\tau_r}\right)^{-\beta_r}, \tag{3.7}$$

which may be cast as

$$\varepsilon^{\left(\frac{1}{\alpha_c}-\alpha_r\right)}\Gamma\,(\beta_r)\,\Gamma\,(1+\beta_c)^{1/\alpha_c}=\left(\frac{\tau_c}{\tau_c}\right)^{-\beta_r}, \tag{3.8}$$

which holds true for any value of the strain ε as $\alpha_r=1/\alpha_c$ so that a relation among the material characteristic times observed in creep and relaxation may be established as

$$\tau_r=\tau_c\Gamma(\beta_c+1)^{1/(\alpha_c\beta_r)}\Gamma(\beta_r)^{1/\beta_r}, \tag{3.9}$$

which, in conjunction with the relation $\alpha_r=\dfrac{1}{\alpha_c}$ allows the characteristic time of the relaxation to be estimated upon measure of the characteristic time observed in creep once a relation among the decay β_r and the order β_c has been established.

This latter condition may be obtained as we search the estimates of creep parameters with direct measures of the relaxation parameters, namely α_r, β_r, τ_r. Under this condition, the relation among the characteristic time in creep estimate τ_c and the characteristic time observed in relaxation reads

$$\tau_c=\tau_r\left[\Gamma\,(\beta_r)^{\dfrac{1}{\alpha_r\beta_c}}\,\Gamma\,(\beta_c+1)^{\dfrac{1}{\beta_c}}\right]^{-1}, \tag{3.10}$$

yielding

$$\tau_r=\tau_c\left[\Gamma\,(\beta_r)^{\dfrac{1}{\alpha_r\beta_c}}\,\Gamma\,(\beta_c+1)^{\dfrac{1}{\beta_c}}\right]. \tag{3.11}$$

Direct comparison of equation (3.11) with equation (3.9) yields the relation among the orders

$$\beta_c=\alpha_c\beta_r \tag{3.12a}$$

and

$$\beta_r=\alpha_r\beta_c. \tag{3.12b}$$

Equations (3.12) allow for a relation among the decaying order of the relaxation, given the creep parameters as

$$\beta_r=\frac{\beta_c}{\alpha_c}, \tag{3.13}$$

which corresponds, in conjunction with $\alpha_r=1/\alpha_c$, to equation (3.12*b*) namely $\beta_r=\alpha_r\beta_c$.

Relaxation order β_r of the stress $s(t)$ yields assuming $\alpha_c\leq 1$ the order of the relaxation $\beta_r\geq\beta_c$ according to the well-established paradigms that *relaxation run faster than creep*, as reported by several authors [4].

In tables 1 and 2, values of the parameters obtained by best fitting of the experimental data in §3 for creep and relaxation tests have been contrasted with the results of the proposed equations used to relate creep and relaxation parameters. The columns of the tables report the estimates of creep parameters for measured values of the relaxation parameters at different levels of applied stress assuming $G_0=1MPa$, respectively, for hamstring (table 3) and patellar tendons (table 4). The table reports the percentage absolute value of the mean error e_i, with $i=\beta,\alpha,\tau$, among estimated and measured parameters and direct inspections shows that errors are less than 5%.

The observation of the experimental data reported in table 2 as well as of the relations among creep and relaxation coefficients, shows that the order of the power-law $[\bullet]^{-\beta_r}$ and $[\bullet]^{\beta_c}$ depends, nonlinearly, on the level of the initial strain ε (relaxation) or the initially assigned stress σ (creep).

In such circumstances, the multiplicative decomposition of the material functions $J(\sigma,t)$ and $G(\varepsilon,t)$ does not hold. However, statistical analysis on the experimental data shows that assuming

Table 3. Average values for hamstring tendons obtained from parameters relationships.

hamstring tendons						
σ	$\beta_c = \beta_r \alpha_r$	e_β	$\alpha_c = 1/\alpha_r$	e_α	τ_c	e_τ
2.61	0.1251	0.0118	1.247	0.006	5.1	0.033
5.65	0.0833	0.0012	1.831	0.0109	3.8	0.386
8.61	0.0738	0.008	1.641	0.005	4.141	0.0352
9.98	0.0722	0.018	1.477	0.072	5.75	0.008
16.26	0.065	0.0299	1.555	0.009	5.391	0.033

Table 4. Average values for patellar tendons obtained from parameters relationships.

patellar tendons						
σ	$\beta_c = \beta_r \alpha_r$	e_β	$\alpha_c = 1/\alpha_r$	e_α	τ_c	e_τ
2.21	0.1397	0.033	0.614	0.033	4.01	0.008
4.55	0.1235	0.018	0.71	0.024	11.34	0.006
5.37	0.109	0.029	0.765	0.03	8.9	0.0023
6.88	0.012	0.09	0.673	0.002	4.07	0.004
7.57	0.089	0.091	0.732	0.004	11.11	0.01

the average value of the order $\beta_c \to \bar{\beta}_c$ and $\beta_r \to \bar{\beta}_r$ for creep and relaxation may be assumed in an engineering context whereas the scattering of material-time data, namely τ_c and τ_r, with respect to a mean value among the different level of strains has been modelled as random fluctuations [44].

4. The fractional-order nonlinear hereditariness

Data analysis reported in the previous section shows that the mechanics of the fibrous tissues of the knee tendons must be modelled, in the course of creep and relaxation tests, with material functions $\phi_c(\sigma, t)$ and $\phi_r(\varepsilon, t)$ that do not fulfill the linearity condition namely $\phi_c(\alpha\sigma(\tau)) \neq \alpha\phi_c(\sigma(\tau))$ as well as the condition $\phi_r(\alpha\varepsilon(\tau)) \neq \alpha\phi_r(\varepsilon(\tau))$. Under these circumstances, the Boltzmann superposition principle can not be used, leading one to conclude that material behaviour can not be captured with single integral models as in the case of QLV. From an engineering perspective, however, the material behaviour may be simplified as we consider the averaged values of the time-variation order of creep and relaxations $\bar{\beta}_c$ and $\bar{\beta}_r$, respectively. Under such circumstances, data analysis supplied in §3 shows that material functions for constant stress, namely $\phi_c(\sigma, t)$, and for uniform strain, namely, $\phi_r(\varepsilon, t)$, functions may be expressed in a generalized, separable form as

$$\phi_c(\sigma, t) = J_e(\sigma) J(t) = \frac{p}{\Gamma(1-\bar{\beta}_c)} \left(\frac{t}{\tau_0}\right)^{\bar{\beta}_c} \tag{4.1a}$$

and

$$\phi_r(\varepsilon, t) = G_e(\varepsilon) G(t) = \frac{q}{\Gamma(\bar{\beta}_r)} \left(\frac{t}{\tau_0}\right)^{-\bar{\beta}_r}, \tag{4.1b}$$

where $p = |s|^{\alpha_c} sign(s)$ and $q = |\varepsilon|^{\alpha_r} sign(\varepsilon)$ and the time-dependence expressed by the functions $G(t)$ and $J(t)$ may be estimated from experimental data so that it may be represented with a power-law of time with averaged order $\bar{\beta}_c$ and $\bar{\beta}_r$, respectively, for creep and relaxation as observed in the data analysis section. Close observation of equation (4.1a,b) reveals that the separable form of the material function is assumed as the basis of QLV [16] where the nonlinear auxiliary state variables are expressed in terms of the Helmoltz free energy $\Psi(\varepsilon)$ assumed to represent the elastic behaviour in monotone tensile tests as $p = G_e(\varepsilon) = (1/G_0)\mathrm{d}\Psi(\varepsilon)/\mathrm{d}\varepsilon$ and $q = J_e(\sigma) = G_0(\mathrm{d}\Psi^{(-1)}(\sigma)/\mathrm{d}\sigma)$.

The use of the auxiliary variables allows the Boltzmann superposition principle to be introduced yielding

$$s\,(t) = \frac{(\tau_r)^{\bar{\beta}_r}}{\Gamma\left(\bar{\beta}_r\right)} \int_0^t (t-\tau)^{-\bar{\beta}_r}\, \dot{q}\,(\tau)\, d\tau = (\tau_r)^{\bar{\beta}_r} \left(D_0^{\bar{\beta}_r} q\right)(t) \tag{4.2a}$$

and

$$\varepsilon\,(t) = \frac{(\tau_c)^{-\bar{\beta}_c}}{\Gamma\left(1-\bar{\beta}_c\right)} \int_0^t (t-\tau)^{\bar{\beta}_c-1}\, \dot{p}\,(\tau)\, d\tau = (\tau_c)^{-\bar{\beta}_c} \left(I_0^{\bar{\beta}_c} p\right)(t)\,, \tag{4.2b}$$

which may be cast in inverse form using the formalism of fractional calculus yielding, with the knowledge of the relaxation function parameters:

$$s\,(t) = (\tau_r)^{\bar{\beta}_r} \left(D_{0^+}^{\bar{\beta}_r} \left[\varepsilon\,(t)^{\alpha_r}\right]\right)(t) \tag{4.3a}$$

and

$$\varepsilon\,(t) = \left[\frac{1}{\tau_r^{\bar{\beta}_r}} \left(I_{0^+}^{\bar{\beta}_r} s\right)(t)\right]^{1/\alpha_r}, \tag{4.3b}$$

or involving the knowledge of the creep functions

$$\varepsilon\,(t) = (\tau_c)^{-\bar{\beta}_c} \left(I_{0^+}^{\bar{\beta}_c} \left[s\,(t)^{\alpha_c}\right]\right)(t) \tag{4.4a}$$

and

$$s\,(t) = \left[(\tau_c)^{\bar{\beta}_c} \left(D_{0^+}^{\bar{\beta}_c} \varepsilon\right)(t)\right]^{1/\alpha_c}. \tag{4.4b}$$

Observation of equation (4.2*a*,*b*) shows three main features.

(i) The constitutive equations for the non-dimensional stress involves a nonlinear transform of the strain $\varepsilon(t) \to q(t)$ a relaxation time τ_r and the time-decay order $\bar{\beta}_r$.

(ii) The constitutive equation for the strain evolution involves the nonlinear transform of the stress $s(t) \to p(t)$, a creep characteristic time τ_c and an evolution order $\bar{\beta}_c$, with $\bar{\beta}_c \neq \bar{\beta}_r$.

(iii) No relations among the parameters used for the stress evolution and the parameters used for strain evolution have been established so far, yielding one to conclude that no advantages in the use of fractional calculus formalism exists to deal with nonlinear material hereditariness.

Summing up, previous considerations yield the conclusion that the use of single-integral model of nonlinear hereditariness does not allow for a fractional-order description of the material hereditariness, also in presence of the nonlinear transforms of the state variables. Indeed, it is widely accepted that the use of fractional-order calculus proves to be very efficient as the fundamental relation of linear models is satisfied and, in order to respect this latter condition, $\bar{\beta}_c$ and $\bar{\beta}_r$ must satisfy the condition $\bar{\beta}_c = \bar{\beta}_r = \beta$. If this latter condition is not satisfied and/or equivalence relations are not provided, then no advantages in the use of fractional-order calculus is relevant for the mathematical modelling of nonlinear hereditariness.

In the previous section, the relation among creep and relaxation parameters has been established for the first time to the best of the authors' knowledge, and it will be used for the numerical assessment of the proposed formulation in the next section.

5. Numerical validation

The use of single-integral model of nonlinear hereditariness in the context of fractional-order calculus has not been exploited, mainly for the lack of equivalence among creep and relaxation parameters. Analytical and experimental arguments in the previous section showed that, as far as

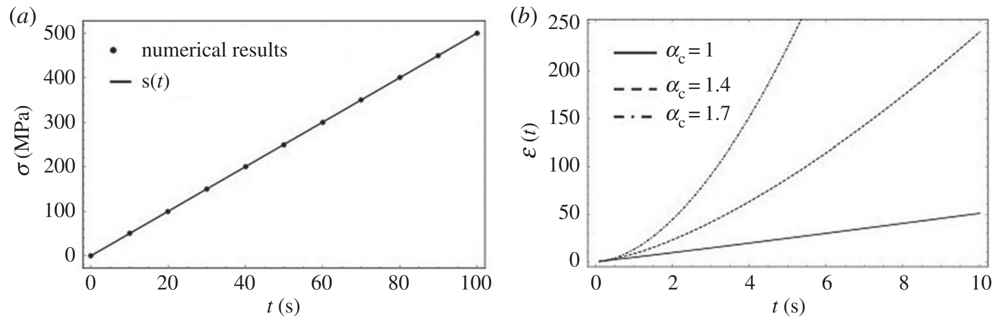

Figure 12. Numerical results between $s(t)$ assigned and numerical results.

the creep parameters $(\alpha_c, \beta_c, \tau_c)$ have been measured by experimental tests, the corresponding relaxation parameters $(\alpha_r, \beta_r, \tau_r)$ may be estimated by means of the relations reported in equation (4.11a,b, 4.12) and vice versa.

In this section, we aim to show that the proposed equivalence relations hold true also in the presence of non-constant stress or strain histories and that the constitutive equations in equation (4.4a,b) are completely general.

In more detail, in this section, we consider a specific form of the strain history $\varepsilon(t)$ so that the non-dimensional stress $s(t)$ is obtained by means of equation (4.4*b*). The stress history $s(t)$ is then used as the stress applied to evaluate the strain function by means of the creep material function returning, exactly, the initially assigned strain function $\varepsilon(t)$.

Similar arguments hold true as we assign a specific stress history, namely $s(t)$, and we evaluate the strain $\varepsilon(t)$ by means of equation (4.4*a*). In such a case, the initially assigned expression for the stress $s(t)$ is obtained by equation (4.4*a*) as we introduce the equivalences among relaxation and creep parameters in equations (4.11*a*,*b*).

(a) Linear class of stress and strain histories

In this section, we can show the numerical results for a polynomial class of stress and strain. In this application, a function $s(t)$ of type has been assigned

$$s(t) = 5t, \tag{5.1}$$

we have studied the problem by applying equation (4.2*b*) and equation (4.4*b*), and we have considered three different values of α_c, in particular $\alpha_c = 1$ to show the particular case of linear behaviour, $\beta_c = \beta_r = \beta = 0.45$, and $\alpha_c = 1, 1.4, 1.7$ and other parameters are fixed, $\tau_c = 4.5$; The solutions were obtained by developed a numerical code implementing Grunwald–Letnikov's differintegral, we considered 2000 steps with a time step of 0.1. Figure 12*a*,*b* shows the numerical results, respectively, of equation (4.4*b*) and equation (4.2*b*). Figure 12*a* shows the perfect match between assigned function of stress history and numerical results of equation (4.4*b*) and figure 12*b* highlights the effect of nonlinearity in the $\varepsilon(t)$ function, which increases for increasing values α_c.

(b) Harmonic-type stress and strain histories

In this application, a function $s(t)$ of type has been assigned

$$s(t) = s_i Sin(\omega_j t) \quad \text{with } i, j = 0, 1, 2, \tag{5.2}$$

where $s_0 = 10$, $s_1 = 15$, $s_2 = 20$. In the first application, we have considered to show the effect of amplitude on the nonlinearity of the function $s(t)$ equation (4.4b), so we fixed the value of ω, in particular $\omega_0 = 1$, and also we established values for $\beta_c = 0.045$, $\tau_c = 7.5$ and $\alpha_c = 0.77$. A numerical code has been developed for solving equations (4.4*a*,*b*); in particular, we used the solution of

royalsocietypublishing.org/journal/rsta Phil. Trans. R. Soc. A 378: 20190294

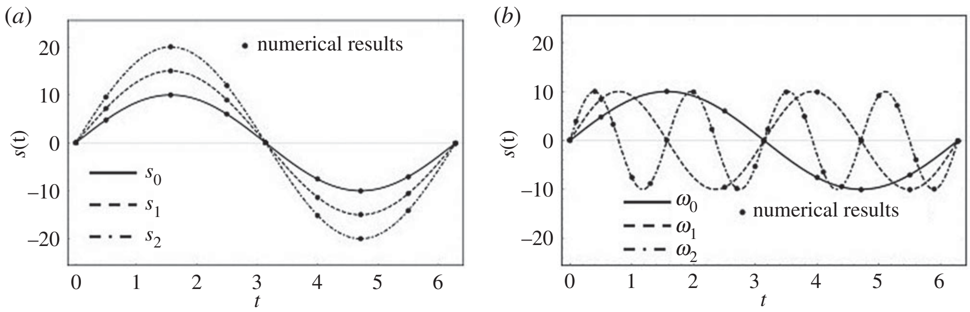

Figure 13. (*a*) Numerical results between *s*(*t*) assigned ad *s*(*t*) obtained for three different amplitude; (*b*) Numerical results between *s*(*t*) assigned ad *s*(*t*) obtained for three different frequency levels.

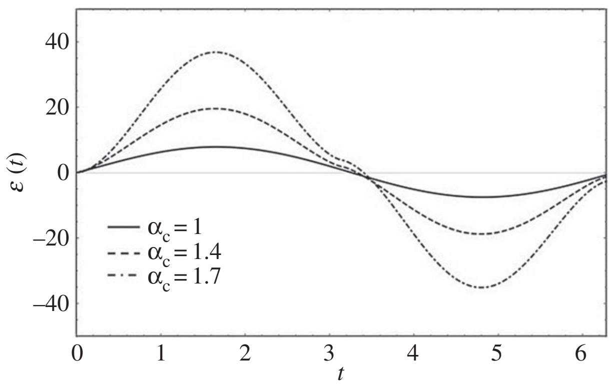

Figure 14. Numerical results of strain for assigned amplitude $s_0 = 10$ and frequency ω_1 and three different value of α_c.

fractional integral and fractional derivative by Grunwald–Letnikov's diferintegral, we considered 5000 steps with a time step of 0.01. Figure 13*a* shows that numerical application of equation (4.4*b*) perfectly overlaps the assigned functions of $s(t)$.

The effect of nonlinearity in the $\varepsilon(t)$ function for three different values of s_i is shown in figure 14. In the following, we have fixed the values of amplitude and have changed ω_j; the three parameters chosen are $\omega_0 = 1, \omega_1 = 2, \omega_2 = 4$. The function $s(t)$ is calculated considering $\beta_c = 0.045$, $\tau_c = 7.5$ and $\alpha_c = 0.77$. Figure 13*b* shows the evaluation of

$$s(t) = s_1 Sin(\omega_j t), \tag{5.3}$$

while figure 13*b* shows the perfect overlap between assigned stress history and the stress functions from equation (4.4*b*) for each value of ω.

To further highlight the effect of nonlinearity in the case of nonlinear viscoelastic behaviour, the value of the function to $\varepsilon(t)$ in equation (4.2*b*) was also calculated for three different values of α_c, in particular $\alpha_c = 1$, to show the particular case of linear behaviour, $\beta_c = \beta_r$, and $\alpha_c = 1.4$ and $\alpha_c = 1.7$. For each analysis, we have considered a fixed value of amplitude and ω. In the following, figures 14–16 show the results of equation (4.2*b*). Equation (3.2*b*) was calculated by implementing Grunwald–Letnikov's diferintegral for calculation of the fractional integral, in which 5000 steps were carried out with a time step of 0.01.

In the numerical tests carried out, the effect of the nonlinearity on the strain function both by keeping the amplitude constant and by keeping the frequency constant, the amplitude of the function is increased in both cases as the parameter increases, α_c.

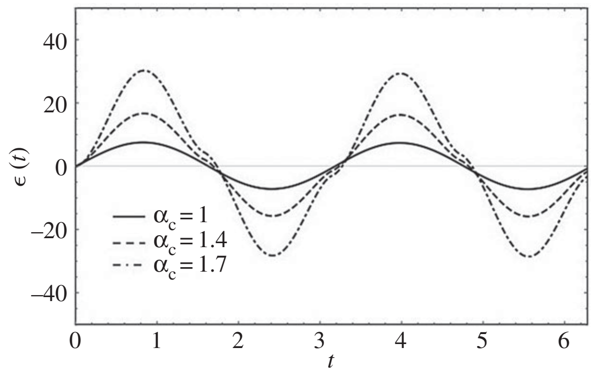

Figure 15. Numerical results of strain for assigned amplitude $s_0 = 10$ and frequency ω_2 and three different value of α_c.

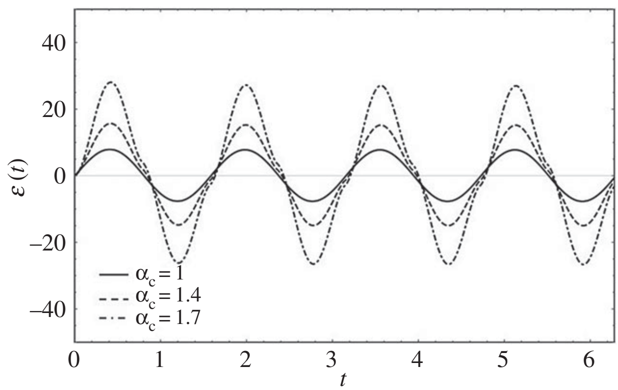

Figure 16. Numerical results of strain for assigned amplitude $s_0 = 10$ and frequency ω_3 and three different value of α_c.

6. Conclusion

In this paper, the authors pointed out the main features of a wide experimental campaign devoted to investigate the mechanical behaviour of knee tendons subjected to long-standing loads. The results of the experimental campaign show that creep and relaxation behaviour of the tendons may be well captured by power laws in the ranges of strain (1–5%) and stress (0–16 Mpa) considered.

The order of the power law proved to be different between creep and relaxation proving that some form of nonlinearity is involved in the micro mechanics of the tissue so that no linear theory of fractional hereditariness may be used to capture the mechanics of tendons. Moreover, it has been observed that the material parameters are significative dependent on the applied stress in creep tests as well as on the applied strain during relaxation.

These features, already observed in other mechanical tests in the last 20 years, have never been conducted in combination on human tendons, yielding one to conclude that the finding *relaxation runs faster than creep* is valid also on human knee patellar and hamstring tendons.

Based on this observation, the paper was devoted to the introduction of an analytical model to describe creep and relaxation showing that some closed-form expression relating creep and relaxation parameters could be established.

Direct comparison among results of such expressions and the measured values showed excellent matches with slight coefficient of variations and, in order to show that such relations hold whatever kind of test is considered, a numerical validation has been introduced with other than constant value of the applied strain (stress), namely linear and harmonically varying strain (stress).

The obtained results showed excellent match among the initial and the recovered values of the applied stress (strain) leading one to conclude that the proposed relations may be a benchmark to provide clinical support to the surgeons that apply pre-stress to the tendons before surgical replacement to reconstruct anterior cruciate ligaments functionality.

Data accessibility. This article has no additional data.
Competing interests. We declare we have no competing interest.
Funding. The authors are very grateful to the MIUR grant no. PON FSE-FESR ricerca e innovazione 2014–2020 DOT1320558, with coordinator Prof. Massimiliano Zingales; this financial support is gratefully acknowledged. K.D. acknowledges support from ARO Numerical Analysis (W911NF-17-1-0084), ONR Applied and Computational Analysis (N00014-18-1-2528), NSF Mechanics of Materials and Structures (1635407) and ARO MURI Program (W911NF-19-1-0245). M.Z. thanks the Center for Nonlinear Analysis at Carnegie Mellon University for hospitality during his visit to Carnegie Mellon University.

References

1. Nutting PG. 1936 Adsorption and pycnometry. *J. Wash. Acad. Sci.* **26.1**, 1–6.
2. Gemant A. 1936 A method of analyzing experimental results obtained from elasto-viscous bodies. *Physics* **7**, 311–317. (doi:10.1063/1.1745400)
3. Volterra V, Peres J. 1936 *Théorie générale des fonctionnelles*. Paris, France: Gauthier-Villars.
4. Lakes RS, Vanderby R. 1999 Interrelation of creep and relaxation: a modeling approach for ligaments. *J. Biomech. Eng.* **121**, 612–615. (doi:10.1115/1.2800861)
5. Mainardi F. 2010 *Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models*. Singapore: World Scientific.
6. Flugge W. 1967 *Viscoelasticity blaisdell publishing company*. Massachussetts: Waltham.
7. Nutting PG. 1946 Deformation in relation to time, pressure and temperature. *J. Franklin Inst.* **242**, 449–458. (doi:10.1016/0016-0032(46)90636-9)
8. Mainardi F, Spada G. 2011 Creep, relaxation and viscosity properties for basic fractional models in rheology. *Eur. Phys. J. Spec. Top.* **193**, 133–160. (doi:10.1140/epjst/e2011-01387-1)
9. Bologna E, Graziano F, Deseri L, Zingales M. 2019 Power-laws hereditariness of biomimetic ceramics for cranioplasty neurosurgery. *Int. J. Non-Linear Mech.* **115**, 61–67. (doi:10.1016/j.ijnonlinmec.2019.01.008)
10. Di Paola M, Zingales M. 2012 Exact mechanical models of fractional hereditary materials. *J. Rheol.* **56**, 983–1004. (doi:10.1122/1.4717492)
11. Di Paola M, Pinnola FP, Zingales M. 2013 A discrete mechanical model of fractional hereditary materials. *Meccanica* **48**, 1573–1586. (doi:10.1007/s11012-012-9685-4)
12. Deseri L, Di Paola M, Zingales M. 2014 Free energy and states of fractional-order hereditariness. *Int. J. Solids Struct.* **51**, 3156–3167. (doi:10.1016/j.ijsolstr.2014.05.008)
13. Deseri L, Zingales M, Pollaci P. 2014 The state of fractional hereditary materials (FHM). *Discr. Contin. Dyn. Syst. Ser. B* **19**, 2065–2089.
14. Pioletti DP, Rakotomanana LR. 2000 On the independence of time and strain effects in the stress relaxation of ligaments and tendons. *J. Biomech.* **33**, 1729–1732. (doi:10.1016/S0021-9290(00)00128-7)
15. Castile RM, Skelley NW, Babaei B, Brophy RH, Lake SP. 2016 Microstructural properties and mechanics vary between bundles of the human anterior cruciate ligament during stress-relaxation. *J. Biomech.* **49**, 87–93. (doi:10.1016/j.jbiomech.2015.11.016)
16. Fung YC. 1993 *Biomechanics: mechanical properties of living tissues*. New York, NY: Springer.
17. Provenzano P, Lakes R, Keenan T, Vanderby R. 2001 Nonlinear ligament viscoelasticity. *Ann. Biomed. Eng.* **29**, 908–914. (doi:10.1114/1.1408926)
18. Vena P, Gastaldi D, Contro R. 2005 A constituent-based model for the nonlinear viscoelastic behavior of ligaments. *J. Biomech. Eng.* **128**, 449–457. (doi:10.1115/1.2187046)
19. Abramowitch SD, Zhang X, Curran M, Kilger R. 2010 A comparison of the quasi-static mechanical and nonlinear viscoelastic properties of the human semitendinosus and gracilis tendons. *Clin. Biomech.* **25**, 325–331. (doi:10.1016/j.clinbiomech.2009.12.007)
20. Marchiori G *et al.* Submitted. How preconditioning and pretensioning of grafts used in ACL surgical reconstruction are influenced by their mechanical time-dependent characteristics: Can we optimize their initial loading state? *Clin. Biomech.*

21. Gurtin ME, Sternberg E. 1962 On the linear theory of viscoelasticity. *Arch. Ration. Mech. Analysis* **11**, 291–356. (doi:10.1007/BF00253942)
22. Gurtin ME, Hrusa WJ. 1988 On energies for nonlinear viscoelastic materials of single-integral type. *Q. Appl. Math.* **46**, 381–392. (doi:10.1090/qam/950610)
23. Pipkin AC, Rogers TG. 1968 A non-linear integral representation for viscoelastic behaviour. *J. Mech. Phys. Solids* **16**, 59–72. (doi:10.1016/0022-5096(68)90016-1)
24. Suckochi C. 2013 A Quasi-linear viscoelastic rehological model for thermoplastics and resins. *J. Theor. Appl. Mech.* **51**, 117–129.
25. Drapaca CS, Sivaloganathan S, Tenti G. 2007 Nonlinear constitutive laws in viscoelasticity. *Math. Mech. Solids* **12**, 475–501. (doi:10.1177/1081286506062450)
26. Muliana A, Rajagopal KR, Wineman AS. 2013 A new class of quasi-linear models for describing the nonlinear viscoelastic response of materials. *Acta Mech.* **224**, 2169–2183. (doi:10.1007/s00707-013-0848-8)
27. Findley WN, Lai JS, Onaran K. 1976 *Creep and relaxation of non-linear viscoelastic materials*. Dover, UK: North Holland.
28. Park SW, Schapery RA. 1999 Methods of interconversion between linear viscoelastic material functions. Part I-A numerical method based on Prony series. *Int. J. Solids Struct.* **36**, 1653–1675. (doi:10.1016/S0020-7683(98)00055-9)
29. Schmidt A, Gaul L. 2001 FE implementation of viscoelastic constitutive stress-strain relations involving fractional time derivatives. *Constitutive Models Rubber* **2**, 79–92.
30. Deseri L, Paola MD, Zingales M, Pollaci P. 2013 Power-law hereditariness of hierarchical fractal bones. *Int. J. Numer. Methods Biomed. Eng.* **29**, 1338–1360. (doi:10.1002/cnm.2572)
31. Bagley, Ronald L, Torvik PJ. 1983 A theoretical basis for the application of fractional calculus to viscoelasticity. *J. Rheol.* **27**, 201–210. (doi:10.1122/1.549724)
32. Bagley RL, Torvik J. 1983 Fractional calculus-a different approach to the analysis of viscoelastically damped structures. *AIAA J.* **21**, 741–748. (doi:10.2514/3.8142)
33. Behnke R, Hüsnü D, Michael K. 2011 An extended tube model for thermo-viscoelasticity of rubberlike materials: parameter identification and examples. *Proc. Appl. Math. Mech.* **11**, 353–354. (doi:10.1002/pamm.201110168)
34. Zingales M. 2016 An exact thermodynamical model of power-law temperature time scaling. *Ann. Phys.* **365**, 24–37. (doi:10.1016/j.aop.2015.08.014)
35. Alaimo G, Zingales M. 2015 Laminar flow through fractal porous materials: the fractional-order transport equation. *Commun. Nonlinear Sci. Numer. Simul.* **22**, 889–902. (doi:10.1016/j.cnsns.2014.10.005)
36. Deseri L, Zingales M. 2015 A mechanical picture of fractional-order Darcy equation. *Commun. Nonlinear Sci. Numer. Simul.* **20**, 940–949. (doi:10.1016/j.cnsns.2014.06.021)
37. Zingales M. 2011 Wave propagation in 1D elastic solids in presence of long-range central interactions. *J. Sound Vib.* **330**, 3973–3989. (doi:10.1016/j.jsv.2010.10.027)
38. Di Paola M, Zingales M. 2008 Long-range cohesive interactions of non-local continuum faced by fractional calculus. *Int. J. Solids Struct.* **45**, 5642–5659. (doi:10.1016/j.ijsolstr.2008.06.004)
39. Di Paola M, Pinnola FP, Zingales M. 2013 Fractional differential equations and related exact mechanical models. *Comput. Math. Appl.* **66**, 608–620. (doi:10.1016/j.camwa.2013.03.012)
40. Bologna E, Zingales M. 2018 Stability analysis of Beck's column over a fractional-order hereditary foundation. *Proc. R. Soc. A* **474**, 20180315. (doi:10.1098/rspa.2018.0315)
41. Blair GWS, Caffyn JE. 1949 VI. An application of the theory of quasi-properties to the treatment of anomalous strain-stress relations. *London, Edinburgh, Dublin Phil. Mag. J. Sci.* **40**, 80–94. (doi:10.1080/14786444908561213)
42. Abramowitch SD, Woo SL-Y. 2004 An improved method to analyze the stress relaxation of ligaments following a finite ramp time based on the quasi-linear viscoelastic theory. *J. Biomech. Eng.* **126**, 92–97. (doi:10.1115/1.1645528)
43. Hashemi J, Chandrashekar N, Slauterbeck J. 2005 The mechanical properties of the human patellar tendon are correlated to its mass density and are independent of sex. *Clin. Biomech.* **20**, 645–652. (doi:10.1016/j.clinbiomech.2005.02.008)
44. Bologna E, Lopomo N, Marchiori G, Zingales M. 2020 A non-linear stochastic approach of ligaments and tendons fractional-order hereditariness. *Probab. Eng. Mech.* **60**, 103034. (doi:10.1016/j.probengmech.2020.103034)