CLASSIFYING ALGEBRAS WITH GRADED INVOLUTIONS OR SUPERINVOLUTIONS WITH MULTIPLICITIES OF THEIR COCHARACTER BOUNDED BY ONE

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Abstract. Let A be superalgebra over a field of characteristic zero and let $*$ be either a graded involution or a superinvolution defined on A. In this paper we characterize the $*$-algebras whose $*$-cocharacter has multiplicities bounded by one, showing a set of $*$-polynomial identities satisfied by such algebras.

1. Introduction

Let A be an algebra, over a field F of characteristic zero, graded by $\mathbb{Z}_2$, the cyclic group of order 2, and let $*$ be either a graded involution or a superinvolution on A. The set of all $*$-polynomial identities is denoted by $\mathrm{Id}^*(A)$ and it has a natural structure of T^*-ideal, i.e. an ideal invariant under all $*$-endomorphism of the free associative $*$-superalgebra. Moreover, it is well-known that in case $\mathrm{char}F = 0$, $\mathrm{Id}^*(A)$ is determined by the multilinear polynomials it contains. Thus we can consider $P^*_{n_1,n_2,n_3,n_4}$, the space of multilinear polynomials of degree n in n_1 fixed symmetric even variables, n_2 fixed skew–symmetric even variables, n_3 fixed symmetric odd variables and n_4 fixed skew–symmetric odd variables, where $n = n_1 + n_2 + n_3 + n_4$. In order to simplify the notation, let $P^*_{\langle n\rangle} = P^*_{n_1,n_2,n_3,n_4}$, where $\langle n\rangle = (n_1, n_2, n_3, n_4)$. Hence, if one sets

$$P^*_{\langle n\rangle}(A) = \frac{P^*_{\langle n\rangle}}{P^*_{\langle n\rangle} \cap \mathrm{Id}^*(A)},$$

we can define $c^*_{\langle n\rangle}(A) = \dim_F P^*_{\langle n\rangle}(A)$ and

$$c^*_n(A) = \sum_{n_1+\ldots+n_4=n} \binom{n}{n_1, \ldots, n_4} c^*_{\langle n\rangle}(A),\ \ n \geq 1,$$

where $\binom{n}{n_1,\ldots,n_4}$ is the multinomial coefficient.

Such a sequence is called the $*$-codimension sequence of the superalgbera A and it was introduced firstly in the setting of associative algebras without any additional structure by Regev in [13]. In some sense, it gives a quantitative measure of the identities satisfied by a given algebra. In the same paper, the author showed that if A satisfies a non-trivial polynomial identity, then the codimension sequence is exponentially bounded.

It turned out that the study of the asymptotic behavior of the codimensions is a powerful tool that one can use in order to give a sort of classification of the algebras. In fact, a celebrated theorem of Kemer establishes that in case of ordinary polynomial identities, $c_n(A)$ is exponentially bounded of grows polynomially (see [11]). Similar results hold also for algebras with graded involution ([10]) and superinvolution ([6]).

Unluckily often it is not simple to determine exactly $c^*_n(A)$, thus one settles for a bound that allows to figure out at least the asymptotic behavior of the codimension sequence. In order to reach this goal, an important role is played by the representation theory of symmetric groups. In fact, we can naturally act on $P^*_{\langle n\rangle}$ with $S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4}$ by permuting the variables of the same homogenous degree and the same symmetry with respect to $*$. Since $\mathrm{Id}^*(A)$ is invariant under all $*$-endomorphism, then the previous action is inherited by $P^*_{\langle n\rangle}(A)$ that becomes a left $S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4}$-module.

2010 *Mathematics Subject Classification.* Primary 16R10, 16R50; Secondary 20C30.

Key words and phrases. Superinvolution, Cocharacters, Multiplicities, Graded Involution.

Given a 4-tuple $\langle n\rangle = (n_1, n_2, n_3, n_4)$, the corresponding $S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4}$–character, called the $\langle n\rangle$-th cocharacter of A and denoted by $\chi^*_{\langle n\rangle}(A)$, decomposes into irreducibles

$$\chi^*_{\langle n\rangle}(A) = \sum_{\langle\lambda\rangle\vdash\langle n\rangle} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \chi_{\lambda(2)} \otimes \chi_{\lambda(3)} \otimes \chi_{\lambda(4)},$$

where $\langle\lambda\rangle = (\lambda(1), \lambda(2), \lambda(3), \lambda(4))$, $\chi_{\lambda(i)}$ is the irreducible S_{n_i}-character corresponding to the partition $\lambda(i) \vdash n_i$ and $m_{\langle\lambda\rangle} \geq 0$ are the multiplicities. It is clear that if one knows $\chi^*_{\langle n\rangle}(A)$ for all $n \geq 1$, then he can gather informations about $c^*_n(A)$.

One of the main objectives of the theory is to determine such multiplicities. This is in general a very difficult problem and in the past years several authors have given quite a few contributions in various settings (see for instance [2, 3, 4, 7, 12]).

The purpose of this paper is to classify the T^*-ideals of identities such that the multiplicities $m_{\langle\lambda\rangle}$ are bounded by one, in case $*$ is a graded involution or a superinvolution. In particular, we show that these multiplicities are bounded by one if and only if the $*$-superalgebra A satisfies a suitable list of $*$-polynomial identities.

The corresponding result for ordinary polynomial identities was given by Ananin and Kemer in [1] and, later on, it was extended by Giambruno and Mishchenko in [8] in case of superalgebras or algebras with involution and by Giambruno, Polcino and Valenti in [9] in case of algebras graded by a finite group G.

2. Preliminaries

Throughout this paper, F will denote a field of characteristic zero and $A = A_0 \oplus A_1$ an associative algebra over F graded by $\mathbb{Z}_2$, the cyclic group of order two, satisfying a non-trivial polynomial identity. In this setting, the elements of A_0 and A_1 are called homogeneous elements of degree zero (or even elements) and of degree one (or odd elements), respectively.

We now assume that the superalgebra A is endowed either with a graded involution, i.e. an involution preserving the grading, or with a superinvolution. Recall that a superinvolution $*$ is a graded linear map $* : A \to A$ such that $(a^*)^* = a$ and $(ab)^* = (-1)^{|a||b|}b^*a^*$, for all $a, b \in A_0 \cup A_1$. Here $|c|$ stands for the homogeneous degree of c.

Notice that if the grading on A is trivial, i.e. $A_1 = 0$, then the graded involutions and the superinvolutions coincide with the involutions on A. Moreover, if $A_1^2 = 0$, then the superinvolutions are in particular graded involutions.

From now on, let $*$ be indifferently a graded involution or a superinvolution on A, thus in order to simplify the exposition, we refer at $*$ as a gs-involution. Since we are assuming that $\mathrm{char}F = 0$, we can write $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$, where for $i \in \{0, 1\}$, $A_i^+ = \{a \in A \mid a^* = a\}$ and $A_i^- = \{a \in A \mid a^* = -a\}$ denote the sets of symmetric and skew-symmetric elements of A, respectively.

If $X = \{x_1, x_2, \ldots\}$ is a countable set of variables, we write $F\langle X\rangle$ to denote the free algebra on X over F, i.e., the algebra of polynomials in the non-commuting indeterminates of X. As in the case of graded algebras or of algebras with involution, if $*$ is a gs-involution on A, then $F\langle X\rangle$ inherits $*$ in a natural way. We write the set X as the union of two disjoint infinite sets Y and Z, requiring that their elements are of homogeneous degree 0 and 1 respectively. Then each set is written as the disjoint union of two other infinite sets of symmetric and skew elements respectively. The free algebra endowed with $*$ is denoted by $F\langle Y \cup Z, *\rangle$ and it is generated by symmetric and skew elements of even and odd degree. We write

$$F\langle Y \cup Z, *\rangle = F\langle y_1^+, y_1^-, z_1^+, z_1^-, y_2^+, y_2^-, z_2^+, z_2^-, \ldots\rangle,$$

where y_i^+ stands for a symmetric variable of even degree, y_i^- for a skew variable of even degree, z_i^+ for a symmetric variable of odd degree and z_i^- for a skew variable of odd degree.

Let $f(y_1^+, \ldots, y_n^+, y_1^-, \ldots, y_m^-, z_1^+, \ldots, z_t^+, z_1^-, \ldots, z_s^-) \in F\langle Y \cup Z, *\rangle$ be a polynomial. We say that f is a $*$-polynomial identity of A (or simply a $*$-identity), and we write $f \equiv 0$, if for all $u_1^+, \ldots, u_n^+ \in A_0^+$, $u_1^-, \ldots, u_m^- \in A_0^-$, $v_1^+, \ldots, v_t^+ \in A_1^+$ and $v_1^-, \ldots, v_s^- \in A_1^-$, we have

$$f(u_1^+, \ldots, u_n^+, u_1^-, \ldots, u_m^-, v_1^+, \ldots, v_t^+, v_1^-, \ldots, v_s^-) = 0.$$

We denote by $\mathrm{Id}^*(A) = \{f \in F\langle Y \cup Z, *\rangle \mid f \equiv 0 \text{ on } A\}$ the T_2^*-ideal of $*$-identities of A, i.e., $\mathrm{Id}^*(A)$ is an ideal of $F\langle Y \cup Z, *\rangle$ invariant under all $\mathbb{Z}_2$-graded endomorphisms of the free superalgebra commuting with $*$. It is well known that in characteristic zero, every $*$-identity is equivalent to a system of multilinear $*$-identities. Hence if we denote by

$$P_n^* = \mathrm{span}_F\Big\{w_{\sigma(1)}\cdots w_{\sigma(n)} \mid \sigma\in S_n,\ \ w_i\in\{y_i^+, y_i^-, z_i^+, z_i^-\},\ i=1,\dots,n\Big\}$$

the space of multilinear polynomials of degree n in $y_1^+, y_1^-, z_1^+, z_1^-, \dots, y_n^+, y_n^-, z_n^+, z_n^-$ (i.e., y_i^+ or y_i^- or z_i^+ or z_i^- appears in each monomial at degree 1) the study of $\mathrm{Id}^*(A)$ is equivalent to the study of $P_n^*\cap \mathrm{Id}^*(A)$, for all $n\geq 1$.

Let $n\geq 1$ and write $n=n_1+\cdots+n_4$ as a sum of four non-negative integers. We denote by $P^*_{\langle n\rangle}\subseteq P_n^*$ the vector space of multilinear $*$-polynomials in which n_1 variables are symmetric of even degree, n_2 variables are skew of even degree, n_3 variables are symmetric of odd degree and n_4 variables are skew of odd degree. The group $S_{n_1}\times\cdots\times S_{n_4}$ acts on the left on the vector space $P^*_{\langle n\rangle}$ by permuting the variables of the same homogeneous degree which are all even or all odd at the same time. Thus S_{n_1} permutes the variables $y_1^+,\dots,y_{n_1}^+$, S_{n_2} permutes the variables $y_1^-,\dots,y_{n_2}^-$, and so on. In this way $P^*_{\langle n\rangle}$ becomes an $S_{n_1}\times\cdots\times S_{n_4}$-left module. Now, $P^*_{\langle n\rangle}\cap\mathrm{Id}^*(A)$ is invariant under this action and so the vector space

$$P^*_{\langle n\rangle}(A)=\frac{P^*_{\langle n\rangle}}{P^*_{\langle n\rangle}\cap\mathrm{Id}^*(A)}$$

is an $S_{n_1}\times\cdots\times S_{n_4}$-left module with induced action. We denote by $\chi_{\langle n\rangle}(A)$ its character and we call it the $\langle n\rangle$-th cocharacter of A.

If $\lambda\vdash n$, we denote by χ_λ the corresponding irreducible S_n-character. Thus, if $\lambda(1)\vdash n_1$, $\dots$, $\lambda(4)\vdash n_4$ are partitions, we write $\langle\lambda\rangle=(\lambda(1),\dots,\lambda(4))\vdash\langle n\rangle$ and we say that $\langle\lambda\rangle$ is a multipartition of $n=n_1+\cdots+n_4$. Since $\mathrm{char}\,F=0$, by complete reducibility, $\chi_{\langle n\rangle}(A)$ can be written as a sum of irreducible characters

$$\chi_{\langle n\rangle}(A)=\sum_{\langle\lambda\rangle\vdash(n_1,\dots,n_4)} m_{\langle\lambda\rangle}\chi_{\lambda(1)}\otimes\cdots\otimes\chi_{\lambda(4)} \tag{1}$$

where $m_{\langle\lambda\rangle}\geq 0$ is the multiplicity of $\chi_{\lambda(1)}\otimes\cdots\otimes\chi_{\lambda(4)}$ in $\chi_{\langle n\rangle}(A)$. Such a sequence is called $*$-cocharacter sequence and for all $n\geq 1$, $\chi_{\langle n\rangle}(A)$ is called $\langle n\rangle$-th $*$-cocharacter. Here recall that the multiplicities in the cocharacter sequence are equal to the maximal number of linearly independent highest weight vectors, according to the representation theory of the general linear group GL_n. Moreover, a highest weight vector is obtained from the polynomial corresponding to an essential idempotent by identifying the variables whose indices lie in the same row of the corresponding Young tableaux (see [5, Chapter 12] for more details).

3. The main result

In this section we will prove the main theorem of the paper. In fact, we will show that the multiplicities in the $*$-cocharacter of a $*$-superalgebra A are bounded by one if and only if A satisfies a list of suitable $*$-identities. Since in [8] the authors dealt with superalgebras and algebras with involution, we may assume that in our case the $\mathbb{Z}_2$-grading and the gs-involution $*$ are always non-trivial.

In what follows, we denote by $[x_1,x_2]=x_1x_2-x_2x_1$, the usual commutator among two variables, and by $x_1\circ x_2=x_1x_2+x_2x_1$, the Jordan product among x_1 and x_2.

Lemma 3.1. *Let A be a superalgebra with gs-involution $*$ and let*

$$\chi^*_{\langle n\rangle}(A)=\sum_{\langle\lambda\rangle\vdash\langle n\rangle} m_{\langle\lambda\rangle}\chi_{\lambda(1)}\otimes\ldots\otimes\chi_{\lambda(4)}$$

be its $\langle n\rangle$-th $*$-cocharacter. If $m_{\langle\lambda\rangle}\leq 1$ for all $\langle\lambda\rangle\vdash\langle n\rangle$ and for all $n\geq 1$, then A satisfies

$$\alpha_1 y_1^+[y_1^+,y_2^+]+\beta_1[y_1^+,y_2^+]y_1^+ \tag{2}$$

$$\alpha_2 y_1^-[y_1^-,y_2^-]+\beta_2[y_1^-,y_2^-]y_1^-, \tag{3}$$

for some $\alpha_i,\beta_i\in F$, $(\alpha_i,\beta_i)\neq 0$, $i=1,2$, plus at least one identity of each group:

(i) $y_1^+z_2^+\equiv 0$ *or* $y_1^+\circ z_2^+\equiv 0$ *or* $[y_1^+,z_2^+]\equiv 0$;
(ii) $y_1^+z_2^-\equiv 0$ *or* $y_1^+\circ z_2^-\equiv 0$ *or* $[y_1^+,z_2^-]\equiv 0$;

(iii) $y_1^- z_2^+ \equiv 0$ *or* $y_1^- \circ z_2^+ \equiv 0$ *or* $[y_1^-, z_2^+] \equiv 0$;
(iv) $y_1^- z_2^- \equiv 0$ *or* $y_1^- \circ z_2^- \equiv 0$ *or* $[y_1^-, z_2^-] \equiv 0$;
(v) $z_1^+ z_2^- \equiv 0$ *or* $z_1^+ \circ z_2^- \equiv 0$ *or* $[z_1^+, z_2^-] \equiv 0$;
(vi) $y_1^+ y_2^- \equiv 0$ *or* $y_1^+ \circ y_2^- \equiv 0$ *or* $[y_1^+, y_2^-] \equiv 0$.

Proof. First let us consider the regular representation $P^*_{3,0,0,0}$ and its decomposition into irreducible submodules. Since $m_{\langle\lambda\rangle} \leq 1$ for all multipartitions, we get that in particular the submodule $M_{(2,1),\emptyset,\emptyset,\emptyset}$ has multiplicity less or equal to one, thus the highest weight vectors corresponding to standard tableaux of shape $(2,1)$ must be linearly dependent modulo $\mathrm{Id}^*(A)$. This implies that there exist $\alpha_1, \beta_1 \in F$ such that

$$\alpha_1 y_1^+[y_1^+, y_2^+] + \beta_1 [y_1^+, y_2^+] y_1^+ \pmod{\mathrm{Id}^*(A)}.$$

This proves identity (2). By considering the regular representation $P^*_{0,3,0,0}$ and its irreducible submodule $M_{\emptyset,(2,1),\emptyset,\emptyset}$, with similar arguments one can also prove that (3) also holds in A.
Let now consider the regular representation $P^*_{\langle 2\rangle}$ and its decomposition into irreducible $S_{n_1} \times \ldots \times S_{n_4}$- submodules. In particular, if one takes into account the submodule $M_{((1),\emptyset,(1),\emptyset)}$, since the multiplicity corresponding to the multipartition $((1), \emptyset, (1), \emptyset)$ is less or equal to 1, then there exist $\alpha, \beta, \gamma, \delta \in F$, $(\alpha, \beta) \neq (0,0)$, $(\gamma, \delta) \neq (0,0)$, such that

$$\alpha y_1^+ z_2^+ + \beta z_2^+ y_1^+ + \gamma y_2^+ z_1^+ + \delta z_1^+ y_2^+ \equiv 0 \pmod{\mathrm{Id}^*(A)}.$$

Due to the fact that $\mathrm{Id}^*(A)$ is multihomogeneous, we get that

$$\alpha y_1^+ z_2^+ + \beta z_2^+ y_1^+ \equiv 0 \pmod{\mathrm{Id}^*(A)}. \tag{4}$$

By applying the gs-involution $*$ we also obtain

$$\alpha z_2^+ y_1^+ + \beta y_1^+ z_2^+ \equiv 0 \pmod{\mathrm{Id}^*(A)}. \tag{5}$$

If either $\alpha = 0$ or $\beta = 0$, then $y_1^+ z_2^+ \equiv 0$ that is equivalent to $z_2^+ y_1^+ \equiv 0$. Otherwise, by summing (4) to (5) we get

$$(\alpha + \beta) y_1^+ z_2^+ + (\alpha + \beta) z_2^+ y_1^+ \equiv 0 \pmod{\mathrm{Id}^*(A)}.$$

If $\alpha + \beta \neq 0$, we get $y_1^+ z_2^+ + z_2^+ y_1^+ = y_1^+ \circ z_2^+ \equiv 0$, otherwise notice that $\alpha - \beta$ must be non-zero, hence by subtracting (5) to (4) we obtain

$$(\alpha - \beta) y_1^+ z_2^+ + (\alpha - \beta) z_2^+ y_1^+ \equiv 0 \pmod{\mathrm{Id}^*(A)}$$

and so $y_1^+ z_2^+ - z_2^+ y_1^+ = [y_1^+, z_2^+] \equiv 0$. Thus we proved that A satisfies at least one identity of the group (i) of the statement.
Following step-by-step the previous arguments also for the irreducible submodules $M_{(1),(1),\emptyset,\emptyset}$, $M_{(1),\emptyset,\emptyset,(1)}$, $M_{\emptyset,(1),(1),\emptyset}$, $M_{\emptyset,(1),\emptyset,(1)}$and $M_{\emptyset,\emptyset,(1),(1)}$ we get that A must satisfy at least one identity of the groups (ii) – (vi) of the statement and we are done. □

We can also prove the following.

Lemma 3.2. *Let A be a superalgebra with gs-involution $*$ and let*

$$\chi^*_{\langle n\rangle} = \sum_{\langle\lambda\rangle \vdash \langle n\rangle} m_{\langle n\rangle} \chi_{\lambda(1)} \otimes \ldots \otimes \chi_{\lambda(4)}$$

be its $\langle n\rangle$-th $*$-cocharacter. If A satisfies at least one identity of each group:
(i) $y_1^+ z_2^+ \equiv 0$ *or* $y_1^+ \circ z_2^+ \equiv 0$ *or* $[y_1^+, z_2^+] \equiv 0$;
(ii) $y_1^+ z_2^- \equiv 0$ *or* $y_1^+ \circ z_2^- \equiv 0$ *or* $[y_1^+, z_2^-] \equiv 0$;
(iii) $y_1^- z_2^+ \equiv 0$ *or* $y_1^- \circ z_2^+ \equiv 0$ *or* $[y_1^-, z_2^+] \equiv 0$;
(iv) $y_1^- z_2^- \equiv 0$ *or* $y_1^- \circ z_2^- \equiv 0$ *or* $[y_1^-, z_2^-] \equiv 0$;
(v) $z_1^+ z_2^- \equiv 0$ *or* $z_1^+ \circ z_2^- \equiv 0$ *or* $[z_1^+, z_2^-] \equiv 0$,
then $m_{\langle n\rangle} \leq 1$ if either $\langle\lambda\rangle = (\emptyset, \emptyset, \lambda(3), \emptyset)$ or $\langle\lambda\rangle = (\emptyset, \emptyset, \emptyset, \lambda(4))$, for all $n \geq 1$.

Proof. We will prove the statement providing that $*$ is a graded involution and $\langle\lambda\rangle = (\emptyset, \emptyset, \lambda(3), \emptyset)$. The cases $\langle\lambda\rangle = (\emptyset, \emptyset, \emptyset, \lambda(4))$ or $*$ superinvolution will follow with similar arguments.
Let us start by considering the first group of polynomials. In order to simplify the notation, let us sumarizing them by writing that A must satisfies

$$\alpha y_1^+ z_2^+ + \beta z_2^+ y_1^+ \equiv 0 \pmod{\mathrm{Id}^*(A)}, \tag{6}$$

where either $(\alpha,\beta)=(1,0)$ or $(\alpha,\beta)=(1,1)$ or $(\alpha,\beta)=(1,-1)$.
Analogously, let us write the polynomials of the third group as

$$\alpha' y_1^- z_2^+ + \beta' z_2^+ y_1^- \equiv 0 \pmod{\mathrm{Id}^*(A)}, \tag{7}$$

where either $(\alpha',\beta')=(1,0)$ or $(\alpha',\beta')=(1,1)$ or $(\alpha',\beta')=(1,-1)$.
Let us assume first that $(\alpha,\beta)=(1,0)$, thus $y_1^+z_2^+\equiv 0$. Since $z_1^+\circ z_3^+$ is an even symmetric variable, let us substitute it inside the previous identity. We get

$$z_1^+z_3^+z_2^+ + z_3^+z_1^+z_2^+ \equiv 0 \pmod{\mathrm{Id}^*(A)}.$$

By taking the $*$ of the previous one, we also get

$$z_2^+z_3^+z_1^+ + z_2^+z_1^+z_3^+ \equiv 0 \pmod{\mathrm{Id}^*(A)}.$$

It readily follows that

$$P^*_{0,0,n,0}(A)=\mathrm{span}_F\{z_1^+\dots z_n^+\},\ \text{if } n\geq 3 \text{ and } P^*_{0,0,2,0}(A)=\mathrm{span}_F\{z_1^+z_2^+, z_2^+z_1^+\}.$$

It is clear that if $n=2$ we have nothing to prove, moreover if $n\geq 3$ then $\dim_F P^*_{0,0,n,0}(A)\leq 1$ and only $\chi_{\emptyset,\emptyset,(n),\emptyset}$ can participate in the cocharacter. Since $m_{\emptyset,\emptyset,(n),\emptyset}\leq \deg\chi_{\emptyset,\emptyset,(n),\emptyset}=1$, we are done.
Let now assume $(\alpha,\beta)=(1,1)$, thus $y_1^+z_2^+ + z_2^+y_1^+\equiv 0$ and so

$$(z_1^+\circ z_3^+)z_2^+ + z_2^+(z_1^+\circ z_3^+)\equiv 0 \pmod{\mathrm{Id}^*(A)}. \tag{8}$$

Since $[z_1^+,z_3^+]$ is an even skew-symmetric variable, we can replace it in (7) instead of y_1^-.
If $(\alpha',\beta')=(1,0)$, then it follows that $z_1^+z_3^+z_2^+ - z_3^+z_1^+z_2^+\equiv 0$ and $z_2^+z_3^+z_1^+ - z_2^+z_1^+z_3^+\equiv 0$ modulo $\mathrm{Id}^*(A)$. As in the previous case, we can prove the claim.
If $(\alpha',\beta')=(1,1)$, then $[z_1^+,z_3^+]z_2^+ + z_2^+[z_1^+,z_3^+]\equiv 0$. Summing the latter one to identity (8), one gets

$$z_1^+z_3^+z_2^+ + z_2^+z_1^+z_3^+\equiv 0 \pmod{\mathrm{Id}^*(A)}$$

and an easy computation shows that

$$P^*_{0,0,n,0}(A)=\mathrm{span}_F\{z_1^+\dots z_{n-1}^+z_n^+, z_1^+\dots z_{n-2}^+z_n^+z_{n-1}^+\}$$

and so $\dim_F P^*_{0,0,n,0}(A)\leq 2$. This says that only the characters $\chi_{\emptyset,\emptyset,(n),\emptyset}$ and $\chi_{\emptyset,\emptyset,(1^n),\emptyset}$ can partecipate in the character of $P^*_{0,0,n,0}(A)$. Since $m_{\emptyset,\emptyset,(n),\emptyset}\leq\deg\chi_{\emptyset,\emptyset,(n),\emptyset}=1$ and $m_{\emptyset,\emptyset,(1^n),\emptyset}\leq \deg\chi_{\emptyset,\emptyset,(1^n),\emptyset}=1$, we are done.
If $(\alpha',\beta')=(1,-1)$, then $[z_1^+,z_3^+]z_2^+ - z_2^+[z_1^+,z_3^+]\equiv 0$. By summing the latter identity to (8), we get

$$z_1^+z_3^+z_2^+ + z_2^+z_3^+z_1^+\equiv 0 \pmod{\mathrm{Id}^*(A)}. \tag{9}$$

Notice that using (9), we get $(z_1^+z_2^+z_3^+)^* = z_3^+z_2^+z_1^+ \equiv -z_1^+z_2^+z_3^+$, thus any product of three odd symmetric variables is an odd skew-symmetric variable. So we have to consider three more possibilities. Recall that in addition to (9), we are assuming

$$y_1^+z_2^+ + z_2^+y_1^+\equiv 0 \tag{10}$$

and

$$y_1^-z_2^+ - z_2^+y_1^-\equiv 0. \tag{11}$$

If $z_1^+z_2^-\equiv 0$, then $z_1^+z_2^+z_3^+z_4^+\equiv 0$ and $P^*_{0,0,n,0}(A)=0$ for all $n\geq 4$. Furthermore, if $n=2$ we have nothing to prove and if $n=3$, then $P^*_{0,0,n,0}(A)=\mathrm{span}_F\{z_1^+z_2^+z_3^+, z_1^+z_3^+z_2^+, z_2^+z_1^+z_3^+\}$. This implies that the characters that can appear in the cocharacter decomposition of $P^*_{0,0,n,0}(A)$ are $\chi_{\emptyset,\emptyset,(3),\emptyset}$, $\chi_{\emptyset,\emptyset,(1^3),\emptyset}$ and $\chi_{\emptyset,\emptyset,(2,1),\emptyset}$. Notice that the last one cannot appear with multiplicity equal to 2 since its degree is 2 and $\dim_F P^*_{0,0,n,0}(A)\leq 3$. Moreover, $\deg\chi_{\emptyset,\emptyset,(3),\emptyset}=\deg\chi_{\emptyset,\emptyset,(1^3),\emptyset}=1$ and also in this case the claim is established.
Now let $[z_1^+,z_2^-]\equiv 0$, then

$$z_1^+z_2^+z_3^+z_4^+\equiv z_4^+z_1^+z_2^+z_3^+.$$

If $n\leq 3$ then the previous argument applies, so let suppose $n\geq 4$. In particular, if $n>4$ then due to the previous identity plus identity (9), one can always reorder all the variables modulo $\mathrm{Id}^*(A)$, i.e.

$$P^*_{0,0,n,0}(A)=\mathrm{span}_F\{z_1^+\dots z_n^+\}$$

and we have nothing to prove. In case $n=4$, then it is easily checked that

$$P^*_{0,0,n,0}(A)=\mathrm{span}_F\{z_1^+z_2^+z_3^+z_4^+,\ z_1^+z_3^+z_2^+z_4^+,\ z_1^+z_2^+z_4^+z_3^+,\ z_2^+z_1^+z_4^+z_3^+,\ z_3^+z_1^+z_4^+z_2^+,\ z_2^+z_1^+z_3^+z_4^+\}$$

and so $\dim_F P^*_{0,0,n,0}(A)\le 6$. Furthermore, in the corresponding character, $\chi_{\emptyset,\emptyset,(4),\emptyset}$, $\chi_{\emptyset,\emptyset,(1^4),\emptyset}$, $\chi_{\emptyset,\emptyset,(2,2),\emptyset}$, $\chi_{\emptyset,\emptyset,(2,1^2),\emptyset}$ and $\chi_{\emptyset,\emptyset,(3,1),\emptyset}$ can occur. It is clear that we have to check only the multiplicities of the last three irreducible characters, since $\deg\chi_{\emptyset,\emptyset,(4),\emptyset}=\deg\chi_{\emptyset,\emptyset,(1^4),\emptyset}=1$.
Let start with $\chi_{\emptyset,\emptyset,(2,2),\emptyset}$. If we fill the corresponding Young diagram in all possible standard ways, we get two highest weight vectors

$$f_1=\bar z_1^+\tilde z_1^+\bar z_2^+\tilde z_2^+ \text{ and}$$
$$f_2=\bar z_1^+\bar z_2^+\tilde z_1^+\tilde z_2^+,$$

where $^-$and$^\sim$mean alternation on the corresponding variables. Let use identities (9) and (10) in order to reduce f_1 modulo $\mathrm{Id}^*(A)$. Here recall that $(z_i^+)^2$ is an even symmetric variable. Moreover notice that due to identity (9), we get $z_1^+z_2^+z_1^+z_3^+\equiv 0$ and $z_1^+z_2^+z_3^+z_2^+\equiv 0$. Hence

$$\begin{aligned} f_1 =&(z_1^+)^2(z_2^+)^2-z_2^+(z_1^+)^2z_2^+-z_1^+(z_2^+)^2z_1^++(z_2^+)^2(z_1^+)^2\equiv\\ &(z_1^+)^2(z_2^+)^2+(z_1^+)^2(z_2^+)^2+(z_1^+)^2(z_2^+)^2-z_2^+(z_1^+)^2z_2^+\equiv\\ &3(z_1^+)^2(z_2^+)^2+(z_1^+)^2(z_2^+)^2=4(z_1^+)^2(z_2^+)^2.\end{aligned}$$

Similarly,

$$\begin{aligned} f_2 =&z_1^+z_2^+z_1^+z_2^+-z_2^+(z_1^+)^2z_2^+-z_1^+(z_2^+)^2z_1^++z_2^+z_1^+z_2^+z_1^+\equiv\\ &(z_1^+)^2(z_2^+)^2+(z_1^+)^2(z_2^+)^2=2(z_1^+)^2(z_2^+)^2.\end{aligned}$$

This shows that f_1 and f_2 are linearly dependent modulo $\mathrm{Id}^*(A)$ and so $m_{\emptyset,\emptyset,(2,2),\emptyset}\le 1$.
Let now consider $\chi_{\emptyset,\emptyset,(3,1),\emptyset}$. In this case we may have three highest weight vectors:

$$g_1=\bar z_1^+(z_1^+)^2\bar z_2^+,$$
$$g_2=\bar z_1^+\bar z_2^+(z_1^+)^2,$$
$$g_3=\bar z_1^+z_1^+\bar z_2^+z^+.$$

It turns out that since $z_1^+z_2^+z_1^+z_3^+\equiv 0$ and $z_1^+z_2^+z_3^+z_2^+\equiv 0$, then $g_i\in\mathrm{Id}^*(A)$ for all $1\le i\le 3$. Hence $m_{\emptyset,\emptyset,(3,1),\emptyset}=0$.
Finally, let us consider $\chi_{\emptyset,\emptyset,(2,1^2),\emptyset}$ and the corresponding highest weight vectors

$$h_1=\bar z_1^+z_1^+\bar z_2^+\bar z_3^+,$$
$$h_2=\bar z_1^+\bar z_2^+z_1^+\bar z_3^+,$$
$$h_3=\bar z_1^+\bar z_2^+\bar z_3^+z_1^+.$$

Using the same identities as before, it readily follows that $h_1\equiv h_2\equiv h_3\equiv 4z_2^+z_3^+(z_1^+)^2$, thus they are linearly dependent and $m_{\emptyset,\emptyset,(2,1^2),\emptyset}\le 1$.
We are left with the case $z_1^+\circ z_2^+\equiv 0$. A straightforward computation with similar arguments as in the case $[z_1^+,z_2^+]\equiv 0$, shows that even here the multiplicities are bounded by one.
Hence $m_{\emptyset,\emptyset,\lambda(3),\emptyset}\le 1$ for all $\lambda(3)\vdash n$ and for all $n\ge 1$ and the proof is complete. □

We are now in a position to prove the main theorem of the paper.

Theorem 3.1. *Let A be a superalgebra with gs-involution $*$ and let*

$$\chi^*_{\langle n\rangle}=\sum_{\langle\lambda\rangle\vdash\langle n\rangle} m_{\langle n\rangle}\chi_{\lambda(1)}\otimes\dots\otimes\chi_{\lambda(4)}$$

be its $\langle n\rangle$-th cocharacter. Then $m_{\langle\lambda\rangle}\le 1$ for all $\langle\lambda\rangle\vdash\langle n\rangle$ and for all $n\ge 1$, if and only if A satisfies the identities (2), (3) *plus at least one identity of each group:*

(i) $y_1^+z_2^+\equiv 0$ *or* $y_1^+\circ z_2^+\equiv 0$ *or* $[y_1^+,z_2^+]\equiv 0$;
(ii) $y_1^+z_2^-\equiv 0$ *or* $y_1^+\circ z_2^-\equiv 0$ *or* $[y_1^+,z_2^-]\equiv 0$;
(iii) $y_1^-z_2^+\equiv 0$ *or* $y_1^-\circ z_2^+\equiv 0$ *or* $[y_1^-,z_2^+]\equiv 0$;
(iv) $y_1^-z_2^-\equiv 0$ *or* $y_1^-\circ z_2^-\equiv 0$ *or* $[y_1^-,z_2^-]\equiv 0$;
(v) $z_1^+z_2^-\equiv 0$ *or* $z_1^+\circ z_2^-\equiv 0$ *or* $[z_1^+,z_2^-]\equiv 0$;
(vi) $y_1^+y_2^-\equiv 0$ *or* $y_1^+\circ y_2^-\equiv 0$ *or* $[y_1^+,y_2^-]\equiv 0$.

Proof. If $m_{\langle\lambda\rangle} \leq 1$ for all $\langle\lambda\rangle \vdash \langle n\rangle$ and for all $n \geq 1$, then the statement follows directly from Lemma 3.1.

Conversely, let us assume that A satisfies the identites (2), (3) plus at least one identity of each group (i)–(vi). From Lemma 3.2, it follows that $m_{\langle\lambda\rangle} \leq 1$ if either $\langle\lambda\rangle = (\emptyset, \emptyset, \lambda(3), \emptyset)$ or $\langle\lambda\rangle = (\emptyset, \emptyset, \emptyset, \lambda(4))$. Moreover, since A satisfies (2) and (3), following the lines of [1], one can prove that $m_{\langle\lambda\rangle} \leq 1$ if either $\langle\lambda\rangle = (\lambda(1), \emptyset, \emptyset, \emptyset)$ or $\langle\lambda\rangle = (\emptyset, \lambda(2), \emptyset, \emptyset)$.
Let now analyze irreducibles corresponding to multipartition of the type $\langle\lambda\rangle = (\lambda(1), \lambda(2), \emptyset, \emptyset) \vdash (n_1, n_2, 0, 0)$, $n_1 \neq 0$ and $n_2 \neq 0$. It is clear that if $y_1^+ y_2^- \equiv 0$, then also $y_2^- y_1^+ \equiv 0$, in fact $(y_1^+ y_2^-)^* = -y_2^- y_1^+$. Hence $m_{\langle\lambda\rangle} = 0$. If we suppose that either $y_1^+ \circ y_2^- \equiv 0$ or $[y_1^+, y_2^-] \equiv 0$, then the variables y^+'s and y^-'s can be separated modulo $\mathrm{Id}^*(A)$. Hence

$$P^*_{n_1,n_2,0,0} \equiv \mathrm{span}_F\{y^+_{i_1} \cdots y^+_{i_{n_1}} y^-_{j_1} \cdots y^-_{j_{n_2}}\}$$

or

$$P^*_{n_1,n_2,0,0} \equiv \mathrm{span}_F\{y^-_{i_1} \cdots y^-_{i_{n_1}} y^+_{j_1} \cdots y^+_{j_{n_2}}\}$$

modulo $\mathrm{Id}^*(A)$. It readily follows that for all $\langle\lambda\rangle = (\lambda(1), \lambda(2), \emptyset, \emptyset)$ we get

$$m_{\langle\lambda\rangle} \leq \max\{m_{(\lambda(1),\emptyset,\emptyset,\emptyset)}, m_{(\emptyset,\lambda(2),\emptyset,\emptyset)}\} \leq 1.$$

Similar arguments hold also if either $\langle\lambda\rangle = (\lambda(1), \emptyset, \lambda(3), \emptyset)$ or $\langle\lambda\rangle = (\lambda(1), \emptyset, \emptyset, \lambda(4))$ or $\langle\lambda\rangle = (\emptyset, \lambda(2), \lambda(3), \emptyset)$ or $\langle\lambda\rangle = (\emptyset, \lambda(2), \lambda(3), \emptyset)$ or $\langle\lambda\rangle = (\emptyset, \emptyset, \lambda(3), \lambda(4))$.
Finally, let $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(4))$. By using the identities of the groups (i)–(iv), it is clear that, as in the previous case, in $P^*_{\langle n\rangle}(A)$ we can separe the even variables from the odd ones, thus

$$m_{\langle\lambda\rangle} \leq \max\{m_{(\lambda(1),\lambda(2),\emptyset,\emptyset)}, m_{(\emptyset,\emptyset,\lambda(3),\lambda(4))}\} \leq 1$$

and we are done. □

We conclude the paper by making some short considerations about the lattice of the subvarieties of a $*$-variety.
Recall that the lattice of subvarieties is said to be distributive if for any subvarieties $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of $\mathcal{V}$, $(\mathcal{A} \cap \mathcal{B}) \cup \mathcal{C} = (\mathcal{A} \cup \mathcal{C}) \cap (\mathcal{B} \cup \mathcal{C})$. Let $\mathcal{V}$ be a variety of algebras (without any addictional structure). As a consequence of [1], we have that $\mathcal{V}$ has distributive lattice of the subvarieties if and onfly if in the corresponding cocharacter sequence, all the multiplicities are bounded by one.

This result was extended by Giambruno and Mishchenko in [8] for superalgebras, whereas in the same paper the authors proved that the same extension does not hold for algebras with involution, by showing a $*$-variety whose multiplicities of the cocharacter are not bounded by one but with a distributive lattice.

Concerning superalgebras with gs-involution, a similar result to that of algebras with involution holds. Indeed, it is possible to construct a $*$-variety with multiplicities equal to two, for some suitable multipartition, even if its lattice is distributive. Since this construction is very similar to that of the involution case, we remand the reader to [8, Section 4] for more details.

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