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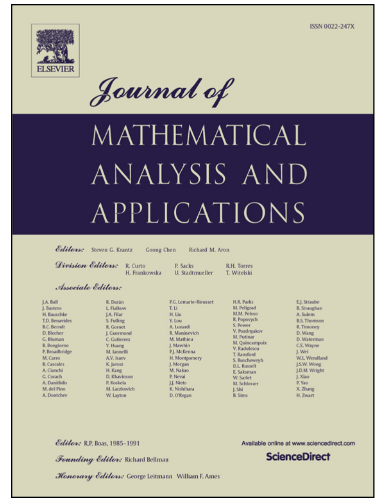

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A New Descriptive Characterization of the HK_r-Integral and its Inclusion in Burkill's Integrals

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Abstract

We introduce a new class of functions, ACG_r^*, and compare it to the class of ACG_r-functions which had been previously introduced by Musial and Sagher to characterize their Henstock–Kurzweil-type integral, the HK_r-integral. We show that these two classes coincide and thereby we obtain a new descriptive characterization of the class of HK_r-integrable functions. We then compare the HK_r-integral with Burkill's CP-integral and obtain a de la Vallée Poussin-type theorem for the HK_r-integral.

Keywords: Non-absolute integral, L^r-derivative, Henstock–Kurzweil-type integral, Cesàro–Perron integral, trigonometric series, Fourier coefficients.
2000 MSC: 26A36, 26A39

1. Introduction

We consider a Henstock–Kurzweil-type integral, the HK_r-integral, which was introduced by Musial and Sagher in [14] to solve a problem of recovering a function from its derivative defined in terms of L^r integral average, $1 \le r < \infty$ (see its equivalent definitions in [16] and in [20]). Originally this L^r-derivative appeared in an earlier paper [8] by Calderón and Zygmund and

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was used to obtain some estimates for solutions of elliptic partial differential equations.

The HK_r-integral belongs to the family of non-absolute generalizations of the Lebesgue integral. The topic which links the theory of generalized integrals to harmonic analysis is the so-called coefficients problem for orthogonal series, i.e., the problem of recovering the coefficients of an orthogonal series from its sum, under condition that the coefficients are uniquely defined by the sum. In the case of point-wise convergence and trigonometric series, the following classical de la Vallée Poussin's theorem (in a simplified version) is known (see [1]): *if a trigonometric series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

converges to a Lebesgue integrable function f nearly everywhere then this series is the Fourier series of its sum f.

At the same time a sum of an everywhere convergent trigonometric series, for example the sum of the series

$$\sum_{n=2}^{\infty} \frac{\sin nx}{\ln n}, \tag{2}$$

can fail to be Lebesgue integrable. So de la Vallée Poussin's theorem does not solve the coefficients problem for such a series. Moreover, the example (2) shows that no integral with continuous primitive can recover the coefficients of this series (to see this it is enough to consider the sum of term-wise integrated series). So it is natural to try to obtain a de la Vallée Poussin-type theorem for generalized integrals, which are continuous only in some general sense, for example for the HK_r-integral, we are considering here.

An essential step in developing this theory was made by introducing the so-called trigonometric integrals [2] which solve the coefficients problem without assuming a priori that the sum of the series is integrable in one or another sense. In the case of convergence everywhere or nearly everywhere the sum of the series (1) automatically becomes integrable in the sense of any of trigonometric integrals with the coefficients being the Fourier coefficients in the respective generalized sense. Starting with Denjoy's construction [9] called totalization T_{2s}, several Perron-type integrals were introduced to handle this coefficients problem. One of them is Burkill's Symmetric Cesàro–Perron integral [3, 5, 6, 7] (SCP-integral for short) which seems to be the most useful and easy to deal with. A little bit earlier Burkill introduced in [4] the more narrow CP-integral, which solves the coefficients problem under some additional assumptions.

Having trigonometric integrals, to obtain a de la Vallée Poussin-type theorem for some integral it is enough to check whether this integral is included in one of the trigonometric integrals. In the case of the Lebesgue integral it is easy because all the trigonometric integrals, as it follows directly from their definitions, are generalizations of the Lebesgue integral. In the case of some other integral to be compared with a trigonometric integral, one has to find and use some similarity in the way they are defined.

Recently in [17], we proved a de la Vallée Poussin-type theorem for the P_r-integral which was introduced by L. Gordon in [10] and which, similarly to the HK_r-integral, recovers a function from its L^r-derivative. As the P_r-integral is defined by Perron's method, it was natural to compare it with SCP-integral which is also a Perron-type integral. In fact we have proved in [17] that the P_r-integral is strictly contained in the CP- and (so also) in the SCP-integral.

Our intention here is to extend the results of [17] to the case of HK_r-integral. This integral was defined originally by the Henstock–Kurzweil method (see Definition 2.3 below) and it has no known Perron-type equivalent definition. It was shown in [15] that its natural Perron-type counterpart, the P_r-integral, is not equivalent to it, being strictly included in it. So Perron's method is not suitable for comparing the HK_r-integral with Burkill's trigonometric integrals. Luckily, the CP-integral has also an equivalent descriptive definition. It was given in [13] in terms of a Cesàro-type analogue of the classical Lusin concept of the ACG_*-function [12].

So one of our main problems here is to find a Lusin-type descriptive characterization of the HK_r-integral. We obtain it in terms of the ACG_r^* property. A version of a descriptive definition of this integral was defined earlier in [14], but it was given in terms of the property called ACG_r. This is an analogue of the property ACG_δ, defined by R. Gordon in [11] which essentially depends on so-called gauges and does not correspond to the class used in the above mentioned definition of CP-integral.

The main technical task we are solving in Section 3 is to show that the class of ACG_r^*-functions which we introduce there coincides with the class of ACG_r-functions given in [14] and hence gives a new descriptive definition of the HK_r-integral. In Section 4 we use this result to prove that the HK_r-integral is included in the CP-integral and so in the SCP-integral. We show by an example that this inclusion is proper. At last, in the final Subsection 4.3 we obtain a de la Vallée Poussin-type theorem for the HK_r-integral.

2. Preliminaries

2.1. Tagged intervals, partitions, and the HK_r-integral

We start with the following definitions.

Definition 2.1. [10] A function $F \in L^r[a,b]$ is said to be *L^r-continuous* at $x \in (a,b)$ if

$$\lim_{h\to 0} \frac{1}{h}\int_{x-h}^{x+h} |F(y) - F(x)|^r \, dy = 0.$$

If F is L^r-continuous for all $x \in E$, we say that F is *L^r-continuous on E*.

Definition 2.2. [10] Let $F \in L^r[a,b]$. We say that F is L^r-differentiable at $x \in (a,b)$ if there exists a real number α such that

$$\left(\frac{1}{h}\int_{-h}^{h} |F(x+t) - F(x) - \alpha t|^r \, dt\right)^{1/r} = o(h)$$

as $h \to 0$. It can easily be checked that if such an α exists, it is unique. If F is L^r-differentiable at x we denote the L^r-derivative of F at x by $F_r'(x) = \alpha$.

Both notions above are defined accordingly in the unilateral cases $x = a$ and $x = b$.

A *tagged interval* is a pair (I, x) where I is a compact interval $I \subset \mathbb{R}$, $x \in I$ is a *tag*. Any finite collection of tagged intervals, $\{(I_i, x_i)\}_{i=1}^q$, such that I_i and I_j are nonoverlapping for $i \neq j$, is called a *partition*. A partition is said to be *in* a segment $[a,b]$ provided $\bigcup_{i=1}^q I_i \subset [a,b]$, while it is a partition *of* $[a,b]$ if $\bigcup_{i=1}^q I_i = [a,b]$. We say this partition is *tagged in* a set E if $x_i \in E$ for all i. Given a *gauge*, i.e., a strictly positive function, δ on $\mathbb{R}$, we say a tagged interval (I, x) is *δ-fine* if $I \subset (x - \delta(x), x + \delta(x))$. A δ-fine partition consists of δ-fine tagged intervals.

We recall the definition of the L^r-Henstock–Kurzweil integral given in [14].

Definition 2.3. A function $f\colon [a,b] \to \mathbb{R}$ is L^r-Henstock–Kurzweil integrable (HK_r-integrable) on $[a,b]$ if there exists a function $F \in L^r[a,b]$ such that for any $\varepsilon > 0$ there exists a gauge δ such that for any δ-fine partition $\{([c_i, d_i], x_i)\}_{i=1}^q$ in $[a, b]$ we have

$$\sum_{i=1}^{q} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r} < \varepsilon.$$

By [14, Theorem 5], the function F in Definition 2.3 is unique up to an additive constant, so we can consider the indefinite HK_r-integral

$$F(x) = (HK_r) \int_a^x f, \quad \text{for each } x \in [a, b].$$

It was also proved in [14] that the indefinite HK_r-integral F is L^r-continuous on $[a, b]$ with $F_r'(x) = f(x)$ a.e. on $[a, b]$.

2.2. A descriptive characterization of the HK_r-integral

Definition 2.4. Let $E \subset [a, b]$. We write $F \in AC(E)$ if for each $\varepsilon > 0$ there exists $\eta > 0$ such that for any finite collection of nonoverlapping intervals $\{[c_i, d_i]\}_{i=1}^q$ having endpoints in E and such that

$$\sum_{i=1}^{q} (d_i - c_i) < \eta$$

we have

$$\sum_{i=1}^{q} |F(d_i) - F(c_i)| < \varepsilon.$$

We say that F is an *ACG-function* if $[a, b] = \bigcup_{n=1}^{\infty} E_n$, where $F \in AC(E_n)$ for all n. If, moreover, all E_n can be chosen closed, we say F is an *$[ACG]$-function*.

Let $1 \le r < \infty$. For $F \in L^r[a, b]$ and a tagged interval (I, x), $I \subset [a, b]$, we denote

$$\Delta_r F(I, x) = \left(\frac{1}{|I|} \int_I |F(y) - F(x)|^r \, dy \right)^{1/r}.$$

Definition 2.5. [14, Definitions 11 and 12] Let $E \subset [a, b]$. We write $F \in AC_r(E)$ if for all $\varepsilon > 0$ there exist $\eta > 0$ and a gauge δ defined on E such that for any δ-fine partition $\{(I_i, x_i)\}_{i=1}^q$ tagged in E and such that

$$\sum_{i=1}^{q} |I_i| < \eta \tag{3}$$

we have

$$\sum_{i=1}^{q} \Delta_r F(I_i, x_i) < \varepsilon. \tag{4}$$

We say that F is an ACG_r-function if $[a,b] = \bigcup_{n=1}^{\infty} E_n$ where $F \in AC_r(E_n)$ for all n.

Definition 2.6. Let $E \subset [a,b]$. We write $F \in BV_r(E)$ if there exist a constant M_F and a gauge δ defined on E such that for any δ-fine partition $\{(I_i, x_i)\}_{i=1}^{q}$ tagged in E we have

$$\sum_{i=1}^{q} \Delta_r F(I_i, x_i) < M_F. \tag{5}$$

We say that F is a BVG_r-*function* if $[a,b] = \bigcup_{n=1}^{\infty} E_n$ and $F \in BV_r(E_n)$ for all n.

The following descriptive characterization of the HK_r-integral was given in [14, Theorem 14].

Theorem 2.7. *A function f is HK_r-integrable on $[a,b]$ if and only if there exists an ACG_r-function F such that $F'_r = f$ a.e.; the function $F(x) - F(a)$ being the indefinite HK_r-integral of f.*

Remark 2.8. In [14] it had not been proved that an ACG_r-function is L^r-differentiable a.e. This property was proved only recently in [21, Theorem 12]. A slight modification of the proof of that result shows that each BVG_r-function, in fact, has this property.

In the next subsection we obtain a new descriptive characterization of the HK_r-integral which will be more suitable for comparing it with the CP-integral.

3. New descriptive characterization of the HK_r-integral

Definition 3.1. Let $E \subset [a,b]$. We write $F \in AC_r^*(E)$ if for each $\varepsilon > 0$ there exists $\eta > 0$ such that for any partition $\{(I_i, x_i)\}_{i=1}^{q}$ tagged in E and such that (3) holds we have (4). We say that F is an ACG_r^*-*function* if $[a,b] = \bigcup_{n=1}^{\infty} E_n$ where $F \in AC_r^*(E_n)$ for all n.

Definition 3.2. Let $E \subset [a,b]$. We write $F \in BV_r^*(E)$ if there exists a constant M_F such that (5) holds for all partitions $\{(I_i, x_i)\}_{i=1}^q$ in $[a,b]$ that are tagged in E. We say that F is a BVG_r^**-function* if $[a,b] = \bigcup_{n=1}^\infty E_n$ where $F \in BV_r^*(E_n)$ for all n.

Remark 3.3. By some technical reason it is useful to put some additional condition on partition in Definition 3.2. Namely we assume that the length of intervals I_i is bounded by the diameter of the set E.

We need the following obvious inequality for positive numbers which is true for finite or infinite sums provided the series are convergent:

$$\frac{\sum_j A_j}{\sum_j B_j} \leq \sum_j \frac{A_j}{B_j}. \tag{6}$$

Lemma 3.4. *If F is continuous on a closed set P (w.r. to this set) and $F \in BV_r^*(E)$, $E \subset P$, then $F \in BV_r^*(\overline{E})$, $\overline{E}$ being the closure of E.*

Proof. As P is closed and $E \subset P$ then $\overline{E} \subset P$. So F is continuous with respect to P at any point of $\overline{E}$.

Let $\{([c_i, d_i], x_i)\}_{i=1}^q$ be any partition in $[a,b]$ tagged in $\overline{E}$. Note that if $c_i < x_i < d_i$ then due to inequality (6) and convexity upwards of the function $x^{1/r}$ we have

$$\Delta_r F([c_i, d_i], x_i) \leq \Delta_r F([c_i, x_i], x_i) + \Delta_r F([x_i, d_i], x_i). \tag{7}$$

So it is enough to obtain a required estimate for a partition in which each tag is an endpoint of its interval. Moreover, we may assume that intervals $[c_i, d_i]$ are disjoint (considering separately intervals with even and odd indexes) and that each tag is the right endpoint of its interval, i.e., $x_i = d_i$. The case of intervals satisfying $x_i = c_i$ is similar.

Having fixed such a partition $\{([c_i, d_i], d_i)\}_{i=1}^q$, we construct a new partition tagged in E for which, due to condition, (5) holds.

Using the continuity of the function F at d_i we find a point $x_i^* \in E \cap (c_i, c_{i+1})$ so close to d_i that

$$|F(d_i) - F(x_i^*)| < \frac{1}{q}. \tag{8}$$

If $x_i^* < d_i$ then we include $([c_i, d_i], x_i^*)$ in the new partition, getting

$$\begin{aligned}\Delta_r F([c_i, d_i], d_i) &\le \Delta_r F([c_i, d_i], x_i^*) + \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(d_i) - F(x_i^*)|^r \, dy \right)^{1/r} \\ &< \Delta_r F([c_i, d_i], x_i^*) + \frac{1}{q}. \end{aligned} \tag{9}$$

If a point $x_i^* \in E$ with property (8) can be found only to the right of d_i, then we use the continuity of the function $\psi(t) := \Delta_r F([c_i, t], d_i)$ at $t = d_i$ and put an additional requirement on the choice of x_i^*:

$$|\Delta_r F([c_i, d_i], d_i) - \Delta_r F([c_i, x_i^*], d_i)| < \frac{1}{q}. \tag{10}$$

In this case a member of the new partition tagged in E will be the pair $([c_i, x_i^*], x_i^*)$ and we obtain

$$\begin{aligned}\Delta_r F([c_i, d_i], d_i) &\le |\Delta_r F([c_i, d_i], d_i) - \Delta_r F([c_i, x_i^*], d_i)| \\ &\quad + |\Delta_r F([c_i, x_i^*], d_i) - \Delta_r F([c_i, x_i^*], x_i^*)| + \Delta_r F([c_i, x_i^*], x_i^*) \\ &< \frac{2}{q} + \Delta_r F([c_i, x_i^*], x_i^*). \end{aligned} \tag{11}$$

To unify the notation it is convenient to denote the new partition, tagged in E, as $\{([c_i, d_i^*], x_i^*)\}_{i=1}^q$, with $d_i^* = d_i$ for the intervals taking part in the estimate (9), and with $d_i^* = x_i^*$ for the intervals taking part in the estimate (11). Summing up the above estimates over i we finally get

$$\sum_{i=1}^{q} \Delta_r F([c_i, d_i], d_i) < \sum_{i=1}^{q} \Delta_r F([c_i, d_i^*], x_i^*) + 2 < M_F + 2.$$

$\square$

Theorem 3.5. *If an L^r-continuous function F is BVG_r^* and $[ACG]$ then it is ACG_r^*.*

Proof. We have $[a, b] = \bigcup_{n=1}^{\infty} A_n$ with $F \in BV_r^*(A_n)$ and $[a, b] = \bigcup_{m=1}^{\infty} B_m$ with each B_m closed and $F \in AC(B_m)$. By Lemma 3.4, $F \in BV_r^*(\overline{A_n \cap B_m})$. So we want to prove that $F \in AC_r^*(\overline{A_n \cap B_m})$. Having fixed n and m, we put $D = \overline{A_n \cap B_m}$ for short.

By subtracting an appropriate function which is absolutely continuous on $[a,b]$, we can assume that $F(x)=0$ if $x\in D$. Thus, for any segment $I\subset[a,b]$ with $I\cap D\neq\emptyset$ we can write, without ambiguity, $\Delta_rF(I)$ for $\Delta_rF(I,x)$, $x\in I\cap D$.

Let $J_j=[a_j,b_j]$ be the closures of contiguous intervals to D which are ordered by decreasing length. Denote

$$M_j:=\max\left\{\max_{x\in[a_j,b_j]}\Delta_rF([a_j,x]),\ \max_{x\in[a_j,b_j]}\Delta_rF([x,b_j])\right\}$$

As $F\in BV_r^*(D)$ we have

$$\sum_{j=1}^{\infty}M_j<\infty. \tag{12}$$

Take any $\varepsilon>0$ and choose n so that

$$\sum_{j\geq n+1}M_j<\frac{\varepsilon}{2}. \tag{13}$$

In particular,

$$\sum_{j\geq n+1}\Delta_rF(J_j)<\frac{\varepsilon}{2}.$$

Now using the L^r-continuity of F we choose for each $j\leq n$ an $h_j<|J_j|$ such that

$$\Delta_rF([a_j,a_j+h])<\frac{\varepsilon}{4n}\quad\text{and}\quad\Delta_rF([b_j-h,b_j])<\frac{\varepsilon}{4n} \tag{14}$$

for each $0<h\leq h_j$. Now we choose $\eta<\min\{h_j:1\leq j\leq n\}$ and take any partition $\{(L_k,x_k)\}_{k=1}^q$ tagged in D such that $\sum_{k=1}^q|L_k|<\eta$. Denote $K_{jk}=J_j\cap L_k$. Note that $|L_k|\geq\sum_{j=1}^{\infty}|K_{jk}|$ and

$$\int_{L_k}|F|^r=\sum_{j=1}^{\infty}\int_{K_{jk}}|F|^r.$$

Applying (6) for each fixed k we obtain

$$\begin{aligned}\Delta_rF(L_k,x_k)&=\left(\frac{1}{|L_k|}\int_{L_k}|F|^r\right)^{1/r}\leq\left(\frac{\sum_{j=1}^{\infty}\int_{K_{jk}}|F|^r}{\sum_{j=1}^{\infty}|K_{jk}|}\right)^{1/r}\\&\leq\left(\sum_{j=1}^{\infty}\frac{1}{|K_{jk}|}\int_{K_{jk}}|F|^r\right)^{1/r}\leq\sum_{j=1}^{\infty}\left(\frac{1}{|K_{jk}|}\int_{K_{jk}}|F|^r\right)^{1/r}.\end{aligned}$$

Note that the mean values

$$\left(\frac{1}{|K_{jk}|}\int_{K_{jk}}|F|^r\right)^{1/r}$$

for all j and k are estimated either in (13) or in (14) and all intervals of the family $\{K_{jk}\}_{j,k}$ are non-overlapping pairwise. Summing up these estimates of $\Delta_r F(L_k, x_k)$ over all k and using (13) and (14) we get the final estimate

$$\sum_{k=1}^{q}\Delta_r F(L_k, x_k) < \frac{\varepsilon}{2} + \frac{2n\varepsilon}{4n} = \varepsilon.$$

□

Lemma 3.6. *If $F \in AC_r(E)$, $E \subset [a,b]$, then $F \in BV_r(E)$.*

Proof. Let η and a gauge δ_1 be found for $\varepsilon = 1$ according to Definition 2.5. We subdivide $[a,b]$ into $s := [(b-a)/\eta] + 1$ equal intervals $[a_j, a_{j+1}]$, $j = 0, \dots, s-1$. Note that $a_{j+1} - a_j < \eta$ for each $j = 0, \dots, s-1$. Now we define a new gauge on E by putting $\delta(x) = \min\{\delta_1(x), x - a_j, a_{j+1} - x\}$ if $a_j < x < a_{j+1}$, $j = 0, \dots, s-1$, and $\delta(a_j) = \min\{\delta_1(a_j), (b-a)/s\}$ in case $a_j \in E$, $j = 0, \dots, s$.

Let $\mathcal{P} := \{(I_i, x_i)\}_{i=1}^q$ be any δ-fine partition tagged in E. In case $x_i = a_j$ for some i and j we can represent I_i as a union of two intervals $I_{i,j} \subset [a_{j-1}, a_j]$ and $I_{i,j+1} \subset [a_j, a_{j+1}]$. Then due to the way δ is defined we have a representation $\mathcal{P} = \bigcup_{j=0}^{s-1}\mathcal{P}_j$ where $\mathcal{P}_j := \{(I_i, x_i) : I_i \subset [a_j, a_{j+1}]\}$. Each partition $\mathcal{P}_j$ is δ_1-fine and $\sum_{(I_i,x_i)\in\mathcal{P}_j}|I_i| < \eta$. Hence $\sum_{(I_i,x_i)\in\mathcal{P}_j}\Delta_r F(I_i, x_i) < 1$. Summing over j and having in mind the following estimate in case $x_i = a_j$,

$$\Delta_r F(I_j, x_i) \le \Delta_r F(I_{i,j}, x_i) + \Delta_r F(I_{i,j+1}, x_i),$$

cf. (7), we eventually obtain (5) with $M_F = s$. □

Theorem 3.7. *If an L^r-continuous function F is BVG_r and $[ACG]$ then it is ACG_r^*.*

Proof. Having Theorem 3.5 it is enough to prove that F is BVG_r^*. Let T be one of the sets on which F is BV_r. According to Definition 2.6 we find a gauge δ on T such that (5) holds for any δ-fine partition $\{(I_i, x_i)\}_{i=1}^q$ tagged in T. We define

$$T_n = \left\{x \in T : \delta(x) > \frac{1}{n}\right\}$$

and $T_{nj} = T_n \cap [j/(3n), (j+1)/(3n)]$, $n \in \mathbb{N}$, $j \in \mathbb{Z}$, getting $T = \bigcup_{n,j} T_{nj}$. Having fixed n and j consider any partition $\{(I_i, x_i)\}_i$ tagged in T_{nj}. This partition is automatically δ-fine (see Remark 3.3) and hence the inequality (5) holds for this partition. So we obtain that F is BV_r^* on T_{nj} and therefore BVG_r^*. □

Remark 3.8. Note that the implications from Theorems 3.5 and 3.7 are in fact equivalences. Indeed, the above proof demonstrates that BVG_r and BVG_r^* are equivalent, so the opposite implications follow from [24, Corollary 13].

Remark 3.9. It was proved in [24, Corollary 12] that the HK_r-integral is included in the so-called approximate Henstock integral (AH-integral). Not giving details, just note that, using Theorems 3.5 and 3.7, it is possible to obtain a criterion, in terms of the BVG_r^* property, when an AH-integrable function is HK_r-integrable. Namely, if f is AH-integrable, then it is HK_r-integrable iff its indefinite integral is BVG_r^* and L^r-continuous.

Theorem 3.10. *Each ACG_r-function F is ACG_r^*.*

Proof. By [24, Corollary 13], F is $[ACG]$. Lemma 3.6 implies F is BVG_r and it is clear that an ACG_r function is L^r-continuous. So, Theorem 3.7 implies that F is ACG_r^*. □

The opposite inclusion is obvious: any AC_r^* function is AC_r, since if (4) holds for an arbitrary partition then it holds also for a δ-fine partition (subject to (3)). So we obtain

Theorem 3.11. *The classes ACG_r and ACG_r^* coincide.*

The above result allows to obtain a new descriptive definition of the HK_r-integral

Theorem 3.12. *The class of indefinite HK_r-integrals coincides with the class of all ACG_r^*-functions.*

4. Comparison with the CP-integral and application to Fourier series

4.1. The CP-integral and its descriptive definition

In the following definitions and theorems taken from [23] it is assumed that F is a Perron integrable (P-integrable). We need not recall the definition of the P-integral because when we apply these definitions and results in our case, a function F is in fact always Lebesgue integrable.

Definition 4.1. The *Cesàro mean* of F over $[a,b]$ is given by

$$C(F,a,b) = C(F,b,a) = \frac{1}{b-a}\int_a^b F.$$

Definition 4.2. If $C(F,\xi,x)$ tends to a finite limit l as $x \to \xi$, we say that $F(x)$ tends to l in C-sense as $x \to \xi$, and write

$$\text{C-}\lim_{x\to\xi} F(x) = l.$$

Definition 4.3. If $F(\xi)$ is finite and $\text{C-}\lim_{x\to\xi} F(x) = F(\xi)$, we say that F is *C-continuous* at the point $\xi \in [a,b]$.

Definition 4.4. If $C(F,\xi,\xi+h)$ exists for sufficiently small values of h, the *C-derivative* $\mathrm{CD}\,F(\xi)$ at ξ is defined by

$$\mathrm{CD}\,F(\xi) = 2\lim_{h\to 0}\frac{C(F,\xi,\xi+h) - F(\xi)}{h}.$$

Definition 4.5. A function F is said to be AC^* *in C-sense* on a set E if it is integrable over an interval which contains E, and if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any collection of nonoverlapping intervals $\{[a_k,b_k]\}_{k=1}^q$ with both endpoints in E satisfying

$$\sum_{k=1}^q (b_k - a_k) \le \delta,$$

we have

$$\begin{aligned} &\sum_{k=1}^q \sup_{a_k<x<b_k} |C(F,a_k,x) - F(a_k)| < \varepsilon \qquad \text{and} \\ &\sum_{k=1}^q \sup_{a_k<x<b_k} |C(F,x,b_k) - F(b_k)| < \varepsilon. \end{aligned} \tag{15}$$

The function F is said to be ACG^* in C-sense if it is C-continuous at all points of $[a,b]$, and $[a,b] = \bigcup_{n=1}^\infty E_n$ where F is AC^* in C-sense on each E_n.

Remark 4.6. Definition 4.5 has been introduced and investigated by Sargent in [23] (in a more general setting, namely, under any given *rank*) as a descriptive characterization for Burkill's corresponding *Cesàro–Perron integrals* [4]. In particular, she proved the following theorems.

Theorem 4.7. *[23, Theorem III] If F is AC^* in C-sense on a measurable set $E \subset [a,b]$, then $\mathrm{CD}\, F(x)$ exists and is finite a.e. in E. Also, $\mathrm{CD}\, F(x)$ is equal to the approximate derivative $F'_{\mathrm{ap}}(x)$ a.e. in E, and is absolutely integrable on any measurable subset of E.*

Theorem 4.8. *[23, Theorems VIII and XI] A function f over $[a,b]$ is CP-integrable iff there is a function F that is ACG^* in C-sense and such that $\mathrm{CD}\, F(x) = f(x)$ a.e. in $[a,b]$. In this case, moreover, $F(b) - F(a)$ coincides with the value of the CP-integral of f.*

4.2. Comparison of the HK_r-integral with the CP-integral

We compare now the CP-integral (and by this also the SCP-integral) with the HK_r-integral. We will need the following two lemmas.

Lemma 4.9. *If a function is L^r-continuous at a point x then it is C-continuous at x.*

Proof. The condition for C-continuity can be rewritten in the form

$$C(F, x, x+h) - F(x) = \frac{1}{h}\int_x^{x+h} (F(t) - F(x))\, dt = o(1)$$

as $h \to 0$. Comparing the definition of L^r-continuity and C-continuity at a point x and using Hölder's inequality we obtain

$$\begin{aligned} |C(F, x, x+h) - F(x)| &\leq \frac{1}{h}\int_x^{x+h} |F(t) - F(x)|\, dt \\ &\leq \frac{1}{|h|}\left|\int_x^{x+h} |F(t) - F(x)|^r\, dt\right|^{1/r} |h|^{1/r'} \\ &= \left(\frac{1}{h}\int_x^{x+h} |F(t) - F(x)|^r\, dt\right)^{1/r} = o(1) \end{aligned}$$

as $h \to 0$. $\square$

Lemma 4.10. *If a function is L^r-differentiable at a point x then it is C-differentiable at x to the same derivative.*

Proof. The C-derivative $f(x) = \mathrm{CD}\, F(x)$ of a function F at a point x, according to Definition 4.4, is defined by the condition

$$\frac{1}{h}\int_x^{x+h} (F(t) - F(x) - f(x)(t-x))\, dt = o(h)$$

as $h \to 0$. Comparing this expression with the condition in Definition 2.2 and using Hölder's inequality like above we obtain

$$\begin{aligned}
&\left|\frac{1}{h}\int_x^{x+h} (F(t) - F(x) - f(x)(t-x))\, dt\right| \\
&\quad \le \frac{1}{h}\int_x^{x+h} |F(t) - F(x) - f(x)(t-x)|\, dt \\
&\quad \le \left(\frac{1}{h}\int_x^{x+h} |F(t) - F(x) - f(x)(t-x)|^r\, dt\right)^{1/r} = o(h)
\end{aligned}$$

as $h \to 0$. □

Theorem 4.11. *Any ACG_r^*-function is ACG^* in C-sense.*

Proof. In view of Lemma 4.9, the proof boils down to showing that if $F \in AC_r^*(E)$, then F is AC^* on E in C-sense. Let $\{[a_k, b_k]\}_{k=1}^q$ be any collection of nonoverlapping intervals with both endpoints in E. Further computation, based on Hölder's inequality, is similar to the one used in the proof of the previous two lemmas. Indeed, for every k and $a_k < x < b_k$, we have

$$\begin{aligned}
|C(F, a_k, x) - F(a_k)| &\le \frac{1}{x - a_k}\int_{a_k}^{x} |F(t) - F(a_k)|\, dt \\
&\le \left(\frac{1}{x - a_k}\int_{a_k}^{x} |F(t) - F(a_k)|^r\, dt\right)^{1/r} = \Delta_r F([a_k, x], a_k).
\end{aligned}$$

Analogously we obtain

$$|C(F, x, b_k) - F(b_k)| \le \Delta_r F([x, b_k], b_k).$$

Thus, provided (4) holds for partitions $\{([a_k, x], a_k)\}_{k=1}^q$ and $\{([x, b_k], b_k)\}_{k=1}^q$, also condition (15) is satisfied. □

By Theorem 4.8, Lemma 4.10 and Theorem 4.11 we can state the following

Theorem 4.12. *For any $r \geq 1$, if $f : [a, b] \to \mathbb{R}$ is HK_r-integrable then it is CP-integrable and the values of integrals coincide.*

We show now that the inclusion of the HK_r-integral in the CP-integral is proper.

The following property of the CP-integral (the so-called Hake property) is known (see [13])

Theorem 4.13. *If a function f defined on $[a, b]$ is CP-integrable on $[\alpha, \beta]$ whenever $a < \alpha < \beta < b$, and if*

$$\operatorname*{C-lim}_{\substack{\alpha \to a^+ \\ \beta \to b^-}} \int_\alpha^\beta f \tag{16}$$

exists and is finite, then f is CP-integrable on $[a, b]$ to the value (16).

Theorem 4.14. *There exists a function which is CP-integrable on $[0, 1]$ but which is HK_r-integrable for no $r \geq 1$.*

Proof. We can use the property that the indefinite HK_r-integral is L^r-continuous everywhere and the fact that C-continuity is strictly more general than L^r-continuity. So it is enough to construct a CP-integrable function whose indefinite CP-integral is not L^1-continuous (and so not L^r-continuous) at a single point. Consider the following example.

For each integer $n \geq 6$, put

$$c_n = \frac{1}{n}, \quad b_n = \frac{1}{n} - \frac{1}{n^{2.5}}, \quad a_n = \frac{1}{n} - \frac{2}{n^{2.5}}$$

and define a function F on $[0, 1]$

$$F(x) = \begin{cases} n, & \text{if } x \in [a_n, b_n], \\ -n, & \text{if } x \in (b_n, c_n], \\ 0, & \text{if } x \in [0, 1] \setminus \bigcup_{n=1}^{\infty} [a_n, c_n]. \end{cases}$$

Note that the choice of $n \geq 6$ insures that F is well defined and that always $c_{n+1} < a_n$. It is clear how to make F to be smooth on $(0, 1]$ changing it in small neighborhoods of the points of discontinuity, without influencing further estimations. So, we can consider F to be differentiable everywhere on $(0,1]$ with F' being in L^1 on each interval $[\alpha, 1]$ for $0 < \alpha \leq 1$.

It is easy to check that

$$\int_{a_n}^{c_n} |F| = \frac{2}{n^{1.5}}, \tag{17}$$

so $F \in L^1[0,1]$. F is C-continuous at 0. Indeed, assume that $a_n \leq h < a_{n-1}$. We compute

$$0 \leq \frac{1}{h}\int_0^h F \leq \frac{1}{c_{n+1}}\int_{a_n}^{b_n} F \;=\; \frac{n+1}{n^{1.5}} \to 0 \tag{18}$$

as $h \to 0$. So, F is C-continuous at 0 and by Theorem 4.13 is the indefinite CP-integral for its derivative. At the same time (see (17))

$$\begin{aligned} \Delta_1 F([0,1/n],0) &= n\int_0^{1/n} |F| \\ &= n\sum_{k=n}^{\infty}\frac{2}{k^{1.5}} \geq 4n^{0.5} \to \infty. \end{aligned} \tag{19}$$

If F' were HK_r-integrable, then, according to Theorem 4.12, F would be its HK_r-integral. But by (19), F is not L^r-continuous resulting in a contradiction. □

4.3. Application to Fourier Series

We are going to obtain here a de la Vallée Poussin-type theorem for the HK_r-integral. As it was already mentioned in the introduction, it is enough for this purpose to show that the HK_r-integral is included in the SCP-integral. For our argument here we need not to recall the definition of the SCP-integral. It is enough to know that the CP-integral is included in the SCP-integral and to use Theorem 4.12 of the previous subsection.

The following statement was in fact proved in [17, Theorem 4.3 and the proof of Theorem 4.4].

Theorem 4.15. *If the series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{20}$$

converges to a CP-integrable function f almost everywhere on $[0, 2\pi]$ and the partial sums of (20) *are bounded for each x except for a countable set, then*

$$\begin{aligned} a_n &= \frac{1}{\pi}(CP)\int_0^{2\pi} f(x)\cos nx\,dx, \qquad n = 0, 1, 2, \ldots; \\ b_n &= \frac{1}{\pi}(CP)\int_0^{2\pi} f(x)\sin nx\,dx, \qquad n = 1, 2, \ldots \end{aligned} \tag{21}$$

We need also the following result on integration by parts for the HK_r-integral:

Theorem 4.16. *[18] Suppose that, on an interval $[a, b]$, f is HK_r-integrable, G is absolutely continuous and $g := G'$ is in $L^{r'}, r' = r/(r-1)$ (if $r = 1$, the condition on G becomes G is Lipschitz). Then fG is HK_r-integrable on $[a, b]$ and if $F(x) = C + \int_a^x f$, then*

$$(HK_r)\int_a^b fG = FG|_a^b - \int_a^b Fg,$$

where the integral on the right exists as a Lebesgue integral.

Now we apply the above theorems to obtain the main result of this section:

Theorem 4.17. *If the series* (20) *converges to an HK_r-integrable function f almost everywhere on $[0, 2\pi]$ and the partial sums of* (20) *are bounded for each x except on a countable set, then*

$$\begin{aligned} a_n &= \frac{1}{\pi}(HK_r)\int_0^{2\pi} f(x)\cos nx\,dx, \qquad n = 0, 1, 2, \ldots; \\ b_n &= \frac{1}{\pi}(HK_r)\int_0^{2\pi} f(x)\sin nx\,dx, \qquad n = 1, 2, \ldots \end{aligned} \tag{22}$$

Proof. By Theorem 4.12, the function f is CP-integrable and by Theorem 4.15 formulas (21) hold. Then by Theorems 4.16 and 4.12 formulas (21) imply formulas (22) giving the desired result. □

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