HOMOCLINIC SOLUTIONS OF NONLINEAR LAPLACIAN DIFFERENCE EQUATIONS WITHOUT AMBROSETTI-RABINOWITZ CONDITION

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Abstract. The aim of this paper is to establish the existence of at least two non-zero homoclinic solutions for a nonlinear Laplacian difference equation without using Ambrosetti-Rabinowitz type-conditions. The main tools are mountain pass theorem and Palais-Smale compactness condition involving suitable functionals.

1. Introduction

This paper is motivated by some manuscripts concerning with the nonlinear Laplacian difference equations. Here, we consider the boundary value problem

$$\begin{gathered}\Delta^2_{p_2} u(z-2) - a\Delta_{p_1} u(z-1) + V(z)\phi_q(u(z)) = \lambda g(z, u(z)),\\ \text{for all } z \in \mathbb{Z}\ (a, \lambda \in]0, +\infty[,\ 1 < q \le p_i \text{ for } i = 1, 2),\\ |u(z)| \to 0 \quad \text{as } |z| \to +\infty,\end{gathered} \tag{1.1}$$

driven by q-Laplacian type operator $\phi_q(u) = |u|^{q-2}u$ with $u \in \mathbb{R}$. In (1.1) we use the following notations:

- $\Delta u(z-1) = u(z) - u(z-1)$ is the forward difference operator,
- $\Delta^2 u(z-2) = \Delta u(z-1) - \Delta u(z-2)$ is the second order forward difference operator,
- $\Delta_{p_i} u(z-1) := \Delta(\phi_{p_i}(\Delta u(z-1))) = \phi_{p_i}(\Delta u(z)) - \phi_{p_i}(\Delta u(z-1))$ is the discrete p_i-Laplace operator.

Let $V : \mathbb{Z} \to \mathbb{R}$ and $g : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$. We assume that the function g is continuous. Moreover, we denote by $G : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ the function

$$G(z, t) = \int_0^t g(z, \xi)d\xi, \quad \text{for all } t \in \mathbb{R},\ z \in \mathbb{Z}.$$

The assumptions on problem (1.1) are as follows:

(H1) $\lim_{t\to 0} \frac{|g(z,t)|}{|t|^{q-1}} = 0$ uniformly for all $z \in \mathbb{Z}$;
(H2) $\sup_{|t|\le T} |G(\cdot, t)| \in l^1$ for all $T > 0$;
(H3) $\limsup_{|t|\to+\infty} \frac{G(z,t)}{|t|^q} \le 0$ uniformly for all $z \in \mathbb{Z}$;
(H4) $G(h, b) > 0$ for some $(h, b) \in \mathbb{Z} \times \mathbb{R}$;
(H5) $V(z) \in [V_0, +\infty[$ for all $z \in \mathbb{Z}$, $V_0 > 0$ and $V(z) \to +\infty$ as $|z| \to +\infty$.

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We approach the problem (1.1) by using its variational formulation. We note that there is a relevant literature on differential equations driven by a q-Laplacian operator or (p_1, p_2)-Laplacian operator (which is the sum of a p_1-Laplacian and of a p_2-Laplacian). Indeed, by using specific nonlinearities ($g : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$), we are able to study the dynamical behaviour of real phenomena in biological and physical settings (for more details we refer to Diening-Harjulehto-Hästö-Růžička [4] and Motreanu-Motreanu-Papageorgiou [8]). In addition, Mugnai-Papageorgiou [11] and Motreanu-Vetro-Vetro [9, 10] focused on (p_1, p_2)-Laplacian equations with $p_1 < p_2$.

On the other hand the use of difference operators leads to discrete versions of continuous differential equations (as shown in Agarwal [1] and Kelly-Peterson [7]). This specific topic of research links mathematical analysis to numerical analysis. In particular, we think to the approximation of solutions and the study of convergence and stability. In this direction, we mention the works of Cabada-Iannizzotto-Tersian [2], Iannizzotto-Tersian [5], Jiang-Zhou [6] (for discrete q-Laplacian operator) and Nastasi-Vetro-Vetro [13] (for discrete (p_1, p_2)-Laplacian operator). Also, Cabada-Li-Tersian [3], Nastasi-Vetro [12] and Saavedra-Tersian [17] investigated the existence of homoclinic solutions (that is, solutions satisfying $|u(z)| \to 0$ as $|z| \to +\infty$).

We continue this study by using the variational formulation of (1.1). Thus, we establish the existence of at least two non-zero homoclinic solutions for such a problem. The hypotheses $(H1)$ - $(H5)$ imply that the energy functional associated to (1.1) satisfies mountain-pass geometry and Palais-Smale compactness condition (Pucci-Serrin [16]). Here, we do not use the Ambrosetti-Rabinowitz condition (that is, there exist $\mu > p_i \geq q > 1$ and $M > 0$ such that $0 < \mu G(z,t) \leq g(z,t)t$ for all $z \in \mathbb{Z}$, $|t| > M$). We point out that the Ambrosetti-Rabinowitz condition is a useful tool in establishing the existence of nontrivial solutions for elliptic equations driven by the Laplacian, q-Laplacian and (p_1, p_2)-Laplacian operators, via variational methods. Indeed, it easily ensures that the above mentioned energy functional has a mountain-pass geometry and the associated Palais-Smale property holds. The interest for our results, it is motivated by the fact that in many practical problems the nonlinearity $g : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ does not satisfy the Ambrosetti-Rabinowitz condition (for example, $g(z,t) = z^{-\mu}|t|^{p-2}t \ln(1 + |t|^{\alpha})$ with $\mu > 1$ and $\alpha \geq 1$). So, it is of a certain interest to prove the existence and multiplicity of solutions in such situations. We mention the paper of Papageorgiou-Vetro-Vetro [14], where the authors employ a weaker condition which permits the consideration of both convex and concave nonlinearities.

2. Mathematical background

On the same lines of Saavedra-Tersian [17] and the references therein, we denote by l^q the set of sequences $u : \mathbb{Z} \to \mathbb{R}$ such that

$$\|u\|_q^q := \sum_{z \in \mathbb{Z}} |u(z)|^q < +\infty.$$

So, $(l^q, \|\cdot\|_q)$ is a reflexive Banach space. Similarly, by l^∞ we mean the set of sequences $u : \mathbb{Z} \to \mathbb{R}$ such that

$$\|u\|_\infty := \sup_{z \in \mathbb{Z}} |u(z)| < +\infty.$$

Now, $l^q \hookrightarrow l^\infty$ is a continuous embedding, since $\|u\|_\infty \leq \|u\|_q$ for all $u \in l^q$, and $l^q \subseteq l^{p_i}$ with $1 < q < p_i < +\infty$, $i = 1, 2$.

We construct the functional $\mathcal{J}_\lambda : l^q \to \mathbb{R}$, associated to problem (1.1). Precisely, we consider the functional $\mathcal{G} : l^q \to \mathbb{R}$ given as

$$\mathcal{G}(u) = \sum_{z\in\mathbb{Z}} G(z, u(z)), \quad \text{for all } u \in l^q,$$

so that $\mathcal{G} \in C^1(l^q, \mathbb{R})$ and

$$\langle \mathcal{G}'(u), v\rangle = \sum_{z\in\mathbb{Z}} g(z, u(z))v(z), \quad \text{for all } u, v \in l^q.$$

Also, we need the functional $\mathcal{L} : l^q \to \mathbb{R}$ defined by

$$\mathcal{L}(u) = \sum_{z\in\mathbb{Z}} \left[\frac{1}{p_2}|\Delta^2 u(z-2)|^{p_2} + a\frac{1}{p_1}|\Delta u(z-1)|^{p_1} + \frac{1}{q}V(z)|u(z)|^q\right], \text{ for all } u \in l^q.$$

Also, we have

$$\begin{aligned}\langle \mathcal{L}'(u), v\rangle = &\sum_{z\in\mathbb{Z}}[\Delta^2(\phi_{p_2}(\Delta^2 u(z-2))) - a\Delta(\phi_{p_1}(\Delta u(z-1)))]v(z)\\ &+ \sum_{z\in\mathbb{Z}} V(z)\phi_q(u(z))v(z), \quad \text{for all } u, v \in l^q.\end{aligned}$$

Finally, we put

$$\mathcal{J}_\lambda(u) = \mathcal{L}(u) - \lambda\mathcal{G}(u), \quad \text{for all } u \in l^q. \tag{2.1}$$

We point out that $\mathcal{J}_\lambda(0) = 0$. By Lemma 2.2 of Saavedra-Tersian [17], the functional $\mathcal{J}_\lambda$ is well defined and C^1-differentiable. Moreover, its critical points are solutions of problem (1.1).

Now, we consider the Banach space

$$X := \left\{u \in l^q : \sum_{z\in\mathbb{Z}} V(z)|u(z)|^q < +\infty\right\}$$

endowed with the norm

$$\|u\|_X := \left(\sum_{z\in\mathbb{Z}} V(z)|u(z)|^q\right)^{\frac{1}{q}}.$$

Clearly, $V_0\|u\|_q^q \leq \|u\|_X^q$. We point out that $(X, \|\cdot\|_X)$ is a reflexive Banach space and the embedding $X \hookrightarrow l^q$ is compact in view of [5, Proposition 3].

To get our result, we have to show that the functional $\mathcal{J}_\lambda$ satisfies the Palais-Smale condition (Definition 2.1) and has the mountain-pass geometry (Theorem 2.2), that we state for our setting $(X, \|\cdot\|_X)$.

Definition 2.1. Given the reflexive Banach space X and its topological dual X^*. Then, $\mathcal{J}_\lambda : X \to \mathbb{R}$ satisfies the Palais-Smale condition if any sequence $\{u_n\}$ such that

(i) $\{\mathcal{J}_\lambda(u_n)\}$ is bounded;
(ii) $\lim_{n\to+\infty} \|\mathcal{J}_\lambda'(u_n)\|_{X^*} = 0$,

has a convergent subsequence.

We recall the mountain-pass theorem due to Pucci-Serrin [16, Theorem 1]:

Theorem 2.2. *Consider the reflexive Banach space X. Let $\mathcal{J}_\lambda \in C^1(X,\mathbb{R})$ be a functional satisfying the Palais-Smale condition and suppose that there are $\widetilde{u} \in X$ and positive real numbers γ_1, γ_2 with $\gamma_1 < \gamma_2 \leq \|\widetilde{u}\|_X$ such that*

$$\inf_{\gamma_1 \leq \|u\|_X \leq \gamma_2} \mathcal{J}_\lambda(u) = m \geq \max\{\mathcal{J}_\lambda(0), \mathcal{J}_\lambda(\widetilde{u})\}.$$

Then the functional $\mathcal{J}_\lambda$ has a critical point $\widehat{u} \in X$ with $\mathcal{J}_\lambda(\widehat{u}) \geq m$. Moreover, if $\mathcal{J}_\lambda(\widehat{u}) = m$ then $\gamma_1 \leq \|\widehat{u}\|_X \leq \gamma_2$.

To show that Theorem 2.2 applies to the functional $\mathcal{J}_\lambda$ in (2.1), we give the following auxiliary proposition.

Proposition 2.3. *If (H1), (H2), (H3) and (H5) hold, then $\mathcal{J}_\lambda : X \to \mathbb{R}$ is coercive and satisfies the Palais-Smale condition.*

Proof. We split the proof in two steps.
Claim 1: $\mathcal{J}_\lambda : X \to \mathbb{R}$ is coercive.
To be coercive, the functional $\mathcal{J}_\lambda$ has to satisfy

$$\mathcal{J}_\lambda(u) \to +\infty \quad \text{as } \|u\|_X \to +\infty.$$

We fix $\lambda > 0$ in such a way that, for all $\nu \in]0, \dfrac{V_0}{\lambda q}[$, by hypothesis (H3) there is $T > 0$ with

$$G(z,t) \leq \nu |t|^q, \quad \text{for all } |t| > T.$$

Let $\omega \in l^1$ with

$$|G(z,t)| \leq \omega(z), \quad \text{for all } k \in \mathbb{Z}, \quad |t| \leq T \quad \text{(by hypothesis (H2))}.$$

For all $u \in X$, we have

$$\begin{aligned}
\mathcal{J}_\lambda(u) &= \mathcal{L}(u) - \lambda \mathcal{G}(u) \\
&\geq \frac{\|u\|_X^q}{q} - \lambda \sum_{|u(z)| \leq T} G(z,u(z)) - \lambda \sum_{|u(z)| > T} G(z,u(z)) \\
&\geq \frac{\|u\|_X^q}{q} - \lambda \|\omega\|_1 - \lambda \nu \|u\|_q^q \\
&\geq \left(\frac{1}{q} - \lambda \frac{\nu}{V_0}\right) \|u\|_X^q - \lambda \|\omega\|_1 \quad (\text{recall } V_0 \|u\|_q^q \leq \|u\|_X^q),
\end{aligned}$$

which tends to $+\infty$ as $\|u\|_X \to +\infty$, and hence $\mathcal{J}_\lambda$ has the coercivity property.
Claim 2: $\mathcal{J}_\lambda : X \to \mathbb{R}$ has the Palais-Smale property.
From Lemma A.0.5 of Peral [15], we derive two technical conditions. Precisely, we assume that there exists $c > 0$ satisfying

$$(\phi_r(x) - \phi_r(y))(x-y) \geq c|x-y|^r \quad \text{for all } x,y \in \mathbb{R}, \text{ if } r \geq 2, \tag{2.2}$$

$$(\phi_r(x) - \phi_r(y))(x-y) \geq c(|x|+|y|)^{r-2}|x-y|^2 \quad \text{for all } x,y \in \mathbb{R}, \text{ if } 1 < r < 2. \tag{2.3}$$

We consider a sequence $\{u_n\}$ in X such that the sequence $\{\mathcal{J}_\lambda(u_n)\}$ is bounded in $\mathbb{R}$ and $\|\mathcal{J}'_\lambda(u_n)\|_{X^*} \to 0$ as $n \to +\infty$. So, Claim 1 (the coercivity of $\mathcal{J}_\lambda$) implies that our sequence $\{u_n\}$ is bounded. Moreover, the fact that X is compactly embedded on l^q, ensures that

$$u_n \rightharpoonup u \text{ in } X \quad \text{and} \quad u_n \to u \text{ in } l^q, \quad \text{for some } u \in X.$$

It follows that

$$\langle \mathcal{J}_\lambda'(u), u_n - u\rangle \to 0 \text{ as } n \to +\infty.$$

In special case (2.2) (that is, $2 \le q$), we have

$$\begin{aligned}
&\langle \mathcal{L}'(u_n) - \mathcal{L}'(u), u_n - u\rangle - \sum_{z\in\mathbb{Z}}[\Delta^2(\phi_{p_2}(\Delta^2 u_n(z-2))) - \Delta^2(\phi_{p_2}(\Delta^2 u(z-1))) \\
&\quad + a\phi_{p_1}(\Delta u_n(z-1)) - a\phi_{p_1}(\Delta u(z-1))](\Delta u_n(z-1) - \Delta u(z-1)) \\
&= V(z)[\phi_q((u_n(z)) - \phi_q(u(z))](u_n(z) - u(z)) \qquad (2.4)\\
&\ge c\|u_n - u\|_X^q.
\end{aligned}$$

The special case (2.3) (that is, $1 < q < 2$) involves a similar reasoning as above, and hence we omit the details. By easy calculations we can check that

$$\begin{aligned}
&\lim_{n\to+\infty}\sum_{z\in\mathbb{Z}}[\phi_{p_i}(\Delta u_n(z-1)) - \phi_{p_i}(\Delta u(z-1))](\Delta u_n(z-1) - \Delta u(z-1)) = 0,\\
&\lim_{n\to+\infty}\langle \mathcal{G}'(u_n) - \mathcal{G}'(u), u_n - u\rangle = 0,\\
&\lim_{n\to+\infty}\langle \mathcal{L}'(u_n) - \mathcal{L}'(u), u_n - u\rangle = 0.
\end{aligned}$$

Therefore, taking the limit as $n \to +\infty$ in (2.4), we conclude that $u_n \to u$ in X. Thus, $\mathcal{J}_\lambda : X \to \mathbb{R}$ satisfies the Palais-Smale condition. □

3. Main results

To get our goal, the following value of the parameter λ in (1.1), say $\widehat{\lambda}$, plays a crucial role. Denote

$$\widehat{\lambda} := \frac{|b|^q}{G(h,b)}\left(\frac{4}{p_2}|b|^{p_2-q} + \frac{2a}{p_1}|b|^{p_1-q} + \frac{V(h)}{q}\right),$$

where $h \in \mathbb{Z}$ and $b \in \mathbb{R}$ satisfy hypothesis (H4).

We give the existence theorem which produces two non-zero homoclinic solutions.

Theorem 3.1. *If (H1) - (H5) hold, then the problem* (1.1) *has two non-zero homoclinic solutions, for each* $\lambda > \widehat{\lambda} \ge 0$ *sufficiently large.*

Proof. We recall that $\mathcal{J}_\lambda(0) = 0$. So, we prove that $\mathcal{J}_\lambda$ has zero as strict local minimizer, for all positive real value of the parameter λ. Using hypothesis (H1) for all $\varepsilon \in]0, \frac{V_0}{\lambda q}[$, we have that there is a positive real number $\delta > 0$ for which

$$|G(z,t)| \le \varepsilon|t|^q \quad \text{for all } z \in \mathbb{Z},\ |t| \le \delta.$$

As pointed out in the preliminaries (see also [5]), we know that $X \hookrightarrow l^q \hookrightarrow l^\infty$, and hence there is a positive real number ρ such that $\delta > \|u\|_\infty$ for all u in the open ball of center zero and radius ρ. Thus, for all $u \neq 0, \|u\|_X < \rho$, we have

$$\mathcal{J}_\lambda(u) \ge \frac{\|u\|_X^q}{q} - \lambda\varepsilon\|u\|_q^q \ge \left(\frac{1}{q} - \lambda\frac{\varepsilon}{V_0}\right)\|u\|_X^q > 0.$$

Next, we prove that zero is not a global minimizer of $\mathcal{J}_\lambda$. To this aim, we set $\widehat{u} = be_h$ ($e_h(z) = \delta_{hz}$ for all $z \in \mathbb{Z}$, with $\delta_{hz} = 1$ if $h = z$, and $\delta_{hz} = 0$ otherwise).

It follows that

$$\mathcal{J}_\lambda(\widehat{u}) = \frac{4}{p_2}|b|^{p_2} + \frac{2a}{p_1}|b|^{p_1} + \frac{V(h)}{q}|b|^q - \lambda G(h,b) < 0.$$

Let $\rho \in \mathbb{R}$ be such that $\mathcal{J}_\lambda(\widehat{u}) < \rho < 0$. We consider the set $M := \{u \in X : \mathcal{J}_\lambda(u) < \rho\} \neq \emptyset$, which is bounded since the functional $\mathcal{J}_\lambda$ is coercive (see Claim 1 of Proposition 2.3). Now, $\mathcal{J}_\lambda$ is bounded from below on M. Indeed, let $\{u_n\} \subseteq M$ be a sequence for which $\mathcal{J}_\lambda(u_n) \to -\infty$ as $n \to +\infty$. The sequence $\{u_n\}$ is bounded by coercivity of $\mathcal{J}_\lambda$. So, using the compactness of $X \hookrightarrow l^q$ and continuity of $l^q \hookrightarrow l^\infty$, and by passing to a subsequence if necessary, we suppose

$$u_n \rightharpoonup u \text{ in } X \quad \text{and} \quad u_n \to u \text{ in } l^q.$$

Consequently, we get

$$\mathcal{J}_\lambda(u) \leq \liminf_{n\to+\infty} \mathcal{J}_\lambda(u_n),$$

which leads to contradiction with the fact that $\{\mathcal{J}_\lambda(u_n)\}$ is unbounded from below. Next, let $\{u_n\} \subseteq M$ be such that

$$\mathcal{J}_\lambda(u_n) \to \inf_{u\in M} \mathcal{J}_\lambda(u) = \inf_{u\in X} \mathcal{J}_\lambda(u) := \eta.$$

Again passing to a subsequence if necessary, we get

$$u_n \rightharpoonup \overline{u} \text{ in } X \quad \text{and} \quad u_n \to \overline{u} \text{ in } l^q \quad \text{for some } \overline{u} \in X.$$

We deduce that $\mathcal{J}_\lambda(\overline{u}) = \eta < 0$ which implies $\overline{u} \neq 0$. So, $\overline{u}$ is a first critical point of the functional $\mathcal{J}_\lambda$. Now, we need another critical point of $\mathcal{J}_\lambda$. Since there are numbers $\gamma_1, \gamma_2 > 0$ for which $\rho < \gamma_1 < \gamma_2 \leq \|\overline{u}\|_X$ and

$$\inf_{\gamma_1 \leq \|u\|_X \leq \gamma_2} \mathcal{J}_\lambda(u) = m \geq 0 \quad \text{(recall that zero is a local minimizer)}.$$

The mountain-pass theorem (recall Theorem 2.2) give us that $\mathcal{J}_\lambda : X \to \mathbb{R}$ has a critical point $\widehat{u} \in X$ with $m \leq \mathcal{J}_\lambda(\widehat{u})$. So, if we also recall that $\mathcal{J}_\lambda(\overline{u}) = \eta < 0$, then we conclude that $\widehat{u} \neq \overline{u}$ ($\widehat{u} \neq 0$) are two non-zero solutions of (1.1). □

Example 3.2. *The following function $g : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ satisfies hypotheses $(H1)$ - $(H4)$, but does not satisfy the Ambrosetti-Rabinowitz condition:*

$$g(z,t) = \begin{cases} f(z)|t|^{\kappa_1-2}t & \text{if } t < -1 \text{ or } t > 1, \\ f(z)|t|^{\kappa_2-2}t & \text{if } -1 \leq t \leq 1, \end{cases}$$

for which

$$G(z,t) = \begin{cases} f(z)(\frac{1}{\kappa_2} - \frac{1}{\kappa_1} + \frac{|t|^{\kappa_1}}{\kappa_1}) & \text{if } t < -1 \text{ or } t > 1, \\ \\ f(z)\frac{|t|^{\kappa_2}}{\kappa_2} & \text{if } -1 \leq t \leq 1, \end{cases}$$

where $1 < \kappa_1 < q < \kappa_2 < +\infty$ and $f : \mathbb{Z} \to \mathbb{R}_+$ is any bounded l^1-function.
We note that the boundedness of f implies that, in establishing assumptions $(H1)$ and $(H3)$, only the polynomial terms play a role. So, $(H1)$ holds true since $q < \kappa_2$ (recall that we pass to the limit as $t \to 0$), and $(H3)$ follows easily since $\kappa_1 < q$ (recall that we pass to the limit as $|t| \to +\infty$). In establishing $(H4)$, we have to consider the fact that f assumes positive values. So, $(H4)$ holds true for all $b \in [-1,1]$. Clearly, $(H2)$ holds true as f is a l^1-function.

Remark 3.3. We can suppose a more general assumption (H3)$'$, instead of (H3):

(H3)$'$ $\limsup_{|t|\to+\infty} \frac{G(z,t)}{|t|^q} \le L$ uniformly for all $z \in \mathbb{Z}$, where $L > 0$.

Then, in the proof of Claim 1 (Proposition 2.3), one can fix $\nu \in]0, \min\{L, \dfrac{V_0}{\lambda q}\}[$ and in Theorem 3.1 one can take $\lambda > \max\{\dfrac{V_0}{qL}, \widehat{\lambda}\}$.

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