CARDINAL INVARIANTS FOR THE G_δ TOPOLOGY

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Abstract. We prove upper bounds for the spread, the Lindelöf number and the weak Lindelöf number of the G_δ topology on a topological space and apply a few of our bounds to give a short proof to a recent result of Juhász and van Mill regarding the cardinality of a σ-countably tight homogeneous compactum.

1. Introduction

All spaces are assumed to be T_1. The word *compactum* indicates a compact Hausdorff space.

Given a topological space X we can consider a finer topology on X by declaring countable intersections of open subsets of X to be a base. The new space is called the G_δ *topology of* X and is denoted with X_δ.

There are various papers in the literature investigating what properties of X are preserved when passing to X_δ and presenting bounds for cardinal invariants on X_δ in terms of the cardinal invariants of X (see for example [15], [13], [21], [18]). Moreover, results of that kind have found applications to central topics in general topology like the study of covering properties in box products (see, for example, [19]), cardinal invariants for homogeneous compacta (see, for example [2], [7], [8] and [23]) and spaces of continuous functions (See [1]).

Two of the early results on this topic are Juhász's bound $c(X_\delta) \leq 2^{c(X)}$ for every compact Hausdorff space X, where $c(X)$ denotes the *cellularity* of X and Arhangel'skii's result that the G_δ topology on a Lindelöf regular scattered space is Lindelöf. Juhász's bound is tight in the sense that it's not possible to prove that $c(X_\delta) \leq c(X)^\omega$ for every compact space X (see [12]) and the scattered property is essential in Arhangel'skii's result because there are compact Hausdorff spaces whose G_δ-topology even has (weak) Lindelöf number $\mathfrak{c}^+$ (see [23]).

In this paper we prove various new bounds for cardinal invariants on the G_δ topology. For example we prove that $s(X_\delta) \leq 2^{s(X)}$ for every space

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X, where $s(X)$ is the spread of X (that is, the supremum of the cardinalities of the discrete subsets of X). For a regular space we prove that $L(X_\delta) \leq \min\{psw(X)^{d(X)}, 2^{s(X)}\}$, where $L(X)$ denotes the Lindelöf degree of X, $psw(X)$ denotes the point-separating weight of X and $d(X)$ denotes the density of X.

Many questions are left open. For example we don't know whether the inequality $t(X_\delta) \leq 2^{t(X)}$ is true, where t denotes the *tightness*, even when X is a compact space.

Finally, we exploit a few of our results to give a short proof of a recent result of Juhász and van Mill on the cardinality of homogeneous compacta.

Our notation regarding cardinal functions follows [16]. The remaining undefined notions can be found in [11].

In our proofs we often use elementary submodels of the structure $(H(\mu), \epsilon)$. Dow's survey [10] is enough to read our paper, and we give a brief informal refresher here. Recall that $H(\mu)$ is the set of all sets whose transitive closure has cardinality smaller than μ. When μ is regular uncountable, $H(\mu)$ is known to satisfy all axioms of set theory, except the power set axiom. We say, informally, that a formula is satisfied by a set S if it is true when all existential quantifiers are restricted to S. A set $M \subset H(\mu)$ is said to be an elementary submodel of $H(\mu)$ (and we write $M \prec H(\mu)$) if a formula with parameters in M is satisfied by $H(\mu)$ if and only if it is satisfied by M.

The downward Löwenheim-Skolem theorem guarantees that for every $S \subset H(\mu)$, there is an elementary submodel $M \prec H(\mu)$ such that $|M| \leq |S| \cdot \omega$ and $S \subset M$. This theorem is sufficient for many applications, but it is often useful (especially in cardinal bounds for topological spaces) to have the following closure property. We say that M is *κ-closed* if for every $S \subset M$ such that $|S| \leq \kappa$ we have $S \in M$. For large enough regular μ and for every countable set $S \subset H(\mu)$ there is always a κ-closed elementary submodel $M \prec H(\mu)$ such that $|M| = 2^\kappa$ and $S \subset M$.

The following theorem is also used often: let $M \prec H(\mu)$ such that $\kappa+1 \subset M$ and $S \in M$ be such that $|S| \leq \kappa$. Then $S \subset M$.

2. Cardinal invariants for the G_δ topology

Let's start by listing the simplest bounds for cardinal functions of the G_κ topology. They are probably folklore, and we include them just for the convenience of the reader.

Proposition 2.1.

(1) $w(X_\kappa) \leq (w(X))^\kappa$.
(2) $\chi(X_\kappa) \leq (\chi(X))^\kappa$.
(3) If X *is regular then* $d(X_\kappa) \leq 2^{d(X)\cdot\kappa}$.
(4) If X *is regular, then* $\pi w(X_\kappa) \leq 2^{\pi w(X)\cdot\kappa}$.

Proof. The first two items are easy.

As for the third item, recalling that $w(X) \leq 2^{d(X)}$ for regular spaces, we have that $d(X_\kappa) \leq w(X_\kappa) \leq w(X)^\kappa \leq 2^{d(X)\cdot\kappa}$.

To prove the fourth item, recall that $w(X) \leq (\pi w(X))^{c(X)}$ for every regular space X. Hence $\pi w(X_\kappa) \leq w(X_\kappa) \leq w(X)^\kappa \leq (\pi w(X))^{c(X)\cdot\kappa} \leq (\pi w(X))^{\pi w(X)\cdot\kappa} \leq 2^{\pi w(X)\cdot\kappa}$.

□

Regularity is essential in both the third and the fourth item, as the following example shows.

Example 2.2. A Hausdorff space X such that:

$$\pi w(X_\delta) \geq d(X_\delta) > 2^{\pi w(X)} \geq 2^{d(X)}$$

Proof. Let $X = \beta\omega$, provided with the following topology: every principal ultrafilter is isolated. A basic neighbourhood of a non-principal ultrafilter p has the form $\{p\} \cup A \setminus F$, where $A \in p$ and F is a finite set. The space X has a countable π-base, but X_δ is a discrete set of cardinality $2^{\mathfrak{c}}$. □

The following example shows that, unlike in the case of the π-weight, there is no bound on the π-character of the G_δ-topology on a regular space of countable π-character.

Example 2.3. For every cardinal κ, there is a hereditarily normal space of countable π-character $X(\kappa)$ such that $\pi\chi(X(\kappa)_\delta) \geq \kappa$.

Proof. Let $X(\kappa)$ be the space obtained by taking the sum of a convergent sequence and the one-point compactification of a discrete set of size κ and then collapsing the limit points to a single point ∞. In the resulting space, every point is isolated except for ∞, which nevertheless has a countable π-base. So $\pi\chi(X(\kappa)) = \omega$. However, $X(\kappa)_\delta$ is homeomorphic to the one-point Lindelöfication of a discrete set of size κ. So its π-character is no smaller than κ. □

One of the early results regarding cardinal invariants for the G_δ topology was proved by Juhász in [15] and was originally motivated by a problem of Arhangel'skii regarding the weak Lindelöf number of the G_δ topology on a compactum. Its proof is an application of the Erdös-Rado theorem from infinite combinatorics.

Theorem 2.4. *(Juhász,* [15]*) Let X be a countably compact regular space. Then $c(X_\delta) \leq 2^{c(X)}$.*

Note that regularity is essential in the above theorem as Vaughan [25] constructed a countably compact Hausdorff space with points G_δ and cardinality larger than the continuum which is even separable.

We also exploit the Erdös-Rado theorem in our next result. Recall that $s(X) = \sup\{|D| : D \text{ is a discrete subset of } X\}$.

Theorem 2.5. *Let X be any space and κ be a cardinal. Then $s(X_\kappa) \leq 2^{s(X)\cdot\kappa}$.*

Proof. Without loss of generality we can assume that $s(X) \leq \kappa$. Suppose by contradiction that there is a discrete set $D \subset X_\kappa$ of cardinality $\geq (2^\kappa)^+$. For every $x \in D$ we can find a G_κ set G_x in X such that $G_x \cap D = \{x\}$. Let $\{U^x_\alpha : \alpha < \kappa\}$ be a sequence of open sets such that $G_x = \bigcap\{U^x_\alpha : \alpha < \kappa\}$. Let $\prec$ be a linear ordering on X. For every $\alpha, \beta < \kappa$ let $C_{\alpha,\beta} = \{\{x, y\} \in [D]^2 : x \prec y \wedge x \notin U^y_\alpha \wedge y \notin U^x_\beta\}$. Then $\{C_{\alpha,\beta} : (\alpha, \beta) \in \kappa^2\}$ is a coloring of $[D]^2$ into κ many colors. By the Erdös-Rado theorem we can find a set $T \subset D$ of cardinality κ^+ and a pair of ordinals $(\gamma, \delta) \in \kappa^2$ such that $[T]^2 \subset C_{\gamma,\delta}$. Note now that $U^x_\gamma \cap U^x_\delta \cap T = \{x\}$ for every $x \in T$. Hence T is a discrete subset of X of cardinality κ^+, which contradicts $s(X) = \kappa$. □

Corollary 2.6. *(Hajnal and Juhász) Let X be a T_1 space. Then $|X| \leq 2^{s(X)\cdot\psi(X)}$.*

Proof. Set $\kappa = s(X) \cdot \psi(X)$. By the above theorem we have $s(X_\kappa) \leq 2^\kappa$, but since X_κ is discrete we must have $|X| \leq 2^\kappa$. □

The next example shows that $2^{s(X)\cdot\kappa}$ cannot be replaced with $s(X)^\kappa$ in Theorem 2.5, even for compact LOTS.

Example 2.7. There is a compact linearly ordered space L such that $s(L_\delta) > s(L)^\omega$.

Proof. Fleissner constructed in [12] a compact linearly ordered space L such that $c(L) \leq \mathfrak{c}$ and L has a $\mathfrak{c}^+$-sized subset S consisting of G_δ points. Since $c(X) = s(X)$ for every linearly ordered space X we must have $s(L) \leq \mathfrak{c}$, but it's clear that $s(L_\delta) \geq \mathfrak{c}^+$. □

Recall that the Lindelöf degree of a topological space X ($L(X)$) is defined as the minimum cardinal κ such that for every open cover of X has a κ-sized subcover.

The weak Lindelöf degree of X ($wL(X)$) is defined as the minimum cardinal κ such that, for every open cover $\mathcal{U}$ of X there is a κ-sized subcollection $\mathcal{V} \subset \mathcal{U}$ such that $X \subset \overline{\bigcup \mathcal{V}}$.

At the 1970 International Congress of Mathematicians in Nice, France, Arhangel'skii asked whether the weak Lindelöf degree of a compact space with its G_δ topology is always bounded by the continuum. A counterexample has recently been given in [23] but various related bounds for the (weak) Lindelöf number of the G_δ topology have been presented in the literature (see, for example [13], [21], [15] and [8]).

A set $G \subset X$ is called a G^c_κ set if there is a family $\{U_\alpha : \alpha < \kappa\}$ of open subsets of X such that $G = \bigcap \{U_\alpha : \alpha < \kappa\} = \bigcap \{\overline{U_\alpha} : \alpha < \kappa\}$.

Given a space X, we denote with X^c_κ the topology generated by the G^c_κ subsets of X. Obviously if X is regular, then $X_\kappa = X^c_\kappa$.

Theorem 2.8. *Let X be any space and κ be a cardinal. Then $L(X^c_\kappa) \leq 2^{s(X) \cdot \kappa}$.*

Proof. Without loss we can assume $s(X) \leq \kappa$. Fix a cover $\mathcal{F}$ of X by G^c_κ sets.

Let θ be a large enough regular cardinal and M be a κ-closed elementary submodel of $H(\theta)$ such that $X, \mathcal{F} \in M$, $2^\kappa + 1 \subset M$ and $|M| = 2^\kappa$.

For every $F \in \mathcal{F}$ choose open sets $\{U_\alpha(F) : \alpha < \kappa\}$ witnessing that F is a G^c_κ-set. Note that when $F \in \mathcal{F} \cap M$ we can assume that $\{U_\alpha(F) : \alpha < \kappa\} \in M$ and hence $\{U_\alpha(F) : \alpha < \kappa\} \subset M$.

Claim 1. $\mathcal{F} \cap M$ covers $\overline{X \cap M}$.

Proof of Claim 1. Suppose this is not true and let $p \in \overline{X \cap M} \setminus \bigcup(\mathcal{F} \cap M)$. For every $x \in X \cap M$ we can find $F_x \in \mathcal{F} \cap M$ such that $x \in F_x$. Moreover, there must be $\alpha(x) < \kappa$ such that $p \notin U_{\alpha(x)}(F_x)$. Now, $\mathcal{O} = \{U_{\alpha(x)}(F_x) : x \in X \cap M\}$ is an open cover of $X \cap M$. By Shapirovskii's Lemma (see [16]) there is a discrete set $D \subset X \cap M$ and a subcollection $\mathcal{U} \subset \mathcal{O}$ with $|\mathcal{U}| = |D| \leq \kappa$ such that $X \cap M \subset \overline{D} \cup \bigcup \mathcal{U}$. By κ closedness of M we have $D, \mathcal{U} \in M$ hence $M \models X \subset \overline{D} \cup \bigcup \mathcal{U}$. Therefore by elementarity $H(\theta) \models X \subset \overline{D} \cup \bigcup \mathcal{U}$. Since $p \notin \bigcup \mathcal{U}$ we must have $p \in \overline{D}$.

Let now F be an element of $\mathcal{F}$ such that $p \in F$. We have $p \in \overline{U_\alpha(F) \cap D}$ for every $\alpha < \kappa$ and $\overline{U_\alpha(F) \cap D} \in M$, by κ-closedness of M. Define $B = \bigcap \{\overline{U_\alpha(F) \cap D} : \alpha < \kappa\}$. Then $B \in M$. Note that we have $H(\theta) \models (\exists G \in$

$\mathcal{F})(B \subset G)$, hence by elementarity $M \models (\exists G \in \mathcal{F})(B \subset G)$, which implies the existence of $H \in \mathcal{F} \cap M$ such that $p \in B \subset H$. But this contradicts the fact that $p \notin \bigcup(\mathcal{F} \cap M)$. Hence $\mathcal{F} \cap M$ covers $\overline{X \cap M}$ and the claim is proved.

△

Claim 2. $\mathcal{F} \cap M$ covers X.

Proof of Claim 2. Suppose this is not true and let p be a point of $X \setminus \bigcup(\mathcal{F} \cap M)$. For every $F \in \mathcal{F} \cap M$ we can find $\beta(F) < \kappa$ such that $p \notin U_{\beta(F)}(F)$.

It follows from Claim 1 that the family $\mathcal{V} := \{U_{\beta(F)}(F) : F \in \mathcal{F} \cap M\}$ is an open cover of $\overline{X \cap M}$. By Shapirovskii's Lemma we can find a discrete $D \subset \overline{X \cap M}$ and a family $\mathcal{W} \subset \mathcal{V}$ such that $|\mathcal{W}| = |D| < \kappa$ and $X \cap M \subset \overline{X \cap M} \subset \overline{D} \cup \bigcup \mathcal{W}$. Note that $D, \mathcal{W} \in M$, by $< \kappa$-closedness of M. This implies that $M \models X \subset \overline{D} \cup \bigcup \mathcal{W}$ and hence $H(\theta) \models X \subset \overline{D} \cup \bigcup \mathcal{W}$ by elementarity. But this is a contradiction because $p \notin W$, for every $W \in \mathcal{W}$ and since $\overline{D} \subset \overline{X \cap M}$ we also have that $p \notin \overline{D}$. △

Since $|M| \leq 2^\kappa$ it follows that $\mathcal{F} \cap M$ is a 2^κ-sized subfamily of $\mathcal{F}$ covering X and hence we are done.

□

It's not possible to replace X^c_δ with X_δ in the above result, as the following example shows.

Example 2.9. There are T_1 spaces X of countable spread where $L(X_\delta)$ can be arbitrarily large.

Proof. Let κ be a cardinal of uncountable cofinality and $\mu = cf(\kappa)$. Define a topology on $X = \kappa$ by declaring sets of the form $[0, \alpha] \setminus F$ to be a base, where α is an ordinal less than κ and F is a finite set. It is easy to see that $s(X) = \omega$. Moreover $\{[0, \alpha] : \alpha < \kappa\}$ is an open cover of X without subcovers of cardinality less than μ and hence $L(X_\delta) \geq \mu$. □

However, for regular spaces, the G_δ modification and the G^c_δ modification coincide, so we obtain the following result:

Theorem 2.10. *Let X be a regular space. Then $L(X_\kappa) \leq 2^{s(X) \cdot \kappa}$.*

Recall that the tightness of a point x in the space X ($t(x, X)$) is defined as the minimum cardinal κ such that for every subset A of X with $x \in \overline{A} \setminus A$ there is a subset $B \subset A$ such that $|B| \leq \kappa$ and $x \in \overline{B}$. The tightness of the space X is then defined as $t(X) = \sup\{t(x, X) : x \in X\}$. A space of countable tightness is also called *countably tight*.

Recall that for a topological space X, the cardinal invariant $wL_c(X)$ is defined as the minimum cardinal κ such that, for every closed set $F \subset X$ and for every open family $\mathcal{U}$ covering F there is a κ-sized subfamily $\mathcal{V} \subset \mathcal{U}$ such that $F \subset \overline{\bigcup \mathcal{V}}$.

Theorem 2.11. *Let X be a countably compact space with a dense set of points of countable character. Then $wL(X^c_\kappa) \leq 2^{t(X)\cdot wL_c(X)\cdot \kappa}$.*

Proof. Without loss of generality we can assume that $wL_c(X) \cdot t(X) \leq \kappa$. Fix a cover $\mathcal{F}$ of X by G^c_κ sets

Let θ be a large enough regular cardinal and M be a κ-closed elementary submodel of $H(\theta)$ such that $X, \mathcal{F} \in M$ and $|M| = 2^\kappa$.

For every $F \in \mathcal{F}$ choose open sets $\{U_\alpha : \alpha < \kappa\}$ witnessing that F is a G^c_κ set.

Claim 1. $\mathcal{F} \cap M$ covers $\overline{X \cap M}$.

Proof of Claim 1. Let $x \in \overline{X \cap M}$ and use $t(X) \leq \kappa$ to fix a κ-sized set $A \subset X \cap M$ such that $x \in \overline{A}$. Note $A \in M$. Let $F \in \mathcal{F}$ be such that $x \in F$ and let $\{U_\alpha : \alpha < \kappa\}$ be a sequence of open sets witnessing that F is a G^c_κ set.

Note that the set $B = \bigcap \{\overline{A \cap U_\alpha} : \alpha < \kappa\}$ is in M and $x \in B \subset F$. Now $H(\theta) \models (\exists F \in \mathcal{F})(B \subset F)$. Hence $M \models (\exists F \in \mathcal{F})(B \subset F)$. Therefore we can find $G \in \mathcal{F} \cap M$ such that $x \in B \subset G$, which is what we wanted. △

Claim 2. $\mathcal{F} \cap M$ has dense union in X.

Proof of Claim 2. Suppose not and let $p \in X \setminus \overline{\bigcup \mathcal{F} \cap M}$ be a point of countable character. Fix a local base $\{V_n : n < \omega\}$ at p.

For every $x \in \overline{X \cap M}$ pick $F_x \in \mathcal{F} \cap M$ such that $x \in F_x$ and let $\{V^x_\alpha : \alpha < \kappa\} \in M$ be a sequence of open sets witnessing that F_x is a G^c_κ set. Since $p \notin F_x$, there must be $\alpha < \kappa$ such that $p \notin \overline{V^x_\alpha}$. Hence there must be $n_x < \omega$ such that $V_{n_x} \cap V^x_\alpha = \emptyset$. let $U_n = \bigcup \{V^x_\alpha : n_x = n\}$. Then $\{U_n : n < \omega\}$ is a countable open cover of the countably compact space $\overline{X \cap M}$. So there is $k < \omega$ such that $\{U_n : n < k\}$ covers $\overline{X \cap M}$. Let now $\mathcal{U} = \{U^x_\alpha : n_x < k\}$. Then $\mathcal{U}$ covers $\overline{X \cap M}$, hence $wL_c(X) \leq \kappa$ implies the existence of $\mathcal{V} \in [\mathcal{U}]^\kappa$ such that $\overline{X \cap M} \subset \overline{\bigcup \mathcal{V}}$. But that implies $M \models X \subset \overline{\bigcup \mathcal{V}}$ and hence $H(\theta) \models X \subset \overline{\bigcup \mathcal{V}}$, which contradicts $V_k \cap (\bigcup \mathcal{V}) = \emptyset$. △

□

Corollary 2.12. *(Alas, [3]) Let X be a countably compact T_2 space with a dense set of points of countable character. Then $|X| \leq 2^{\psi_c(X) t(X) wL_c(X)}$.*

Corollary 2.13. *Let X be a regular countably compact space with a dense set of points of countable character. Then $wL(X_\kappa) \leq 2^{wL_c(X)\cdot t(X)\cdot\kappa}$.*

Corollary 2.14. *Let X be a normal countably compact space with a dense set of points of countable character. Then $wL(X_\kappa) \leq 2^{wL(X)\cdot t(X)\cdot\kappa}$.*

In a similar way we can prove the following theorem:

Theorem 2.15. *Let X be a space with a dense set of isolated points. Then $wL(X^c_\kappa) \leq 2^{wL_c(X)\cdot t(X)\cdot\kappa}$.*

Question 1. *Is it true that $wL(X^c_\kappa) \leq 2^{wL_c(X)\cdot t(X)\cdot\kappa}$ for any Hausdorff space X?*

The referee pointed out that a positive answer to the above question would lead to a common generalization of the Hajnal-Juhász inequality $|X| \leq 2^{c(X)\cdot\chi(X)}$ and the Arhangel'skii-Shapirovskii bound $|X| \leq 2^{L(X)\cdot\psi(X)\cdot t(X)}$. No such generalization is known at the moment.

We call a cover $\mathcal{U}$ of a space X, strongly point-separating if $\bigcap\{\overline{U} : U \in \mathcal{U} \wedge x \in U\} = \{x\}$.

We define $psw_s(X)$ to be the least cardinal κ such that X admits a strongly point-separating open cover of order κ. Obviously $psw_s(X) = psw(X)$ for every regular space X.

Theorem 2.16. *Let X be a T_2 space. Then $L(X_\kappa) \leq psw_s(X)^{L(X)\cdot\kappa}$.*

Proof. Let $\lambda = psw_s(X)$ and fix a strongly point-separating open cover $\mathcal{U}$ of X of order $\leq \lambda$. We can assume $L(X) \leq \kappa$. Let $\mathcal{F}$ be a G_κ cover of X. Since $L(X) \leq \kappa$ we can assume that $\mathcal{F}$ is made up of κ-sized intersections of elements of $\mathcal{U}$. Let M be a κ-closed elementary submodel of $H(\theta)$ such that $\lambda^\kappa \subset M$, $X, \mathcal{U}, \mathcal{F} \in M$ and $|M| = \lambda^\kappa$.

Claim 1. $\mathcal{F} \cap M$ covers $\overline{X \cap M}$.

Proof of Claim 1. Let $p \in \overline{X \cap M}$. Let $F \in \mathcal{F}$ be such that $p \in F$. Let $\{U_\alpha : \alpha < \kappa\} \subset \mathcal{U}$ be a family of open sets such that $F = \bigcap\{U_\alpha : \alpha < \kappa\}$. Let x_α be any point in $U_\alpha \cap M$. Note that for every $\alpha < \kappa$ we have that $\{U \in \mathcal{U} : x_\alpha \in U\}$ is an element of M of cardinality λ. Therefore $\{U \in \mathcal{U} : x_\alpha \in U\} \subset M$ and hence $U_\alpha \in M$, for every $\alpha < \kappa$. By κ-closedness of M we have $F = \bigcap\{U_\alpha : \alpha < \kappa\} \in M$, as we wanted. △

Claim 2. $\mathcal{F} \cap M$ actually covers X.

Proof of Claim 2. Suppose that is not true and let $p \in X \setminus \bigcup(\mathcal{F} \cap M)$. For every $x \in \overline{X \cap M}$, let $F_x \in \mathcal{F} \cap M$ such that $x \in F_x$ and let $\{U^x_\alpha :$

$\alpha < \kappa\} \in M$ be a sequence of open sets such that $\bigcap\{U^x_\alpha : \alpha < \kappa\} = F_x$. We again have that $\{U^x_\alpha : \alpha < \kappa\} \subset M$ and hence, for every $x \in \overline{X \cap M}$ we can find an open neighbourhood $U_x \in M$ of x such that $p \notin U_x$. The family $\mathcal{V} := \{U_x : x \in \overline{X \cap M}\}$ is an open cover of the space $\overline{X \cap M}$, which has Lindelöf number at most κ and hence we can find $\mathcal{C} \in [\mathcal{V}]^\kappa$ such that $X \cap M \subset \overline{X \cap M} \subset \bigcup \mathcal{C}$. By κ-closedness of M we have $\mathcal{C} \in M$ and hence the previous formula implies $M \models X \subset \bigcup \mathcal{C}$. By elementarity we get that $H(\theta) \models X \subset \bigcup \mathcal{C}$, which contradicts the fact that $p \notin \bigcup \mathcal{C}$. △

□

Corollary 2.17. *Let X be a regular space. Then $L(X_\kappa) \leq psw(X)^{L(X) \cdot \kappa}$.*

Question 2. *Is $t(X_\delta) \leq 2^{t(X)}$ true for every (compact) T_2 space X?*

3. An application to homogeneous compacta

Definition 3.1. Let X be a topological space. A set $S \subset X$ is called *subseparable* if there is a countable set $C \subset X$ such that $S \subset \overline{C}$.

Since $w(X) \leq 2^{d(X)}$ for every regular space X and the weight is hereditary, every subseparable subspace of a regular topological space has weight at most continuum.

Lemma 3.2. *(Juhász and van Mill,* [17]*) Let X be a σ-countably tight homogeneous compactum. Then X contains a non-empty subseparable G_δ-subset and has a point of countable π-character.*

Corollary 3.3. *Every σ-countably tight homogeneous compactum has character at most continuum.*

Proof. Let $x \in X$ be any point. By homogeneity we can find a subseparable G_δ set G containing x. Then $w(G) \leq 2^\omega$. So we can fix a continuum-sized family $\mathcal{U}$ of open neighbourhoods of x such that $G \cap \bigcap \mathcal{U}_x = \{x\}$. Let $\{U_n : n < \omega\}$ be a countable family of open sets such that $G = \bigcap\{U_n : n < \omega\}$. Then $\mathcal{V} = \{U_n : n < \omega\} \cup \mathcal{U}$ is a continuum sized family of open subsets of X such that $\bigcap \mathcal{V} = \{x\}$. Since X is compact, this implies that $\chi(x, X) \leq 2^\omega$. □

Theorem 3.4. *Let X be a homogeneous compactum which is the union of countably many countably tight dense subspaces. Then $L(X_\delta) \leq 2^\omega$.*

Proof. Let $\{X_n : n < \omega\}$, be a countable family of countably tight subspaces covering X. Let $\mathcal{U}$ be a G_δ-cover of X. Without loss we can assume that for every $U \in \mathcal{U}$ there are open sets $\{O_n(U) : n < \omega\}$ such that $\overline{O_{n+1}(U)} \subset O_n(U)$, for every $n < \omega$ and $U = \bigcap\{O_n(U) : n < \omega\}$.

Let θ be a large enough regular cardinal and let M be an ω-closed elementary submodel of $H(\theta)$ such that $|M| = 2^\omega$ and M contains everything we need.

Claim. $\mathcal{U} \cap M$ covers $\overline{X \cap M}$.

Proof of Claim. Let $x \in \overline{X \cap M}$.

We claim that $x \in \overline{X_n \cap M}$, for every $n < \omega$. Indeed, fix $n < \omega$ and let V be a neighbourhood of x. Pick $y \in V \cap (X \cap M)$. Then y has a local base $\mathcal{U}_y \in M$ having cardinality continuum. By the assumptions on $M, \mathcal{U}_y \subset M$. Since X_n is dense in X, M reflects this and therefore for every $U \in \tau \cap M$ we have $U \cap X_n \cap M \neq \emptyset$. Hence for every $U \in \mathcal{U}_y$ we have $U \cap X_n \cap M \neq \emptyset$. It turns out that $V \cap X_n \cap M \neq \emptyset$, for every open neighbourhood V of x, as we wanted.

Let $k < \omega$ be such that $x \in X_k$. Using the fact that X_k has countable tightness we can choose a countable set $C_k \subset X_k \cap M$ such that $x \in \overline{C_k}$. Note that, since M is countably closed, every subset of C is an element of M. Since $\mathcal{U}$ covers X there is $U \in \mathcal{U}$ such that $x \in U$. Note that $x \in \overline{O_i(U) \cap C}$, for every $i < \omega$. Let $B = \bigcap\{\overline{O_i(U) \cap C} : i < \omega\}$ and note that $B \in M$. We have $H(\theta) \models (\exists U \in \mathcal{U})(B \subset U)$. Since every free variable in the previous formula belongs to M, by elementarity we have $M \models (\exists U \in \mathcal{U})(B \subset U)$ and hence there is $U \in \mathcal{U} \cap M$ such that $x \in B \subset U$, which finishes the proof of the Claim. △

Let us now prove that $\mathcal{U} \cap M$ actually covers X, which will finish the proof.

Suppose this is not the case and let $p \in X \setminus \bigcup(\mathcal{U} \cap M)$. By the claim, for every $x \in \overline{X \cap M}$ we can pick a $U_x \in \mathcal{U} \cap M$ containing x. Then we can choose $m < \omega$ such that $p \notin O_m(U_x) \in M$. This means that we can cover $\overline{X \cap M}$ by an open family $\mathcal{V} \subset M$ such that $p \notin \bigcup \mathcal{V}$. By compactness we can then take a finite subfamily $\mathcal{F}$ of $\mathcal{U}$ such that $X \cap M \subset \bigcup \mathcal{F}$. Since $\mathcal{F} \in M$ this is equivalent to $M \models X \subset \bigcup \mathcal{F}$, which implies, by elementarity, $H(\theta) \models X \subset \bigcup \mathcal{F}$, and that is a contradiction because $p \in H(\theta) \setminus \bigcup \mathcal{F}$.

□

Lemma 3.5. *Let X be a compact homogeneous space which is the union of finitely many countably tight subspaces. Then $L(X_\delta) \leq 2^\omega$.*

Proof. Let $\mathcal{F}$ be a finite cover of X by countably tight subspaces. We can find a non-empty open subset V of X such that $V \cap F$ is dense in V, whenever

$V \cap F \neq \emptyset$ and $F \in \mathcal{F}$. Applying the argument proving Lemma 3.4 to $\overline{V}$ we obtain that $L(\overline{V}_\delta) \leq 2^{\aleph_0}$. Using the homogeneity of X we can find an open cover $\mathcal{V}$ of X such that $L(\overline{V}_\delta) \leq 2^{\aleph_0}$, for every $V \in \mathcal{V}$. Choosing a finite subcover of $\mathcal{V}$ we see that $L(X_\delta) \leq 2^\omega$: □

The following lemma was noted independently by de la Vega and Ridderbos (see [6] and [22] for the proof of much more general statements).

Lemma 3.6. *Let X be a homogeneous space. Then $|X| \leq d(X)^{\pi\chi(X)}$.*

Theorem 3.7. *(Juhász and van Mill) Let X be a compact homogeneous space which is the union of countably many dense countably tight subspaces or of finitely many countably tight subspaces. Then $|X| \leq 2^\omega$.*

Proof. Use homogeneity to fix, for every $x \in X$, a subseparable G_δ set G_x containing x. We have $w(G_x) \leq 2^\omega$. Note that $\mathcal{U} = \{G_x : x \in X\}$ is a G_δ cover of X, so there is $C \in [X]^{2^\omega}$ such that $X \subset \bigcup\{G_x : x \in C\}$. For every $x \in C$, we can fix a continuum-sized $D_x \subset G_x$, dense in G_x. Then $D = \bigcup\{D_x : x \in C\}$ is a dense subset of X having cardinality at most continuum, proving that $d(X) \leq 2^\omega$. Using the above lemmas we obtain that $|X| \leq 2^\omega$. □

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