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Nonlinear elliptic p-Laplacian equations in the whole space

Eleonora Amoroso [a], Gabriele Bonanno [a,*], Kanishka Perera [b]

[a] *Department of Engineering, University of Messina, 98166 Messina, Italy*
[b] *Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901, USA*

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ABSTRACT

In this paper the existence and multiplicity of non-zero solutions for nonlinear Dirichlet problems involving the p-Laplacian operator and which are defined in the whole space is established. In particular, the existence of two non-zero solutions, one with negative energy and other with positive one for equations having combined effects of concave and convex nonlinearities is obtained. The approach is based on variational methods.

1. Introduction

This paper is devoted to the study of existence and multiplicity of solutions for the following problem

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda f(x,u) \quad \text{in} \quad \mathbb{R}^N, \\ \lim\limits_{|x|\to+\infty} u(x) = 0; \end{cases}$$

where $\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian differential operator, $p > N$, where N is the dimension of the space, λ is a real positive parameter, $a : \mathbb{R}^N \to \mathbb{R}$ is such that $a_- = \operatorname{ess\,inf}_{\mathbb{R}^N} a > 0\,, a \in L^\infty(\mathbb{R}^N)$, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is L^1-Carathéodory. Precisely, we establish the existence of one non-zero solution when the primitive of the nonlinearity $f(x,u)$ with respect to u has a growth which is less than p-linearity in a precise set $[c_1, c_2]$ (see (b) of Theorem 3.1) and if, in addition, it satisfies the classical Ambrosetti–Rabinowitz condition (that is, (AR) in Lemma 2.4), a distinct second non-zero solution is obtained. It is worth noticing that the above growth is satisfied when the nonlinearity f is $(p-1)-$sublinear at zero uniformly with respect to x and, as it is already known, the (AR) condition implies

* Corresponding author.
E-mail addresses: eleonora.amoroso@unime.it (E. Amoroso), bonanno@unime.it (G. Bonanno), kperera@fit.edu (K. Perera).

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the $(p-1)-$superlinearity at infinity. This situation contains the combined effects of convex and concave nonlinearities, as shown by the following particular case of our results, given here as an example.

Theorem 1.1. *Let $\alpha \in L^1(\mathbb{R}^N)$ be a nonnegative function and s, q two nonnegative constants such that $s+1 < p < q+1$. Then, there is $\eta^* > 0$ such that for each $\eta \in]0, \eta^*[$ the problem*

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \alpha(x)\eta u^s + \alpha(x)u^q \quad \text{in} \quad \mathbb{R}^N, \\ \lim\limits_{|x|\to+\infty} u(x) = 0; \end{cases}$$

admits at least two non-zero and nonnegative distinct solutions.

There is a wide literature on elliptic p-Laplacian problems in bounded domains while it is less abundant for equations in the whole space (we refer to [1,2,9,10,12] and references therein for a general overview). This is due to the lack of compactness of the embedding of Sobolev spaces into suitable spaces and therefore the difficulty of verifying the condition of Palais–Smale, that is $(PS)-$condition recalled in Section 2. To the best of our knowledge, there are no results in the case $p > N$ in the whole space and the purpose of this note is to fill this gap. In this case, thanks to Morrey's Theorem and Cantor's diagonal process, despite the lack of compactness of the embedding, it is possible to prove the Palais Smale condition of the functional (see proof of Lemma 2.3) and thus obtain results in line with the bounded domains. In particular, as seen with Theorem 1.1, concave–convex nonlinearities can be studied and, in this case, it is worth noting that thanks to [5, Theorem 2.1], which is one of our approaches, we avoid using regularity results and lower–upper solution method which are usually the main tool for solving problems of this type.

This paper is organized as follows. In Section 2, we give some auxiliary results to insert the problem in a variational framework. Precisely, we provide a numerical value of the constant of the Sobolev–Morrey inequality and prove both a weak Palais–Smale condition and Palais–Smale condition of the energy functional associated to the problem. In Section 3, we present our main results which guarantee the existence of one non-zero solution to our problem under a suitable behavior of the nonlinear term, and the existence of a second distinct non-zero solution when also $(AR)-$condition is assumed.

2. Variational setting and preliminaries

Given the Euclidean space $\mathbb{R}^N$ and fixed $p > N$, denote with $W^{1,p}(\mathbb{R}^N)$ the usual Sobolev space endowed with the norm

$$\|u\| = \|u\|_{L^p} + \|\nabla u\|_{L^p}.$$

The Morrey Theorem (see [6, Theorem 9.12, p.282]) ensures that there is a constant $K > 0$ such that

$$\|u\|_\infty \leq K\|u\| \qquad \forall u \in W^{1,p}(\mathbb{R}^N),$$

that is, the space $L^\infty(\mathbb{R}^N)$ is continuously embedded in $W^{1,p}(\mathbb{R}^N)$.

For our aim is useful to obtain an explicit numerical value of K for which the previous inequality holds. To this end, we recall the following result for bounded convex domains deduced from Burenkov–Gusakov [8, Theorem 1, p.1293] (see also [7, Remark 33, p.184]).

Proposition 2.1. *Let Ω be a convex open bounded set in $\mathbb{R}^N$ with a generally regular boundary $\partial\Omega$. Fix $x \in \Omega$ and put $d_x = \sup\Big\{d(x,y) : y \in \partial\Omega\Big\}$. Then, one has*

$$|u(x)| \leq C_x\|u\| \quad \forall u \in C^1(\Omega),$$

where

$$C_x = \frac{1}{|\Omega|^{1/p}} \max\left\{1; \frac{d_x}{N^{1/p}}\left(\frac{p-1}{p-N}\right)^{(p-1)/p}\right\}$$

and $\|u\|$ *is the usual norm in the Sobolev space* $W^{1,p}(\Omega)$, *that is,* $\|u\| = \|u\|_p + \|\nabla u\|_p$.

Proof. Arguing as in the proof of Theorem 1 of [8] we obtain

$$|u(x)| \le \frac{1}{|\Omega|^{1/p}}\left(\|u\|_{L^p} + \frac{d_x}{N^{1/p}}\left(\frac{p-1}{p-N}\right)^{(p-1)/p}\|\nabla u\|_{L^p}\right).$$

Hence, the conclusion follows. □

As a consequence of Proposition 2.1 we point out the following result.

Proposition 2.2. *Put*

$$K = \left[\frac{\Gamma\left(1+\frac{N}{2}\right)}{\pi^{N/2}}\right]^{1/p} \frac{1}{N^{N/p^2}}\left(\frac{p-1}{p-N}\right)^{\frac{N(p-1)}{p^2}}.$$

Then one has

$$\|u\|_\infty \le K\|u\| \quad \forall u \in W^{1,p}(\mathbb{R}^N).$$

Proof. Fix $u \in C_0^1(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$ and consider the ball $B(x,r_0)$ of center x and radius $r_0 = N^{1/p}\left(\frac{p-N}{p-1}\right)^{\frac{p-1}{p}}$. From Proposition 2.1, applied to $\Omega = B(x,r_0)$ and taking into account that $d_x = r_0$, one has

$$\begin{aligned}
|u(x)| &\le \frac{1}{|B(x,r_0)|^{1/p}} \max\left\{1; \frac{r_0}{N^{1/p}}\left(\frac{p-1}{p-N}\right)^{(p-1)/p}\right\}(\|u\|_{L^p} + \|\nabla u\|_{L^p}) \\
&= \left[\frac{\Gamma(1+N/2)}{\pi^{N/2}}\left(\frac{1}{N^{1/p}}\left(\frac{p-1}{p-N}\right)^{\frac{p-1}{p}}\right)^N\right]^{1/p}(\|u\|_{L^p} + \|\nabla u\|_{L^p}) \\
&= K\left(\left(\int_{B(x,r_0)} |u(t)|^p dt\right)^{1/p} + \left(\int_{B(x,r_0)} |\nabla u(t)|^p dt\right)^{1/p}\right) \\
&\le K\left(\left(\int_{\mathbb{R}^N} |u(t)|^p dt\right)^{1/p} + \left(\int_{\mathbb{R}^N} |\nabla u(t)|^p dt\right)^{1/p}\right),
\end{aligned}$$

that is,

$$|u(x)| \le K\left(\left(\int_{\mathbb{R}^N} |u(t)|^p dt\right)^{1/p} + \left(\int_{\mathbb{R}^N} |\nabla u(t)|^p dt\right)^{1/p}\right).$$

Hence, from the arbitrary of x and the density of $C_0^1(\mathbb{R}^N)$ in $W^{1,p}(\mathbb{R}^N)$, the conclusion is achieved. □

Remark 2.1. We explicitly observe that from the classical proof of Morrey's Theorem a numerical value of the embedding constant can be easily deduced (see for instance [6, p. 283]), which is

$$\frac{p}{p-N}.$$

Moreover, an explicit value is reported in [11, Lemma II.3.4, formula II.3.13, p. 56], which is

$$\left[\frac{\Gamma\left(1+\frac{N}{2}\right)}{\pi^{N/2}}\right]^{1/p} \max\left\{1, \left(\frac{p-1}{p-N}\right)^{(p-1)/p}\right\}.$$

Clearly, the value provided by Proposition 2.2 is more precise than the previous values. We note that, as a consequence, the value η^* guaranteed in Theorem 1.1 is numerically greater than the one obtained through the aforementioned estimates (see Remark 3.3).

In this paper, we consider $X = W^{1,p}(\mathbb{R}^N)$ endowed with the equivalent norm

$$\|u\|_X = \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p dx + \int_{\mathbb{R}^N} a(x)|u(x)|^p dx\right)^{\frac{1}{p}},$$

where $a \in L^\infty(\mathbb{R}^N)$ is such that $a_- = \operatorname{ess\,inf}_{\mathbb{R}^N} a > 0$.

We now adapt Proposition 2.2 to the previous norm. Precisely, we have the following result which we will use in the sequel.

Lemma 2.1. *One has*

$$\|u\|_\infty \le L_a \|u\|_X \qquad \forall u \in W^{1,p}(\mathbb{R}^N), \tag{2.1}$$

where

$$L_a = \left(\frac{1}{a_-}\right)^{\frac{p-N}{p^2}} 2^{\frac{p-1}{p}} \left[\frac{\Gamma\left(1+\frac{N}{2}\right)}{\pi^{N/2}}\right]^{1/p} \frac{1}{N^{N/p^2}} \left(\frac{p-1}{p-N}\right)^{\frac{N(p-1)}{p^2}}. \tag{2.2}$$

Proof. As seen in the proof of Proposition 2.1 we have

$$\begin{aligned}
|u(x)| \le & \frac{1}{|\Omega|^{1/p}}\left(\|u\|_{L^p} + \frac{d_x}{N^{1/p}}\left(\frac{p-1}{p-N}\right)^{(p-1)/p} \|\nabla u\|_{L^p}\right) \\
= & \frac{1}{|\Omega|^{1/p}}\left(\left(\int_\Omega |u(t)|^p dt\right)^{1/p} + \frac{d_x}{N^{1/p}}\left(\frac{p-1}{p-N}\right)^{(p-1)/p}\left(\int_\Omega |\nabla u(t)|^p dt\right)^{1/p}\right) \\
\le & \frac{1}{|\Omega|^{1/p}}\left(\frac{1}{a_-^{1/p}}\left(\int_\Omega a(t)|u(t)|^p dt\right)^{1/p} + \frac{d_x}{N^{1/p}}\left(\frac{p-1}{p-N}\right)^{(p-1)/p}\left(\int_\Omega |\nabla u(t)|^p dt\right)^{1/p}\right) \\
\le & \frac{1}{|\Omega|^{1/p}} \max\left\{\frac{1}{(a_-)^{1/p}}; \frac{d_x}{N^{1/p}}\left(\frac{p-1}{p-N}\right)^{(p-1)/p}\right\} \times \\
& \times \left(\left(\int_\Omega a(t)|u(t)|^p dt\right)^{1/p} + \left(\int_\Omega |\nabla u(t)|^p dt\right)^{1/p}\right).
\end{aligned}$$

At this point, arguing as in Proposition 2.2 by choosing $\Omega = B(x, r_0)$, $d_x = r_0$ and $r_0 = \left(\frac{N}{a_-}\right)^{1/p}\left(\frac{p-N}{p-1}\right)^{\frac{p-1}{p}}$, one has

$$\begin{aligned}
|u(x)| \le & \frac{1}{|B(x,r_0)|^{1/p}} \frac{1}{(a_-)^{1/p}}\left(\left(\int_{B(x,r_0)} a(t)|u(t)|^p dt\right)^{1/p} + \left(\int_{B(x,r_0)} |\nabla u(t)|^p dt\right)^{1/p}\right) \\
= & \left[\frac{\Gamma(1+N/2)}{\pi^{N/2}}\left((a_-)^{1/p}\frac{1}{N^{1/p}}\left(\frac{p-1}{p-N}\right)^{\frac{p-1}{p}}\right)^N\right]^{1/p} \frac{1}{(a_-)^{1/p}} \times
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\left(\int_{B(x,r_0)} a(t)|u(t)|^p dt \right)^{1/p} + \left(\int_{B(x,r_0)} |\nabla u(t)|^p dt \right)^{1/p} \right) \\
\leq & 2^{\frac{p-1}{p}} \left[\frac{\Gamma(1+N/2)}{\pi^{N/2}} \left(\frac{1}{(a_-)^{\frac{p-N}{pN}}} \frac{1}{N^{1/p}} \left(\frac{p-1}{p-N} \right)^{\frac{p-1}{p}} \right)^N \right]^{1/p} \times \\
&\times \left(\int_{B(x,r_0)} a(t)|u(t)|^p dt + \int_{B(x,r_0)} |\nabla u(t)|^p dt \right)^{1/p} \\
&\leq L_a \left(\int_{\mathbb{R}^N} a(t)|u(t)|^p dt + \int_{\mathbb{R}^N} |\nabla u(t)|^p dt \right)^{1/p} .
\end{aligned}
$$

Hence, arguing again as in the proof of Proposition 2.2, we obtain

$$
\|u\|_\infty \leq L_a \left(\int_{\mathbb{R}^N} a(t)|u(t)|^p dt + \int_{\mathbb{R}^N} |\nabla u(t)|^p dt \right)^{1/p}
$$

for all $u \in W^{1,p}(\mathbb{R}^N)$, that is, the conclusion. □

Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function, that is a function such that

(i) $x \to f(x,t)$ is measurable for all $t \in \mathbb{R}$;
(ii) $x \to f(x,t)$ is continuous for almost every $x \in \mathbb{R}^N$;
(iii) for all $\rho > 0$ the function $\sup_{|t|\leq\rho} |f(\cdot,t)|$ belongs to $L^1(\mathbb{R}^N)$.

Consider the following nonlinear differential problem on the entire space

$$
\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda f(x,u) & \text{in} \quad \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \end{cases} \tag{P_λ}
$$

with $\lambda > 0$. We recall that $u \in X$ is a *weak solution* of problem (P_λ) if

$$
\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2} \nabla u \nabla v + a(x)|u|^{p-2}uv \right) dx = \lambda \int_{\mathbb{R}^N} f(x,u)v \, dx \,, \tag{2.3}
$$

for all $v \in X$. In order to find this type of solution for problem (P_λ), put

$$
F(x,t) = \int_0^t f(x,\xi) d\xi \qquad \forall \, (x,t) \in \mathbb{R}^N \times \mathbb{R}.
$$

Thanks to the assumptions on the function f, one has

(i') $x \to F(x,t)$ is measurable for all $t \in \mathbb{R}$;
(ii') $t \to F(x,t)$ belongs to $C^1(\mathbb{R}^N)$ for almost every $x \in \mathbb{R}^N$;
(iii') for all $\rho > 0$ the function $\sup_{|t|\leq\rho} |F(\cdot,t)|$ belongs to $L^1(\mathbb{R}^N)$.

Moreover, we define Φ, $\Psi : X \to \mathbb{R}$ by

$$
\Phi(u) = \frac{1}{p} \|u\|_X^p \,, \quad \Psi(u) = \int_{\mathbb{R}^N} F(x,u(x)) dx \,, \quad \forall \, u \in X,
$$

and

$$
I_\lambda(u) = \Phi(u) - \lambda \Psi(u) \qquad \forall \, u \in X, \forall \, \lambda > 0.
$$

It is well known that Φ and Ψ are Gâteaux differentiable and one has

$$\Phi'(u)(v) = \int_{\mathbb{R}^N} |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\,dx + \int_{\mathbb{R}^N} a(x)|u(x)|^{p-2}u(x)v(x)\,dx\,,$$
$$\Psi'(u)(v) = \int_{\mathbb{R}^N} f(x,u(x))v(x)\,dx,$$

for all $u, v \in X$. Hence, it follows that u is a critical point of I_λ, namely $I'_\lambda(u)(v) = 0$ for every $v \in X$, if and only if u is a weak solution for problem (P_λ), see (2.3). Noticing that we present results on the existence of non-negative solutions, we can assume, without loss of generality, that one has

$$f(x,t) = f(x,0) \qquad \forall t \le 0, \forall x \in \mathbb{R}^N.$$

Indeed, we have the following proposition.

Lemma 2.2. *Assume that $f(x,0) \ge 0$ for a.e. $x \in \Omega$. Then, any weak solution of problem (P_λ) is nonnegative.*

Proof. Let $u \in X$ be a weak solution of (P_λ). One has

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u\nabla v + a(x)|u|^{p-2}uv\right)dx = \lambda\int_{\mathbb{R}^N} f(x,u)v\,dx,$$

for all $v \in X$. Therefore, by choosing as v the function $u^- = \min\{u, 0\}$ and setting $A = \{x \in \mathbb{R}^N : u(x) < 0\}$, it follows

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u\nabla u^- + a(x)|u|^{p-2}uu^-\right)dx = \lambda\int_{\mathbb{R}^N} f(x,u)u^-\,dx,$$
$$\int_{A} \left(|\nabla u|^{p-2}\nabla u\nabla u^- + a(x)|u|^{p-2}uu^-\right)dx = \lambda\int_{A} f(x,u)u^-\,dx,$$
$$0 \le \int_A (|\nabla u|^p + a(x)|u|^p)\,dx = \lambda\int_A f(x,0)u\,dx \le 0.$$

Hence, one has $u = 0$ in A for which $A = \emptyset$ and so $u(x) \ge 0$ for all $x \in \mathbb{R}^N$. □

Our main tools are a local minimum theorem proved in [3], given as in [4, Theorem 2.6], and two non-zero critical points theorem established in [5], which is a non immediate consequence of the local minimum theorem in combination with the Ambrosetti–Rabinowitz theorem. Therefore, we recall some definitions. Let $(X, \|\cdot\|)$ be a Banach space, X^* its dual and $I_\lambda : X \to \mathbb{R}$ a Gâteaux differentiable functional, with $\lambda > 0$.

Definition 2.1. We say that I_λ satisfies the Palais–Smale condition (in short, (PS)-condition), if any sequence $\{u_n\} \subseteq X$ such that

(P_1) $I_\lambda(u_n)$ is bounded,
(P_2) $\lim_{n\to\infty} \|I_\lambda(u_n)\|_{X^*} = 0$,

has a convergent subsequence in X.

Definition 2.2. Fix $r \in]-\infty, \infty]$. We say that I_λ satisfies the Palais–Smale condition cut-off upper at r (in short, $(PS)^{[r]}$-condition), if any sequence $\{u_n\} \subseteq X$ such that

(P_1) $I_\lambda(u_n)$ is bounded,
(P_2) $\lim_{n\to\infty} \|I_\lambda(u_n)\|_{X^*} = 0$,
(P_3) $\Phi(u_n) < r$,

has a convergent subsequence in X.

Clearly, if I_λ satisfies the (PS)-condition, then it satisfies also the $(PS)^{[r]}$-condition. Here we give two preliminary results on the energy functional I_λ.

Lemma 2.3. *For each $\lambda > 0$, I_λ satisfies the $(PS)^{[r]}$-condition for every $r > 0$.*

Proof. Fix $\lambda > 0, r > 0$ and $\{u_n\} \subseteq X$ such that $(P_1), (P_2)$ and (P_3) hold. From (P_3), taking the coercivity of Φ into account, it follows that $\{u_n\}$ is bounded in X. Since X is reflexive, then up to a subsequence, one has $u_n \rightharpoonup u$ in X. Moreover, taking into account that for all open ball $B(0,R)$ one has that $W^{1,p}(B(0,R))$ is compactly embedded in $C(\bar{B}(0,R))$, by Cantor's diagonal process it follows that $u_n(x) \to u(x)$ for every $x \in \mathbb{R}^N$.

Now, recalling that X is continuously embedded in $L^\infty(\mathbb{R}^N)$, one has that $u_n \rightharpoonup u$ in $L^\infty(\mathbb{R}^N)$, then $|u_n(x)| \le \rho$ for all $n \in \mathbb{N}$, for a.e. $x \in \mathbb{R}^N$. So, from the assumptions (ii) and (iii) on function f, we have that $f(x, u_n(x)) \to f(x, u(x))$ for a. e. $x \in \mathbb{R}^N$ and $f(\cdot, u_n)$ belongs to $L^1(\mathbb{R}^N)$ for all $n \in \mathbb{N}$. Therefore, Lebesgue dominated convergence theorem ensures that $f(\cdot, u_n(\cdot))$ strongly converges to $f(\cdot, u(\cdot))$ in $L^1(\mathbb{R}^N) \subset (L^\infty(\mathbb{R}^N))^*$ and from [6, proposition III.5 (iv)] it follows that

$$\langle f(\cdot, u_n(\cdot)), u_n(\cdot)\rangle \to \langle f(\cdot, u(\cdot)), u(\cdot)\rangle\,,$$

which leads to

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x, u_n(x))\,(u_n(x) - u(x))\; dx = 0\,. \tag{2.4}$$

Now, exploiting (P_2), there exists a sequence $\{\varepsilon_n\}$, with $\varepsilon_n \to 0^+$, such that $\|I'_\lambda(u_n)\|_{X^*} \le \varepsilon_n$ for all $n \in \mathbb{N}$, which implies $|\langle I'_\lambda(u_n), v\rangle| \le \|I'_\lambda(u_n)\|_{X^*}\|v\|_X \le \varepsilon_n$ for every $n \in \mathbb{N}$ and for each $v \in X$ such that $\|v\|_X \le 1$. Then, choosing $v = \frac{u_n - u}{\|u_n - u\|_X}$ one has

$$\begin{aligned}&\int_{\mathbb{R}^N} \Big(|\nabla u_n|^{p-2}\nabla u_n(\nabla u_n - \nabla u) + a(x)|u_n|^{p-2}u_n(u_n - u)\Big)\, dx \\ &\quad -\lambda \int_{\mathbb{R}^N} f(x, u_n)(u_n - u)\, dx \quad \le \varepsilon_n\|u_n - u\|_X\,,\end{aligned} \tag{2.5}$$

for all $n \in \mathbb{N}$. Focusing on the first integral, we have that

$$\begin{aligned}&\int_{\mathbb{R}^N} \Big(|\nabla u_n|^{p-2}\nabla u_n(\nabla u_n - \nabla u) + a(x)|u_n|^{p-2}u_n(u_n - u)\Big)\, dx \\ \ge\;& \|u_n\|_X^p - \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}|\nabla u_n||\nabla u|\, dx - \int_{\mathbb{R}^N} a(x)|u_n|^{p-2}|u_n||u|\, dx \\ \ge\;& \|u_n\|_X^p - \int_{\mathbb{R}^N} |\nabla u_n|^{p-1}|\nabla u|\, dx - \int_{\mathbb{R}^N} a(x)|u_n|^{p-1}|u|\, dx\,.\end{aligned}$$

Using the following inequality (see [6, page 92])

$$|a|^{p-1}|b| \le \frac{p-1}{p}|a|^p + \frac{1}{p}|b|^p,$$

we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^N} \Big(|\nabla u_n|^{p-2}\nabla u_n(\nabla u_n - \nabla u) + a(x)|u_n|^{p-2}u_n(u_n - u)\Big)\, dx \\ \ge\;& \|u_n\|_X^p - \int_{\mathbb{R}^N} \left(\frac{p-1}{p}|\nabla u_n|^p + \frac{1}{p}|\nabla u^p|\right)\, dx - \int_{\mathbb{R}^N} a(x)\left(\frac{p-1}{p}|u_n|^p + \frac{1}{p}|u|^p\right)\, dx \\ =\;& \|u_n\|_X^p - \frac{p-1}{p}\|u_n\|_X^p - \frac{1}{p}\|u\|_X^p \\ =\;& \frac{1}{p}\|u_n\|_X^p - \frac{1}{p}\|u\|_X^p\,.\end{aligned}$$

Thus, from (2.5), one has

$$\frac{1}{p}\|u_n\|_X^p - \varepsilon_n\|u_n - u\|_X \leq \lambda \int_{\mathbb{R}^N} f(x, u_n(x))(u_n - u)\, dx + \frac{1}{p}\|u\|_X^p.$$

Taking (2.4) into account, the latter implies that

$$\limsup_{n\to+\infty} \|u_n\|_X \leq \|u\|_X.$$

Finally, thanks to [6, Proposition 3.32, p.78], it follows that $\{u_n\}$ strongly converges to $u \in X$ and the proof is complete. □

Lemma 2.4. *Suppose that $f(x,0) \geq 0$ for all $x \in \mathbb{R}^N$ and assume that*

$$\text{there are } s > 0,\ \mu > p: \quad 0 < \mu F(x,t) \leq t f(x,t) \quad \forall x \in \mathbb{R}^N, \forall t \geq s. \tag{AR}$$

Then, I_λ satisfies the (PS)-condition and it is unbounded from below for each $\lambda > 0$.

Proof. Fix $\lambda > 0$ and $\{u_n\} \subseteq X$ such that (P_1) and (P_2) hold. We prove the thesis in different steps.

Claim 1. *There exists $k \geq 0$ such that $u_n(x) \geq -k$ for a.e. $x \in \mathbb{R}^N$, for all $n \in \mathbb{N}$.*

From (P_2), there exists $\{\varepsilon_n\}$, with $\varepsilon_n \to 0^+$, such that $|I'_\lambda(u_n)(v)| \leq \varepsilon_n$ for all $n \in \mathbb{N}$, for each $v \in X$, with $\|v\|_X \leq 1$. So, in correspondence of $v = \frac{u_n^-}{\|u_n^-\|_X}$, the following inequality hold

$$\int_{\mathbb{R}^N} \left(|\nabla u_n|^{p-2}\nabla u_n \nabla u_n^- + a(x)|u_n|^{p-2}u_n u_n^-\right) dx - \lambda \int_{\mathbb{R}^N} f(x,u_n)u_n^-\, dx \leq \varepsilon_n \|u_n^-\|_X,$$

for every $n \in \mathbb{N}$. Now, since $\int_{\mathbb{R}^N} f(x,u_n)u_n^-\, dx \leq 0$, being $\int_{\{u_n<0\}} f(x,0)u_n^-\, dx \leq 0$ and $\int_{\{u_n\geq 0\}} f(x,u_n)u_n^-\, dx = 0$, one has

$$0 \leq \|u_n^-\|_X^p \leq \|u_n^-\|_X^p - \lambda \int_{\mathbb{R}^N} f(x,u_n)u_n^-(x)\, dx \leq \varepsilon_n \|u_n^-\|_X,$$

that is, $\|u_n^-\|_X \leq (\varepsilon_n)^{\frac{1}{p-1}}$, for which passing to the limit for $n \to \infty$, one has that $\|u_n^-\|_X \to 0$, then it is bounded in X. Taking Lemma 2.1 into account, there exists $k \geq 0$ such that $|u_n^-(x)| \leq k$ for a.e. $x \in \mathbb{R}^N$, which means $u_n(x) \geq -k$ for a.e. $x \in \mathbb{R}^N$.

Claim 2. *$\{u_n\}$ is bounded in X.*

Arguing as in Claim 1, in correspondence of $v = \frac{u_n}{\|u_n\|_X}$, one has

$$-I'_\lambda(u_n)(u_n) \leq \varepsilon_n \|u_n\|_X, \tag{2.6}$$

for every $n \in \mathbb{N}$. Also, from (P_1) there exists $M > 0$ such that

$$|I_\lambda(u_n)| \leq M, \tag{2.7}$$

for all $n \in \mathbb{N}$. On the other hand, one has

$$\begin{aligned}\frac{1}{\mu}I'_\lambda(u_n)(u_n) &= \frac{1}{\mu}\|u_n\|_X^p - \frac{\lambda}{\mu}\int_{\mathbb{R}^N} f(x,u_n)u_n\, dx \\ &\quad + \lambda\int_{\mathbb{R}^N} F(x,u_n)\, dx - \lambda\int_{\mathbb{R}^N} F(x,u_n)\, dx \\ &= \frac{1}{\mu}\|u_n\|_X^p - \frac{\lambda}{\mu}\int_{\mathbb{R}^N}\Big(f(x,u_n)u_n - \mu F(x,u_n)\Big)dx - \lambda\int_{\mathbb{R}^N} F(x,u_n)\, dx,\end{aligned}$$

Therefore, it follows that

$$\begin{aligned} I_\lambda(u_n) - \frac{1}{\mu} I'_\lambda(u_n)(u_n) &= \frac{1}{p}\|u_n\|_X^p - \lambda \int_{\mathbb{R}^N} F(x,u_n)\,dx \\ &\quad - \frac{1}{\mu}\|u_n\|_X^p + \frac{\lambda}{\mu}\int_{\mathbb{R}^N}\Big(f(x,u_n)u_n - \mu F(x,u_n)\Big)dx \\ &\quad + \lambda \int_{\mathbb{R}^N} F(x,u_n)\,dx \\ &= \left(\frac{1}{p}-\frac{1}{\mu}\right)\|u_n\|_X^p + \frac{\lambda}{\mu}\int_{\mathbb{R}^N}\Big(f(x,u_n)u_n - \mu F(x,u_n)\Big)dx. \end{aligned} \tag{2.8}$$

Focusing on the integral, taking Claim 1 and (AR) into account, one has

$$\begin{aligned} \int_{\mathbb{R}^N}\Big(f(x,u_n)u_n - \mu F(x,u_n)\Big)dx &= \int_{u_n(x)\geq -k}\Big(f(x,u_n)u_n - \mu F(x,u_n)\Big)dx \\ = \int_{-k\leq u_n(x)\leq s}\Big(f(x,u_n)u_n - \mu F(x,u_n)\Big)dx &+ \int_{u_n(x)>s}\Big(f(x,u_n)u_n - \mu F(x,u_n)\Big)dx \\ \geq \int_{-k\leq u_n(x)\leq s}\Big(f(x,u_n)u_n - \mu F(x,u_n)\Big)dx &\geq \int_{|u_n(x)|\leq c}\Big(f(x,u_n)u_n - \mu F(x,u_n)\Big)dx, \end{aligned}$$

with $0 \leq c \leq \min\{k,s\}$. Moreover, by some computations, using (iii) and $(iii)'$, we obtain that

$$\begin{aligned} \int_{\mathbb{R}^N}\Big(f(x,u_n)u_n - \mu F(x,u_n)\Big)dx &\geq -c\int_{|u_n(x)|\leq c}\left(\sup_{|\xi|\leq c}|f(x,\xi)|\right)dx \\ &\quad - \mu\int_{|u_n(x)|\leq c}\left(\sup_{|\xi|\leq u_n(x)}|f(x,\xi)|\right)|u_n(x)|\,dx \\ &\geq -c\int_{\mathbb{R}^N}\left(\sup_{|\xi|\leq c}|f(x,\xi)|\right)dx \\ &\quad - c\mu\int_{\mathbb{R}^N}\left(\sup_{|\xi|\leq u_n(x)}|f(x,\xi)|\right)dx \\ &\geq -c(k_1-\mu k_2)\,, \end{aligned} \tag{2.9}$$

where k_1 and k_2 are L^1-norms. Thus, from (2.8), one has

$$I_\lambda(u_n) - \frac{1}{\mu}I'_\lambda(u_n)(u_n) \geq \left(\frac{1}{p}-\frac{1}{\mu}\right)\|u_n\|_X^p - \frac{\lambda}{\mu}c(k_1-\mu k_2)\,,$$

which, using (2.6) and (2.7), leads to

$$\left(\frac{1}{p}-\frac{1}{\mu}\right)\|u_n\|_X^p \leq M + \frac{1}{\mu}\varepsilon_n\|u_n\|_X + \frac{\lambda}{\mu}c(k_1-\mu k_2).$$

Now, if we assume by a contradiction that $\|u_n\|_X$ is unbounded, there is a subsequence $\{u_{n_k}\}$ such that $\lim_{k\to\infty}\|u_{n_k}\|_X = +\infty$. Therefore, by passing to the limit for $k\to\infty$ the previous inequality, it follows that $\lim_{k\to\infty}\|u_{n_k}\|_X^{p-1} = 0$ and this is an absurd. Hence, $\|u_n\|_X$ is bounded and our claim is proved.

At this point, the same proof of Lemma 2.3 proves that I_λ satisfies (PS)-condition.

Finally, we prove that I_λ is unbounded from below.

To this aim, using (AR), we obtain that

$$\int_s^t \frac{\mu}{\xi}d\xi \leq \int_s^t \frac{f(x,\xi)}{F(x,\xi)}d\xi, \qquad \forall x\in\mathbb{R}^N, \forall t\geq s,$$

which implies

$$\ln\left(\frac{t}{s}\right)^{\mu} \leq \ln\left(\frac{F(x,t)}{F(x,s)}\right),$$

namely

$$\frac{F(x,t)}{t^{\mu}} \geq \frac{F(x,s)}{s^{\mu}} = A(x) > 0 \qquad \forall x \in \mathbb{R}^N, \forall t \geq s. \tag{2.10}$$

Clearly, owing to (iii'), one has $A \in L^1(\mathbb{R}^N)$. On the other hand, for every $x \in \mathbb{R}^N, 0 \leq t \leq s$ one has

$$\begin{aligned} F(x,t) &\geq \min_{\xi\in[0,s]} F(x,\xi) \geq \min_{\xi\in[0,s]} F(x,\xi) + A(x)t^{\mu} - A(x)s^{\mu} \\ &= A(x)t^{\mu} - \left(A(x)s^{\mu} - \min_{\xi\in[0,s]} F(x,\xi)\right) = A(x)t^{\mu} - B(x) \end{aligned} \tag{2.11}$$

where $B(x) = A(x)s^{\mu} - \min_{\xi\in[0,s]} F(x,\xi)$ and, owing to (iii'), $B \in L^1(\mathbb{R}^N)$. Hence, putting (2.10) and (2.11) together, we have that

$$F(x,t) \geq A(x)t^{\mu} - B(x)$$

for all $x \in \mathbb{R}^N$ and for every $t \geq 0$. Therefore, fixed $\bar{u} \in X$ such that $\|\bar{u}\|_X \neq 0$ and $\bar{u}(x) \geq 0$ in $\mathbb{R}^N$, one has $\Psi(t\bar{u}) = \int_{\mathbb{R}^n} F(x, t\bar{u}(x))dx \geq t^{\mu}\int_{\mathbb{R}^n} A(x)|\bar{u}|^{\mu}dx - \int_{\mathbb{R}^N} B(x)dx = C_1t^{\mu} - C_2$. Moreover, one has $\Phi(t\bar{u}) = \frac{1}{p}\|t\bar{u}\|_X^p = \frac{1}{p}t^{\frac{1}{p}}\|\bar{u}\|_X^p = C_3t^p$. Hence, one has $I_\lambda(t\bar{u}) \leq C_3t^p - \lambda C_1t^{\mu} + \lambda C_2$ and so $\lim_{t\to+\infty} I_\lambda(t\bar{u}) = -\infty$ for which I_λ is unbounded from below and the proof is completed. ☐

3. Main results

Fix an open ball of center x_0 and radius R, which we denote by $B(x_0, R)$, and put

$$K_R := \frac{1}{L_a^p}\frac{\Gamma(1+N/2)}{\pi^{N/2}}\left(\frac{R^{p-N}}{2^p - 2^{p-N} + R^p\|a\|_\infty}\right), \tag{3.1}$$

where L_a is given in (2.2).

We explicitly observe that the constant K is independent by the choice of x_0. Moreover, it can often be convenient, for simplicity, to take $R = 1$. So, by choosing as ball $B(0, 1)$ the constant becomes

$$K_1 := \frac{1}{L_a^p}\frac{\Gamma(1+N/2)}{\pi^{N/2}}\left(\frac{1}{2^p - 2^{p-N} + \|a\|_\infty}\right).$$

The constant K_R plays an important role in the following statements since it regulates the growth less than $(p-1)-$linear of the nonlinearity in a suitable range (see (b) below), which is a fundamental assumption of our results.

In the following, we present our first result on the existence of one non-trivial and non-negative solution for problem (P_λ).

Theorem 3.1. *Suppose that $f(x,0) \geq 0$ for all $x \in \mathbb{R}^N$ and assume that there exist a ball $B(x_0,R)$ and two positive constants c_1, c_2, with $0 < c_1 < c_2$, such that*

(a) $F(x,t) \geq 0$ *for all* $(x,t) \in \mathbb{R}^N \times [0, c_1]$,

(b) $\frac{\int_{\mathbb{R}^N} \max_{|\xi|\leq c_2} F(x,\xi)\,dx}{c_2^p} < K_R \frac{\int_{B(x_0,\frac{R}{2})} F(x,c_1)\,dx}{c_1^p}$.

Then, for each $\lambda \in \Lambda_{c_1,c_2}$, where

$$\Lambda_{c_1,c_2} := \left]\frac{1}{pL_a^p}\frac{1}{K_R}\frac{c_1^p}{\int_{B(x_0,\frac{R}{2})} F(x,c_1)\,dx}, \frac{1}{pL_a^p}\frac{c_2^p}{\int_{\mathbb{R}^N}\max_{|\xi|\leq c_2} F(x,\xi)\,dx}\right[,$$

problem (P_λ) admits at least one non-trivial and non-negative solution u_λ such that $\|u_\lambda\|_\infty < c_2$.

Proof. Our aim is to apply [4, Theorem 2.3]. To this end, consider $(X, \|\cdot\|_X)$, Φ, $\Psi : X \to \mathbb{R}$ as defined in Section 2, which verify the required regularity assumptions. In addition, thanks to Lemma 2.3, one has that $\Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$ condition for every $r > 0, \lambda > 0$. So, one has to prove that there exist $r > 0$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\frac{\sup_{\Phi(u)<r} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}.$$

Put $r = \frac{c_2^p}{pL_a^p}$; taking Lemma 2.1 into account, for each $u \in X$ with $\Phi(u) = \frac{1}{p}\|u\|_X^p < r$ one has $\|u\|_\infty < c_2$. Therefore, it follows that

$$\frac{\sup_{\Phi(u)<r} \Psi(u)}{r} \leq p\, L_a^p \, \frac{\int_{\mathbb{R}^N} \max_{|\xi|\leq c_2} F(x,\xi)\, dx}{c_2^p}. \tag{3.2}$$

Now, consider $\tilde{u} \in X$ defined by

$$\tilde{u}(x) := \begin{cases} 0 & \text{if } x \in \mathbb{R}^N \setminus B(x_0, R), \\ \frac{2c_1}{R}(R - |x - x_0|) & \text{if } x \in B(x_0, R) \setminus B(x_0, \frac{R}{2}), \\ c_1 & \text{if } x \in B(x_0, \frac{R}{2}). \end{cases}$$

For simplicity, put $S := B(x_0, R) \setminus B(x_0, \frac{R}{2})$ and $m_R := |B(x_0, R)| = \dfrac{\pi^{N/2}}{\Gamma(1 + N/2)} R^N$. Clearly, $\tilde{u} \in X$ and

$$\begin{aligned}
\Phi(\tilde{u}) &= \frac{1}{p}\left(\int_S (|\nabla \tilde{u}(x)|^p + a(x)|\tilde{u}(x)|^p)\, dx + \int_{B(x_0,\frac{R}{2})} a(x) c_1^p \, dx\right) \\
&= \frac{1}{p}\left[\left(\frac{2c_1}{R}\right)^p \int_S (1 + a(x)|R - |x - x_0||^p)\, dx + c_1^p \int_{B(x_0,\frac{R}{2})} a(x)\, dx\right] \\
&\leq \frac{1}{p}\left[\left(\frac{2c_1}{R}\right)^p \int_S \left(1 + \|a\|_\infty \left(\frac{R}{2}\right)^p\right) dx + c_1^p \|a\|_\infty \left|B\left(x_0, \frac{R}{2}\right)\right|\right] \\
&\leq \frac{1}{p}\left[\left(\frac{2c_1}{R}\right)^p |S| \left(1 + \|a\|_\infty \left(\frac{R}{2}\right)^p\right) + c_1^p \|a\|_\infty \frac{m_R}{2^N}\right] \\
&= \frac{1}{p}\left[\left(\frac{2c_1}{R}\right)^p m_R \frac{2^N - 1}{2^N} \left(1 + \|a\|_\infty \frac{R^p}{2^p}\right) + c_1^p \|a\|_\infty \frac{m_R}{2^N}\right] \\
&= \frac{m_R}{p} c_1^p \left(\frac{2^p - 2^{p-N} + \|a\|_\infty R^p}{R^p}\right) \\
&= \frac{1}{p}\frac{m_R}{R^N}\left(\frac{2^p - 2^{p-N} + \|a\|_\infty R^p}{R^{p-N}}\right) c_1^p \\
&= \frac{1}{p}\frac{1}{L_a^p K_R} c_1^p,
\end{aligned}$$

that is,

$$\Phi(\tilde{u}) \leq \frac{1}{p}\frac{1}{L_a^p K_R} c_1^p. \tag{3.3}$$

Furthermore,

$$\Psi(\tilde{u}) = \int_S F\left(x, \frac{2c_1}{R}(R - |x - x_0|)\right) dx + \int_{B(x_0,\frac{R}{2})} F(x, c_1) dx,$$

and from assumption (a) it follows that

$$\Psi(\tilde{u}) \geq \int_{B(x_0,\frac{R}{2})} F(x, c_1) dx. \tag{3.4}$$

Therefore, putting (3.3) and (3.4) together, we have

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \geq pL_a^p K_R \frac{\int_{B(x_0,\frac{R}{2})} F(x,c_1)dx}{c_1^p}. \tag{3.5}$$

Taking (3.2) into account, from our assumption (b) it follows that

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} > p\,L_a^p \frac{\int_{\mathbb{R}^N} \max_{|\xi|\leq c_2} F(x,\xi)\,dx}{c_2^p} \geq \frac{\sup_{\Phi(u)<r} \Psi(u)}{r},$$

that is,

$$\frac{\sup_{\Phi(u)<r} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}.$$

Hence, it remains to prove that $\Phi(\tilde{u}) < r$. Taking into account (3.3), that is,

$$\Phi(\tilde{u}) \leq \frac{1}{p}\frac{1}{K_R L_a^p} c_1^p$$

and $r = \frac{c_2^p}{pL_a^p}$, one has

$$\Phi(\tilde{u}) < r \qquad \textit{when} \qquad \left(\frac{1}{K_R}\right)^{1/p} c_1 < c_2.$$

This inequality holds thanks to the assumption that $0 < c_1 < c_2$. In fact,

reasoning by absurd we suppose that $\left(\frac{1}{K_R}\right)^{1/p} c_1 \geq c_2$, which implies that

$$\frac{\int_{\mathbb{R}^N} \max_{|\xi|\leq c_2} F(x,\xi)\,dx}{c_2^p} \geq K_R \frac{\int_{\mathbb{R}^N} \max_{|\xi|\leq c_1} F(x,\xi)\,dx}{c_1^p}$$

$$\geq K_R \frac{\int_{\mathbb{R}^N} F(x,c_1)\,dx}{c_1^p} \geq K_R \frac{\int_{B(x_0,\frac{R}{2})} F(x,c_1)\,dx}{c_1^p},$$

that is in contradiction with assumption (b).

All the hypotheses of [4, Theorem 2.3] are verified, then for each $\lambda \in \Lambda_{c_1,c_2}$ there exists $u_\lambda \in X$, with $0 < \Phi(u_\lambda) < r$, which is a non-zero local minimum of functional I_λ in $\Phi^{-1}(]0,r[)$, namely u_λ is a non-zero weak solution for problem (P_λ) and one has that $\|u_\lambda\|_\infty < c_2$. Moreover, Lemma 2.2 ensures that it is nonnegative and the proof is completed. □

Theorem 3.2. *Suppose that $f(x,0) \geq 0$ for all $x \in \mathbb{R}^N$. Assume that (AR)-condition is satisfied and there exist two positive constant c_1, c_2 such that condition (a) and (b) hold. Then, for each $\lambda \in \Lambda_{c_1,c_2}$ problem (P_λ) admits at least two non-trivial and non-negative solution $u_{\lambda,1}, u_{\lambda,2}$ such that*

$$\frac{1}{p}\int_{\mathbb{R}^N} |u_{\lambda,1}|^p dx + \lambda \int_{\mathbb{R}^N} F(x,u_{\lambda,1}(x))dx < 0 < \frac{1}{p}\int_{\mathbb{R}^N} |u_{\lambda,2}|^p dx + \lambda \int_{\mathbb{R}^N} F(x,u_{\lambda,2}(x))dx.$$

Proof. Our aim is to apply [5, Theorem 2.1]. To this end, put $(X, \|\cdot\|_X)$, Φ, $\Psi : X \to \mathbb{R}$ as defined in Section 2, which verify the required regularity assumptions. In addition, thanks to Lemma 2.4, one has that $I_\lambda = \Phi - \lambda\Psi$ satisfies the (PS) condition and it is unbounded from below. Moreover, the same proof of Theorem 3.1 ensures that there exist $r > 0$ and $\tilde{u} \in X$, with $0 < \Phi(u) < r$, such that

$$\frac{\sup_{\Phi(u)<r} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}.$$

All the hypotheses of [5, Theorem 2.1] are verified, then for each $\lambda \in \Lambda_{c_1,c_2}$ there exist two critical points of I_λ $u_{\lambda,1}, u_{\lambda,2} \in X$, which are two non-zero weak solutions of Problem (P_λ) and, owing to Lemma 2.2, they are nonnegative. Finally, again from [5, Theorem 2.1], one has

$$\frac{1}{p}\int_{\mathbb{R}^N} |u_{\lambda,1}|^p dx + \lambda \int_{\mathbb{R}^N} F(x, u_{\lambda,1}(x))dx < 0 < \frac{1}{p}\int_{\mathbb{R}^N} |u_{\lambda,2}|^p dx + \lambda \int_{\mathbb{R}^N} F(x, u_{\lambda,2}(x))dx.$$

and the proof is completed. □

Now we point out some consequences of previous main results when the nonlinear term is with separated variables. Let $\alpha \in L^1(\mathbb{R}^N)$ be a nonnegative and non-zero function. Moreover, let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function and put $G(t) = \int_0^\xi g(\xi)d\xi$, $t \in \mathbb{R}$. Finally, put

$$H_R := \frac{\int_{B(0,R/2)} \alpha(x)dx}{\|\alpha\|_1} K_R \tag{3.6}$$

where K_R is given in (3.1). Consider the following nonlinear differential problem on the entire space

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda\alpha(x)g(u(x)) & \text{in} \quad \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \end{cases} \tag{D_λ}$$

We have the following result.

Theorem 3.3. *Assume that there exist two positive constants c_1, c_2, with $0 < c_1 < c_2$, such that*

(b') $\frac{G(c_2)}{c_2^p} < H_R \frac{G(c_1)}{c_1^p}$.

Then, for each $\lambda \in \Lambda'_{c_1,c_2}$, where

$$\Lambda'_{c_1,c_2} := \left] \frac{1}{pL_a^p\|\alpha\|_1} \frac{1}{H_R} \frac{c_1^p}{G(c_1)}, \frac{1}{pL_a^p\|\alpha\|_1} \frac{c_2^p}{G(c_2)} \right[,$$

problem (D_λ) admits at least one non-trivial and non-negative solution $u_{\lambda,1}$ such that $\|u_{\lambda,1}\|_\infty < c_2$.

Further, in addition, assume that

$$\text{there are } s > 0,\ \mu > p : \quad 0 < \mu G(t) \le tg(t) \quad \forall t \ge s. \tag{AR'}$$

Then, for each $\lambda \in \Lambda'_{c_1,c_2}$ problem (D_λ) admits a second distinct non-zero and non-negative solution $u_{\lambda,2}$ such that

$$\frac{1}{p}\int_{\mathbb{R}^N} |u_{\lambda,1}|^p dx + \lambda \int_{\mathbb{R}^N} F(x, u_{\lambda,1}(x))dx < 0 < \frac{1}{p}\int_{\mathbb{R}^N} |u_{\lambda,2}|^p dx + \lambda \int_{\mathbb{R}^N} F(x, u_{\lambda,2}(x))dx.$$

Proof. It follows from Theorems 3.1 and 3.2. □

Remark 3.1. Assumption (b') of Theorem 3.3 is verified when

$$\lim_{t\to 0^+} \frac{g(t)}{t^{p-1}} = +\infty. \tag{3.7}$$

Put $\lambda^* = \frac{1}{pL_a^p\|\alpha\|_1} \sup_{c>0} \frac{c^p}{G(c)}$, the assumption (3.7) ensures the existence of at least one non-zero and nonnegative weak solution to Problem (D_λ) for each $\lambda \in]0, \lambda^*[$.

Clearly, if we assume (3.7) in addition with (AR'), then for each $\lambda \in]0, \lambda^*[$, problem (D_λ) admits at least two distinct non-zero and nonnegative weak solutions.

Remark 3.2. Theorem 1.1 in Introduction is a special case of Theorem 3.3. Indeed, putting $g(t) = \eta t^s + t^q$, assumption (3.7) in Remark 3.1 is verified as well as condition (AR'). Hence, for each $\lambda \in]0, \lambda^*[$, problem

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda\alpha(x)\left(\eta u^s + u^q\right) & \text{in} \quad \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \end{cases} \tag{$D_{s,q,\lambda}$}$$

admits two non-zero and nonnegative weak solutions. In this case (see Remark 3.1), one has

$$\lambda^* = \frac{1}{pL_a^p\|\alpha\|_1} \sup_{c>0} \frac{c^p}{G(c)} = \frac{1}{pL_a^p\|\alpha\|_1} \frac{\bar{c}^p}{G(\bar{c})} = \frac{1}{pL_a^p\|\alpha\|_1} \left(\eta \frac{1}{s+1}\bar{c}^{s+1-p} + \frac{1}{q+1}\bar{c}^{q+1-p}\right),$$

where

$$\bar{c} = \left(\eta \frac{q+1}{s+1} \frac{p-(s+1)}{(q+1)-p}\right)^{\frac{1}{q-s}}.$$

It follows that

$$\lambda^* > 1 \iff \eta < \eta^*,$$

where

$$\eta^* = \left(\frac{1}{pL_a^p\|\alpha\|_1}\right)^{\frac{q-s}{(q+1)-p}} \frac{(s+1)((q+1)-p)(q+1)^{\frac{p-(s+1)}{(q+1)-p}}(p-(s+1))^{\frac{p-(s+1)}{(q+1)-p}}}{(q-s)^{\frac{(q+1)-p}{q-s}}}. \tag{3.8}$$

Therefore, for each $\eta \in]0, \eta^*[$, problem $(D_{s,q,\lambda})$ admits two non-zero and nonnegative weak solutions for $\lambda = 1$, that is our conclusion.

Remark 3.3. We explicitly observe that the value η^* in Theorem 1.1 is determined numerically by (3.8). We also note that it depends on the embedding constant L_a.

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References

[1] A. Ambrosetti, J. Garcia Azorero, I. Peral, Elliptic variational problems in $\mathbb{R}^N$ with critical growth, J. Differential Equations 168 (2000) 10–32.
[2] G. Barletta, P. Candito, S.A. Marano, K. Perera, Multiplicity results for critical p-Laplacian problems, Ann. Mat. Pura Appl. 196 (2017) 1431–1440.
[3] G. Bonanno, A critical point theorem via the ekeland variational principle, Nonlinear Anal. 75 (2012) 2992–3007.
[4] G. Bonanno, Relations between the mountain pass theorem and local minima, Adv. Nonlinear Anal. 1 (2012) 205–220.
[5] G. Bonanno, G. D'Aguı, Two non-zero solutions for elliptic Dirichlet problems, Z. Anal. Anwend. 35 (4) (2016) 449–464.
[6] H. Brézis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, New York, 2011.
[7] V.I. Burenkov, Sobolev Spaces on Domains, B. G. Teubner Stuttgart, Leipzig, 1998.
[8] V.I. Burenkov, V.A. Gusakov, On exact constants in Sobolev embedding theorems, Dokl. Akad. Nauk SSSR 294 (6) (1987) 1293–1297.
[9] P.L. De Napoli, M.C. Mariani, Equations of p-Laplacian type in unbounded domains, J. Differential Equations 140 (1997) 106–132.
[10] P. Drabek, Y.X. Huang, Multiplicity of positive solutions for some quasilinear elliptic equation in $\mathbb{R}^N$ with critical Sobolev exponent, Demonstr. Math. 55 (2022) 416–428.
[11] G.P. Galdi, An Introduction to the Mathematical Theory of the Navier–Stokes Equations, second ed., Springer, New York, 2011.
[12] U. Guarnotta, R. Livrea, S.A. Marano, Some recent results on singular p-Laplacian equations, Demonstr. Math. 55 (2022) 416–428.