

Compound conditionals, Fréchet-Hoeffding bounds, and Frank t-norms [☆]

Angelo Gilio^{a,1,2}, Giuseppe Sanfilippo^{b,3,*}

^aDepartment of Basic and Applied Sciences for Engineering, University of Rome “La Sapienza”, Via A. Scarpa 14, 00161 Roma, Italy

^bDepartment of Mathematics and Computer Science, Via Archirafi 34, 90123 Palermo, Italy

Abstract

In this paper we consider compound conditionals, Fréchet-Hoeffding bounds and the probabilistic interpretation of Frank t-norms. By studying the solvability of suitable linear systems, we show under logical independence the sharpness of the Fréchet-Hoeffding bounds for the prevision of conjunctions and disjunctions of n conditional events. In addition, we illustrate some details in the case of three conditional events. We study the set of all coherent prevision assessments on a family containing n conditional events and their conjunction, by verifying that it is convex. We discuss the case where the prevision of conjunctions is assessed by Lukasiewicz t-norms and we give explicit solutions for the linear systems; then, we analyze a selected example. We obtain a probabilistic interpretation of Frank t-norms and t-conorms as prevision of conjunctions and disjunctions of conditional events, respectively. Then, we characterize the sets of coherent prevision assessments on a family containing n conditional events and their conjunction, or their disjunction, by using Frank t-norms, or Frank t-conorms. By assuming logical independence, we show that any Frank t-norm (resp., t-conorm) of two conditional events $A|H$ and $B|K$, $T_\lambda(A|H, B|K)$ (resp., $S_\lambda(A|H, B|K)$), is a conjunction $(A|H) \wedge (B|K)$ (resp., a disjunction $(A|H) \vee (B|K)$). Then, we analyze the case of logical dependence where $A = B$ and we obtain the set of coherent assessments on $A|H, A|K, (A|H) \wedge (A|K)$; moreover we represent it in terms of the class of Frank t-norms T_λ , with $\lambda \in [0, 1]$. By considering a family \mathcal{F} containing three conditional events, their conjunction, and all pairwise conjunctions, we give some results on Frank t-norms and coherence of the prevision assessments on \mathcal{F} . By assuming logical independence, we show that it is coherent to assess the previsions of all the conjunctions by means of Minimum and Product t-norms. In this case all the conjunctions coincide with the t-norms of the corresponding conditional events. We verify by a counterexample that, when the previsions of conjunctions are assessed by the Lukasiewicz t-norm, coherence is not assured. Then, the Lukasiewicz t-norm of conditional events may not be interpreted as their conjunction. Finally, we give two sufficient conditions for coherence and incoherence when using the Lukasiewicz t-norm.

Keywords: Coherence, Conditional previsions, Convexity, Conjunction and disjunction, Fréchet-Hoeffding bounds, Frank t-norms

*Corresponding author

Email addresses: angelo.gilio@sbai.uniroma1.it (Angelo Gilio), giuseppe.sanfilippo@unipa.it (Giuseppe Sanfilippo)

¹Both authors equally contributed to this work

²Retired

³Also affiliated with INdAM-GNAMPA, Italy

1. Introduction

In this paper we consider conjunctions and disjunctions of conditional events. These compound conditionals are defined in the setting of coherence as suitable conditional random quantities, with values in the unit interval (see, e.g. [27, 28, 29, 30, 33, 34, 52, 53]). In [30] we proved the sharpness of the Fréchet-Hoeffding bounds for the prevision of the conjunction and disjunction of two conditional events. We recall that such lower and upper bounds are particular Frank t-norms and t-conorms: for the conjunction they are the Lukasiewicz and Minimum t-norms, respectively; for the disjunction they are the dual t-conorms. In this paper we generalize this result to the conjunction $\mathcal{C}_{1\dots n}$ and the disjunction $\mathcal{D}_{1\dots n}$ of n conditional events $E_1|H_1, \dots, E_n|H_n$. To obtain this result we study the solvability of suitable linear systems associated with prevision assessments \mathcal{M} on the family $\{E_1|H_1, \dots, E_n|H_n, \mathcal{C}_{1\dots n}\}$. We provide some explicit solutions for the linear systems and we show that the set of coherent assessments \mathcal{M} is convex. To better illustrate our results, we examine more details in the case of three conditional events.

We discuss the case where the prevision of conjunctions is assessed by Lukasiewicz t-norms and we give explicit solutions for the linear systems; then, we analyze a selected example. We give a probabilistic interpretation of Frank t-norms and t-conorms as prevision of conjunction and disjunction of conditional events, respectively. Then, we characterize the sets of coherent prevision assessments on the families $\{E_1|H_1, \dots, E_n|H_n, \mathcal{C}_{1\dots n}\}$ and $\{E_1|H_1, \dots, E_n|H_n, \mathcal{D}_{1\dots n}\}$ in terms of Frank t-norms and Frank t-conorms, respectively. In addition, by assuming logical independence, we show that any Frank t-norm of two conditional events $A|H$ and $B|K$, $T_\lambda(A|H, B|K)$, is the conjunction $(A|H) \wedge (B|K)$ associated with the assessment $\mathbb{P}[(A|H) \wedge (B|K)] = T_\lambda(P(A|H), P(B|K))$. A dual result is given for the disjunction in terms of dual t-conorm.

We analyze the case of logical dependence where $A = B$ and we determine the set of all coherent assessments (x, y, z) on $\{A|H, A|K, (A|H) \wedge (A|K)\}$, by also showing that $T_\lambda(A|H, A|K)$ represents a conjunction $(A|H) \wedge (A|K)$, only for $\lambda \in [0, 1]$. In particular, when $HK = \emptyset$, we obtain that $(A|H) \wedge (A|K)$ coincides with the Product t-norm $T_1(A|H, A|K) = (A|H) \cdot (A|K)$.

Given three conditional events, we consider all possible conjunctions among them and we show that to make prevision assignments on conjunctions by means of the Product t-norm, or the Minimum t-norm, is coherent. Moreover, the conjoined conditionals can be represented as Product t-norms, or Minimum t-norms, of the involved conditional events. This representation may not hold for the Lukasiewicz t-norm. Indeed, we show by a counterexample that prevision assignments on conjunctions by means of the Lukasiewicz t-norm may be not coherent and we examine some sufficient conditions for coherence and incoherence. Finally, we give two sufficient conditions for coherence and incoherence when using the Lukasiewicz t-norm.

A relevant aspect which would deserve investigation is the application of our results on compound conditionals and t-norms in statistical matching, misclassified data, data fusion, aggregation operators, fuzzy logic, belief and plausibility functions, and description logic ([1, 5, 9, 10, 15, 16, 17, 18, 19, 20, 36, 47, 49]). This paper originated from [32] and the large part of the material is new. In particular all the results given in Section 3, Section 4, and Subsection 5.1 are new. Revised and extended material from [32] is given in Subsections 5.2 and 5.3, and Section 6.

The paper is organized as follows: In Section 2 we recall some preliminary notions and theoretical results on conditional random quantities and coherence. We give some examples and we examine an extended notion of conditional random quantity $X|H$. We recall compound conditionals and Frank t-norms. In Section 3, by studying the solvability of suitable linear systems, we show under logical independence the sharpness of Fréchet-Hoeffding bounds for the prevision of the conjunction $\mathcal{C}_{1\dots n}$ of n conditional events; we illustrate more details in the case $n = 3$. We also give a geometrical characterization of the set Π of all coherent prevision assessments on $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n, \mathcal{C}_{1\dots n}\}$, by showing that Π is convex. In Section

4 we examine in detail the case where the prevision of the conjunction is assessed by means of Lukasiewicz t-norm and we analyze a selected example. In Section 5 we study the representation of the prevision, for the conjunction $\mathcal{C}_{1\dots n}$ and the disjunction $\mathcal{D}_{1\dots n}$ of n conditional events, as a Frank t-norm T_λ and a Frank t-conorm S_λ , respectively. Then, by exploiting Frank t-norms and t-conorms, we characterize the sets of coherent prevision assessments on $\{E_1|H_1, \dots, E_n|H_n, \mathcal{C}_{1\dots n}\}$ and on $\{E_1|H_1, \dots, E_n|H_n, \mathcal{D}_{1\dots n}\}$. We show that, under logical independence, $T_\lambda(A|H, B|K) = (A|H) \wedge (B|K)$ and $S_\lambda(A|H, B|K) = (A|H) \vee (B|K)$ for every $\lambda \in [0, +\infty]$. We also examine the case of logical dependence where $A = B$ and the particular case where $HK = \emptyset$. In Section 6 we give some particular results on Frank t-norms and coherence of prevision assessments on the family $\mathcal{F} = \{E_1|H_1, E_2|H_2, E_3|H_3, (E_1|H_1) \wedge (E_2|H_2), (E_1|H_1) \wedge (E_3|H_3), (E_2|H_2) \wedge (E_3|H_3), (E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)\}$. In particular, we show that, under logical independence, the assessment $\mathcal{M} = (x_1, x_2, x_3, T_\lambda(x_1, x_2), T_\lambda(x_1, x_3), T_\lambda(x_2, x_3), T_\lambda(x_1, x_2, x_3))$ on \mathcal{F} is coherent for every $(x_1, x_2, x_3) \in [0, 1]^3$ when T_λ is the minimum t-norm, or the product t-norm. Moreover, when T_λ is the Lukasiewicz t-norm, the coherence of \mathcal{M} is not assured and hence it may happen that the Frank t-norm of three conditional events is not a conjunction. Finally, we give some sufficient conditions for coherence/incoherence of \mathcal{M} when using the Lukasiewicz t-norm. In Section 7 we give some conclusions.

2. Preliminary notions and results

In this section we recall some basic notions and results which concern conditional events, conditional random quantities, coherence (see, e.g., [2, 3, 4, 7, 12, 26, 43, 46, 48]), and logical operations among conditional events (see [28, 29, 30, 31, 33, 35]).

2.1. Conditional events, conditional random quantities, and coherent prevision assessments

Uncertainty about unknown facts is formalized by events. In formal terms, an event E is a two-valued logical entity which can be *true*, or *false*. The *indicator* of E , denoted by the same symbol, is 1, or 0, according to whether E is true, or false. Thus, a symbol like xE represents the product of the quantity x and the indicator of the event E . The sure event and impossible event are denoted by Ω and \emptyset , respectively. Given two events E_1 and E_2 , we denote by $E_1 \wedge E_2$, or simply by E_1E_2 , (resp., $E_1 \vee E_2$) the logical conjunction (resp., the logical disjunction). The negation of E is denoted \bar{E} . We simply write $E_1 \subseteq E_2$ to denote that E_1 logically implies E_2 , that is $E_1\bar{E}_2 = \emptyset$. We recall that n events E_1, \dots, E_n are logically independent when the number m of constituents, or possible worlds, generated by them is 2^n .

Given two events E, H , with $H \neq \emptyset$, the conditional event $E|H$ is defined as a three-valued logical entity which is *true*, or *false*, or *void*, according to whether EH is true, or $\bar{E}H$ is true, or \bar{H} is true, respectively.

Given a (real) random quantity X and an event $H \neq \emptyset$, we denote by $\mathbb{P}(X|H)$ the prevision of X conditional on H , with $\mathbb{P}(X|H) = P(E|H)$ when X is (the indicator of) an event E . In what follows, for any given conditional random quantity $X|H$, we assume that, when H is true, the set of possible values of X is a finite subset of the set of real numbers \mathbb{R} . In this case we say that $X|H$ is a finite conditional random quantity. In the framework of coherence, to assess $\mathbb{P}(X|H) = \mu$ means that, for every real number s , you are willing to pay an amount $s\mu$ and to receive $s(XH + \mu\bar{H})$, that is to receive sX , or $s\mu$, according to whether H is true, or \bar{H} is true (bet called off), respectively. The random gain is $G = s(XH + \mu\bar{H}) - s\mu = sH(X - \mu)$. In particular, given any conditional event $E|H$, if we assess $P(E|H) = x$, then the random gain is $G = sH(E - x)$.

Given a prevision function \mathbb{P} defined on an arbitrary family \mathcal{K} of finite conditional random quantities, consider a finite subfamily $\mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\} \subseteq \mathcal{K}$ and the vector $\mathcal{M} = (\mu_1, \dots, \mu_n)$, where $\mu_i = \mathbb{P}(X_i|H_i)$ is the assessed prevision for the conditional random quantity $X_i|H_i$, $i \in \{1, \dots, n\}$. With the pair $(\mathcal{F}, \mathcal{M})$ we associate the random gain $G = \sum_{i=1}^n s_i H_i (X_i - \mu_i)$ and we denote by $\mathcal{G}_{\mathcal{H}_n}$ the set of values of G restricted to $\mathcal{H}_n = H_1 \vee \dots \vee H_n$. Then, by the *betting scheme* of de Finetti, coherence is defined as:

Definition 1. The function \mathbb{P} defined on \mathcal{K} is coherent if and only if, $\forall n \geq 1, \forall s_1, \dots, s_n, \forall \mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\} \subseteq \mathcal{K}$, it holds that: $\min \mathcal{G}_{\mathcal{H}_n} \leq 0 \leq \max \mathcal{G}_{\mathcal{H}_n}$.

As it is well known, in Definition 1, the condition $\min \mathcal{G}_{\mathcal{H}_n} \leq 0 \leq \max \mathcal{G}_{\mathcal{H}_n}$ can be equivalently replaced by $\min \mathcal{G}_{\mathcal{H}_n} \leq 0$, or by $\max \mathcal{G}_{\mathcal{H}_n} \geq 0$.

A conditional prevision assessment \mathbb{P} on \mathcal{K} is not coherent, or incoherent, if and only if there exists a finite combination of n bets such that $\min \mathcal{G}_{\mathcal{H}_n} \cdot \max \mathcal{G}_{\mathcal{H}_n} > 0$, that is such that the values in $\mathcal{G}_{\mathcal{H}_n}$ are all positive, or all negative (*Dutch Book*). In other words, \mathbb{P} is incoherent if and only if there exists a finite combination of n bets such that, after discarding the case where all the bets are called off, the values of the random gain are all positive or all negative. In the particular case where \mathcal{K} is a family of conditional events, then Definition 1 becomes the well known definition of coherence for a probability function, denoted as P , defined on \mathcal{K} .

By Definition 1, given any (finite) conditional random quantity $X|H$ and denoting by x_1, \dots, x_r the possible values of X when H is true, a prevision assessment μ on $X|H$ is coherent if and only if $\min\{x_1, \dots, x_r\} \leq \mu \leq \max\{x_1, \dots, x_r\}$. When X is (the indicator of) an event E , with $P(E|H) = x$, the coherence of x amounts to $0 \leq x \leq 1$, or $x = 0$, or $x = 1$, according to whether $\emptyset \neq EH \neq H$, or $EH = \emptyset$, or $EH = H$, respectively.

Given a family $\mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\}$, for each $i = 1, \dots, n$, we denote by $\{x_{i1}, \dots, x_{ir_i}\}$ the set of possible values for the restriction of X_i to H_i ; then, for each $i = 1, \dots, n$, and $j = 1, \dots, r_i$, we set $A_{ij} = (X_i = x_{ij})$. Of course, for each i , the family $\{\bar{H}_i, A_{ij}H_i, j = 1, \dots, r_i\}$ is a partition of the sure event Ω , with $A_{ij}H_i = A_{ij}$ and $\bigvee_{j=1}^{r_i} A_{ij} = H_i$, that is $A_{11} \vee \dots \vee A_{1r_1} \vee \bar{H}_1 = \dots = A_{n1} \vee \dots \vee A_{nr_n} \vee \bar{H}_n = \Omega$, or more explicitly $(X_1 = x_{11}) \vee \dots \vee (X_1 = x_{1r_1}) \vee \bar{H}_1 = \dots = (X_n = x_{n1}) \vee \dots \vee (X_n = x_{nr_n}) \vee \bar{H}_n = \Omega$. Then,

$$\Omega = (A_{11} \vee \dots \vee A_{1r_1} \vee \bar{H}_1) \wedge \dots \wedge (A_{n1} \vee \dots \vee A_{nr_n} \vee \bar{H}_n). \quad (1)$$

By expanding the expression in (1) and by discarding the logical conjunctions which coincide with \emptyset , we obtain a disjunctive representation of Ω . The elements of this disjunction form a partition of Ω and are called the constituents generated by the family \mathcal{F} . We denote by C_1, \dots, C_m the constituents contained in $\mathcal{H}_n = H_1 \vee \dots \vee H_n$. Moreover, when $\mathcal{H}_n \neq \Omega$, we set $C_0 = \bar{\mathcal{H}}_n = \bar{H}_1 \cdot \dots \cdot \bar{H}_n$. Hence $\Omega = \bigvee_{h=0}^m C_h$. In particular, the constituents generated by a family of n conditional events $\{E_1|H_1, \dots, E_n|H_n\}$ are obtained by expanding the expression $(E_1H_1 \vee \bar{E}_1H_1 \vee \bar{H}_1) \wedge \dots \wedge (E_nH_n \vee \bar{E}_nH_n \vee \bar{H}_n)$, and by discarding the logical conjunctions which are impossible. If $E_1, \dots, E_n, H_1, \dots, H_n$ are logically independent, then the number of constituents for the family $\{E_1|H_1, \dots, E_n|H_n\}$ is 3^n (in which case the conditional events $E_1|H_1, \dots, E_n|H_n$ are said logically independent). Given a prevision assessment $\mathcal{M} = (\mu_1, \dots, \mu_n)$ on $\mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\}$, with each constituent $C_h, h = 1, \dots, m$, we associate a vector

$$Q_h = (q_{h1}, \dots, q_{hn}), \text{ with } q_{hi} = \begin{cases} x_{ij}, & \text{if } C_h \subseteq A_{ij}, j = 1, \dots, r_i, \\ \mu_i, & \text{if } C_h \subseteq \bar{H}_i. \end{cases} \quad (2)$$

With C_0 it is associated $Q_0 = \mathcal{M} = (\mu_1, \dots, \mu_n)$. As, for each i, j , the quantities x_{ij}, μ_i are real numbers, it holds that $Q_h \in \mathbb{R}^n, h = 0, 1, \dots, m$. Denoting by \mathcal{I} the convex hull of Q_0, \dots, Q_m , the condition $\mathcal{M} \in \mathcal{I}$ amounts to the existence of a vector $(\lambda_1, \dots, \lambda_m)$ such that: $\sum_{h=1}^m \lambda_h Q_h = \mathcal{M}, \sum_{h=1}^m \lambda_h = 1, \lambda_h \geq 0, \forall h$; in other words, $\mathcal{M} \in \mathcal{I}$ is equivalent to the solvability of the system (Σ) given below.

$$(\Sigma) \begin{cases} \sum_{h=1}^m \lambda_h q_{hi} = \mu_i, & i = 1, \dots, n, \\ \sum_{h=1}^m \lambda_h = 1, & \lambda_h \geq 0, h = 1, \dots, m. \end{cases} \quad (3)$$

We say that (Σ) is the system associated to the pair $(\mathcal{F}, \mathcal{M})$. By a suitable alternative theorem, it can be shown that the solvability of (Σ) is equivalent to the condition: $\min \mathcal{G}_{\mathcal{H}_n} \leq 0 \leq \max \mathcal{G}_{\mathcal{H}_n}$.

Let $\mathbf{x} = (x_1, \dots, x_m)$, $\mathbf{y} = (y_1, \dots, y_n)^t$ and $A = (a_{hi})$ be, respectively, a row m -vector, a column n -vector and a $m \times n$ -matrix. The vector \mathbf{x} is said *semipositive* if $x_h \geq 0, h = 1, \dots, m$, and $x_1 + \dots + x_m > 0$. Then, we have (cf. [23], Theorem 2.9)

Theorem 1. Exactly one of the following alternatives holds: (i) the equation $\mathbf{x}A = 0$ has a *semipositive* solution; (ii) the inequality $A\mathbf{y} > 0$ has a solution.

We observe that, choosing $a_{hi} = q_{hi} - \mu_i, h = 1, \dots, m, i = 1, \dots, n$, the solvability of $\mathbf{x}A = 0$ means that $\mathcal{M} \in \mathcal{I}$, while the solvability of $A\mathbf{y} > 0$ means that, choosing $s_i = y_i, i = 1, \dots, n$, one has $\min \mathcal{G}_{\mathcal{H}_n} > 0$ (and hence \mathcal{M} would be incoherent). Therefore, by applying Theorem 1 with $A = (q_{hi} - \mu_i)$, we obtain that $\mathcal{M} \in \mathcal{I}$ if and only if $\min \mathcal{G}_{\mathcal{H}_n} \leq 0$, that is, (Σ) is solvable if and only if $\min \mathcal{G}_{\mathcal{H}_n} \leq 0$.

Given a nonempty subset $J \subseteq \{1, \dots, n\}$, we set $\mathcal{F}_J = \{X_j|H_j : j \in J\}$ and $\mathcal{M}_J = (\mu_j : j \in J)$, then we denote by (Σ_J) the system associated to the pair $(\mathcal{F}_J, \mathcal{M}_J)$. Of course, when $J = \{1, \dots, n\}$ it holds that $(\mathcal{F}_J, \mathcal{M}_J) = (\mathcal{F}, \mathcal{M})$ and $(\Sigma_J) = (\Sigma)$. Then, by Definition 1 and Theorem 1, for the prevision assessment \mathcal{M} on \mathcal{F} it holds that

$$\mathcal{M} \text{ is coherent } \iff (\Sigma_J) \text{ is solvable } , \forall J \subseteq \{1, \dots, n\}. \quad (4)$$

In other words \mathcal{M} on \mathcal{F} is coherent if and only if, for every nonempty subset $J \subseteq \{1, \dots, n\}$, the sub-vector \mathcal{M}_J belongs to the convex hull \mathcal{I}_J associated to the pair $(\mathcal{F}_J, \mathcal{M}_J)$.

Given the assessment $\mathcal{M} = (\mu_1, \dots, \mu_n)$ on $\mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\}$, let S be the set of solutions $\Lambda = (\lambda_1, \dots, \lambda_m)$ of system (Σ) defined in (3). We point out that the solvability of system (Σ) (i.e., the condition $\mathcal{M} \in \mathcal{I}$) is a necessary (but not sufficient) condition for coherence of \mathcal{M} on \mathcal{F} . We introduce, for each $i = 1, \dots, n$, the function $\sum_{h: C_h \subseteq H_i} \lambda_h$ of the vector $\Lambda = (\lambda_1, \dots, \lambda_m)$. Moreover, by assuming the system (Σ) solvable, that is $S \neq \emptyset$, we compute the maximum M_i of the function $\sum_{h: C_h \subseteq H_i} \lambda_h$ with respect to $\Lambda \in S$. Then, we define:

$$I_0 = \{i : M_i = 0; i = 1, \dots, n\}, \quad \mathcal{F}_0 = \{X_i|H_i, i \in I_0\}, \quad \mathcal{M}_0 = (\mu_i, i \in I_0). \quad (5)$$

We observe that $i \in I_0$ if and only if the (unique) coherent extension of \mathcal{M} to $H_i|H_n$ is zero.

Of course, the previous notions can be used in the case of conditional events. We observe that, given a probability assessment $\mathcal{P} = (p_1, \dots, p_n)$ on a family of n conditional events $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\}$, we can determine the constituents C_0, C_1, \dots, C_m , where $C_0 = \bar{H}_1 \cdots \bar{H}_n$, and the associated points Q_0, Q_1, \dots, Q_m , where $Q_0 = \mathcal{P}$. We observe that $Q_h = (q_{h1}, \dots, q_{hn})$, with $q_{hi} \in \{1, 0, p_i\}, i = 1, \dots, n$. We also observe that given a subset $J \subset \{1, \dots, n\}$, we can determine the constituents C_{hJ} 's and the corresponding points Q_{hJ} 's associated to the pair $(\mathcal{F}_J, \mathcal{M}_J)$. We set $J^c = \{1, \dots, n\} \setminus J$ and if, for instance, $J = \{1, \dots, r\}$, with $r < n$, then $J^c = \{r+1, \dots, n\}$ and $\mathcal{P} = (\mathcal{P}_J, \mathcal{P}_{J^c})$. Moreover, each point Q_h can be represented, for suitable indexes i_h, k_h , as $Q_h = (Q_{i_h J}, Q_{k_h J^c})$; then, any linear convex combination $\sum_h \lambda_h Q_h$ coincides with $(\sum_h \lambda_h Q_{i_h J}, \sum_h \lambda_h Q_{k_h J^c})$. A similar representation holds for $J = \{i_1, \dots, i_r\}$, after a suitable permutation of indexes. On this basis, we recall three results ([24, Theorems 3.1, 3.2, 3.3]).

Theorem 2. Given a subset $J \subset \{1, \dots, n\}$, if there exist m nonnegative coefficients $\lambda_1, \dots, \lambda_m$, with $\sum_{h=1}^m \lambda_h = 1$, such that $\mathcal{P}_J = \sum_{h=1}^m \lambda_h Q_{i_h J}$, if $\sum_{h: C_h \subseteq H_J} \lambda_h > 0$, where $H_J = \bigvee_{j \in J} H_j$, then $\mathcal{P}_J \in \mathcal{I}_J$.

Theorem 3. If $\mathcal{P} \in \mathcal{I}$, then for every $J \subset \{1, \dots, n\}$ such that $J \setminus I_0 \neq \emptyset$ it holds that $\mathcal{P}_J \in \mathcal{I}_J$.

Theorem 4. The conditional probability assessment $\mathcal{P} = (p_1, \dots, p_n)$ on the family $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\}$ is coherent if and only if the following conditions are satisfied:

(i) $\mathcal{P} \in \mathcal{I}$; (ii) if $I_0 \neq \emptyset$, then \mathcal{P}_0 is coherent.

We remark that when we consider prevision assessments on conditional random quantities results similar to Theorems 2, 3, 4 can be obtained. In particular, by taking into account that $\mathcal{M} \in \mathcal{I}$ amounts to the solvability of system (Σ) , Theorem 4 becomes

Theorem 5. The conditional prevision assessment $\mathcal{M} = (\mu_1, \dots, \mu_n)$ on the family $\mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\}$ is coherent if and only if the following conditions are satisfied:

(i) the system (Σ) in (3) is solvable; (ii) if $I_0 \neq \emptyset$, then \mathcal{M}_0 is coherent.

We observe that, when $I_0 = \emptyset$, coherence of \mathcal{M} amounts to solvability of (Σ) . In order to illustrate the previous results, we examine two examples.

Example 1. Let A, H, K be three events, with A, H, K logically independent. Moreover, let $\mathcal{P} = (x, y)$ be a probability assessment on the family $\mathcal{E} = \{A|H, A|K\}$, where $x = P(A|H)$, $y = P(A|K)$. The constituents generated by \mathcal{E} and contained in $H \vee K$ are: $C_1 = AHK$, $C_2 = \bar{A}HK$, $C_3 = \bar{A}\bar{H}K$, $C_4 = \bar{A}H\bar{K}$, $C_5 = A\bar{H}K$, $C_6 = AH\bar{K}$. Then, the points Q_h 's associated with C_1, \dots, C_6 are: $Q_1 = (1, 1)$, $Q_2 = (0, 0)$, $Q_3 = (x, 0)$, $Q_4 = (0, y)$, $Q_5 = (x, 1)$, $Q_6 = (1, y)$. Moreover $C_0 = \bar{H}\bar{K}$ and $Q_0 = (x, y) = \mathcal{P}$. The condition $\mathcal{P} \in \mathcal{I}$, where \mathcal{I} is the convex hull of the points Q_1, \dots, Q_6 , amounts to the solvability of the system (Σ) below

$$\lambda_1 + \lambda_3 x + \lambda_5 x + \lambda_6 = x, \quad \lambda_1 + \lambda_4 y + \lambda_5 + \lambda_6 y = y, \quad \lambda_1 + \dots + \lambda_6 = 1, \quad \lambda_h \geq 0, \quad \forall h.$$

We observe that, for each $(x, y) \in [0, 1]^2$, the vector $\Lambda = (\lambda_1, \dots, \lambda_6) = (0, 0, \frac{1-y}{2}, \frac{1-x}{2}, \frac{y}{2}, \frac{x}{2})$ is a solution of (Σ) ; indeed $\mathcal{P} = \lambda_1 Q_1 + \dots + \lambda_6 Q_6 = \frac{1-y}{2} Q_3 + \frac{1-x}{2} Q_4 + \frac{y}{2} Q_5 + \frac{x}{2} Q_6 = \frac{1-y}{2}(x, 0) + \frac{1-x}{2}(0, y) + \frac{y}{2}(x, 1) + \frac{x}{2}(1, y) = (x, y)$. Moreover, for this solution it holds that $\sum_{C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_6 = \frac{1}{2} > 0$ and $\sum_{C_h \subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_5 = \frac{1}{2} > 0$. Then, $I_0 = \emptyset$ and by Theorem 5 the assessment (x, y) is coherent, for every $(x, y) \in [0, 1]^2$. Notice that in particular cases, like $x = 0$ or $x = 1$, the number of distinct points Q_h 's is less than 6, anyway the previous analysis is still valid. For instance, when $x = 0$ and $0 < y < 1$ it holds that $Q_2 = Q_3$, $\Lambda = (\lambda_1, \dots, \lambda_6) = (0, 0, \frac{1-y}{2}, \frac{1}{2}, \frac{y}{2}, 0)$, and in geometrical terms it holds that $\mathcal{P} = \frac{1-y}{2} Q_3 + \frac{1}{2} Q_4 + \frac{y}{2} Q_5 = \frac{1-y}{2}(0, 0) + \frac{1}{2}(0, y) + \frac{y}{2}(0, 1) = (0, y)$.

Example 2. Let A, H, K be three events, with $HK = \emptyset$ and A logically independent from H and K . Moreover, let $\mathcal{P} = (x, y)$ be a probability assessment on the family $\mathcal{E} = \{A|H, A|K\}$. The constituents generated by \mathcal{E} are $C_1 = \bar{A}\bar{H}K$, $C_2 = \bar{A}H\bar{K}$, $C_3 = A\bar{H}K$, $C_4 = AH\bar{K}$, $C_0 = \bar{H}\bar{K}$ (which coincide with C_3, C_4, C_5, C_6, C_0 examined in the Example 1, respectively). The associated points Q_h 's are $Q_1 = (x, 0)$, $Q_2 = (0, y)$, $Q_3 = (x, 1)$, $Q_4 = (1, y)$, $\mathcal{P} = (x, y)$. We observe that, for each $(x, y) \in [0, 1]^2$, the vector $(\lambda_1, \dots, \lambda_4) = (\frac{1-y}{2}, \frac{1-x}{2}, \frac{y}{2}, \frac{x}{2})$ is a solution of (Σ) , with $I_0 = \emptyset$. Then, by Theorem 5 the assessment (x, y) is coherent, for every $(x, y) \in [0, 1]^2$.

We recall the following extension theorem for conditional previsions, which is a generalization of de Finetti's fundamental theorem of probability to conditional random quantities (see, e.g., [37, 50, 54])

Theorem 6. Let $\mathcal{M} = (\mu_1, \dots, \mu_n)$ be a coherent prevision assessment on a family of bounded conditional random quantities $\mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\}$. Moreover, let $X|H$ be a further bounded conditional random quantity. Then, there exists a suitable closed interval $[\mu', \mu'']$ such that the extension $\mu = \mathbb{P}(X|H)$ is coherent if and only if $\mu \in [\mu', \mu'']$.

2.2. A deepening on conditional random quantities

The indicator of a conditional event $E|H$ (denoted by the same symbol), with $P(E|H) = x$, is defined as

$$E|H = EH + x\bar{H} = EH + x(1 - H) = \begin{cases} 1, & \text{if } EH \text{ is true,} \\ 0, & \text{if } \bar{E}H \text{ is true,} \\ x, & \text{if } \bar{H} \text{ is true.} \end{cases} \quad (6)$$

Of course, the third value of the random quantity $E|H$ (subjectively) depends on the assessed probability $P(E|H) = x$. Notice that, when $H \subseteq E$ (i.e., $EH = H$), by coherence $P(E|H) = 1$ and hence for the indicator it holds that $E|H = H + P(E|H)\bar{H} = 1$. Moreover, when $EH = \emptyset$, by coherence $P(E|H) = 0$ and hence $E|H = EH + P(E|H)\bar{H} = 0$. The negation of a conditional event $E|H$ is defined as $\bar{E}|\bar{H} = \bar{E}|H = 1 - E|H$. We recall that, in the subjective approach to probability, if you assess $\mathbb{P}(X|H) = \mu$, then you agree to pay μ by knowing that you will receive the amount $XH + \mu\bar{H}$, which coincides with X , if H is true, or with μ , if H is false (bet called off). Usually, in literature the conditional random quantity $X|H$ is defined as the *restriction* of X to H , which coincides with X , when H is true, and it is undefined when H is false. Under this point of view, (when H is false) $X|H$ does not coincide with $XH + \mu\bar{H}$. However, by coherence, it holds that

$$\mathbb{P}(XH + \mu\bar{H}) = \mathbb{P}(XH) + \mu\mathbb{P}(\bar{H}) = \mathbb{P}(X|H)P(H) + \mu P(\bar{H}) = \mu P(H) + \mu P(\bar{H}) = \mu. \quad (7)$$

Therefore, once a coherent assessment $\mu = \mathbb{P}(X|H)$ is specified, we can extend the notion of $X|H$, by defining its value as equal to μ when H is false (for further details see [30]). Then, denoting by x_1, \dots, x_r the possible values of X when H is true, it holds that

$$X|H = XH + \mu\bar{H} \in \{x_1, \dots, x_r, \mu\}. \quad (8)$$

By (7) the prevision of the extended notion of $X|H$, as defined in (8), coincides with the conditional prevision $\mu = \mathbb{P}(X|H)$ where $X|H$ is looked at as the restriction of X to H . From (8) $X|H$ can be interpreted as the amount that you receive when you pay its prevision μ . Then, the random gain G can be also represented as $G = s(X|H - \mu)$. In particular, when X is (the indicator of) an event E , we obtain $E|H = EH + P(E|H)\bar{H}$, that is formula (6). Moreover, the prevision $\mathbb{P}(E|H)$ of (the conditional random quantity) $E|H$ coincides with the conditional probability $P(E|H)$. For related discussions, see also [11, 28, 42].

Remark 1. Given a prevision assessment $\mathcal{M} = (\mu_1, \dots, \mu_n)$ on a family of n conditional random quantities $\{X_1|H_1, \dots, X_n|H_n\}$, based on (8) we observe that for each constituent C_h the corresponding point Q_h represents the value assumed by the random vector $(X_1|H_1, \dots, X_n|H_n)$ when C_h is true. In particular, when C_0 is true the value of the random vector is the prevision point \mathcal{M} .

2.3. Conjunction and disjunction of conditional events

We recall now the notion of conjoined conditional which was introduced in the framework of conditional random quantities ([28, 29, 30, 33]). Given a coherent probability assessment (x, y) on $\{A|H, B|K\}$ we consider the random quantity $AHBK + x\bar{H}BK + y\bar{K}AH$ and we set $\mathbb{P}[(AHBK + x\bar{H}BK + y\bar{K}AH)|(H \vee K)] = z$. Then we define the conjunction $(A|H) \wedge (B|K)$ as follows:

Definition 2. Given a coherent prevision assessment $P(A|H) = x$, $P(B|K) = y$, and $\mathbb{P}[(AHBK + x\bar{H}BK + y\bar{K}AH)|(H \vee K)] = z$, the conjunction $(A|H) \wedge (B|K)$ is the conditional random quantity defined as

$$\begin{aligned} (A|H) \wedge (B|K) &= (AHBK + x\bar{H}BK + y\bar{K}AH)|(H \vee K) = \\ &= (AHBK + x\bar{H}BK + y\bar{K}AH)(H \vee K) + z\bar{H}\bar{K} = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \bar{A}H \vee \bar{B}K \text{ is true,} \\ x, & \text{if } \bar{H}BK \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ z, & \text{if } \bar{H}\bar{K} \text{ is true.} \end{cases} \end{aligned} \quad (9)$$

Of course, by recalling (7), it holds that $\mathbb{P}[(A|H) \wedge (B|K)] = z$. Notice that in (9) the conjunction is represented as $X|H$ is in (8) and, once the (coherent) assessment (x, y, z) is given, the conjunction $(A|H) \wedge (B|K)$ is (subjectively) determined. Conversely, each given conjunction uniquely determines a coherent assessment (x, y, z) . We recall that, in betting terms, z represents the amount you agree to pay, with the proviso that you will receive the quantity $(A|H) \wedge (B|K) = AHBK + x\bar{H}BK + y\bar{K}AH + z\bar{H}\bar{K}$, which assumes one of the following values: 1, if both conditional events are true; 0, if at least one of the conditional events is false; the probability of the conditional event that is void, if one conditional event is void and the other one is true; the payed amount z , if both conditional events are void. We recall that $A|H = B|K$ amounts to $AH = BK$ and $H = K$. Thus, when $A|H = B|K$ it holds that $(AHBK + x\bar{H}BK + y\bar{K}AH)(H \vee K) = AH|H = A|H$, that is $(A|H) \wedge (A|H) = A|H$. Moreover the conjunction is commutative, that is $(A|H) \wedge (B|K) = (B|K) \wedge (A|H)$. The next result shows that the Fréchet-Hoeffding bounds still hold for the conjunction of two conditional events ([30, Theorem 7]).

Theorem 7. Given any coherent assessment (x, y) on $\{A|H, B|K\}$, with A, H, B, K logically independent, and with $H \neq \emptyset, K \neq \emptyset$, the extension $z = \mathbb{P}[(A|H) \wedge (B|K)]$ is coherent if and only if the following Fréchet-Hoeffding bounds are satisfied:

$$\max\{x + y - 1, 0\} = z' \leq z \leq z'' = \min\{x, y\}. \quad (10)$$

From Definition 2 and Theorem 7, it holds that

$$\max\{A|H + B|K - 1, 0\} \leq (A|H) \wedge (B|K) \leq \min\{A|H, B|K\}. \quad (11)$$

Remark 2. We observe that, by logical independence, the assessment (x, y) on $\{A|H, B|K\}$ is coherent for every $(x, y) \in [0, 1]^2$. Then, from Theorems 6 and 7 the set Π of coherent prevision assessments (x, y, z) on $\{A|H, B|K, (A|H) \wedge (B|K)\}$ is

$$\Pi = \{(x, y, z) : (x, y) \in [0, 1]^2, \max\{x + y - 1, 0\} \leq z \leq \min\{x, y\}\}. \quad (12)$$

The set Π is the tetrahedron with vertices the points $(1, 1, 1), (1, 0, 0), (0, 1, 0), (0, 0, 0)$. Notice that, the assumption of logical independence plays a key role for the validity of Theorem 7. Indeed, in case of some logical dependencies, for the interval $[z', z'']$ of coherent extensions z it holds that $\max\{x + y - 1, 0\} \leq z' \leq z'' \leq \min\{x, y\}$. For instance, when $H = K$ and $AB = \emptyset$, the coherence of the assessment (x, y) on $\{A|H, B|H\}$ is equivalent to the condition $x + y - 1 \leq 0$. In this case, it holds that $(A|H) \wedge (B|H) = AB|H$ with $P(AB|H) = 0$; then, the unique coherent extension on $AB|H$ is $z = 0$. As another example, in the case $A = B$, with A, H, K logically independent, it holds that the assessment (x, y) on $\{A|H, A|K\}$ is coherent for every $(x, y) \in [0, 1]^2$. Moreover, as it will be shown by Theorem 17, the extension z is coherent if and only if $xy \leq z \leq \min\{x, y\}$. Finally, we remark that in all cases, for each coherent extension z , it holds that $z \in [z', z''] \subseteq [0, 1]$; thus $(A|H) \wedge (B|K) \in [0, 1]$.

Other approaches to compounded conditionals, which are not based on coherence, can be found in [6, 22, 39, 44]. A study of the lower and upper bounds for other definitions of conjunction, where the conjunction is a conditional event like Adams' quasi conjunction, has been given in [51].

We recall now the notion of disjointed conditional. Given a coherent probability assessment (x, y) on $\{A|H, B|K\}$ we consider the random quantity $(AH \vee BK) + x\bar{H}\bar{B}K + y\bar{K}\bar{A}H$ and we set $\mathbb{P}[(AH \vee BK) + x\bar{H}\bar{B}K + y\bar{K}\bar{A}H)(H \vee K)] = w$. Then we define the disjunction $(A|H) \vee (B|K)$ as follows:

Definition 3. Given a coherent prevision assessment $P(A|H) = x$, $P(B|K) = y$, and $\mathbb{P}[(AH \vee BK) + x\bar{H}\bar{B}K + y\bar{K}\bar{A}H)|(H \vee K)] = w$, the disjunction $(A|H) \vee (B|K)$ is the conditional random quantity

$$\begin{aligned} (A|H) \vee (B|K) &= ((AH \vee BK) + x\bar{H}\bar{B}K + y\bar{K}\bar{A}H)|(H \vee K) = \\ &= ((AH \vee BK) + x\bar{H}\bar{B}K + y\bar{K}\bar{A}H)(H \vee K) + w\bar{H}\bar{K} = \begin{cases} 1, & \text{if } AH \vee BK \text{ is true,} \\ 0, & \text{if } \bar{A}\bar{H}\bar{B}K \text{ is true,} \\ x, & \text{if } \bar{H}\bar{B}K \text{ is true,} \\ y, & \text{if } \bar{A}H\bar{K} \text{ is true,} \\ w, & \text{if } \bar{H}\bar{K} \text{ is true.} \end{cases} \end{aligned} \quad (13)$$

We recall the notion of conjunction of n conditional events ([33]).

Definition 4. Let n conditional events $E_1|H_1, \dots, E_n|H_n$ be given. For each non-empty strict subset S of $\{1, \dots, n\}$, let x_S be a prevision assessment on $\bigwedge_{i \in S} (E_i|H_i)$. Then, the conjunction $(E_1|H_1) \wedge \dots \wedge (E_n|H_n)$ is the conditional random quantity $\mathcal{C}_{1\dots n}$ defined as

$$\begin{aligned} \mathcal{C}_{1\dots n} &= [\bigwedge_{i=1}^n E_i H_i + \sum_{\emptyset \neq S \subset \{1, 2, \dots, n\}} x_S (\bigwedge_{i \in S} \bar{H}_i) \wedge (\bigwedge_{i \notin S} E_i H_i)] | (\bigvee_{i=1}^n H_i) = \\ &= \begin{cases} 1, & \text{if } \bigwedge_{i=1}^n E_i H_i \text{ is true,} \\ 0, & \text{if } \bigvee_{i=1}^n \bar{E}_i H_i \text{ is true,} \\ x_S, & \text{if } (\bigwedge_{i \in S} \bar{H}_i) \wedge (\bigwedge_{i \notin S} E_i H_i) \text{ is true, } \emptyset \neq S \subset \{1, 2, \dots, n\}, \\ x_{1\dots n}, & \text{if } \bigwedge_{i=1}^n \bar{H}_i \text{ is true,} \end{cases} \end{aligned} \quad (14)$$

where

$$x_{1\dots n} = x_{\{1, \dots, n\}} = \mathbb{P}(\mathcal{C}_{1\dots n}) = \mathbb{P}[(\bigwedge_{i=1}^n E_i H_i + \sum_{\emptyset \neq S \subset \{1, 2, \dots, n\}} x_S (\bigwedge_{i \in S} \bar{H}_i) \wedge (\bigwedge_{i \notin S} E_i H_i)) | (\bigvee_{i=1}^n H_i)].$$

Of course, we obtain $\mathcal{C}_1 = E_1|H_1$, when $n = 1$. In Definition 4 each possible value x_S of $\mathcal{C}_{1\dots n}$, $\emptyset \neq S \subset \{1, \dots, n\}$, is evaluated when defining (in a previous step) the conjunction $\mathcal{C}_S = \bigwedge_{i \in S} (E_i|H_i)$. Then, after the conditional prevision $x_{1\dots n}$ is evaluated, $\mathcal{C}_{1\dots n}$ is completely specified. Of course, we require coherence for the prevision assessment $(x_S, \emptyset \neq S \subseteq \{1, \dots, n\})$, so that $\mathcal{C}_{1\dots n} \in [0, 1]$. In the framework of the betting scheme, $x_{1\dots n}$ is the amount that you agree to pay with the proviso that you will receive:

- 1, if all conditional events are true;
- 0, if at least one of the conditional events is false;
- the prevision of the conjunction of that conditional events which are void, otherwise. In particular you receive back $x_{1\dots n}$ when all conditional events are void.

The operation of conjunction is associative and commutative ([33, Proposition 1]). We recall below a necessary condition of coherence related with the Fréchet-Hoeffding bounds ([33, Theorem 13]).

Theorem 8. Let $(x_1, \dots, x_n, x_{1\dots n})$ be a coherent prevision assessment on the family $\{E_1|H_1, \dots, E_n|H_n, \mathcal{C}_{1\dots n}\}$. Then, $\max\{\sum_{i=1}^n x_i - (n-1), 0\} \leq x_{1\dots n} \leq \min\{x_1, \dots, x_n\}$.

2.4. Frank t -norms

We recall below the notion of t -norm (see [36, 41, 40]).

Definition 5. A t -norm is a function $T : [0, 1]^2 \rightarrow [0, 1]$ which satisfies, for all $x, y, z \in [0, 1]$, the following four axioms: $T(x, y) = T(y, x)$ (commutativity); $T(x, T(y, z)) = T(T(x, y), z)$ (associativity); $T(x, y) \leq T(x, z)$ whenever $y \leq z$ (monotonicity); $T(x, 1) = x$ (boundary condition).

Some basic t-norms are the *minimum* T_M (which is the greatest t-norm), the product T_P , the *Lukasiewicz* t-norm T_L , given below:

$$T_M(x, y) = \min(x, y), \quad T_P(x, y) = xy, \quad T_L(x, y) = \max(x + y - 1, 0).$$

Frank t-norms are a relevant class of t-norms to which the previous basic ones belong. The Frank t-norm $T_\lambda : [0, 1]^2 \rightarrow [0, 1]$, with parameter $\lambda \in [0, +\infty]$, is defined as

$$T_\lambda(x, y) = \begin{cases} T_M(x, y) = \min\{x, y\}, & \text{if } \lambda = 0, \\ T_P(x, y) = xy, & \text{if } \lambda = 1, \\ T_L(x, y) = \max\{x + y - 1, 0\}, & \text{if } \lambda = +\infty, \\ \log_\lambda\left(1 + \frac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1}\right), & \text{otherwise.} \end{cases} \quad (15)$$

We recall that T_λ is continuous with respect to λ ; moreover, it is decreasing with respect to the parameter λ . Then, for each given $(x, y) \in [0, 1]^2$, it holds that $T_L(x, y) \leq T_\lambda(x, y) \leq T_M(x, y)$, for every $\lambda \in [0, +\infty]$ (see, e.g., [41],[40]). Frank t-norms provide a gradual transition between Lukasiewicz t-norm ($\lambda = +\infty$) and minimum t-norm ($\lambda = 0$). Frank t-norms have been exploited in [8] (see also [13]) with the aim of obtaining the coherent values for the membership function of the intersection of two fuzzy subsets. Since t-norms are associative they can be extended in a unique way to an n -ary operation for arbitrary integer $n \geq 2$ (see [36, 40]).

3. Sharpness of the Fréchet-Hoeffding bounds for the conjunction of n conditional events

In this section we show, under logical independence, the sharpness of Fréchet-Hoeffding bounds for the prevision of the conjunction $\mathcal{C}_{1\dots n}$ and we illustrate some details by considering the case $n = 3$. We also show that the set of all coherent prevision assessments on $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n, \mathcal{C}_{1\dots n}\}$ is convex.

Let $\mathcal{M} = (x_1, \dots, x_n, x_{1\dots n})$ be a prevision assessment on $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n, \mathcal{C}_{1\dots n}\}$, with $E_1, \dots, E_n, H_1, \dots, H_n$ logically independent. In order to determine the constituents generated by the family \mathcal{F} it is enough to consider the constituents C_0, C_1, \dots, C_m generated by the family $\{E_1|H_1, \dots, E_n|H_n\}$, where by logical independence $m + 1 = 3^n$. Indeed, each C_h uniquely determines the value of $\mathcal{C}_{1\dots n}$, that is for each h there exists a unique x_S such that C_h logically implies the event $(\mathcal{C}_{1\dots n} = x_S)$ and hence $C_h \wedge (\mathcal{C}_{1\dots n} = x_S) = C_h$. Then, C_0, C_1, \dots, C_m also represent the constituents generated by the family \mathcal{F} . By Remark 1, for each C_h the associated point Q_h represents the value of the random vector $(E_1|H_1, \dots, E_n|H_n, \mathcal{C}_{1\dots n})$ when C_h is true. The last component of Q_h is the value of $\mathcal{C}_{1\dots n}$ when C_h is true. By Definition 4, we observe that, when the conditioning events H_1, \dots, H_n are all true, it holds that $\mathcal{C}_{1\dots n} \in \{1, 0\}$. In this section we only need to consider the constituents C_h 's such that $C_h \subseteq \bigwedge_{i=1}^n H_i$. For these constituents the associated Q_h 's have binary components, where the last component is 1, or 0, according to whether $C_h = \bigwedge_{i=1}^n E_i H_i$, or $C_h \subseteq (\bigvee_{i=1}^n \bar{E}_i) \wedge (\bigwedge_{i=1}^n H_i)$. Given any subset $\{i_1, \dots, i_k\}$ of $\{1, \dots, n\}$, we set

$$\{i_{k+1}, \dots, i_n\} = \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}.$$

Then, we denote by

$$\mathcal{K} = \{K_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_n}, \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}\}, \quad (16)$$

where

$$K_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_n} = \left(\bigwedge_{i \in \{i_1, \dots, i_k\}} E_i H_i \right) \wedge \left(\bigwedge_{i \in \{i_{k+1}, \dots, i_n\}} \bar{E}_i H_i \right) = E_{i_1} H_{i_1} \cdots E_{i_k} H_{i_k} \bar{E}_{i_{k+1}} H_{i_{k+1}} \cdots \bar{E}_{i_n} H_{i_n},$$

The Fréchet-Hoeffding bounds are

$$T_L(x_1, \dots, x_n) = \max\left\{\sum_{i=1}^n x_i - n + 1, 0\right\}, \quad T_M(x_1, \dots, x_n) = \min\{x_1, \dots, x_n\}.$$

In the next result (Theorem 10) we show that, when propagating the probability assessment $\mathcal{P} = (x_1, \dots, x_n)$, defined on the family of n conditional events $\{E_1|H_1, \dots, E_n|H_n\}$, to their conjunction $\mathcal{C}_{1\dots n}$, under logical independence the prevision assessment $\mathbb{P}(\mathcal{C}_{1\dots n}) = \mu$ is a coherent extension of \mathcal{P} if and only if $\mu \in [\mu'(x_1, \dots, x_n), \mu''(x_1, \dots, x_n)]$, where

$$\mu'(x_1, \dots, x_n) = T_L(x_1, \dots, x_n), \quad \mu''(x_1, \dots, x_n) = T_M(x_1, \dots, x_n).$$

For the convenience of the reader we sketch the proof.

1. We first observe that in general it holds that

$$T_L(x_1, \dots, x_n) \leq \mu'(x_1, \dots, x_n) \leq \mu''(x_1, \dots, x_n) \leq T_M(x_1, \dots, x_n).$$

2. We show that $\mu'(x_1, \dots, x_n) = T_L(x_1, \dots, x_n)$, by verifying the coherence of the assessment $(x_1, \dots, x_n, T_L(x_1, \dots, x_n))$ on $\{E_1|H_1, \dots, E_n|H_n, \mathcal{C}_{1\dots n}\}$. Based on Theorem 9, we verify the coherence of $(x_1, \dots, x_n, T_L(x_1, \dots, x_n))$ by showing that the associated system (Σ_n^*) is solvable, for each n . We proceed by induction. We assume (Σ_n^*) solvable and then we verify the solvability of (Σ_{n+1}^*) , by separately examining two cases: (i) $T_L(x_1, \dots, x_n) = 0$, (ii) $T_L(x_1, \dots, x_n) > 0$. In the case (i) for the assessment $P(E_{n+1}|H_{n+1}) = x_{n+1}$ we distinguish three sub-cases: $x_{n+1} = 0, x_{n+1} = 1, 0 < x_{n+1} < 1$. In the case (ii) we give an explicit solution of (Σ_n^*) and a related solution for (Σ_{n+1}^*) , by distinguishing two sub-cases which concern x_{n+1} : (ii.a) $0 \leq x_{n+1} \leq n - \sum_{i=1}^n x_i$ (with three further sub-cases); (ii.b) $n - \sum_{i=1}^n x_i < x_{n+1} \leq 1$.

3. We show that $\mu''(x_1, \dots, x_n) = T_M(x_1, \dots, x_n)$. We verify the coherence of the assessment $(x_1, \dots, x_n, T_M(x_1, \dots, x_n))$ on $\{E_1|H_1, \dots, E_n|H_n, \mathcal{C}_{1\dots n}\}$, by providing an explicit solution of (Σ_n^*) and by applying Theorem 9.

Theorem 10. Let $E_1, \dots, E_n, H_1, \dots, H_n$ be logically independent events, with $H_1 \neq \emptyset, \dots, H_n \neq \emptyset$, $n \geq 2$. The set Π of all prevision coherent assessments $\mathcal{M} = (x_1, \dots, x_n, x_{1\dots n})$ on the family $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n, \mathcal{C}_{1\dots n}\}$ is

$$\Pi = \{(x_1, \dots, x_n, x_{1\dots n}) : (x_1, \dots, x_n) \in [0, 1]^n, x_{1\dots n} \in [T_L(x_1, \dots, x_n), T_M(x_1, \dots, x_n)]\}. \quad (21)$$

Proof. Given any integer $n \geq 2$, by logical independence of $E_1, \dots, E_n, H_1, \dots, H_n$ each point $(x_1, \dots, x_n) \in [0, 1]^n$ is a coherent assessment on $\{E_1|H_1, \dots, E_n|H_n\}$ ([25, Proposition 11]). Moreover, by Theorem 6, for each $(x_1, \dots, x_n) \in [0, 1]^n$ there exist two values $\mu'(x_1, \dots, x_n)$ and $\mu''(x_1, \dots, x_n)$ such that $x_{1\dots n}$ is a coherent extension of (x_1, \dots, x_n) if and only if $x_{1\dots n} \in [\mu'(x_1, \dots, x_n), \mu''(x_1, \dots, x_n)]$. Then,

$$\Pi = \{(x_1, \dots, x_n, x_{1\dots n}) : (x_1, \dots, x_n) \in [0, 1]^n, x_{1\dots n} \in [\mu'(x_1, \dots, x_n), \mu''(x_1, \dots, x_n)]\}.$$

By Theorem 8, coherence requires that $x_{1\dots n} \in [T_L(x_1, \dots, x_n), T_M(x_1, \dots, x_n)]$ and hence

$$T_L(x_1, \dots, x_n) \leq \mu'(x_1, \dots, x_n) \leq \mu''(x_1, \dots, x_n) \leq T_M(x_1, \dots, x_n).$$

Thus, $\Pi \subseteq \{(x_1, \dots, x_n, x_{1\dots n}) : (x_1, \dots, x_n) \in [0, 1]^n, x_{1\dots n} \in [T_L(x_1, \dots, x_n), T_M(x_1, \dots, x_n)]\}$. In order to complete the proof it is enough to show that the two assessments $(x_1, \dots, x_n, T_L(x_1, \dots, x_n))$ and $(x_1, \dots, x_n, T_M(x_1, \dots, x_n))$ are coherent, for every $(x_1, \dots, x_n) \in [0, 1]^n$, that is $\mu'(x_1, \dots, x_n) = T_L(x_1, \dots, x_n)$ and $\mu''(x_1, \dots, x_n) = T_M(x_1, \dots, x_n)$, which amounts to the sharpness of the Fréchet-Hoeffding bounds.

Coherence of $(x_1, \dots, x_n, T_L(x_1, \dots, x_n))$. We will proceed by induction on the solvability of the system (Σ_n^*) associated to the assessment $(x_1, \dots, x_n, T_L(x_1, \dots, x_n))$.

($n = 2$). In this case, by Remark 2, the assessment $\mathcal{M} = (x_1, x_2, T_L(x_1, x_2))$ is coherent because \mathcal{M} belongs to the set of coherent prevision assessments, given by the tetrahedron with vertices the points $(1, 1, 1)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 0)$. Moreover, by recalling (16), for $n = 2$ the constituents which logically imply H_1H_2 are

$$K_{11} = E_1H_1E_2H_2, \quad K_{1\bar{2}} = E_1H_1\bar{E}_2H_2, \quad K_{\bar{1}2} = \bar{E}_1H_1E_2H_2, \quad K_{\bar{1}\bar{2}} = \bar{E}_1H_1\bar{E}_2H_2.$$

The associated points are

$$Q_{11} = (1, 1, 1), \quad Q_{1\bar{2}} = (1, 0, 0), \quad Q_{\bar{1}2} = (0, 1, 0), \quad Q_{\bar{1}\bar{2}} = (0, 0, 0);$$

Thus, the convex hull \mathcal{I}^* of $Q_{12}, Q_{1\bar{2}}, Q_{\bar{1}2}, Q_{\bar{1}\bar{2}}$ coincides with the tetrahedron and hence $\mathcal{M} \in \mathcal{I}^*$, that is (Σ_2^*) is solvable. Indeed, (Σ_2^*) is the following system

$$(\Sigma_2^*) \begin{cases} x_1 = \lambda_{12} + \lambda_{1\bar{2}}, \\ x_2 = \lambda_{12} + \lambda_{\bar{1}2}, \\ T_L(x_1, x_2) = \lambda_{12}, \\ \lambda_{12} + \lambda_{1\bar{2}} + \lambda_{\bar{1}2} + \lambda_{\bar{1}\bar{2}} = 1, \\ \lambda_{i_1 i_2} \geq 0, \quad \forall (i_1, i_2) \in \{1, \bar{1}\} \times \{2, \bar{2}\}, \end{cases} \quad (22)$$

with a solution

$$\Lambda_2 = (\lambda_{12}, \lambda_{1\bar{2}}, \lambda_{\bar{1}2}, \lambda_{\bar{1}\bar{2}}) = (T_L(x_1, x_2), x_2 - T_L(x_1, x_2), x_1 - T_L(x_1, x_2), 1 - x_1 - x_2 + T_L(x_1, x_2)).$$

In particular

$$\Lambda_2 = \begin{cases} (0, x_2, x_1, 1 - x_1 - x_2), & \text{if } x_1 + x_2 - 1 \leq 0, \\ (x_1 + x_2 - 1, 1 - x_1, 1 - x_2, 0), & \text{if } x_1 + x_2 - 1 > 0. \end{cases}$$

We now assume the system (Σ_n^*) associated with the assessment $(x_1, \dots, x_n, T_L(x_1, \dots, x_n))$ is solvable and we show that the system (Σ_{n+1}^*) associated with the assessment $(x_1, \dots, x_{n+1}, T_L(x_1, \dots, x_{n+1}))$ is solvable too. Then, (Σ_n^*) is solvable for every $n \geq 2$ and by Theorem 9 it follows the coherence of the assessment $(x_1, \dots, x_n, T_L(x_1, \dots, x_n))$, for every n .

Let the vector $\Lambda_n = (\lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_n} : \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\})$ be a solution of (Σ_n^*) . Then, by (20), $\lambda_{1 \dots n} = x_{1 \dots n} = T_L(x_1, \dots, x_n)$. When necessary we assume that the components of Λ_n are suitably ordered. Given a further conditional event $E_{n+1}|H_{n+1}$, with $P(E_{n+1}|H_{n+1}) = x_{n+1}$, the system (Σ_{n+1}^*) associated with the assessment $(x_1, \dots, x_{n+1}, T_L(x_1, \dots, x_{n+1}))$ on $\{E_1|H_1, \dots, E_{n+1}|H_{n+1}, \mathcal{C}_{1 \dots n+1}\}$ is

$$(\Sigma_{n+1}^*) \begin{cases} x_j = \sum_{\{j\} \subseteq \{i_1, \dots, i_k\} \subseteq \{1, \dots, n+1\}} \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_{n+1}}, \quad j = 1, \dots, n+1, \\ T_L(x_1, \dots, x_{n+1}) = \lambda_{1 \dots n+1}, \\ \sum_{\{i_1, \dots, i_k\} \subseteq \{1, \dots, n+1\}} \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_{n+1}} = 1, \\ \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_{n+1}} \geq 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, n+1\}. \end{cases} \quad (23)$$

Based on Λ_n we will find a solution Λ_{n+1} of (Σ_{n+1}^*) such that

$$\lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_n} = \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_{n+1}} + \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_n \bar{n+1}}, \quad \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}. \quad (24)$$

Then, the system (Σ_{n+1}^*) becomes

$$\left\{ \begin{array}{l} \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_n} = \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_{n+1}} + \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_{n+1}}, \quad \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}, \\ x_j = \sum_{\{j\} \subseteq \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}} \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_{n+1}} + \sum_{\{j\} \subseteq \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}} \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_{n+1}}, \quad j = 1, \dots, n, \\ x_{n+1} = \sum_{\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}} \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_{n+1}}, \\ T_L(x_1, \dots, x_{n+1}) = \lambda_{1 \dots n+1}, \\ \sum_{\{i_1, \dots, i_k\} \subseteq \{1, \dots, n+1\}} \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_{n+1}} = 1, \\ \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_{n+1}} \geq 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, n+1\}. \end{array} \right. \quad (25)$$

We distinguish two cases: (i) $T_L(x_1, \dots, x_n) = 0$, that is $x_1 + \dots + x_n - n + 1 \leq 0$; (ii) $T_L(x_1, \dots, x_n) > 0$, that is $x_1 + \dots + x_n - n + 1 > 0$.

Case (i). As $T_L(x_1, \dots, x_n) = 0$, it holds that $T_L(x_1, \dots, x_{n+1}) = T_L(T_L(x_1, \dots, x_n), x_{n+1}) = T_L(0, x_{n+1}) = 0$. Moreover, for the component $\lambda_{1 \dots n}$ of the vector Λ_n it holds that $\lambda_{1 \dots n} = 0$; hence, in (25), $\lambda_{1 \dots n \bar{n+1}} = 0$ and $\lambda_{1 \dots n n+1} = 0$, which satisfies the equation $T_L(x_1, \dots, x_{n+1}) = \lambda_{1 \dots n+1}$. We first examine the particular cases where $x_{n+1} = 0$, or $x_{n+1} = 1$; then, we consider the case $0 < x_{n+1} < 1$.

If $x_{n+1} = 0$, the system (25) becomes

$$\left\{ \begin{array}{l} \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_n} = \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_{n+1}}, \quad \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}, \\ \lambda_{1 \dots n \bar{n+1}} = 0, \\ x_j = \sum_{\{j\} \subseteq \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}} \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_{n+1}}, \quad j = 1, \dots, n, \\ x_{n+1} = \sum_{\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}} \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_{n+1}} = 0, \\ T_L(x_1, \dots, x_{n+1}) = \lambda_{1 \dots n+1} = 0, \\ \sum_{\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}} \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_{n+1}} = 1 - x_{n+1} = 1, \\ \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_{n+1}} \geq 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, n+1\}, \end{array} \right. \quad (26)$$

with a solution $\Lambda_{n+1,0}$ given by

$$\Lambda_{n+1,0} = (\lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_{n+1}}, \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_{n+1}}, \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}),$$

where $\lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_{n+1}} = 0$ and $\lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_{n+1}} = \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_n}$, $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$, with in particular $\lambda_{1 \dots n \bar{n+1}} = \lambda_{1 \dots n} = 0$. Thus (Σ_{n+1}^*) is solvable when $T_L(x_1, \dots, x_n) = 0$ and $x_{n+1} = 0$. We also observe that $\Lambda_{n+1,0}$ has the following structure

$$(\lambda_{1 \dots n+1}, \dots, \lambda_{\bar{1} \dots \bar{n} n+1}, \lambda_{1 \dots n \bar{n+1}}, \dots, \lambda_{\bar{1} \dots \bar{n} n+1}) = (0, \dots, 0, \lambda_{1 \dots n}, \dots, \lambda_{\bar{1} \dots \bar{n}}) = (\mathbf{0}_n, \Lambda_n), \quad (27)$$

where $\mathbf{0}_n$ is the subvector $(0, \dots, 0)$ of length 2^n and Λ_n is a solution of (Σ_n^*) .

Indeed, Λ_n is a solution of (Σ_n^*) as shown below

$$\begin{cases} x_j = \lambda_{1\dots n} + \sum_{k=1}^n \lambda_{1\dots k-1\bar{k}k+1\dots n} - \lambda_{1\dots j-1\bar{j}j+1\dots n}, & j = 1, 2, \dots, n, \\ T_L(x_1, \dots, x_n) = \lambda_{1\dots n}, \\ \lambda_{i_1\dots i_k\bar{i}_{k+1}\dots\bar{i}_n} = 0, & \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}, \quad k < n-1, \\ \sum_{\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}} \lambda_{i_1\dots i_k\bar{i}_{k+1}\dots\bar{i}_n} = \lambda_{1\dots n} + \sum_{k=1}^n \lambda_{1\dots k-1\bar{k}k+1\dots n} = 1. \end{cases} \quad (32)$$

Based on (31) the system (25) becomes

$$\begin{cases} \lambda_{1\dots n+1} = T_L(x_1, \dots, x_{n+1}), \\ \lambda_{1\dots n+1} + \lambda_{1\dots n\bar{n}+1} = \lambda_{1\dots n} = x_1 + \dots + x_n - n + 1, \\ \lambda_{1\dots r-1\bar{r}r+1\dots n+1} + \lambda_{1\dots r-1\bar{r}r+1\dots n\bar{n}+1} = \lambda_{1\dots r-1\bar{r}r+1\dots n} = 1 - x_r, \quad r = 1, \dots, n, \\ \lambda_{1\dots n+1} + \lambda_{\bar{1}2\dots n+1} + \lambda_{\bar{1}\bar{2}3\dots n+1} + \dots + \lambda_{\bar{1}2\dots n-1\bar{n}n+1} = x_{n+1}, \\ \lambda_{1\dots n\bar{n}+1} + \lambda_{\bar{1}2\dots n\bar{n}+1} + \lambda_{\bar{1}\bar{2}3\dots n\bar{n}+1} + \dots + \lambda_{\bar{1}2\dots n-1\bar{n}n+1} = 1 - x_{n+1}, \\ \lambda_{i_1\dots i_k\bar{i}_{k+1}\dots\bar{i}_n n+1} + \lambda_{i_1\dots i_k\bar{i}_{k+1}\dots\bar{i}_n \bar{n}+1} = \lambda_{i_1\dots i_k\bar{i}_{k+1}\dots\bar{i}_n} = 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}, \quad k < n-1, \\ \lambda_{i_1\dots i_k\bar{i}_{k+1}\dots\bar{i}_n n+1} \geq 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, n+1\}. \end{cases} \quad (33)$$

Based on the solution of (Σ_n^*) given in (31) we find a solution of (33), which of course is also a solution of (Σ_{n+1}^*) . We observe that, as $T_L(x_1, \dots, x_n) > 0$, it holds that $n - \sum_{i=1}^n x_i = 1 - T_L(x_1, \dots, x_n) < 1$; then, we distinguish two sub-cases which concern x_{n+1} :

(ii.a) $0 \leq x_{n+1} \leq n - \sum_{i=1}^n x_i < 1$; (ii.b) $n - \sum_{i=1}^n x_i < x_{n+1} \leq 1$.

Sub-case (ii.a). In this case $T_L(x_1, \dots, x_{n+1}) = 0$. We analyze separately three cases:

(ii.a.1) $x_{n+1} = 0$; (ii.a.2) $x_{n+1} = n - x_1 - \dots - x_n$; (ii.a.3) $0 < x_{n+1} < n - x_1 - \dots - x_n$.

In case (ii.a.1), as $x_{n+1} = 0$, the system (33) becomes

$$\begin{cases} \lambda_{1\dots n+1} = 0, \\ \lambda_{1\dots n\bar{n}+1} = x_1 + \dots + x_n - n + 1, \\ \lambda_{1\dots r-1\bar{r}r+1\dots n\bar{n}+1} = 1 - x_r, \quad r = 1, \dots, n, \\ \lambda_{\bar{1}2\dots n+1} = \lambda_{\bar{1}\bar{2}3\dots n+1} = \dots = \lambda_{\bar{1}2\dots n-1\bar{n}n+1} = x_{n+1} = 0, \\ \lambda_{1\dots n\bar{n}+1} + \lambda_{\bar{1}2\dots n\bar{n}+1} + \lambda_{\bar{1}\bar{2}3\dots n\bar{n}+1} + \dots + \lambda_{\bar{1}2\dots n-1\bar{n}n+1} = 1 - x_{n+1} = 1, \\ \lambda_{i_1\dots i_k\bar{i}_{k+1}\dots\bar{i}_n n+1} = \lambda_{i_1\dots i_k\bar{i}_{k+1}\dots\bar{i}_n \bar{n}+1} = 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}, \quad k < n-1, \\ \lambda_{i_1\dots i_k\bar{i}_{k+1}\dots\bar{i}_n n+1} \geq 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, n+1\}, \end{cases} \quad (34)$$

with a solution $\Lambda_{n+1,0}$ given by

$$\Lambda_{n+1,0} = (\lambda_{1\dots n+1}, \dots, \lambda_{\bar{1}\dots\bar{n}n+1}, \lambda_{1\dots n\bar{n}+1}, \dots, \lambda_{\bar{1}\dots\bar{n}n+1}),$$

where

$$\begin{cases} \lambda_{1\dots n+1} = 0, \\ \lambda_{1\dots r-1\bar{r}r+1\dots n+1} = 0, \quad r = 1, \dots, n, \\ \lambda_{i_1\dots i_k\bar{i}_{k+1}\dots\bar{i}_n n+1} = 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}, \quad k < n-1, \\ \lambda_{1\dots n\bar{n}+1} = x_1 + \dots + x_n - n + 1, \\ \lambda_{1\dots r-1\bar{r}r+1\dots n\bar{n}+1} = 1 - x_r, \quad r = 1, \dots, n, \\ \lambda_{i_1\dots i_k\bar{i}_{k+1}\dots\bar{i}_n n+1} = 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}, \quad k < n-1. \end{cases} \quad (35)$$

Thus, (Σ_{n+1}^*) is solvable when $T_L(x_1, \dots, x_n) > 0$ and $x_{n+1} = 0$. We observe that, from (33) and (35), it holds that

$$\Lambda_{n+1,0} = (0, \dots, 0, \lambda_{1\dots n}, \dots, \lambda_{\bar{1}\dots\bar{n}}) = (\mathbf{0}_n, \Lambda_n). \quad (36)$$

In case (ii.a.2), we preliminarily observe that, if $n - (x_1 + \dots + x_n) = \sum_{r=1}^n (1 - x_r) = 0$, that is $x_r = 1$, $r = 1, \dots, n$ and hence $T_L(x_1, \dots, x_n) = 1$, then $x_{n+1} = n - (x_1 + \dots + x_n) = 0$, which is the case (ii.a.1) considered before. In this case a solution Λ_{n+1} of (Σ_{n+1}^*) is the vector $\Lambda_{n+1,0}$ given in (35), which becomes

$$\begin{cases} \lambda_{1\dots n+1} = 0, \\ \lambda_{1\dots r-1\bar{r}r+1\dots n+1} = 0, \quad r = 1, \dots, n, \\ \lambda_{i_1\dots i_k\bar{i}_{k+1}\dots \bar{i}_n n+1} = 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}, \quad k < n-1, \\ \lambda_{1\dots n\bar{n}+1} = 1, \\ \lambda_{1\dots r-1\bar{r}r+1\dots n\bar{n}+1} = 0, \quad r = 1, \dots, n, \\ \lambda_{i_1\dots i_k\bar{i}_{k+1}\dots \bar{i}_n \bar{n}+1} = 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}, \quad k < n-1. \end{cases} \quad (37)$$

If $0 < x_{n+1} = n - (x_1 + \dots + x_n) < 1$, then the system (33) becomes

$$\begin{cases} \lambda_{1\dots n+1} = 0, \\ \lambda_{1\dots n\bar{n}+1} = x_1 + \dots + x_n - n + 1, \\ \lambda_{1\dots r-1\bar{r}r+1\dots n+1} + \lambda_{1\dots r-1\bar{r}r+1\dots n\bar{n}+1} = 1 - x_r, \quad r = 1, \dots, n, \\ \lambda_{1\bar{1}2\dots n+1} + \lambda_{1\bar{1}2\bar{3}\dots n+1} + \dots + \lambda_{1\bar{1}2\dots n-1\bar{n}n+1} = x_{n+1} = n - (x_1 + \dots + x_n), \\ \lambda_{1\dots n\bar{n}+1} + \lambda_{1\bar{1}2\dots n\bar{n}+1} + \lambda_{1\bar{1}2\bar{3}\dots n\bar{n}+1} + \dots + \lambda_{1\bar{1}2\dots n-1\bar{n}n+1} = 1 - x_{n+1} = x_1 + \dots + x_n - n + 1, \\ \lambda_{i_1\dots i_k\bar{i}_{k+1}\dots \bar{i}_n n+1} + \lambda_{i_1\dots i_k\bar{i}_{k+1}\dots \bar{i}_n \bar{n}+1} = 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}, \quad k < n-1, \\ \lambda_{i_1\dots i_k\bar{i}_{k+1}\dots \bar{i}_n n+1} \geq 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, n+1\}, \end{cases} \quad (38)$$

which, by setting $s = n - (x_1 + \dots + x_n)$, is solvable with a solution given by

$$\Lambda_{n+1,s} = (\lambda_{1\dots n+1}, \dots, \lambda_{1\dots \bar{n}n+1}, \lambda_{1\dots n\bar{n}+1}, \dots, \lambda_{1\dots \bar{n}\bar{n}+1}),$$

where

$$\begin{cases} \lambda_{1\dots n+1} = 0, \\ \lambda_{1\dots r-1\bar{r}r+1\dots n+1} = 1 - x_r, \quad r = 1, \dots, n, \\ \lambda_{i_1\dots i_k\bar{i}_{k+1}\dots \bar{i}_n n+1} = 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}, \quad k < n-1, \\ \lambda_{1\dots n\bar{n}+1} = x_1 + \dots + x_n - n + 1, \\ \lambda_{1\dots r-1\bar{r}r+1\dots n\bar{n}+1} = 0, \quad r = 1, \dots, n, \\ \lambda_{i_1\dots i_k\bar{i}_{k+1}\dots \bar{i}_n \bar{n}+1} = 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}, \quad k < n-1. \end{cases} \quad (39)$$

Thus, (Σ_{n+1}^*) is solvable when $T_L(x_1, \dots, x_n) > 0$ and $x_{n+1} = n - (x_1 + \dots + x_n)$.

In case (ii.a.3), as the vector $(x_1, \dots, x_{n+1}, 0)$ coincides with the linear convex combination

$$\left(1 - \frac{x_{n+1}}{n - (x_1 + \dots + x_n)}\right) \cdot (x_1, \dots, x_n, 0, 0) + \frac{x_{n+1}}{n - (x_1 + \dots + x_n)} \cdot (x_1, \dots, x_n, n - (x_1 + \dots + x_n), 0),$$

the vector

$$\Lambda_{n+1} = \left(1 - \frac{x_{n+1}}{n - (x_1 + \dots + x_n)}\right) \Lambda_{n+1,0} + \frac{x_{n+1}}{n - (x_1 + \dots + x_n)} \Lambda_{n+1,s} \quad (40)$$

is a solution of system (25); thus (Σ_{n+1}^*) is solvable when $T_L(x_1, \dots, x_n) > 0$ and $0 < x_{n+1} < n - (x_1 + \dots + x_n)$. We observe that, from (35) and (39), in explicit terms the components of the solution Λ_{n+1} in (40) are

$$\begin{cases} \lambda_{1\dots n+1} = 0, \\ \lambda_{1\dots r-1\bar{r}r+1\dots n+1} = \frac{x_{n+1}}{n - (x_1 + \dots + x_n)} (1 - x_r), \quad r = 1, \dots, n, \\ \lambda_{i_1\dots i_k\bar{i}_{k+1}\dots \bar{i}_n n+1} = 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}, \quad k < n-1, \\ \lambda_{1\dots n\bar{n}+1} = x_1 + \dots + x_n - n + 1, \\ \lambda_{1\dots r-1\bar{r}r+1\dots n\bar{n}+1} = \left(1 - \frac{x_{n+1}}{n - (x_1 + \dots + x_n)}\right) (1 - x_r), \quad r = 1, \dots, n, \\ \lambda_{i_1\dots i_k\bar{i}_{k+1}\dots \bar{i}_n \bar{n}+1} = 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}, \quad k < n-1. \end{cases} \quad (41)$$

Theorem 11. Let $E_1, \dots, E_n, H_1, \dots, H_n$ be logically independent events, with $H_1 \neq \emptyset, \dots, H_n \neq \emptyset$, $n \geq 2$. The set Π of all prevision coherent assessments $\mathcal{M} = (x_1, \dots, x_n, x_{1\dots n})$ on the family $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n, \mathcal{C}_{1\dots n}\}$ is convex and coincides with the set \mathcal{I}^* .

Proof. Let $\mathcal{M}_1 = (\alpha_1, \dots, \alpha_n, \alpha_{1\dots n})$ and $\mathcal{M}_2 = (\beta_1, \dots, \beta_n, \beta_{1\dots n})$ be two coherent assessments on \mathcal{F} . Given any $a \in [0, 1]$ we show that the assessment

$$\mathcal{M} = a\mathcal{M}_1 + (1-a)\mathcal{M}_2 = (\gamma_1, \dots, \gamma_n, \gamma_{1\dots n})$$

on \mathcal{F} is coherent. From (21) it holds that $(\alpha_1, \dots, \alpha_n) \in [0, 1]^n$ and $(\beta_1, \dots, \beta_n) \in [0, 1]^n$. Then, $(\gamma_1, \dots, \gamma_n) \in [0, 1]^n$. Moreover,

$$T_L(\alpha_1, \dots, \alpha_n) \leq \alpha_{1\dots n} \leq T_M(\alpha_1, \dots, \alpha_n), \quad T_L(\beta_1, \dots, \beta_n) \leq \beta_{1\dots n} \leq T_M(\beta_1, \dots, \beta_n).$$

Then, by taking into account that $\gamma_{1\dots n} = a\alpha_{1\dots n} + (1-a)\beta_{1\dots n}$, it follows

$$aT_L(\alpha_1, \dots, \alpha_n) + (1-a)T_L(\beta_1, \dots, \beta_n) \leq \gamma_{1\dots n} \leq aT_M(\alpha_1, \dots, \alpha_n) + (1-a)T_M(\beta_1, \dots, \beta_n).$$

By recalling (45) and (46), it holds that

$$T_L(\gamma_1, \dots, \gamma_n) \leq aT_L(\alpha_1, \dots, \alpha_n) + (1-a)T_L(\beta_1, \dots, \beta_n) \leq \gamma_{1\dots n}$$

and

$$\gamma_{1\dots n} \leq aT_M(\alpha_1, \dots, \alpha_n) + (1-a)T_M(\beta_1, \dots, \beta_n) \leq T_M(\gamma_1, \dots, \gamma_n).$$

Finally, by observing that

$$(\gamma_1, \dots, \gamma_n) \in [0, 1]^n, \quad T_L(\gamma_1, \dots, \gamma_n) \leq \gamma_{1\dots n} \leq T_M(\gamma_1, \dots, \gamma_n),$$

it follows that \mathcal{M} is coherent, that is $\mathcal{M} \in \Pi$; thus Π is convex.

We now show that $\Pi = \mathcal{I}^*$. For each assessment \mathcal{M} on \mathcal{F} , by Theorem 9, if $\mathcal{M} \in \mathcal{I}^*$, then \mathcal{M} is coherent. Thus $\mathcal{I}^* \subseteq \Pi$. Then, in order to complete the proof we need to show that $\Pi \subseteq \mathcal{I}^*$. Given any $\mathcal{M} = (x_1, \dots, x_n, x_{1\dots n}) \in \Pi$, by Theorem 10 it holds that $x_{1\dots n} \in [T_L(x_1, \dots, x_n), T_M(x_1, \dots, x_n)]$. Then, there exists $\alpha \in [0, 1]$ such that

$$x_{1\dots n} = \alpha T_L(x_1, \dots, x_n) + (1-\alpha)T_M(x_1, \dots, x_n),$$

and hence

$$\mathcal{M} = (x_1, \dots, x_n, x_{1\dots n}) = \alpha(x_1, \dots, x_n, T_L(x_1, \dots, x_n)) + (1-\alpha)(x_1, \dots, x_n, T_M(x_1, \dots, x_n)).$$

We denote by Λ_L , or Λ_M , a solution of the system (Σ_n^*) associated with the assessment $(x_1, \dots, x_n, T_L(x_1, \dots, x_n))$, or the assessment $(x_1, \dots, x_n, T_M(x_1, \dots, x_n))$, respectively. Then, the vector $\Lambda = \alpha\Lambda_L + (1-\alpha)\Lambda_M$ is a solution of the system (Σ_n^*) associated with the assessment $\mathcal{M} = (x_1, \dots, x_n, x_{1\dots n})$; thus $\mathcal{M} \in \mathcal{I}^*$, so that $\Pi \subseteq \mathcal{I}^*$. Therefore $\mathcal{I}^* = \Pi$. \square

3.2. An illustration of sharpness of the Fréchet-Hoeffding bounds in the case of three conditional events

In this section, to better understand the previous general results, we illustrate some details which concern the case of three conditional events. Given any logically independent events $E_1, E_2, E_3, H_1, H_2, H_3$, let $\mathcal{M} = (x_1, x_2, x_3, x_{123})$ be a prevision assessment on $\mathcal{F} = \{E_1|H_1, E_2|H_2, E_3|H_3, \mathcal{C}_{123}\}$. The constituents C_h 's which imply $H_1H_2H_3$ are

$$K_{123} = E_1E_2E_3H_1H_2H_3, K_{1\bar{2}3} = E_1\bar{E}_2E_3H_1H_2H_3, K_{\bar{1}23} = \bar{E}_1E_2E_3H_1H_2H_3, K_{\bar{1}\bar{2}3} = \bar{E}_1\bar{E}_2E_3H_1H_2H_3, \\ K_{12\bar{3}} = E_1E_2\bar{E}_3H_1H_2H_3, K_{\bar{1}2\bar{3}} = E_1\bar{E}_2\bar{E}_3H_1H_2H_3, K_{1\bar{2}\bar{3}} = \bar{E}_1E_2\bar{E}_3H_1H_2H_3, K_{\bar{1}\bar{2}\bar{3}} = \bar{E}_1\bar{E}_2\bar{E}_3H_1H_2H_3.$$

The associated points $Q_{123}, \dots, Q_{\bar{1}\bar{2}\bar{3}}$ are

$$Q_{123} = (1, 1, 1, 1), Q_{1\bar{2}3} = (1, 0, 1, 0), Q_{\bar{1}23} = (0, 1, 1, 0), Q_{\bar{1}\bar{2}3} = (0, 0, 1, 0), \\ Q_{12\bar{3}} = (1, 1, 0, 0), Q_{\bar{1}2\bar{3}} = (1, 0, 0, 0), Q_{1\bar{2}\bar{3}} = (0, 1, 0, 0), Q_{\bar{1}\bar{2}\bar{3}} = (0, 0, 0, 0).$$

In this case system (Σ_n^*) in (20) becomes

$$(\Sigma_3^*) \begin{cases} x_{123} = \lambda_{123}, \\ x_1 = \lambda_{123} + \lambda_{1\bar{2}3} + \lambda_{\bar{1}23} + \lambda_{\bar{1}\bar{2}3}, \\ x_2 = \lambda_{123} + \lambda_{\bar{1}23} + \lambda_{12\bar{3}} + \lambda_{\bar{1}2\bar{3}}, \\ x_3 = \lambda_{123} + \lambda_{1\bar{2}3} + \lambda_{\bar{1}23} + \lambda_{\bar{1}\bar{2}3}, \\ \lambda_{123} + \dots + \lambda_{\bar{1}\bar{2}\bar{3}} = 1, \lambda_{123} \geq 0, \dots, \lambda_{\bar{1}\bar{2}\bar{3}} \geq 0. \end{cases}$$

By recalling the proof of Theorem (10), we illustrate below the structure of the vector $\Lambda_3 = (\lambda_{123}, \lambda_{1\bar{2}3}, \lambda_{\bar{1}23}, \lambda_{\bar{1}\bar{2}3}, \lambda_{12\bar{3}}, \lambda_{\bar{1}2\bar{3}}, \lambda_{1\bar{2}\bar{3}}, \lambda_{\bar{1}\bar{2}\bar{3}})$, solution of (Σ_3^*) , in the different cases.

Assessment (x_1, x_2, x_3, x_{123}) , with $x_{123} = T_L(x_1, x_2, x_3)$.

We have two cases: (i) $T_L(x_1, x_2) = 0$, that is $x_1 + x_2 - 1 \leq 0$; (ii) $T_L(x_1, x_2) > 0$, that is $x_1 + x_2 - 1 > 0$.

In case (i) $T_L(x_1, x_2) = T_L(x_1, x_2, x_3) = 0$ and we have three sub-cases: $x_3 = 0$, or $x_3 = 1$, or $0 < x_3 < 1$.

If $x_3 = 0$ the system (Σ_3^*) has a solution

$$\Lambda_{3,0} = (0, 0, 0, 0, 0, x_1, x_2, 1 - x_1 - x_2).$$

If $x_3 = 1$ the system (Σ_3^*) has a solution

$$\Lambda_{3,1} = (0, x_1, x_2, 1 - x_1 - x_2, 0, 0, 0, 0).$$

If $0 < x_3 < 1$ the system (Σ_3^*) has a solution

$$\Lambda_3 = (1 - x_3)\Lambda_{3,0} + x_3\Lambda_{3,1} = \\ = (0, x_1x_3, x_2x_3, (1 - x_1 - x_2)x_3, 0, x_1(1 - x_3), x_2(1 - x_3), (1 - x_1 - x_2)(1 - x_3)). \quad (47)$$

In case (ii), where $T_L(x_1, x_2) > 0$, we have two sub-cases:

(ii.a) $0 \leq x_3 \leq 2 - x_1 - x_2 < 1$; (ii.b) $2 - x_1 - x_2 < x_3 \leq 1$.

Sub-case (ii.a). We have three cases:

(ii.a.1) $x_3 = 0$; (ii.a.2) $x_3 = 2 - x_1 - x_2$; (ii.a.3) $0 < x_3 < 2 - x_1 - x_2$.

If $x_3 = 0$ the system (Σ_3^*) has a solution

$$\Lambda_{3,0} = (0, 0, 0, 0, x_1 + x_2 - 1, 1 - x_2, 1 - x_1, 0). \quad (48)$$

If $x_3 = 2 - x_1 - x_2 = 0$, a solution of (Σ_3^*) is the vector in (48).

If $x_3 = 2 - x_1 - x_2 > 0$, by setting $s = 2 - x_1 - x_2$, the system (Σ_3^*) has a solution

$$\Lambda_{3,s} = (0, 1 - x_2, 1 - x_1, 0, x_1 + x_2 - 1, 0, 0, 0).$$

If $0 < x_3 < 2 - x_1 - x_2$ the system (Σ_3^*) has a solution

$$\begin{aligned} \Lambda_3 &= \left(1 - \frac{x_3}{2-x_1-x_2}\right)\Lambda_{3,0} + \frac{x_3}{2-x_1-x_2}\Lambda_{3,s} = \\ &= \left(0, \frac{(1-x_2)x_3}{2-x_1-x_2}, \frac{(1-x_1)x_3}{2-x_1-x_2}, 0, x_1 + x_2 - 1, \left(1 - \frac{x_3}{2-x_1-x_2}\right)(1-x_2), \left(1 - \frac{x_3}{2-x_1-x_2}\right)(1-x_1), 0\right). \end{aligned} \quad (49)$$

Sub-case (ii.b). The system (Σ_3^*) has a solution

$$\Lambda_3 = (x_1 + x_2 + x_3 - 2, 1 - x_2, 1 - x_1, 0, 1 - x_3, 0, 0, 0).$$

Assessment (x_1, x_2, x_3, x_{123}) , with $x_{123} = T_M(x_1, x_2, x_3)$. We assume that $0 \leq x_1 \leq x_2 \leq x_3 \leq 1$, so that $x_{123} = T_M(x_1, x_2, x_3) = x_1$. The system (Σ_3^*) has a solution

$$\Lambda_3 = (\lambda_{123}, \lambda_{1\bar{2}3}, \lambda_{\bar{1}23}, \lambda_{\bar{1}\bar{2}3}, \lambda_{12\bar{3}}, \lambda_{1\bar{2}\bar{3}}, \lambda_{\bar{1}2\bar{3}}, \lambda_{\bar{1}\bar{2}\bar{3}}) = (x_1, 0, x_2 - x_1, x_3 - x_2, 0, 0, 0, 1 - x_3).$$

We give below two examples.

Example 3. Let $(x_1, x_2, x_3, T_L(x_1, x_2, x_3)) = (0.4, 0.4, 0.4, 0)$ be a prevision assessment on $\mathcal{F} = \{E_1|H_1, E_2|H_2, E_3|H_3, \mathcal{C}_{123}\}$. We observe that $x_1 + x_2 - 1 = -0.2 < 0$; then, based on (47), the system (Σ_3^*) has a solution

$$\Lambda_3 = (\lambda_{123}, \lambda_{1\bar{2}3}, \lambda_{\bar{1}23}, \lambda_{\bar{1}\bar{2}3}, \lambda_{12\bar{3}}, \lambda_{1\bar{2}\bar{3}}, \lambda_{\bar{1}2\bar{3}}, \lambda_{\bar{1}\bar{2}\bar{3}}) = (0, 0.16, 0.16, 0.08, 0, 0.24, 0.24, 0.12).$$

Example 4. Let $(x_1, x_2, x_3, T_L(x_1, x_2, x_3)) = (0.5, 0.6, 0.7, 0)$ be a prevision assessment on $\mathcal{F} = \{E_1|H_1, E_2|H_2, E_3|H_3, \mathcal{C}_{123}\}$. As $x_1 + x_2 - 1 = 0.1 > 0$, based on (49), the system (Σ_3^*) has a solution

$$\Lambda_3 = (\lambda_{123}, \lambda_{1\bar{2}3}, \lambda_{\bar{1}23}, \lambda_{\bar{1}\bar{2}3}, \lambda_{12\bar{3}}, \lambda_{1\bar{2}\bar{3}}, \lambda_{\bar{1}2\bar{3}}, \lambda_{\bar{1}\bar{2}\bar{3}}) = \left(0, \frac{14}{45}, \frac{7}{18}, 0, \frac{1}{10}, \frac{4}{45}, \frac{1}{9}, 0\right).$$

4. On the computation of Λ_{n+1} when $x_{1\dots n+1} = T_L(x_1, \dots, x_{n+1})$

In this section we examine further aspects which concern the prevision assessment $(x_1, \dots, x_{n+1}, T_L(x_1, \dots, x_{n+1}))$. We observe that in the proof of Theorem 10 the explicit solution Λ_{n+1} of the system (Σ_{n+1}^*) is given for the assessment $(x_1, \dots, x_{n+1}, T_L(x_1, \dots, x_{n+1}))$, when $T_L(x_1, \dots, x_n) > 0$. In the case where $T_L(x_1, \dots, x_n) = 0$, for the vector Λ_{n+1} we only have the representation given in (30) in terms of the solution Λ_n . In what follows we give an explicit formula for Λ_{n+1} when $T_L(x_1, \dots, x_n) = 0$.

Given any integer $n \geq 1$, we distinguish two cases: (i) $T_L(x_1, \dots, x_h) = 0$, for all $h = 1, \dots, n$; (ii) $T_L(x_1) > 0, \dots, T_L(x_1, \dots, x_h) > 0, T_L(x_1, \dots, x_{h+1}) = \dots = T_L(x_1, \dots, x_n) = 0$, for some h such that $1 \leq h < n$.

Case (i). If $n = 1$, the assessment is $(x_1, x_2, T_L(x_1, x_2))$ and $T_L(x_1) = x_1 = 0$. Moreover, the (unique) solution of (Σ_1^*) is $\Lambda_1 = (\lambda_1, \lambda_{\bar{1}}) = (x_1, 1 - x_1) = (0, 1)$. Then, by applying (30) with $n = 1$, we obtain the solution

$$\Lambda_2 = (\lambda_{12}, \lambda_{1\bar{2}}, \lambda_{\bar{1}2}, \lambda_{\bar{1}\bar{2}}) = (x_2\Lambda_1, (1 - x_2)\Lambda_1) = (x_2(0, 1), (1 - x_2)(0, 1)) = (0, x_2, 0, 1 - x_2). \quad (50)$$

If $n = 2$, the assessment is $(x_1, x_2, x_3, T_L(x_1, x_2, x_3))$ and $T_L(x_1) = x_1 = T_L(x_1, x_2) = 0$. Moreover, as $x_1 = 0$, it holds that Λ_2 is the vector given in (50) and by applying (30) with $n = 2$, a solution for (Σ_3^*) is given by

$$\begin{aligned}\Lambda_3 &= (\lambda_{123}, \lambda_{\bar{1}23}, \lambda_{1\bar{2}3}, \lambda_{\bar{1}\bar{2}3}, \lambda_{12\bar{3}}, \lambda_{\bar{1}2\bar{3}}, \lambda_{1\bar{2}\bar{3}}, \lambda_{\bar{1}\bar{2}\bar{3}}) = (x_3\Lambda_2, (1-x_3)\Lambda_2) = \\ &= (0, x_2x_3, 0, (1-x_2)x_3, 0, x_2(1-x_3), 0, (1-x_2)(1-x_3)).\end{aligned}$$

More in general, by iterating (30) we obtain

$$\begin{aligned}\Lambda_{n+1} &= (\lambda_{1\dots n+1}, \dots, \lambda_{\bar{1}\dots\bar{n}n+1}, \lambda_{1\dots n\bar{n}+1}, \dots, \lambda_{\bar{1}\dots\bar{n}\bar{n}+1}) = \\ &= (x_1 \cdots x_n x_{n+1}, \dots, (1-x_1) \cdots (1-x_n) x_{n+1}, x_1 \cdots x_n (1-x_{n+1}), \dots, (1-x_1) \cdots (1-x_n) (1-x_{n+1})),\end{aligned}$$

where $x_1 = 0$. Alternatively, by setting

$$\Lambda_{n+1} = (\lambda_{1^* \dots (n+1)^*}; (1^*, \dots, (n+1)^*) \in \{1, \bar{1}\} \times \cdots \times \{n+1, \overline{n+1}\}),$$

and

$$x_{j^*} = \begin{cases} x_j, & \text{if } j^* = j, \\ x_{\bar{j}} = 1 - x_j, & \text{if } j^* = \bar{j}, \end{cases} \quad (51)$$

it holds that

$$\lambda_{1^* \dots (n+1)^*} = \prod_{j=1}^{n+1} x_{j^*}, \quad (1^*, \dots, (n+1)^*) \in \{1, \bar{1}\} \times \cdots \times \{n+1, \overline{n+1}\}, \quad (52)$$

where $\prod_{j=1}^{n+1} x_{j^*} = 0$, if $1^* = 1$, because $x_1 = 0$, and $\prod_{j=1}^{n+1} x_{j^*} = \prod_{j=2}^{n+1} x_{j^*}$, if $1^* = \bar{1}$, because $x_{\bar{1}} = 1 - x_1 = 1$.

Case (ii). For $t = h+1, \dots, n$, it holds that $T_L(x_1, \dots, x_{t+1}) = 0$ and, from (30), it holds that

$$\begin{aligned}\Lambda_{t+1} &= (\lambda_{1\dots t+1}, \dots, \lambda_{\bar{1}\dots\bar{t}t+1}, \lambda_{1\dots t\bar{t}+1}, \dots, \lambda_{\bar{1}\dots\bar{t}\bar{t}+1}) = \\ &= (x_{t+1}\lambda_{1\dots t}, \dots, x_{t+1}\lambda_{\bar{1}\dots\bar{t}}, (1-x_{t+1})\lambda_{1\dots t}, \dots, (1-x_{t+1})\lambda_{\bar{1}\dots\bar{t}}).\end{aligned}$$

Then, based on the representations

$$\Lambda_t = (\lambda_{1^* \dots t^*}; (1^*, \dots, t^*) \in \{1, \bar{1}\} \times \cdots \times \{t, \bar{t}\}),$$

and

$$\Lambda_{t+1} = (\lambda_{1^* \dots (t+1)^*}; (1^*, \dots, (t+1)^*) \in \{1, \bar{1}\} \times \cdots \times \{t+1, \overline{t+1}\}),$$

based on (51), for the components $\lambda_{1^* \dots (t+1)^*}$ and $\lambda_{1^* \dots t^*}$ it holds that

$$\lambda_{1^* \dots (t+1)^*} = \lambda_{1^* \dots t^*} \cdot x_{(t+1)^*}, \quad (53)$$

that is

$$\begin{cases} \lambda_{1^* \dots t^* t+1} = \lambda_{1^* \dots t^*} \cdot x_{t+1}, \\ \lambda_{1^* \dots t^* \bar{t}+1} = \lambda_{1^* \dots t^*} \cdot (1 - x_{t+1}), \end{cases}$$

for every $(1^*, \dots, t^*) \in \{1, \bar{1}\} \times \cdots \times \{t, \bar{t}\}$, $t = h+1, \dots, n$.

By iterating (53) backward from $t = n$ until $t = h+1$, it follows that

$$\lambda_{1^* \dots (n+1)^*} = \lambda_{1^* \dots n^*} \cdot x_{(n+1)^*} = \lambda_{1^* \dots (n-1)^*} \cdot x_{n^*} \cdot x_{(n+1)^*} = \cdots = \lambda_{1^* \dots (h+1)^*} \prod_{t=h+2}^{n+1} x_{t^*}. \quad (54)$$

Thus, in order to determine the vector Λ_{n+1} we need to compute the vector Λ_{h+1} . We examine below this aspect.

If $0 < T_L(x_1, \dots, x_h) < 1$, as $T_L(x_1, \dots, x_{h+1}) = 0$ it holds that $0 \leq x_{h+1} \leq h - x_1 - \dots - x_h$; then from (41) we obtain

$$\begin{cases} \lambda_{1\dots h+1} = 0, \\ \lambda_{1\dots r-1 \bar{r} r+1 \dots h+1} = \frac{x_{h+1}}{h-(x_1+\dots+x_h)}(1-x_r), \quad r = 1, \dots, h, \\ \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_h h+1} = 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, h\}, \quad k < h-1, \\ \lambda_{1\dots h \bar{h}+1} = x_1 + \dots + x_h - h + 1, \\ \lambda_{1\dots r-1 \bar{r} r+1 \dots h \bar{h}+1} = \left(1 - \frac{x_{h+1}}{h-(x_1+\dots+x_h)}\right)(1-x_r), \quad r = 1, \dots, h, \\ \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_h \bar{h}+1} = 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, h\}, \quad k < h-1. \end{cases} \quad (55)$$

Then, concerning the components of Λ_{n+1} , for every

$$((h+2)^*, \dots, (n+1)^*) \in \{h+2, \bar{h}+2\} \times \dots \times \{n+2, \bar{n}+2\},$$

from (54) and (55) it follows that

$$\begin{cases} \lambda_{1\dots h+1(h+2)^* \dots (n+1)^*} = 0, \\ \lambda_{1\dots r-1 \bar{r} r+1 \dots h+1(h+2)^* \dots (n+1)^*} = (1-x_r) \frac{x_{h+1}}{h-(x_1+\dots+x_h)} \prod_{t=h+2}^{n+1} x_t^*, \quad r = 1, \dots, h, \\ \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_h h+1(h+2)^* \dots (n+1)^*} = 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, h\}, \quad k < h-1, \\ \lambda_{1\dots r-1 \bar{r} r+1 \dots h \bar{h}+1(h+2)^* \dots (n+1)^*} = (1-x_r) \left(1 - \frac{x_{h+1}}{h-(x_1+\dots+x_h)}\right) \prod_{t=h+2}^{n+1} x_t^*, \quad r = 1, \dots, h, \\ \lambda_{1\dots h \bar{h}+1(h+2)^* \dots (n+1)^*} = (x_1 + \dots + x_h - h + 1) \prod_{t=h+2}^{n+1} x_t^*, \\ \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_h \bar{h}+1(h+2)^* \dots (n+1)^*} = 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, h\}, \quad k < h-1. \end{cases} \quad (56)$$

Of course the sum of all the components of Λ_{n+1} is equal to 1, indeed

$$\begin{aligned} (1-x_r) \frac{x_{h+1}}{h-(x_1+\dots+x_h)} \prod_{t=h+2}^{n+1} x_t^* + (1-x_r) \left(1 - \frac{x_{h+1}}{h-(x_1+\dots+x_h)}\right) \prod_{t=h+2}^{n+1} x_t^* &= (1-x_r) \prod_{t=h+2}^{n+1} x_t^*, \\ \sum_{r=1}^h (1-x_r) \prod_{t=h+2}^{n+1} x_t^* &= (h-(x_1+\dots+x_h)) \prod_{t=h+2}^{n+1} x_t^*, \\ (x_1 + \dots + x_h - h + 1) \prod_{t=h+2}^{n+1} x_t^* + (h-(x_1+\dots+x_h)) \prod_{t=h+2}^{n+1} x_t^* &= \prod_{t=h+2}^{n+1} x_t^*; \end{aligned}$$

finally

$$\begin{aligned} \sum_{((h+2)^*, \dots, (n+1)^*)} \prod_{t=h+2}^{n+1} x_t^* &= \sum_{((h+2)^*, \dots, n^*)} \prod_{t=h+2}^n x_t^* (x_{n+1} + 1 - x_{n+1}) = \\ &= \sum_{((h+2)^*, \dots, n^*)} \prod_{t=h+2}^n x_t^* = \dots = x_{h+2} + (1 - x_{h+2}) = 1. \end{aligned}$$

If $T_L(x_1, \dots, x_h) = 1$, as $T_L(x_1, \dots, x_{h+1}) = 0$ it holds that $x_{h+1} = 0$; then, concerning the vector Λ_{h+1} , from (37) we obtain that

$$\begin{cases} \lambda_{1\dots h+1} = 0, \\ \lambda_{1\dots r-1 \bar{r} r+1 \dots h+1} = 0, \quad r = 1, \dots, h, \\ \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_h h+1} = 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, h\}, \quad k < h-1, \\ \lambda_{1\dots h \bar{h}+1} = 1, \\ \lambda_{1\dots r-1 \bar{r} r+1 \dots h \bar{h}+1} = 0, \quad r = 1, \dots, h, \\ \lambda_{i_1 \dots i_k \bar{i}_{k+1} \dots \bar{i}_h \bar{h}+1} = 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, h\}, \quad k < h-1, \end{cases} \quad (57)$$

that is the vector Λ_{h+1} has the component $\lambda_{1\dots h\overline{h+1}}$ equal to 1 and all the other components equal to zero. Then, concerning the components of Λ_{n+1} , for every

$$((h+2)^*, \dots, (n+1)^*) \in \{h+2, \overline{h+2}\} \times \dots \times \{n+2, \overline{n+2}\},$$

from (54) and (57) it follows that

$$\begin{cases} \lambda_{1\dots h+1(h+2)^*\dots(n+1)^*} = 0, \\ \lambda_{1\dots r-1\overline{r}r+1\dots h+1(h+2)^*\dots(n+1)^*} = 0, \quad r = 1, \dots, h, \\ \lambda_{i_1\dots i_k\overline{i_k+1}\dots\overline{i_h}h+1(h+2)^*\dots(n+1)^*} = 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, h\}, \quad k < h-1 \\ \lambda_{1\dots h\overline{h+1}(h+2)^*\dots(n+1)^*} = \prod_{t=h+2}^{n+1} x_t^*, \\ \lambda_{1\dots r-1\overline{r}r+1\dots h\overline{h+1}(h+2)^*\dots(n+1)^*} = 0, \quad r = 1, \dots, h, \\ \lambda_{i_1\dots i_k\overline{i_k+1}\dots\overline{i_h}h+1(h+2)^*\dots(n+1)^*} = 0, \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, h\}, \quad k < h-1. \end{cases} \quad (58)$$

Remark 4. In summary, concerning the problem of giving an explicit solution Λ_{n+1} of the system (Σ_{n+1}^*) for the assessment $(x_1, \dots, x_{n+1}, T_L(x_1, \dots, x_{n+1}))$, we distinguish the following cases:

- (a) $T_L(x_1, \dots, x_n) = 1$ and $T_L(x_1, \dots, x_{n+1}) = 0$ (in which case $x_{n+1} = n - \sum_{i=1}^n x_i = 0$); the solution is given in (37).
- (b) $0 < T_L(x_1, \dots, x_n) < 1$ and $T_L(x_1, \dots, x_{n+1}) = 0$ (in which case $0 \leq x_{n+1} \leq n - \sum_{i=1}^n x_i$, with $0 < n - \sum_{i=1}^n x_i < 1$); the solution is given in (41).
- (c) $T_L(x_1, \dots, x_n) > 0$ and $T_L(x_1, \dots, x_{n+1}) > 0$ (in which case $n - \sum_{i=1}^n x_i < x_{n+1} \leq 1$); the solution is given in (42).
- (d) $T_L(x_1, \dots, x_h) = 0, h = 1, \dots, n+1$ (in which case $x_1 = 0$); the solution is given in (52).
- (e) $0 < T_L(x_1, \dots, x_h) < 1, T_L(x_1, \dots, x_{h+1}) = \dots = T_L(x_1, \dots, x_{n+1}) = 0$, with $1 \leq h < n$; the solution is given in (56).
- (f) $T_L(x_1, \dots, x_h) = 1, T_L(x_1, \dots, x_{h+1}) = \dots = T_L(x_1, \dots, x_{n+1}) = 0$, with $1 \leq h < n$; the solution is given in (58).

We illustrate below the cases (e) and (f) by an example where $n+1 = 5$.

Example 5. Given any logically independent events. Let $E_1, \dots, E_5, H_1, \dots, H_5$ be logically Let $(x_1, \dots, x_5, x_{1\dots 5})$, with $x_{1\dots 5} = T_L(x_1, \dots, x_5)$, be a prevision assessment on $\{E_1|H_1, \dots, E_5|H_5, \mathcal{C}_{1\dots 5}\}$, where $\mathcal{C}_{1\dots 5} = \bigwedge_{i=1}^5 E_i|H_i$. We examine below all the cases of Remark 4.

- (a) $T_L(x_1, \dots, x_4) = 1$ and $T_L(x_1, \dots, x_5) = 0$ (in which case $x_1 = \dots = x_4 = 1$ and $x_5 = 4 - \sum_{i=1}^4 x_i = 0$); the solution obtained from (37) is such that $\lambda_{12345} = 1$, with all the other components of Λ_5 equal to zero.
- (b) $0 < T_L(x_1, \dots, x_4) < 1$ and $T_L(x_1, \dots, x_5) = 0$ (in which case $0 \leq x_5 \leq 4 - \sum_{i=1}^4 x_i$, with $0 < 4 - \sum_{i=1}^4 x_i < 1$); the solution obtained from (41) is

$$\begin{cases} \lambda_{1\overline{2}345} = \frac{x_5}{4-(x_1+x_2+x_3+x_4)}(1-x_1), & \lambda_{\overline{1}2345} = \left(1 - \frac{x_5}{4-(x_1+x_2+x_3+x_4)}\right)(1-x_1), \\ \lambda_{1\overline{1}2345} = \frac{x_5}{4-(x_1+x_2+x_3+x_4)}(1-x_2), & \lambda_{1\overline{1}\overline{2}345} = \left(1 - \frac{x_5}{4-(x_1+x_2+x_3+x_4)}\right)(1-x_2), \\ \lambda_{12\overline{3}45} = \frac{x_5}{4-(x_1+x_2+x_3+x_4)}(1-x_3), & \lambda_{12\overline{3}\overline{4}5} = \left(1 - \frac{x_5}{4-(x_1+x_2+x_3+x_4)}\right)(1-x_3), \\ \lambda_{123\overline{4}5} = \frac{x_5}{4-(x_1+x_2+x_3+x_4)}(1-x_4), & \lambda_{123\overline{4}\overline{5}} = \left(1 - \frac{x_5}{4-(x_1+x_2+x_3+x_4)}\right)(1-x_4), \\ \lambda_{1234\overline{5}} = x_1 + x_2 + x_3 + x_4 - 3, & \lambda_{1^*2^*3^*4^*5^*} = 0, \text{ otherwise.} \end{cases}$$

(c) $T_L(x_1, \dots, x_4) > 0$ and $T_L(x_1, \dots, x_5) > 0$ (in which case $4 - \sum_{i=1}^4 x_i < x_5 \leq 1$); the solution obtained from (42) is

$$\begin{cases} \lambda_{12345} = x_1 + x_2 + x_3 + x_4 + x_5 - 4, \\ \lambda_{\bar{1}2345} = 1 - x_1, \quad \lambda_{1\bar{2}345} = 1 - x_2, \\ \lambda_{12\bar{3}45} = 1 - x_3, \quad \lambda_{123\bar{4}5} = 1 - x_4, \\ \lambda_{1234\bar{5}} = 1 - x_5, \quad \lambda_{1*2*3*4*5*} = 0, \text{ otherwise.} \end{cases}$$

(d) $T_L(x_1) = \dots = T_L(x_1, x_2, x_3, x_4, x_5) = 0$ (in which case $x_1 = 0$); the solution obtained from (52) is

$$\begin{array}{ll} \lambda_{12345} = 0, & \lambda_{1234\bar{5}} = 0, \\ \lambda_{\bar{1}2345} = x_2 x_3 x_4 x_5, & \lambda_{\bar{1}234\bar{5}} = x_2 x_3 x_4 (1 - x_5), \\ \lambda_{1\bar{2}345} = 0, & \lambda_{1\bar{2}34\bar{5}} = 0, \\ \lambda_{\bar{1}\bar{2}345} = (1 - x_2) x_3 x_4 x_5, & \lambda_{\bar{1}\bar{2}34\bar{5}} = (1 - x_2) x_3 x_4 (1 - x_5), \\ \lambda_{12\bar{3}45} = 0, & \lambda_{12\bar{3}4\bar{5}} = 0, \\ \lambda_{\bar{1}2\bar{3}45} = x_2 (1 - x_3) x_4 x_5, & \lambda_{\bar{1}2\bar{3}4\bar{5}} = x_2 (1 - x_3) x_4 (1 - x_5), \\ \lambda_{1\bar{2}\bar{3}45} = 0, & \lambda_{1\bar{2}\bar{3}4\bar{5}} = 0, \\ \lambda_{\bar{1}\bar{2}\bar{3}45} = (1 - x_2)(1 - x_3) x_4 x_5, & \lambda_{\bar{1}\bar{2}\bar{3}4\bar{5}} = (1 - x_2)(1 - x_3) x_4 (1 - x_5), \\ \lambda_{123\bar{4}5} = 0, & \lambda_{123\bar{4}\bar{5}} = 0, \\ \lambda_{\bar{1}23\bar{4}5} = x_2 x_3 (1 - x_4) x_5, & \lambda_{\bar{1}23\bar{4}\bar{5}} = x_2 x_3 (1 - x_4) (1 - x_5), \\ \lambda_{1\bar{2}3\bar{4}5} = 0, & \lambda_{1\bar{2}3\bar{4}\bar{5}} = 0, \\ \lambda_{\bar{1}\bar{2}3\bar{4}5} = (1 - x_2) x_3 (1 - x_4) x_5, & \lambda_{\bar{1}\bar{2}3\bar{4}\bar{5}} = (1 - x_2) x_3 (1 - x_4) (1 - x_5), \\ \lambda_{12\bar{3}\bar{4}5} = 0, & \lambda_{12\bar{3}\bar{4}\bar{5}} = 0, \\ \lambda_{\bar{1}2\bar{3}\bar{4}5} = x_2 (1 - x_3)(1 - x_4) x_5, & \lambda_{\bar{1}2\bar{3}\bar{4}\bar{5}} = x_2 (1 - x_3)(1 - x_4) (1 - x_5), \\ \lambda_{1\bar{2}\bar{3}\bar{4}5} = 0, & \lambda_{1\bar{2}\bar{3}\bar{4}\bar{5}} = 0, \\ \lambda_{\bar{1}\bar{2}\bar{3}\bar{4}5} = (1 - x_2)(1 - x_3)(1 - x_4) x_5, & \lambda_{\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}} = (1 - x_2)(1 - x_3)(1 - x_4) (1 - x_5). \end{array}$$

(e) $0 < T_L(x_1, \dots, x_h) < 1$, $T_L(x_1, \dots, x_{h+1}) = \dots = T_L(x_1, \dots, x_5) = 0$, with $1 \leq h < 4$; the solution is given in (56). If for instance $h = 3$, the components of the vector Λ_5 are

$$\begin{array}{ll} \lambda_{12345} = 0, & \lambda_{1234\bar{5}} = 0, \\ \lambda_{\bar{1}2345} = (1 - x_1) \frac{x_4}{3 - x_1 - x_2 - x_3} x_5, & \lambda_{\bar{1}234\bar{5}} = (1 - x_1) \frac{x_4}{3 - x_1 - x_2 - x_3} (1 - x_5), \\ \lambda_{1\bar{2}345} = (1 - x_2) \frac{x_4}{3 - x_1 - x_2 - x_3} x_5, & \lambda_{1\bar{2}34\bar{5}} = (1 - x_2) \frac{x_4}{3 - x_1 - x_2 - x_3} (1 - x_5), \\ \lambda_{\bar{1}\bar{2}345} = 0, & \lambda_{\bar{1}\bar{2}34\bar{5}} = 0, \\ \lambda_{12\bar{3}45} = (1 - x_3) \frac{x_4}{3 - x_1 - x_2 - x_3} x_5, & \lambda_{12\bar{3}4\bar{5}} = (1 - x_3) \frac{x_4}{3 - x_1 - x_2 - x_3} (1 - x_5), \\ \lambda_{\bar{1}2\bar{3}45} = 0, & \lambda_{\bar{1}2\bar{3}4\bar{5}} = 0, \\ \lambda_{1\bar{2}\bar{3}45} = 0, & \lambda_{1\bar{2}\bar{3}4\bar{5}} = 0, \\ \lambda_{\bar{1}\bar{2}\bar{3}45} = 0, & \lambda_{\bar{1}\bar{2}\bar{3}4\bar{5}} = 0, \\ \lambda_{123\bar{4}5} = (x_1 + x_2 + x_3 - 2) x_5, & \lambda_{123\bar{4}\bar{5}} = (x_1 + x_2 + x_3 - 2) (1 - x_5), \\ \lambda_{\bar{1}23\bar{4}5} = (1 - x_1) \left(1 - \frac{x_4}{3 - x_1 - x_2 - x_3}\right) x_5, & \lambda_{\bar{1}23\bar{4}\bar{5}} = (1 - x_1) \left(1 - \frac{x_4}{3 - x_1 - x_2 - x_3}\right) (1 - x_5), \\ \lambda_{1\bar{2}3\bar{4}5} = (1 - x_2) \left(1 - \frac{x_4}{3 - x_1 - x_2 - x_3}\right) x_5, & \lambda_{1\bar{2}3\bar{4}\bar{5}} = (1 - x_2) \left(1 - \frac{x_4}{3 - x_1 - x_2 - x_3}\right) (1 - x_5), \\ \lambda_{\bar{1}\bar{2}3\bar{4}5} = 0, & \lambda_{\bar{1}\bar{2}3\bar{4}\bar{5}} = 0, \\ \lambda_{12\bar{3}\bar{4}5} = (1 - x_3) \left(1 - \frac{x_4}{3 - x_1 - x_2 - x_3}\right) x_5, & \lambda_{12\bar{3}\bar{4}\bar{5}} = (1 - x_3) \left(1 - \frac{x_4}{3 - x_1 - x_2 - x_3}\right) (1 - x_5), \\ \lambda_{\bar{1}2\bar{3}\bar{4}5} = 0, & \lambda_{\bar{1}2\bar{3}\bar{4}\bar{5}} = 0, \\ \lambda_{1\bar{2}\bar{3}\bar{4}5} = 0, & \lambda_{1\bar{2}\bar{3}\bar{4}\bar{5}} = 0, \\ \lambda_{\bar{1}\bar{2}\bar{3}\bar{4}5} = 0, & \lambda_{\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}} = 0. \end{array}$$

(f) $T_L(x_1, \dots, x_h) = 1$, $T_L(x_1, \dots, x_{h+1}) = \dots = T_L(x_1, \dots, x_5) = 0$, with $1 \leq h < 4$; the solution is given in (58). If for instance $h = 2$, the components of the vector Λ_5 are

$$\begin{cases} \lambda_{12\bar{3}45} = x_4x_5, & \lambda_{12\bar{3}4\bar{5}} = x_4(1 - x_5), & \lambda_{12\bar{3}\bar{4}5} = (1 - x_4)x_5, \\ \lambda_{12\bar{3}\bar{4}\bar{5}} = (1 - x_4)(1 - x_5), & \lambda_{1*2*3*4*5*} = 0, & \text{otherwise.} \end{cases}$$

5. Probabilistic interpretation of Frank t-norms and t-conorms

In this section we show that the previsions of the conjunction and the disjunction of n conditional events can be represented as a Frank t-norm T_λ and a Frank t-conorm S_λ , respectively. Then, we characterize the set of coherent assessments by Frank t-norms and t-conorms. Moreover, when $n = 2$, we show that, under logical independence, $T_\lambda(A|H, B|K) = (A|H) \wedge (B|K)$ and $S_\lambda(A|H, B|K) = (A|H) \vee (B|K)$ for every $\lambda \in [0, +\infty]$. We also examine cases where there are logical dependencies.

5.1. Set of coherent assessments, Frank t-norms and t-conorms

We recall that the n -ary Frank t-norm, with parameter $\lambda \in [0, +\infty]$, is

$$T_\lambda(x_1, \dots, x_n) = \begin{cases} T_M(x_1, \dots, x_n) = \min\{x_1, \dots, x_n\}, & \text{if } \lambda = 0, \\ T_P(x_1, \dots, x_n) = \prod_{i=1}^n x_i, & \text{if } \lambda = 1, \\ T_L(x_1, \dots, x_n) = \max\{\sum_{i=1}^n x_i - n + 1, 0\}, & \text{if } \lambda = +\infty, \\ \log_\lambda\left(1 + \frac{\prod_{i=1}^n (\lambda^{x_i} - 1)}{(\lambda - 1)^{n-1}}\right), & \text{otherwise.} \end{cases} \quad (59)$$

The next result shows that, under logical independence, given any coherent assessment $(x_1, \dots, x_n, x_{1\dots n})$ on $\{E_1|H_1, \dots, E_n|H_n, \mathcal{C}_{1\dots n}\}$, it holds that $x_{1\dots n} = T_\lambda(x_1, \dots, x_n)$ for some $\lambda \in [0, +\infty]$; conversely, for every $\lambda \in [0, +\infty]$ the extension $x_{1\dots n} = T_\lambda(x_1, \dots, x_n)$ is coherent.

Theorem 12. Let $E_1, \dots, E_n, H_1, \dots, H_n$ be logically independent events, with $H_1 \neq \emptyset, \dots, H_n \neq \emptyset$, $n \geq 2$. The set Π of all prevision coherent assessments $\mathcal{M} = (x_1, \dots, x_n, x_{1\dots n})$ on the family $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n, \mathcal{C}_{1\dots n}\}$ coincides with the set

$$\Pi_T = \{(x_1, \dots, x_n, x_{1\dots n}) : (x_1, \dots, x_n) \in [0, 1]^n, x_{1\dots n} = T_\lambda(x_1, \dots, x_n), \lambda \in [0, +\infty]\}. \quad (60)$$

Proof. We show that $\Pi \subseteq \Pi_T$ and $\Pi_T \subseteq \Pi$. For each given $\mathcal{M} = (x_1, \dots, x_n, x_{1\dots n}) \in \Pi$, by Theorem 10, it holds that $(x_1, \dots, x_n) \in [0, 1]^n$ and $x_{1\dots n} \in [T_L(x_1, \dots, x_n), T_M(x_1, \dots, x_n)] = [T_{+\infty}(x_1, \dots, x_n), T_0(x_1, \dots, x_n)]$. Then, by the continuity property of T_λ with respect to λ , there exists $\lambda \in [0, +\infty]$ such that $x_{1\dots n} = T_\lambda(x_1, \dots, x_n)$. Thus, $\Pi \subseteq \Pi_T$.

Conversely, for every $\lambda \in [0, +\infty]$ and for every $(x_1, \dots, x_n) \in [0, 1]^n$, by Theorem 10 the assessment $\mathcal{M} = (x_1, \dots, x_n, T_\lambda(x_1, \dots, x_n))$ is coherent because $T_\lambda(x_1, \dots, x_n) \in [T_L(x_1, \dots, x_n), T_M(x_1, \dots, x_n)]$. Thus $\Pi_T \subseteq \Pi$ and hence $\Pi = \Pi_T$. \square

Remark 5. We observe that in case of some logical dependencies, for each given coherent assessment (x_1, \dots, x_n) , the set of coherent extensions $x_{1\dots n}$ is an interval

$$[\mu'(x_1, \dots, x_n), \mu''(x_1 \cdots x_n)] \subseteq [T_{+\infty}(x_1, \dots, x_n), T_0(x_1, \dots, x_n)].$$

By Theorem 12, there exist λ' and λ'' such that $\mu'(x_1, \dots, x_n) = T_{\lambda'}(x_1, \dots, x_n)$ and $\mu''(x_1, \dots, x_n) = T_{\lambda''}(x_1, \dots, x_n)$, with $+\infty \geq \lambda' \geq \lambda'' \geq 0$ because T_λ is decreasing with respect to the parameter λ . Then,

$$[\mu'(x_1, \dots, x_n), \mu''(x_1 \cdots x_n)] = [T_{\lambda'}(x_1, \dots, x_n), T_{\lambda''}(x_1, \dots, x_n)].$$

Moreover, for each $x_{1\dots n} \in [\mu'(x_1, \dots, x_n), \mu''(x_1 \cdots x_n)]$, there exists $\lambda \in [\lambda'', \lambda']$ such that $x_{1\dots n} = T_\lambda(x_1, \dots, x_n)$. However, the set of all coherent assessments $(x_1, \dots, x_n, x_{1\dots n})$ is in general a subset of the set Π_T given in (60). Then, it may happen that, given any coherent assessment (x_1, \dots, x_n) , the extension $x_{1\dots n} = T_\lambda(x_1, \dots, x_n)$ is not coherent, for some $\lambda \in [0, +\infty]$.

We now show that a result dual of Theorem (12) holds for the disjunction of conditional events, where the Frank t-norm is replaced by the dual Frank t-conorm. The notion of disjunction given in Definition 3 can be extended to the case of n conditional events $E_1|H_1, \dots, E_n|H_n$ ([33]). Moreover, the conjunction $\mathcal{C}_{1\dots n}$ and the disjunction $\mathcal{D}_{1\dots n}$ satisfy De Morgan's Laws; in particular

$$\mathcal{D}_{1\dots n} = \bigvee_{i=1}^n (E_i|H_i) = 1 - \bigwedge_{i=1}^n (\bar{E}_i|H_i) = 1 - \mathcal{C}_{\bar{1}\dots\bar{n}}, \quad (61)$$

where $\mathcal{C}_{\bar{1}\dots\bar{n}} = \bigwedge_{i=1}^n (\bar{E}_i|H_i)$. We set $\mathbb{P}(\mathcal{D}_{1\dots n}) = y_{1\dots n}$ and $\mathbb{P}(\mathcal{C}_{\bar{1}\dots\bar{n}}) = x_{\bar{1}\dots\bar{n}}$. Of course, $y_{1\dots n} = 1 - x_{\bar{1}\dots\bar{n}}$. By Theorem 10, $x_{\bar{1}\dots\bar{n}}$ is a coherent extension of the assessment (x_1, \dots, x_n) on $\{E_1|H_1, \dots, E_n|H_n\}$ if and only if

$$T_L(1 - x_1, \dots, 1 - x_n) \leq x_{\bar{1}\dots\bar{n}} \leq T_M(1 - x_1, \dots, 1 - x_n).$$

that is

$$1 - T_M(1 - x_1, \dots, 1 - x_n) \leq y_{1\dots n} \leq 1 - T_L(1 - x_1, \dots, 1 - x_n).$$

Moreover, denoting by S_L and S_M the Lukasiewicz and Minimum t-conorms, respectively, it holds that

$$S_L(x_1, \dots, x_n) = \min\left\{\sum_{i=1}^n x_i, 1\right\} = 1 - T_L(1 - x_1, \dots, 1 - x_n),$$

and

$$S_M(x_1, \dots, x_n) = \max\{x_1, \dots, x_n\} = 1 - T_M(1 - x_1, \dots, 1 - x_n).$$

Then, from (61), we obtain the result below which establishes the Fréchet-Hoeffding bounds for the disjunction of n conditional events.

Theorem 13. Let $E_1, \dots, E_n, H_1, \dots, H_n$ be logically independent events, with $H_1 \neq \emptyset, \dots, H_n \neq \emptyset, n \geq 2$. The set of all prevision coherent assessments $(x_1, \dots, x_n, y_{1\dots n})$ on the family $\{E_1|H_1, \dots, E_n|H_n, \mathcal{D}_{1\dots n}\}$ is the set

$$\Gamma = \{(x_1, \dots, x_n, y_{1\dots n}) : (x_1, \dots, x_n) \in [0, 1]^n, y_{1\dots n} \in [S_M(x_1, \dots, x_n), S_L(x_1, \dots, x_n)]\}. \quad (62)$$

Based on Theorems 12 and 13, we have

Theorem 14. Let $E_1, \dots, E_n, H_1, \dots, H_n$ be logically independent events, with $H_1 \neq \emptyset, \dots, H_n \neq \emptyset, n \geq 2$. The set Γ of all prevision coherent assessments $\mathcal{M} = (x_1, \dots, x_n, y_{1\dots n})$ on the family $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n, \mathcal{D}_{1\dots n}\}$ coincides with the set

$$\Gamma_S = \{(x_1, \dots, x_n, y_{1\dots n}) : (x_1, \dots, x_n) \in [0, 1]^n, y_{1\dots n} = S_\lambda(x_1, \dots, x_n), \lambda \in [0, +\infty]\}. \quad (63)$$

5.2. Representation of conjunction and disjunction of two conditional events

In this section we examine the representation of conjunction and disjunction of two conditional events in terms of Frank t-norms and Frank t-conorms, respectively.

Theorem 15. For each coherent prevision assessment (x, y, z) on $\{A|H, B|K, (A|H) \wedge (B|K)\}$, it holds that

$$(A|H) \wedge (B|K) = T_\lambda(A|H, B|K), \text{ for some } \lambda \in [0, +\infty]. \quad (64)$$

Proof. From Definition 5 it holds that $T_\lambda(1, 1) = 1$, $T_\lambda(x, 0) = T_\lambda(0, y) = 0$, $T_\lambda(x, 1) = x$, $T_\lambda(1, y) = y$. Then,

$$T_\lambda(A|H, B|K) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \bar{A}H \text{ is true or } \bar{B}K \text{ is true,} \\ x, & \text{if } \bar{H}BK \text{ is true,} \\ y, & \text{if } \bar{K}AH \text{ is true,} \\ T_\lambda(x, y), & \text{if } \bar{H}\bar{K} \text{ is true.} \end{cases} \quad (65)$$

By Remark 5, there exists $\lambda \in [0, +\infty]$ such that $z = T_\lambda(x, y)$. Then, from (9) and (65), for each coherent assessment (x, y, z) on $\{A|H, B|K, (A|H) \wedge (B|K)\}$ there exists $\lambda \in [0, +\infty]$ such that $(A|H) \wedge (B|K) = T_\lambda(A|H, B|K)$. \square

Remark 6. We observe that to define the conjunction $(A|H) \wedge (B|K)$ amounts to specify a coherent assessment (x, y, z) on $\{A|H, B|K, (A|H) \wedge (B|K)\}$. Moreover, we recall that, by Theorem 12 (see also formula (12)), in the particular case of logical independence of A, B, H, K , for each $\lambda \in [0, +\infty]$ the extension $z = T_\lambda(x, y)$ on $(A|H) \wedge (B|K)$ of the assessment (x, y) on $\{A|H, B|K\}$ is coherent, for every $(x, y) \in [0, 1]^2$. Then, for any given assessment $(x, y, T_\lambda(x, y))$ on $\{A|H, B|K, (A|H) \wedge (B|K)\}$, with $(x, y) \in [0, 1]^2$, $\lambda \in [0, +\infty]$ it holds that

$$(A|H) \wedge (B|K) = T_\lambda(A|H, B|K).$$

In other words, for every $\lambda \in [0, +\infty]$, it is possible to define the conjunction as $(A|H) \wedge (B|K) = T_\lambda(A|H, B|K)$, for every $(x, y) \in [0, 1]^2$. Of course, in case of some logical dependencies, given a coherent assessment (x, y) , it may happen that $T_\lambda(A|H, B|K)$ is not a conjunction for some $\lambda \in [0, +\infty]$ because, by Remark 5, the extension $z = T_\lambda(x, y)$ is not coherent. In Section 5 we will give an example where $T_\lambda(A|H, A|K)$, with $\lambda > 1$, does not represent the conjunction $(A|H) \wedge (A|K)$ for some coherent (x, y) .

We recall that the dual Frank t-conorm $S_\lambda(x, y) = 1 - T_\lambda(1 - x, 1 - y)$ is defined as

$$S_\lambda(x, y) = \begin{cases} S_M(x, y) = \max\{x, y\}, & \text{if } \lambda = 0, \\ S_P(x, y) = x + y - xy, & \text{if } \lambda = 1, \\ S_L(x, y) = \min\{x + y, 1\}, & \text{if } \lambda = +\infty, \\ 1 - \log_\lambda\left(1 + \frac{(\lambda^{1-x}-1)(\lambda^{1-y}-1)}{\lambda-1}\right), & \text{otherwise.} \end{cases} \quad (66)$$

Moreover, for every $\lambda \in [0, +\infty]$, the pair (T_λ, S_λ) satisfies the functional equation ([41, Theorem 5.14])

$$S_\lambda(x, y) = x + y - T_\lambda(x, y), \quad (x, y) \in [0, 1]^2. \quad (67)$$

Theorem 16. For each coherent prevision assessment (x, y, w) on $\{A|H, B|K, (A|H) \vee (B|K)\}$, it holds that

$$(A|H) \vee (B|K) = S_\lambda(A|H, B|K), \text{ for some } \lambda \in [0, +\infty]. \quad (68)$$

Proof. From (66) it holds that $S_\lambda(1, 1) = 1$, $S_\lambda(x, 1) = S_\lambda(1, y) = 1$, $S_\lambda(x, 0) = x$, $S_\lambda(0, y) = y$. Then,

$$S_\lambda(A|H, B|K) = \begin{cases} 1, & \text{if } AH \vee BK \text{ is true,} \\ 0, & \text{if } \bar{A}H\bar{B}H \text{ is true,} \\ x, & \text{if } \bar{H}\bar{B}K \text{ is true,} \\ y, & \text{if } \bar{K}\bar{A}H \text{ is true,} \\ S_\lambda(x, y), & \text{if } \bar{H}\bar{K} \text{ is true.} \end{cases} \quad (69)$$

By recalling the prevision sum rule ([30, Section 6]), it holds that

$$\mathbb{P}[(A|H) \vee (B|K)] = P(A|H) + P(B|K) - \mathbb{P}[(A|H) \wedge (B|K)],$$

that is $w = x + y - z$, where $z = \mathbb{P}[(A|H) \wedge (B|K)]$. Moreover, by Theorem 15 there exists $\lambda \in [0, +\infty]$ such that $z = T_\lambda(x, y)$. Then, $w = x + y - T_\lambda(x, y)$ and hence, from (67), there exists $\lambda \in [0, +\infty]$ such that $w = S_\lambda(x, y)$. Finally, from (13) and (69), for each coherent assessment (x, y, z) on $\{A|H, B|K, (A|H) \vee (B|K)\}$ there exists $\lambda \in [0, +\infty]$ such that $(A|H) \vee (B|K) = S_\lambda(A|H, B|K)$. \square

As a further comment, we also observe that, for each coherent assessment (x, y, z, w) on the family $\{A|H, B|K, (A|H) \wedge (B|K), (A|H) \vee (B|K)\}$, there exists $\lambda \in [0, +\infty]$ such that

$$(A|H) \vee (B|K) = (A|H) + (B|K) - T_\lambda(A|H, B|K) = S_\lambda(A|H, B|K).$$

We remark that in the case of some logical dependencies among the basic events A, B, H, K , the Frank t-norm may represent the conjunction only for the values of λ in a subset of $[0, +\infty]$. In the next section we examine a case where the subset is the interval $[0, 1]$.

5.3. The conjunction $(A|H) \wedge (B|K)$, when $A = B$

In this section we examine a case of logical dependencies by considering the conjunction $(A|H) \wedge (B|K)$ when $A = B$, that is $(A|H) \wedge (A|K)$. By setting $P(A|H) = x$, $P(A|K) = y$ and $\mathbb{P}[(A|H) \wedge (A|K)] = z$, it holds that

$$(A|H) \wedge (A|K) = \begin{cases} 1, & \text{if } AHK \text{ is true,} \\ 0, & \text{if } \bar{A}H \vee \bar{A}K \text{ is true,} \\ x, & \text{if } \bar{H}AK \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ z, & \text{if } \bar{H}\bar{K} \text{ is true,} \end{cases}$$

that is

$$(A|H) \wedge (A|K) = AHK + x\bar{H}AK + y\bar{K}AH + z\bar{H}\bar{K}.$$

In the next result we show that, for each coherent assessment (x, y) , the lower bound on z is, not $T_L(x, y)$, but $T_P(x, y)$; the upper bound is still $T_M(x, y)$.

Theorem 17. Let A, H, K be three logically independent events, with $H \neq \emptyset$, $K \neq \emptyset$. The set Π of all coherent assessments (x, y, z) on the family $\mathcal{F} = \{A|H, A|K, (A|H) \wedge (A|K)\}$ is given by

$$\Pi = \{(x, y, z) : (x, y) \in [0, 1]^2, T_P(x, y) = xy \leq z \leq \min\{x, y\} = T_M(x, y)\}. \quad (70)$$

Proof. We recall that, by Example 1, the assessment (x, y) is coherent for every $(x, y) \in [0, 1]^2$. Given any coherent assessment (x, y) , by Theorem 6 there exists an interval $[z', z'']$ of coherent extensions z to $(A|H) \wedge (A|K)$. We will show that $z' = xy$ and $z'' = \min\{x, y\}$. Let $\mathcal{M} = (x, y, z)$ be a prevision assessment on \mathcal{F} , with $(x, y) \in [0, 1]^2$. The constituents associated with the pair $(\mathcal{F}, \mathcal{M})$ and contained in $H \vee K$ are: $C_1 = AHK, C_2 = \bar{A}HK, C_3 = \bar{A}\bar{H}K, C_4 = \bar{A}H\bar{K}, C_5 = A\bar{H}K, C_6 = AH\bar{K}$. The associated points Q_h 's are $Q_1 = (1, 1, 1), Q_2 = (0, 0, 0), Q_3 = (x, 0, 0), Q_4 = (0, y, 0), Q_5 = (x, 1, x), Q_6 = (1, y, y)$. With the further constituent $C_0 = \bar{H}\bar{K}$ it is associated the point $Q_0 = \mathcal{M} = (x, y, z)$. Considering the convex hull \mathcal{I} (see Figure 1) of Q_1, \dots, Q_6 , a necessary condition for the coherence of the prevision assessment $\mathcal{M} = (x, y, z)$ on \mathcal{F} is that $\mathcal{M} \in \mathcal{I}$, that is the following system must be solvable

$$(\Sigma) \begin{cases} \lambda_1 + x\lambda_3 + x\lambda_5 + \lambda_6 = x, & \lambda_1 + y\lambda_4 + \lambda_5 + y\lambda_6 = y, & \lambda_1 + x\lambda_5 + y\lambda_6 = z, \\ \sum_{h=1}^6 \lambda_h = 1, & \lambda_h \geq 0, & h = 1, \dots, 6. \end{cases}$$

First of all, we observe that solvability of (Σ) requires that $z \leq x$ and $z \leq y$, that is $z \leq \min\{x, y\}$; thus $z'' \leq \min\{x, y\}$. We now verify that (x, y, z) , with $(x, y) \in [0, 1]^2$ and $z = \min\{x, y\}$, is coherent, from which it follows that $z'' = \min\{x, y\}$. We distinguish two cases: (i) $x \leq y$ and (ii) $x > y$.

Case (i). In this case $z = \min\{x, y\} = x$. If $y = 0$ the system (Σ) becomes

$$\lambda_1 + \lambda_6 = 0, \quad \lambda_1 + \lambda_5 = 0, \quad \lambda_1 = 0, \quad \lambda_2 + \lambda_3 + \lambda_4 = 1, \quad \lambda_h \geq 0, \quad h = 1, \dots, 6.$$

which is clearly solvable. In particular there exist solutions with $\lambda_2 > 0, \lambda_3 > 0, \lambda_4 > 0$, by Theorem 5, as the set I_0 is empty the solvability of (Σ) is sufficient for coherence of the assessment $(0, 0, 0)$. If $y > 0$ the system (Σ) is solvable and a solution is $\Lambda = (\lambda_1, \dots, \lambda_6) = (x, \frac{x(1-y)}{y}, 0, \frac{y-x}{y}, 0, 0)$. We observe that, if $x > 0$, then $\lambda_1 > 0$ and $I_0 = \emptyset$ because $\mathcal{C}_1 = HK \subseteq H \vee K$, so that $\mathcal{M} = (x, y, x)$ is coherent. If $x = 0$ (and hence $z = 0$), then $\lambda_4 = 1$ and $I_0 \subseteq \{2\}$. Then, as the sub-assessment $P(A|K) = y$ is coherent, it follows that the assessment $\mathcal{M} = (0, y, 0)$ is coherent too.

Case (ii). The system is solvable and a solution is $\Lambda = (\lambda_1, \dots, \lambda_6) = (y, \frac{y(1-x)}{x}, \frac{x-y}{x}, 0, 0, 0)$. We observe that, if $y > 0$, then $\lambda_1 > 0$ and $I_0 = \emptyset$ because $\mathcal{C}_1 = HK \subseteq H \vee K$, so that $\mathcal{M} = (x, y, y)$ is coherent. If $y = 0$ (and hence $z = 0$), then $\lambda_3 = 1$ and $I_0 \subseteq \{1\}$. Then, as the sub-assessment $P(A|H) = x$ is coherent, it follows that the assessment $\mathcal{M} = (x, 0, 0)$ is coherent too. Thus, for every $(x, y) \in [0, 1]^2$, the assessment $(x, y, \min\{x, y\})$ is coherent and hence the upper bound on z is $z'' = \min\{x, y\} = T_M(x, y)$.

We now verify that (x, y, xy) , with $(x, y) \in [0, 1]^2$ is coherent; moreover we show that (x, y, z) , with $z < xy$, is not coherent and the lower bound for z is $z' = xy$. First of all, we observe that $\mathcal{M} = (1-x)Q_4 + xQ_6$, so that a solution of (Σ) is $\Lambda_1 = (0, 0, 0, 1-x, 0, x)$. Moreover, $\mathcal{M} = (1-y)Q_3 + yQ_5$, so that another solution is $\Lambda_2 = (0, 0, 1-y, 0, y, 0)$. Then $\Lambda = \frac{\Lambda_1 + \Lambda_2}{2} = (0, 0, \frac{1-y}{2}, \frac{1-x}{2}, \frac{y}{2}, \frac{x}{2})$ is a solution of (Σ) such that $I_0 = \emptyset$. Thus the assessment (x, y, xy) is coherent for every $(x, y) \in [0, 1]^2$. In order to verify that xy is the lower bound on z we observe that the points Q_3, Q_4, Q_5, Q_6 belong to a plane π of equation: $yX + xY - Z = xy$, where X, Y, Z are the axis' coordinates. Now, by considering the function $f(X, Y, Z) = yX + xY - Z$, we observe that for each constant k the equation $f(X, Y, Z) = k$ represents a plane which is parallel to π and coincides with π when $k = xy$. We also observe that $f(Q_1) = f(1, 1, 1) = x + y - 1 = T_L(x, y) \leq xy = T_P(x, y)$, $f(Q_2) = f(0, 0, 0) = 0 \leq xy = T_P(x, y)$, and $f(Q_3) = f(Q_4) = f(Q_5) = f(Q_6) = xy = T_P(x, y)$. Then, for every $\mathcal{P} = \sum_{h=1}^6 \lambda_h Q_h$, with $\lambda_h \geq 0$ and $\sum_{h=1}^6 \lambda_h = 1$, that is $\mathcal{P} \in \mathcal{I}$, it holds that $f(\mathcal{P}) = f(\sum_{h=1}^6 \lambda_h Q_h) = \sum_{h=1}^6 \lambda_h f(Q_h) \leq xy$. On the other hand, given any $a > 0$, by considering $\mathcal{P} = (x, y, xy - a)$ it holds that $f(\mathcal{P}) = f(x, y, xy - a) = xy + xy - xy + a = xy + a > xy$. Therefore, for any given $a > 0$ the assessment $(x, y, xy - a)$ is not coherent because $(x, y, xy - a) \notin \mathcal{I}$. Then, the lower bound on z is $z' = xy = T_P(x, y)$. Thus, the set of all coherent assessments (x, y, z) on \mathcal{F} is the set Π in (70). \square

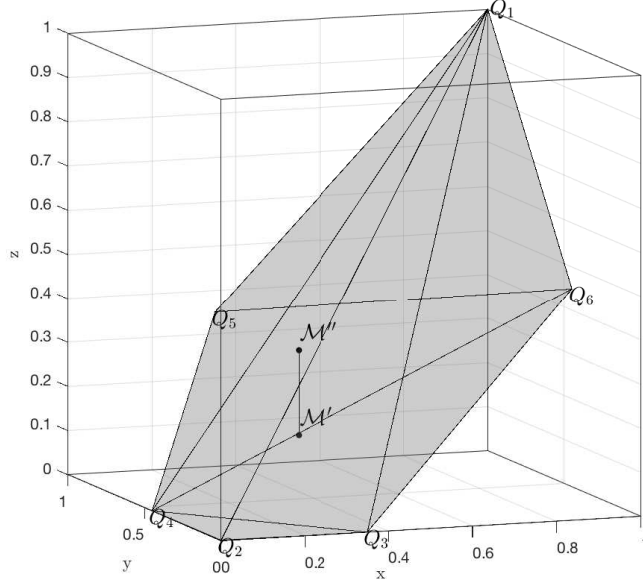


Figure 1: Convex hull I of the points $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6$. $\mathcal{M}' = (x, y, z')$, $\mathcal{M}'' = (x, y, z'')$, where $(x, y) \in [0, 1]^2$, $z' = xy$, $z'' = \min\{x, y\}$. In the figure the numerical values are: $x = 0.35$, $y = 0.45$, $z' = 0.1575$, and $z'' = 0.35$.

Based on Theorem 17, we can give a result which is similar to Theorem 12, with $n = 2$; but in this case λ belongs to the interval $[0, 1]$.

Theorem 18. Let A, H, K be logically independent events, with $H \neq \emptyset$, $K \neq \emptyset$. The set of all prevision coherent assessments $\mathcal{M} = (x, y, z)$ on the family $\mathcal{F} = \{A|H, A|K, (A|H) \wedge (A|K)\}$ is the set

$$\{(x, y, z) : (x, y) \in [0, 1]^2, z = T_\lambda(x, y), \lambda \in [0, 1]\}. \quad (71)$$

Proof. By exploiting Theorem 17, the proof is the same as in Theorem 12. \square

We observe that for every $\lambda \in [0, 1]$, it is possible to define the conjunction as $(A|H) \wedge (A|K) = T_\lambda(A|H, A|K)$, for every $(x, y) \in [0, 1]^2$. Moreover, for some coherent (x, y) , it may happen that $T_\lambda(A|H, A|K)$ is not a conjunction when $\lambda \in (1 + \infty]$, as shown by the example below.

Example 6. Let A, H, K be logically independent events, with $H \neq \emptyset$, $K \neq \emptyset$. If, for instance, $(x, y) = (\frac{1}{2}, \frac{1}{2})$. Then, by Theorem 17, the extension z is coherent if and only if $z \in [T_1(\frac{1}{2}, \frac{1}{2}), T_0(\frac{1}{2}, \frac{1}{2})] = [\frac{1}{4}, \frac{1}{2}]$. Moreover, as $T_\lambda(\frac{1}{2}, \frac{1}{2})$ is decreasing with respect to λ , it holds that

$$T_\lambda(\frac{1}{2}, \frac{1}{2}) < T_1(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}, \quad \forall \lambda \in (1 + \infty].$$

Then, the extension $z = T_\lambda(\frac{1}{2}, \frac{1}{2})$, with $\lambda \in (1 + \infty]$, is not coherent; thus, $T_\lambda(A|H, A|K)$, with $\lambda \in (1 + \infty]$ and $(x, y) = (\frac{1}{2}, \frac{1}{2})$, is not a conjunction.

We also observe that, in particular cases, $T_\lambda(A|H, A|K)$ is a conjunction when $\lambda \in (1 + \infty]$. For instance, if $x = 0$, or $y = 0$, it holds that, for every $\lambda \in [0, +\infty]$, $z = T_\lambda(x, y) = 0$ is the unique coherent extension of the assessment (x, y) . Then, $T_\lambda(A|H, A|K)$, with $x = 0$, or $y = 0$, is a conjunction for every $\lambda \in [0, +\infty]$.

The next result consider the particular case where H and K are incompatible.

Theorem 19. Let A, H, K be three events, with A logically independent from both H and K , with $H \neq \emptyset$, $K \neq \emptyset$, $HK = \emptyset$. The set of all coherent assessments (x, y, z) on the family $\mathcal{F} = \{A|H, A|K, (A|H) \wedge (A|K)\}$ is given by $\{(x, y, z) : (x, y) \in [0, 1]^2, z = xy = T_P(x, y)\}$.

Proof. Let $\mathcal{M} = (x, y, z)$ be a prevision assessment on \mathcal{F} . We recall that, by Example 2, the coherence of (x, y) amounts to $(x, y) \in [0, 1]^2$. Moreover, we observe that $\bar{H}K = K$ and $H\bar{K} = H$ and

$$(A|H) \wedge (A|K) = (xA\bar{H}K + yAH\bar{K})|(H \vee K) = xAK|(H \vee K) + yAH|(H \vee K).$$

Then,

$$z = xP(AK|(H \vee K)) + yP(AH|(H \vee K)) = xyP(K|(H \vee K)) + xyP(H|(H \vee K)) = xy = T_P(x, y).$$

Thus, the set of all coherent assessments (x, y, z) on the family $\mathcal{F} = \{A|H, A|K, (A|H) \wedge (A|K)\}$ is given by $\{(x, y, z) : (x, y) \in [0, 1]^2, z = xy = T_P(x, y)\}$. \square

Remark 7. We remark that, when $HK = \emptyset$, $T_\lambda(A|H, A|K)$ represents the conjunction $(A|H) \wedge (A|K)$ only if $\lambda = 1$. Indeed, from Theorems 15 and 19, it holds that

$$(A|H) \wedge (A|K) = (A|H) \cdot (A|K) = T_P(A|H, A|K) = T_1(A|H, A|K), \quad \text{when } HK = \emptyset.$$

We point out again that, as shown by Theorem 18 and by Remark 7, in case of some logical dependencies to assign conditional previsions and to represent conjunctions by means of a Frank t-norm T_λ is consistent only for some values of λ . For instance, given any assessment (x, y) on $\{A|H, B|K\}$, with $0 < x < 1$, $0 < y < 1$, the assessment $\mathbb{P}[(A|H) \wedge (B|K)] = T_L(x, y)$ is not coherent, because $\max\{x + y - 1, 0\} < xy$. Moreover, $T_L(A|H, A|K) = T_{+\infty}(A|H, A|K)$ is not a conjunction.

6. Some further results on Frank t-norms

In this section we give some particular results on Frank t-norms and coherence of prevision assessments on the family $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}, \mathcal{C}_{123}\}$, where $\mathcal{C}_i = E_i|H_i$, $\mathcal{C}_{ij} = (E_i|H_i) \wedge (E_j|H_j)$, and $\mathcal{C}_{123} = (E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)$. We set $\mathbb{P}(\mathcal{C}_i) = x_i$, $i = 1, 2, 3$, $\mathbb{P}(\mathcal{C}_{ij}) = x_{ij}$, $\{i, j\} \subset \{1, 2, 3\}$, and $\mathbb{P}(\mathcal{C}_{123}) = x_{123}$. In particular, we show that, under logical independence, the assessment

$$\mathcal{M} = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}) = (x_1, x_2, x_3, T_\lambda(x_1, x_2), T_\lambda(x_1, x_3), T_\lambda(x_2, x_3), T_\lambda(x_1, x_2, x_3))$$

on \mathcal{F} is coherent for every $(x_1, x_2, x_3) \in [0, 1]^3$ when T_λ is the minimum t-norm $T_M = T_0$, or T_λ is the product t-norm, $T_P = T_1$. Moreover, when T_λ is the Lukasiewicz t-norm $T_L = T_{+\infty}$, the coherence of \mathcal{M} is not assured. We first observe that, by Definition 4, the conjunction $\mathcal{C}_{123} = (E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)$ is

$$\mathcal{C}_{123} = \begin{cases} 1, & \text{if } E_1H_1E_2H_2E_3H_3 \text{ is true} \\ 0, & \text{if } \bar{E}_1H_1 \vee \bar{E}_2H_2 \vee \bar{E}_3H_3 \text{ is true,} \\ x_1, & \text{if } \bar{H}_1E_2H_2E_3H_3 \text{ is true,} \\ x_2, & \text{if } \bar{H}_2E_1H_1E_3H_3 \text{ is true,} \\ x_3, & \text{if } \bar{H}_3E_1H_1E_2H_2 \text{ is true,} \\ x_{12}, & \text{if } \bar{H}_1\bar{H}_2E_3H_3 \text{ is true,} \\ x_{13}, & \text{if } \bar{H}_1\bar{H}_3E_2H_2 \text{ is true,} \\ x_{23}, & \text{if } \bar{H}_2\bar{H}_3E_1H_1 \text{ is true,} \\ x_{123}, & \text{if } \bar{H}_1\bar{H}_2\bar{H}_3 \text{ is true.} \end{cases} \quad (72)$$

The next result characterizes the set of all coherent assessments on \mathcal{F} ([33, Theorem 15]).

Theorem 20. Assume that the events $E_1, E_2, E_3, H_1, H_2, H_3$ are logically independent, with $H_1 \neq \emptyset, H_2 \neq \emptyset, H_3 \neq \emptyset$. Then, the set Π of all coherent assessments $\mathcal{M} = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ on $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}, \mathcal{C}_{123}\}$ is the set of points $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ which satisfy the following conditions

$$\begin{cases} (x_1, x_2, x_3) \in [0, 1]^3, \\ \max\{x_1 + x_2 - 1, x_{13} + x_{23} - x_3, 0\} \leq x_{12} \leq \min\{x_1, x_2\}, \\ \max\{x_1 + x_3 - 1, x_{12} + x_{23} - x_2, 0\} \leq x_{13} \leq \min\{x_1, x_3\}, \\ \max\{x_2 + x_3 - 1, x_{12} + x_{13} - x_1, 0\} \leq x_{23} \leq \min\{x_2, x_3\}, \\ 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23} \geq 0, \\ x_{123} \geq \max\{0, x_{12} + x_{13} - x_1, x_{12} + x_{23} - x_2, x_{13} + x_{23} - x_3\}, \\ x_{123} \leq \min\{x_{12}, x_{13}, x_{23}, 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23}\}. \end{cases} \quad (73)$$

Remark 8. As it can be verified, the last two inequalities in (73), that is

$$\begin{aligned} \max\{0, x_{12} + x_{13} - x_1, x_{12} + x_{23} - x_2, x_{13} + x_{23} - x_3\} &\leq x_{123} \leq \\ &\leq \min\{x_{12}, x_{13}, x_{23}, 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23}\}, \end{aligned} \quad (74)$$

imply all the other inequalities in (73). For instance, in order to prove that $x_1 \leq 1$, we observe that from (74) it holds that

$$x_{12} + x_{23} - x_2 \leq 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23},$$

from which it follows

$$x_{12} + x_{23} - x_2 + x_2 + x_3 - x_{12} - x_{13} - x_{23} \leq 1 - x_1,$$

that is $x_3 - x_{13} \leq 1 - x_1$. Moreover, still from (74), it holds that $x_{13} + x_{23} - x_3 \leq x_{23}$, that is $x_3 - x_{13} \geq 0$. Thus $x_1 \leq 1$. Then, as (73) and (74) are equivalent, the set Π in Theorem 20 is the set of points $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ which satisfy (74).

Then, by Theorem 20 it follows [33, Corollary 1]

Corollary 1. For any coherent assessment $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23})$ on $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}\}$ the extension x_{123} on \mathcal{C}_{123} is coherent if and only if $x_{123} \in [x'_{123}, x''_{123}]$, where

$$\begin{aligned} x'_{123} &= \max\{0, x_{12} + x_{13} - x_1, x_{12} + x_{23} - x_2, x_{13} + x_{23} - x_3\}, \\ x''_{123} &= \min\{x_{12}, x_{13}, x_{23}, 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23}\}. \end{aligned} \quad (75)$$

We recall that in case of logical dependencies, the set of all coherent assessments may be a strict subset of the set Π associated with the case of logical independence. However, the next result shows that the set of coherent assessments is still Π in the case where $H_1 = H_2 = H_3 = H$ (with possibly $H = \Omega$, see also [38, p. 232]).

Theorem 21. Let be given any logically independent events E_1, E_2, E_3, H , with $H \neq \emptyset$. Then, the set Π of all coherent assessments $\mathcal{M} = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ on $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}, \mathcal{C}_{123}\}$ is the set of points $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ which satisfy the conditions in formula (73).

A corollary similar to Corollary 1 could be associated to Theorem 21. For a similar result based on copulas see [21].

In the next subsection we examine the coherence of the prevision assessment $\mathcal{M} = (x_1, x_2, x_3, T_\lambda(x_1, x_2), T_\lambda(x_1, x_3), T_\lambda(x_2, x_3), T_\lambda(x_1, x_2, x_3))$ on $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}, \mathcal{C}_{123}\}$ in the cases where T_λ is the minimum t-norm, or the product t-norm, or the Lukasiewicz t-norm.

6.1. On the minimum t-norm

We recall that the Frank t-norm T_0 is the minimum t-norm T_M .

Theorem 22. Assume that the events $E_1, E_2, E_3, H_1, H_2, H_3$ are logically independent, with $H_1 \neq \emptyset, H_2 \neq \emptyset, H_3 \neq \emptyset$. The assessment $\mathcal{M} = (x_1, x_2, x_3, T_M(x_1, x_2), T_M(x_1, x_3), T_M(x_2, x_3), T_M(x_1, x_2, x_3))$ on $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}, \mathcal{C}_{123}\}$, with $(x_1, x_2, x_3) \in [0, 1]^3$, is coherent. Moreover, when $\mathcal{M} = (x_1, x_2, x_3, T_M(x_1, x_2), T_M(x_1, x_3), T_M(x_2, x_3), T_M(x_1, x_2, x_3))$, it holds that $\mathcal{C}_{ij} = T_M(\mathcal{C}_i, \mathcal{C}_j) = \min\{\mathcal{C}_i, \mathcal{C}_j\}$, $i \neq j$, and $\mathcal{C}_{123} = T_M(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) = \min\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$.

Proof. From Remark 8, the coherence of \mathcal{M} amounts to the inequalities in (74). Without loss of generality, we assume that $0 \leq x_1 \leq x_2 \leq x_3 \leq 1$. Then $x_{12} = T_M(x_1, x_2) = x_1$, $x_{13} = T_M(x_1, x_3) = x_1$, $x_{23} = T_M(x_2, x_3) = x_2$, and $x_{123} = T_M(x_1, x_2, x_3) = x_1$. The inequalities (74) become

$$\max\{0, x_1, x_1 + x_2 - x_3\} = x_1 \leq x_1 \leq x_1 = \min\{x_1, x_2, 1 - x_3 + x_1\}. \quad (76)$$

Thus, the inequalities are satisfied and hence \mathcal{M} is coherent. By Remark 6, it holds that $\mathcal{C}_{ij} = T_M(\mathcal{C}_i, \mathcal{C}_j) = \min\{\mathcal{C}_i, \mathcal{C}_j\}$, $i \neq j$. Moreover, based on (72), it can be easily verified that $\mathcal{C}_{123} = T_M(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) = \min\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$. \square

Remark 9. As we can see from (76) and Corollary 1, the assessment $x_{123} = \min\{x_1, x_2, x_3\}$ is the unique coherent extension on \mathcal{C}_{123} of the assessment $(x_1, x_2, x_3, \min\{x_1, x_2\}, \min\{x_1, x_3\}, \min\{x_2, x_3\})$ on $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}\}$.

6.2. On the Product t-norm

We recall that the Frank t-norm T_1 is the product t-norm T_P .

Theorem 23. Assume that the events $E_1, E_2, E_3, H_1, H_2, H_3$ are logically independent, with $H_1 \neq \emptyset, H_2 \neq \emptyset, H_3 \neq \emptyset$. The assessment $\mathcal{M} = (x_1, x_2, x_3, T_P(x_1, x_2), T_P(x_1, x_3), T_P(x_2, x_3), T_P(x_1, x_2, x_3))$ on $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}, \mathcal{C}_{123}\}$, with $(x_1, x_2, x_3) \in [0, 1]^3$, is coherent. Moreover, when $\mathcal{M} = (x_1, x_2, x_3, T_P(x_1, x_2), T_P(x_1, x_3), T_P(x_2, x_3), T_P(x_1, x_2, x_3))$, it holds that $\mathcal{C}_{ij} = T_P(\mathcal{C}_i, \mathcal{C}_j) = \mathcal{C}_i \mathcal{C}_j$, $i \neq j$, and $\mathcal{C}_{123} = T_P(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) = \mathcal{C}_1 \mathcal{C}_2 \mathcal{C}_3$.

Proof. From Remark 8, the coherence of \mathcal{M} amounts to the inequalities in (74). As $x_{ij} = T_P(x_i, x_j) = x_i x_j$, $i \neq j$, and $x_{123} = T_P(x_1, x_2, x_3) = x_1 x_2 x_3$, the inequalities (74) become

$$\begin{aligned} \max\{0, x_1(x_2 + x_3 - 1), x_2(x_1 + x_3 - 1), x_3(x_1 + x_2 - 1)\} &\leq x_1 x_2 x_3 \leq \\ &\leq \min\{x_1 x_2, x_1 x_3, x_2 x_3, (1 - x_1)(1 - x_2)(1 - x_3) + x_1 x_2 x_3\}. \end{aligned} \quad (77)$$

As $(x_1, x_2, x_3) \in [0, 1]^3$ it holds that $x_i + x_j - 1 \leq x_i x_j$ because $x_i(1 - x_j) \leq 1 - x_j$. Then, the first inequality in (77) is satisfied. Moreover, the second inequality is trivial. Thus, \mathcal{M} is coherent. By Remark 6, it holds that $\mathcal{C}_{ij} = T_P(\mathcal{C}_i, \mathcal{C}_j) = \mathcal{C}_i \mathcal{C}_j$, $i \neq j$. Finally, based on (72), it can be easily verified that $\mathcal{C}_{123} = T_P(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) = \mathcal{C}_1 \mathcal{C}_2 \mathcal{C}_3$. \square

6.3. On Lukasiewicz t-norm

We recall that the Frank t-norm $T_{+\infty}$ is the Lukasiewicz t-norm T_L . We show that the assessment $\mathcal{M} = (x_1, x_2, x_3, T_L(x_1, x_2), T_L(x_1, x_3), T_L(x_2, x_3), T_L(x_1, x_2, x_3))$ on $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}, \mathcal{C}_{123}\}$ may be not coherent for some $(x_1, x_2, x_3) \in [0, 1]^3$, as shown in the example below.

Example 7. Given any logically independent events $E_1, E_2, E_3, H_1, H_2, H_3$, the assessment $(x_1, x_2, x_3) = (0.5, 0.6, 0.7)$ on $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$ is coherent. However, the prevision assessment $(x_1, x_2, x_3, T_L(x_1, x_2), T_L(x_1, x_3), T_L(x_2, x_3), T_L(x_1, x_2, x_3)) = (0.5, 0.6, 0.7, 0.1, 0.2, 0.3, 0)$ on the family $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}, \mathcal{C}_{123}\}$ is not coherent. Indeed, formula (74) becomes

$$\max\{0, 0.1+0.2-0.5, 0.1+0.3-0.6, 0.2+0.3-0.7\} \leq 0 \leq \min\{0.1, 0.2, 0.3, 1-0.5-0.6-0.7+0.1+0.2+0.3\},$$

that is:

$$\max\{0, -0.2\} = 0 \leq 0 \leq -0.2 = \min\{0.1, 0.2, 0.3, -0.2\};$$

thus, the inequalities in (74) are not satisfied and by Remark 8 the assessment $(0.5, 0.6, 0.7, 0.1, 0.2, 0.3, 0)$ is not coherent. Then, the results of Theorems 22 and 23 do not hold for the Lukasiewicz t-norm.

In the next result we illustrate further details on coherence of the prevision assessment $(x_1, x_2, x_3, T_L(x_1, x_2), T_L(x_1, x_3), T_L(x_2, x_3), T_L(x_1, x_2, x_3))$.

Theorem 24. Assume that the events $E_1, E_2, E_3, H_1, H_2, H_3$ are logically independent, with $H_1 \neq \emptyset, H_2 \neq \emptyset, H_3 \neq \emptyset$. Let $\mathcal{M} = (x_1, x_2, x_3, T_L(x_1, x_2), T_L(x_1, x_3), T_L(x_2, x_3), T_L(x_1, x_2, x_3))$ be a prevision assessment on the family $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}, \mathcal{C}_{123}\}$. If $(x_1, x_2, x_3) \in [0, 1]^3$ and $x_1 + x_2 + x_3 - 2 \geq 0$ then \mathcal{M} is coherent. If $(x_1, x_2, x_3) \in [0, 1]^3$, $x_1 + x_2 - 1 > 0$, $x_1 + x_3 - 1 > 0$, $x_2 + x_3 - 1 > 0$, and $x_1 + x_2 + x_3 - 2 < 0$, then \mathcal{M} is not coherent.

Proof. We observe that the set of points $(x_1, x_2, x_3) \in [0, 1]^3$ such that $x_1 + x_2 + x_3 - 2 \geq 0$ is the convex hull \mathcal{T} of the points $(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$, which is a tetrahedron. If $(x_1, x_2, x_3) \in \mathcal{T}$, then $x_1 + x_2 + x_3 - 2 \geq 0$, and it holds that

$$x_1 + x_2 - 1 \geq 0, \quad x_1 + x_3 - 1 \geq 0, \quad x_2 + x_3 - 1 \geq 0,$$

with

$$0 \leq x_1 + x_2 + x_3 - 2 \leq \min\{x_1 + x_2 - 1, x_1 + x_3 - 1, x_2 + x_3 - 1\}. \quad (78)$$

Thus, the assessment becomes $\mathcal{M} = (x_1, x_2, x_3, x_1 + x_2 - 1, x_1 + x_3 - 1, x_2 + x_3 - 1, x_1 + x_2 + x_3 - 2)$. Moreover, from (78), the conditions of coherence on \mathcal{M} given in (74) become

$$\max\{0, x_1 + x_2 + x_3 - 2\} \leq x_1 + x_2 + x_3 - 2 \leq \min\{x_1 + x_2 - 1, x_1 + x_3 - 1, x_2 + x_3 - 1, x_1 + x_2 + x_3 - 2\},$$

that is

$$x_1 + x_2 + x_3 - 2 \leq x_1 + x_2 + x_3 - 2 \leq x_1 + x_2 + x_3 - 2,$$

which are trivially satisfied. Then, by Remark 8, \mathcal{M} is coherent.

If $(x_1, x_2, x_3) \in [0, 1]^3$, $x_1 + x_2 - 1 > 0$, $x_1 + x_3 - 1 > 0$, $x_2 + x_3 - 1 > 0$, and $x_1 + x_2 + x_3 - 2 < 0$, then

$$1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23} = 1 - x_1 - x_2 - x_3 + T_L(x_1, x_2) + T_L(x_1, x_3) + T_L(x_2, x_3);$$

moreover

$$1 - x_1 - x_2 - x_3 + T_L(x_1, x_2) + T_L(x_1, x_3) + T_L(x_2, x_3) = x_1 + x_2 + x_3 - 2 < 0.$$

Then the inequality $1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23} \geq 0$ in (74) is not satisfied. Therefore, by Remark 8, \mathcal{M} is not coherent. This is the case, for instance, in Example 7. \square

Remark 10. Notice that, if we consider the assessment $(x_1, x_2, x_3, T_L(x_1, x_2), T_L(x_1, x_3), T_L(x_2, x_3), x_{123})$ on the family $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}, \mathcal{C}_{123}\}$, under the condition $x_1 + x_2 + x_3 - 2 \geq 0$, the conditions of coherence given in (74) become

$$\max\{0, x_1 + x_2 + x_3 - 2\} \leq x_{123} \leq \min\{x_1 + x_2 - 1, x_1 + x_3 - 1, x_2 + x_3 - 1, x_1 + x_2 + x_3 - 2\},$$

that is the conditions of coherence on \mathcal{M} given in (74) become

$$x_1 + x_2 + x_3 - 2 \leq x_{123} \leq x_1 + x_2 + x_3 - 2.$$

Thus, the unique coherent extension on \mathcal{C}_{123} is $x_{123} = x_1 + x_2 + x_3 - 2 = T_L(x_1, x_2, x_3)$. In this case, it holds that $\mathcal{C}_{ij} = T_L(\mathcal{C}_i, \mathcal{C}_j) = \mathcal{C}_i + \mathcal{C}_j - 1$, $i \neq j$, and $\mathcal{C}_{123} = T_L(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 - 2$.

Finally, we point out again that when T_λ is the Lukasiewicz t-norm $T_L = T_{+\infty}$, it may happen that the assessment $\mathcal{M} = (x_1, x_2, x_3, T_L(x_1, x_2), T_L(x_1, x_3), T_L(x_2, x_3), T_L(x_1, x_2, x_3))$ is not coherent, that is for some values x_1, x_2 , and x_3 , the assessment \mathcal{M} , with $x_{12} = T_L(x_1, x_2)$, $x_{13} = T_L(x_1, x_3)$, $x_{23} = T_L(x_2, x_3)$, and $x_{123} = T_L(x_1, x_2, x_3)$, is not coherent. Then, to assign conditional previsions by means of Lukasiewicz t-norm may be inconsistent. In Theorem 24 we gave some sufficient conditions for coherence/incoherence of \mathcal{M} when using T_L .

7. Conclusions

In this paper we studied conjoined and disjoined conditionals, Frank t-norms and t-conorms, and the sharpness of Fréchet-Hoeffding bounds. By studying the solvability of suitable linear systems, we showed that, under logical independence, the Fréchet-Hoeffding bounds for the prevision of the conjunction and the disjunction of n conditional events are sharp. In particular we illustrated some details in the case $n = 3$. We gave a geometrical characterization of the set Π of all coherent prevision assessments on $\{E_1|H_1, \dots, E_n|H_n, \mathcal{C}_{1\dots n}\}$, by verifying that Π is convex. We discussed the case where previsions of conjunctions are assessed by Lukasiewicz t-norms and we found explicit solutions for the relevant linear systems; then, we analyzed a selected example. We studied the representation of the prevision of $\mathcal{C}_{1\dots n}$ and $\mathcal{D}_{1\dots n}$ by a Frank t-norm T_λ and a Frank t-conorm S_λ , respectively. Then, we characterized the sets of coherent prevision assessments on $\{E_1|H_1, \dots, E_n|H_n, \mathcal{C}_{1\dots n}\}$ and on $\{E_1|H_1, \dots, E_n|H_n, \mathcal{D}_{1\dots n}\}$ by using T_λ and S_λ . We showed that, under logical independence, $T_\lambda(A|H, B|K)$ is a conjunction $(A|H) \wedge (B|K)$ and $S_\lambda(A|H, B|K)$ is a disjunction $(A|H) \vee (B|K)$, for every $\lambda \in [0, +\infty]$. We also examined the case of logical dependence where $A = B$, by obtaining the set of coherent assessments on $A|H, A|K, (A|H) \wedge (A|K)$ and its representation in terms of T_λ , with $\lambda \in [0, 1]$. We obtained some particular results on Frank t-norms and coherence of prevision assessments on the family $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}, \mathcal{C}_{123}\}$. In particular, we verified that, under logical independence, the assessment $\mathcal{M} = (x_1, x_2, x_3, T_\lambda(x_1, x_2), T_\lambda(x_1, x_3), T_\lambda(x_2, x_3), T_\lambda(x_1, x_2, x_3))$ on \mathcal{F} is coherent for every $(x_1, x_2, x_3) \in [0, 1]^3$ when T_λ is the minimum t-norm T_M , or the product t-norm T_P . We showed that in these cases the conjunction \mathcal{C}_{123} coincides with $T_M(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$, or $T_P(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$, respectively. Based on a counterexample, we verified that, when T_λ is the Lukasiewicz t-norm T_L , the coherence of \mathcal{M} is not assured. Then, we remarked that the Lukasiewicz t-norm of three conditional events may not be a conjunction. Finally, we gave two sufficient conditions for coherence and incoherence of \mathcal{M} , respectively, when using the Lukasiewicz t-norm. Future work could concern possible applications to fuzzy logic in the setting of coherence (see, e.g., [8, 13, 14]) by interpreting multidimensional membership functions as previsions of conjunctions of conditional events.

Declaration of competing interest

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

Acknowledgments

We thank the anonymous reviewers for their comments and suggestions which were very helpful in improving this paper. Giuseppe Sanfilippo has been partially supported by the INdAM–GNAMPA Project (2020 Grant U-UFMBAZ-2020-000819).

References

- [1] Batyrshin, I., Kaynak, O., Rudas, I., 2002. Fuzzy modeling based on generalized conjunction operations. *IEEE Transactions on Fuzzy Systems* 10, 678–683. doi:10.1109/TFUZZ.2002.803500.
- [2] Berti, P., Miranda, E., Rigo, P., 2017. Basic ideas underlying conglomerability and disintegrability. *International Journal of Approximate Reasoning* 88, 387 – 400. doi:https://doi.org/10.1016/j.ijar.2017.06.009.
- [3] Biazzo, V., Gilio, A., Lukasiewicz, T., Sanfilippo, G., 2005. Probabilistic logic under coherence: Complexity and algorithms. *Annals of Mathematics and Artificial Intelligence* 45, 35–81. doi:https://doi.org/10.1007/s10472-005-9005-y.
- [4] Biazzo, V., Gilio, A., Sanfilippo, G., 2012. Coherent conditional previsions and proper scoring rules, in: Greco, S., Bouchon-Meunier, B., Coletti, G., Fedrizzi, M., Matarazzo, B., Yager, R.R. (Eds.), *Advances in Computational Intelligence. IPMU 2012 Part IV*. Springer Heidelberg. volume 300 of *CCIS*, pp. 146–156. doi:https://doi.org/10.1007/978-3-642-31724-8_16.
- [5] Brozzi, A., Capotorti, A., Vantaggi, B., 2012. Incoherence correction strategies in statistical matching. *Int. J. Approx. Reasoning* 53, 1124 – 1136. doi:https://doi.org/10.1016/j.ijar.2012.06.009.
- [6] Calabrese, P., 2017. *Logic and Conditional Probability: A Synthesis*. College Publications.
- [7] Capotorti, A., Lad, F., Sanfilippo, G., 2007. Reassessing accuracy rates of median decisions. *The American Statistician* 61, 132–138. doi:http://dx.doi.org/10.1198/000313007X190943.
- [8] Coletti, G., Gervasi, O., Tasso, S., Vantaggi, B., 2012. Generalized bayesian inference in a fuzzy context: From theory to a virtual reality application. *Computational Statistics and Data Analysis* 56, 967 – 980. doi:http://dx.doi.org/10.1016/j.csda.2011.06.020.
- [9] Coletti, G., Petturiti, D., Vantaggi, B., 2017. Fuzzy memberships as likelihood functions in a possibilistic framework. *International Journal of Approximate Reasoning* 88, 547 – 566. doi:10.1016/j.ijar.2016.11.017.
- [10] Coletti, G., Petturiti, D., Vantaggi, B., 2020. A dutch book coherence condition for conditional completely alternating choquet expectations. *Bollettino dell’Unione Matematica Italiana* 13, 585–593. doi:10.1007/s40574-020-00251-8.
- [11] Coletti, G., Scozzafava, R., 1999. Conditioning and inference in intelligent systems. *Soft Computing* 3, 118–130. doi:https://doi.org/10.1007/s0050000050060.
- [12] Coletti, G., Scozzafava, R., 2002. *Probabilistic logic in a coherent setting*. Kluwer, Dordrecht.
- [13] Coletti, G., Scozzafava, R., 2004. Conditional probability, fuzzy sets, and possibility: A unifying view. *Fuzzy Sets and Systems* 144, 227–249. doi:https://doi.org/10.1016/j.fss.2003.10.022.
- [14] Coletti, G., Scozzafava, R., 2006. Conditional probability and fuzzy information. *Computational Statistics and Data Analysis* 51, 115 – 132.
- [15] Coletti, G., Vantaggi, B., 2018. Coherent conditional plausibility: A tool for handling fuzziness and uncertainty under partial information, in: Collan, M., Kacprzyk, J. (Eds.), *Soft Computing Applications for Group Decision-making and Consensus Modeling*. Springer International Publishing, Cham, pp. 129–152. doi:10.1007/978-3-319-60207-3_9.
- [16] Di Zio, M., Vantaggi, B., 2017. Partial identification in statistical matching with misclassification. *International Journal of Approximate Reasoning* 82, 227 – 241.
- [17] D’Orazio, M., Di Zio, M., Scanu, M., 2006. *Statistical Matching: Theory and Practice*. Wiley, New York.
- [18] Dubois, D., Faux, F., Prade, H., 2020. Prejudice in uncertain information merging: Pushing the fusion paradigm of evidence theory further. *International Journal of Approximate Reasoning* 121, 1 – 22. doi:10.1016/j.ijar.2020.02.012.
- [19] Dubois, D., Liu, W., Ma, J., Prade, H., 2016. The basic principles of uncertain information fusion. an organised review of merging rules in different representation frameworks. *Information Fusion* 32, 12 – 39. doi:10.1016/j.inffus.2016.02.006.

- [20] Dujmović, J.J., Legind Larsen, H., 2007. Generalized conjunction/disjunction. *International Journal of Approximate Reasoning* 46, 423 – 446. doi:<https://doi.org/10.1016/j.ijar.2006.12.011>. special Section: Aggregation Operators.
- [21] Durante, F., Klement, E.P., Quesada-Molina, J.J., 2008. Bounds for trivariate copulas with given bivariate marginals. *Journal of Inequalities and Applications* 2008, 9 pages. Article ID 161537.
- [22] Flaminio, T., Godo, L., Hosni, H., 2020. Boolean algebras of conditionals, probability and logic. *Artificial Intelligence* 286, 103347. doi:[10.1016/j.artint.2020.103347](https://doi.org/10.1016/j.artint.2020.103347).
- [23] Gale, D., 1960. *The theory of linear economic models*. McGraw-Hill, New York.
- [24] Gilio, A., 1993. Probabilistic consistency of knowledge bases in inference systems, in: Clarke, M., Kruse, R., Moral, S. (Eds.), *Symbolic and Quantitative Approaches to Reasoning and Uncertainty*. Springer, Berlin, Heidelberg. volume 747 of *LNCS*, pp. 160–167.
- [25] Gilio, A., Ingrassia, S., 1998. Totally coherent set-valued probability assessments. *Kybernetika* 34, 3–15.
- [26] Gilio, A., Pfeifer, N., Sanfilippo, G., 2016. Transitivity in coherence-based probability logic. *Journal of Applied Logic* 14, 46–64. doi:<https://doi.org/10.1016/j.jal.2015.09.012>.
- [27] Gilio, A., Pfeifer, N., Sanfilippo, G., 2020. Probabilistic entailment and iterated conditionals, in: Elqayam, S., Douven, I., Evans, J.S.B.T., Cruz, N. (Eds.), *Logic and Uncertainty in the Human Mind: A Tribute to David E. Over*. Routledge, Oxon, pp. 71–101. doi:<https://doi.org/10.4324/9781315111902-6>.
- [28] Gilio, A., Sanfilippo, G., 2013a. Conditional random quantities and iterated conditioning in the setting of coherence, in: van der Gaag, L.C. (Ed.), *ECSQARU 2013*. Springer, Berlin, Heidelberg. volume 7958 of *LNCS*, pp. 218–229. doi:[10.1007/978-3-642-39091-3_19](https://doi.org/10.1007/978-3-642-39091-3_19).
- [29] Gilio, A., Sanfilippo, G., 2013b. Conjunction, disjunction and iterated conditioning of conditional events, in: *Synergies of Soft Computing and Statistics for Intelligent Data Analysis*. Springer, Berlin. volume 190 of *AISC*, pp. 399–407. doi:[10.1007/978-3-642-33042-1_43](https://doi.org/10.1007/978-3-642-33042-1_43).
- [30] Gilio, A., Sanfilippo, G., 2014. Conditional random quantities and compounds of conditionals. *Studia Logica* 102, 709–729. doi:[10.1007/s11225-013-9511-6](https://doi.org/10.1007/s11225-013-9511-6).
- [31] Gilio, A., Sanfilippo, G., 2017. Conjunction and disjunction among conditional events, in: Benferhat, S., Tabia, K., Ali, M. (Eds.), *IEA/AIE 2017, Part II*. Springer, Cham. volume 10351 of *LNCS*, pp. 85–96. doi:[10.1007/978-3-319-60045-1_11](https://doi.org/10.1007/978-3-319-60045-1_11).
- [32] Gilio, A., Sanfilippo, G., 2019a. Conjunction of conditional events and t-norms, in: Kern-Isberner, G., Ognjanović, Z. (Eds.), *Symbolic and Quantitative Approaches to Reasoning with Uncertainty, ECSQARU 2019*. Springer International Publishing. volume 11726 of *LNCS*, pp. 199–211. doi:[10.1007/978-3-030-29765-7_17](https://doi.org/10.1007/978-3-030-29765-7_17).
- [33] Gilio, A., Sanfilippo, G., 2019b. Generalized logical operations among conditional events. *Applied Intelligence* 49, 79–102. doi:[10.1007/s10489-018-1229-8](https://doi.org/10.1007/s10489-018-1229-8).
- [34] Gilio, A., Sanfilippo, G., 2020. Algebraic aspects and coherence conditions for conjoined and disjointed conditionals. *International Journal of Approximate Reasoning* 126, 98 – 123. doi:<https://doi.org/10.1016/j.ijar.2020.08.004>.
- [35] Gilio, A., Sanfilippo, G., To Appear. On compound and iterated conditionals. *Argumenta* .
- [36] Grabisch, M., Marichal, J., Mesiar, R., Pap, E., 2009. *Aggregation functions*. Cambridge University Press.
- [37] Holzer, S., 1985. On coherence and conditional prevision. *Bollettino dell’Unione Matematica Italiana* 4, 441–460.
- [38] Joe, H., 1997. *Multivariate Models and Multivariate Dependence Concepts*. Chapman and Hall/CRC, New York.
- [39] Kaufmann, S., 2009. Conditionals right and left: Probabilities for the whole family. *Journal of Philosophical Logic* 38, 1–53. doi:[10.1007/s10992-008-9088-0](https://doi.org/10.1007/s10992-008-9088-0).
- [40] Klement, E., Mesiar, R., Pap, E., 2005. Triangular norms: Basic notions and properties, in: Klement, E., Mesiar, R. (Eds.), *Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms*. Elsevier Science B.V., Amsterdam, pp. 17 – 60.
- [41] Klement, E.P., Mesiar, R., Pap, E., 2000. *Triangular Norms*. Springer.
- [42] Lad, F., 1996. *Operational subjective statistical methods: A mathematical, philosophical, and historical introduction*. Wiley, New York.
- [43] Lad, F., Sanfilippo, G., 2020. Predictive distributions that mimic frequencies over a restricted subdomain. *Decisions in Economics and Finance* 43, 17–41. doi:<https://doi.org/10.1007/s10203-020-00281-z>.
- [44] McGee, V., 1989. Conditional probabilities and compounds of conditionals. *Philosophical Review* 98, 485–541.
- [45] Mundici, D., 2021. Deciding koopman’s qualitative probability. *Artificial Intelligence* 299, 103524. URL: <https://www.sciencedirect.com/science/article/pii/S0004370221000758>, doi:<https://doi.org/10.1016/j.artint.2021.103524>.
- [46] Petturiti, D., Vantaggi, B., 2017. Envelopes of conditional probabilities extending a strategy and a prior probability. *International Journal of Approximate Reasoning* 81, 160 – 182.
- [47] Petturiti, D., Vantaggi, B., 2020. Modeling agent’s conditional preferences under objective ambiguity in dempster-shafer theory. *International Journal of Approximate Reasoning* 119, 151–176. doi:<https://doi.org/10.1016/j.ijar.2019.12.019>.
- [48] Pfeifer, N., Sanfilippo, G., 2017. Probabilistic squares and hexagons of opposition under coherence. *International Journal of*

- Approximate Reasoning 88, 282–294. doi:10.1016/j.ijar.2017.05.014.
- [49] Rawal, R., Goel, K., Gupta, C., 2020. Covid-19: Disease pattern study based on semantic-web approach using description logic, in: 2020 IEEE International Conference for Innovation in Technology (INOCON), pp. 1–5. doi:10.1109/INOCON50539.2020.9298278.
- [50] Regazzini, E., 1985. Finitely additive conditional probabilities. *Rendiconti del Seminario Matematico e Fisico di Milano* 55, 69–89.
- [51] Sanfilippo, G., 2018. Lower and upper probability bounds for some conjunctions of two conditional events, in: SUM 2018. Springer International Publishing, Cham. volume 11142 of *LNCS*, pp. 260–275.
- [52] Sanfilippo, G., Gilio, A., Over, D., Pfeifer, N., 2020. Probabilities of conditionals and previsions of iterated conditionals. *International Journal of Approximate Reasoning* 121, 150 – 173. URL: <http://www.sciencedirect.com/science/article/pii/S0888613X19304451>, doi:<https://doi.org/10.1016/j.ijar.2020.03.001>.
- [53] Sanfilippo, G., Pfeifer, N., Over, D., Gilio, A., 2018. Probabilistic inferences from conjoined to iterated conditionals. *International Journal of Approximate Reasoning* 93, 103 – 118. doi:10.1016/j.ijar.2017.10.027.
- [54] Williams, P., 2007. Notes on conditional previsions. *International Journal of Approximate Reasoning* 44, 366–383. URL: <https://www.sciencedirect.com/science/article/pii/S0888613X06001034>, doi:<https://doi.org/10.1016/j.ijar.2006.07.019>.