The effects of convolution and gradient dependence on a parametric Dirichlet problem

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Abstract Our objective is to study a new type of Dirichlet boundary value problem consisting of a system of equations with parameters, where the reaction terms depend on both the solution and its gradient (i.e., they are convection terms) and incorporate the effects of convolutions. We present results on existence, uniqueness and dependence of solutions with respect to the parameters involving convolutions.

Keywords Dirichlet problem · convolution · system of elliptic equations · (p, q)-Laplacian · parametric problems

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. Consider the following system of nonlinear elliptic equations with Dirichlet boundary

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condition and parameters

$$(P_{\mu_1,\mu_2,\rho_1,\rho_2}) \quad \begin{cases} -\Delta_{p_1} u_1 - \mu_1 \Delta_{q_1} u_1 \\ \qquad = f_1(x, \rho_1 * u_1, \rho_2 * u_2, \nabla(\rho_1 * u_1), \nabla(\rho_2 * u_2)) & \text{in } \Omega \\ -\Delta_{p_2} u_2 - \mu_2 \Delta_{q_2} u_2 \\ \qquad = f_2(x, \rho_1 * u_1, \rho_2 * u_2, \nabla(\rho_1 * u_1), \nabla(\rho_2 * u_2)) & \text{in } \Omega \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

where μ_1, μ_2 are non-negative real numbers and $\rho_1, \rho_2 \in L^1(\mathbb{R}^N)$ enter the convolutions. In statement $(P_{\mu_1,\mu_2,\rho_1,\rho_2})$, Δ_{p_i} and Δ_{q_i} denote the p_i-Laplacian and q_i-Laplacian, with $1 < q_i < p_i < +\infty$ for $i = 1, 2$, given as

$$\Delta_{p_i} u = \operatorname{div}(|\nabla u|^{p_i - 2} \nabla u) \quad \text{for all } u \in W_0^{1,p_i}(\Omega),$$
$$\Delta_{q_i} u = \operatorname{div}(|\nabla u|^{q_i - 2} \nabla u) \quad \text{for all } u \in W_0^{1,q_i}(\Omega),$$

where ∇u means the weak gradient of u. The parameters (μ_1, μ_2) appear in the leading operators of the partial differential equations. Important cases of operator $-\Delta_{p_i} u_i - \mu_i \Delta_{q_i} u_i$ are when $\mu_i = 0$ and $\mu_i = 1$ reducing to p_i-Laplacian and (p_i, q_i)-Laplacian, respectively.

The reaction terms in system $(P_{\mu_1,\mu_2,\rho_1,\rho_2})$ are expressed via Carathéodory functions $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ (that is, $x \mapsto f_i(x, s_1, s_2, \xi_1, \xi_2)$ is measurable for all $(s_1, s_2, \xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ and $(s_1, s_2, \xi_1, \xi_2) \mapsto f_i(x, s_1, s_2, \xi_1, \xi_2)$ is continuous for a.e. $x \in \Omega$), $i = 1, 2$. We emphasize that in system $(P_{\mu_1,\mu_2,\rho_1,\rho_2})$ there is full dependence of the right-hand sides on the solution and its gradient. Such expressions are called convection. The presence of convection terms determines the loss of the variational structure. This brings serious technical difficulty since we cannot use variational methods.

A novel trait of problem $(P_{\mu_1,\mu_2,\rho_1,\rho_2})$ is that it incorporates convolutions, which are generated by the functions $\rho_1, \rho_2 \in L^1(\mathbb{R}^N)$. Precisely, identifying any $v \in W_0^{1,p_i}(\Omega)$ with its extension to $\mathbb{R}^N$ by zero outside Ω, the convolution $\rho_i * v$ $(i = 1, 2)$ stands for

$$\rho_i * v(x) = \int_{\mathbb{R}^N} \rho_i(x - y) v(y) dy \quad \text{for all } x \in \mathbb{R}^N.$$

It is well-known that $\nabla(\rho_i * v) = \rho_i * \nabla v$ with the convolution acting componentwise.

We point out that the presence of convolution appears frequently in various applications, taking different meanings in practical problems of computer science and engineering. To provide a detailed survey of such publications belonging to applied sciences is beyond the scope of the present paper, which is a mathematical work studying the new problem $(P_{\mu_1,\mu_2,\rho_1,\rho_2})$ in partial differential equations. Concerning the field of real life applications, we mention for instance the fact that convolution in deep learning gives the cross-correlation in signal and image processing. Specifically, in the field of signal processing, the convolution is useful to smooth out the noise in the original signal. In the related field of image processing, the result of convolution is to smooth out the rough edges in the values taken by a mapping representing mathematically the model of the image under study, so that we have a blurring effect. A huge technical literature is aimed to implement concrete procedures, for example filter operations, in digital image processing that we illustrate by citing [10]. A few mathematically meaningful comments on such physical phenomena can be found in [9]. In view of its general and clear formulation, it is expected that, in addition to its mathematical interest, our problem $(P_{\mu_1,\mu_2,\rho_1,\rho_2})$ be useful in modeling and rigorously dealing with relevant situations in applied sciences.

From the point of view of mathematics, the convolution is a fundamental method for smoothing functions and approximating by smooth functions in functional analysis and operator theory. For essential results in this direction we refer to [2,9]. In the field of differential equations with convolution there were mainly exploited special structures with terms of particular form. As an illustration of the type of problems considered until now, we indicate [1] where a special differential equation exhibiting convolution can be transformed into an algebraic equation. Here our objective is completely different: we formulate and study a novel problem in partial differential equations where a system is driven by (p,q)-Laplacians and there is superposition of the nonlocal operators given by convolutions and arbitrary Nemytskii mappings in the unknown functions and their gradients.

As usual, we denote by p_i^* the Sobolev critical exponent corresponding to p_i $(i=1,2)$. We assume that the following condition is verified:

(H) There are constants $a_i \geq 0$, $b_i \geq 0$, $\alpha_i \geq 0$, $\beta_i \geq 0$ with

$$\alpha_i, \beta_i < p_i - 1, \quad i = 1,2, \tag{1}$$

and functions $\sigma_i \in L^{\gamma_i'}(\Omega)$ with $\gamma_i \in [1, p_i^*)$, where $\frac{1}{\gamma_i} + \frac{1}{\gamma_i'} = 1$, such that

$$|f_1(x,s_1,s_2,\xi_1,\xi_2)| \leq \sigma_1(x) + a_1(|s_1|^{\alpha_1} + |s_2|^{\frac{\alpha_1 p_2}{p_1}}) + b_1(|\xi_1|^{\beta_1} + |\xi_2|^{\frac{\beta_1 p_2}{p_1}}),$$

$$|f_2(x,s_1,s_2,\xi_1,\xi_2)| \leq \sigma_2(x) + a_2(|s_1|^{\frac{\alpha_2 p_1}{p_2}} + |s_2|^{\alpha_2}) + b_2(|\xi_1|^{\frac{\beta_2 p_1}{p_2}} + |\xi_2|^{\beta_2})$$

for a.e. $x \in \Omega$ and all $s_1, s_2 \in \mathbb{R}$, $\xi_1, \xi_2 \in \mathbb{R}^N$.

By a (weak) solution to problem $(P_{\mu_1,\mu_2,\rho_1,\rho_2})$ we mean a pair $(u_1,u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ satisfying

$$\int_\Omega |\nabla u_1|^{p_1-2}\nabla u_1 \nabla v_1 dx + \mu_1 \int_\Omega |\nabla u_1|^{q_1-2}\nabla u_1 \nabla v_1 dx \tag{2}$$

$$= \int_\Omega f_1(x, \rho_1 * u_1, \rho_2 * u_2, \nabla(\rho_1 * u_1), \nabla(\rho_2 * u_2)) v_1 dx,$$

$$\int_\Omega |\nabla u_2|^{p_2-2}\nabla u_2 \nabla v_2 dx + \mu_2 \int_\Omega |\nabla u_2|^{q_2-2}\nabla u_2 \nabla v_2 dx \tag{3}$$

$$= \int_\Omega f_2(x, \rho_1 * u_1, \rho_2 * u_2, \nabla(\rho_1 * u_1), \nabla(\rho_2 * u_2)) v_2 dx$$

for all $(v_1,v_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$. Note that the growth conditions in hypothesis (H) imply that the integrals in (2), (3) exist.

The starting point of the present work is the equation investigated in [11] with homogeneous Dirichlet boundary condition

$$\begin{cases} -\Delta_p u - \mu \Delta_q u = f(x, \rho * u, \nabla(\rho * u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

corresponding to $\rho \in L^1(\mathbb{R}^N)$. Our goal is to study the system situation giving rise to problem $(P_{\mu_1,\mu_2,\rho_1,\rho_2})$. It is a challenging task by passing from a single equation to a system due to the interaction of variables. Recent results on systems with convection but without convolution can be found in [13]. A sub-supersolution approach for systems with convection without convolution and when $\mu_1 = \mu_2 = 0$ is developed in [4] (for other results in this direction, see [3,14]). For the study of equations involving p-Laplacian and convection terms without convolutions we refer to [5–8,19,20]. We also mention the recent contributions on (p,q)-Laplacian equations without gradient dependence in references [16–18] and [15], the latter dealing with a variable exponent space.

In the present work we prove results guaranteeing existence, uniqueness and upper semi-continuity with respect to the parameters $\rho_1, \rho_2 \in L^1(\mathbb{R}^N)$ for solutions of system $(P_{\mu_1,\mu_2,\rho_1,\rho_2})$. Fundamental tools in the proofs of our results are the theory of pseudomonotone operators as given, e.g., in [3,12,21] and properties of convolution operation (see, e.g., [2,9]).

2 Existence result

We identify f_i with the function $\tilde{f}_i : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ obtained by extending $f_i(\cdot, s_1, s_2, \xi_1, \xi_2)$ by 0 outside Ω for $i = 1,2$. Denote $p_i' = p_i/(p_i - 1)$ (the Hölder conjugate of p_i), for $i = 1,2$. Our existence result is as follows.

Theorem 1 *If condition (H) holds, then problem $(P_{\mu_1,\mu_2,\rho_1,\rho_2})$ admits a (weak) solution for each $\mu_1, \mu_2 \geq 0$ and $\rho_1, \rho_2 \in L^1(\mathbb{R}^N)$.*

Proof Consider the product space $W := W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ endowed with the norm $\|u\| = \||\nabla u_1|\|_{L^{p_1}(\Omega)} + \||\nabla u_2|\|_{L^{p_2}(\Omega)}$ for all $u = (u_1, u_2) \in W$. With fixed $\mu_1, \mu_2 \geq 0$ and $\rho_1, \rho_2 \in L^1(\mathbb{R}^N)$, we introduce the nonlinear operator $A : W \to W^* = W^{-1,p_1'}(\Omega) \times W^{-1,p_2'}(\Omega)$ as

$$\begin{aligned} A(u_1, u_2) = (&-\Delta_{p_1} u_1 - \mu_1 \Delta_{q_1} u_1 - E_1^* N_1(\rho_1 * E_1 u_1, \rho_2 * E_2 u_2), \\ &- \Delta_{p_2} u_2 - \mu_2 \Delta_{q_2} u_2 - E_2^* N_2(\rho_1 * E_1 u_1, \rho_2 * E_2 u_2)), \end{aligned} \tag{4}$$

where $N_i : W^{1,p_i}(\mathbb{R}^N) \to W^{-1,p_i'}(\mathbb{R}^N)$ given by

$$N_i(u_1, u_2) = f_i(\cdot, u_1(\cdot), u_2(\cdot), \nabla u_1(\cdot), \nabla u_2(\cdot))$$

is the Nemytskii operator associated to the function f_i (actually, $\tilde{f}_i$), and $E_i : W_0^{1,p_i}(\Omega) \to W^{1,p_i}(\mathbb{R}^N)$ stands for the bounded linear operator obtained by extension with zero outside Ω, $i = 1,2$. Consider the adjoint operator

$E_i^* : W^{1,p_i}(\mathbb{R}^N)^* \to W^{-1,p_i'}(\Omega)$ of E_i between the respective dual spaces for $i = 1, 2$, which is defined by

$$\langle E_i^*(g), h\rangle = \langle g, E_i(h)\rangle, \ \forall g \in W^{1,p_i}(\mathbb{R}^N)^*, \ h \in W_0^{1,p_i}(\Omega).$$

Therefore for $w \in W_0^{1,p_i}(\Omega)$ we have

$$\begin{aligned}&\langle E_i^* N_i(\rho_1 * E_1 u_1, \rho_2 * E_2 u_2), w\rangle \\ &= \int_\Omega f_i(x, \rho_1 * u_1, \rho_2 * u_2, \nabla(\rho_1 * u_1), \nabla(\rho_2 * u_2)) w dx.\end{aligned}$$

The growth required in condition (H) ensures that the map A in (4) is well defined.

On the basis of Tonelli's and Fubini's theorems and of Hölder's inequality we have the fundamental inequalities

$$\|\rho_i * v\|_{L^{p_i}(\mathbb{R}^N)} \le \|\rho_i\|_{L^1(\mathbb{R}^N)} \|v\|_{L^{p_i}(\Omega)}, \tag{5}$$

$$\left\|\rho_i * \frac{\partial v}{\partial x_j}\right\|_{L^{p_i}(\mathbb{R}^N)} \le \|\rho_i\|_{L^1(\mathbb{R}^N)} \left\|\frac{\partial v}{\partial x_j}\right\|_{L^{p_i}(\Omega)}, \quad j = 1, \cdots, N \tag{6}$$

(see [2]). From (5) and (6) we note that $v \mapsto \rho_i * v$ is a linear continuous operator $W_0^{1,p_i}(\Omega) \to W^{1,p_i}(\mathbb{R}^N)$.

An important consequence of estimate (6) is the following inequality

$$\||\nabla(\rho_i * v)|\|_{L^{p_i}(\mathbb{R}^N)} \le N \|\rho_i\|_{L^1(\mathbb{R}^N)} \||\nabla v|\|_{L^{p_i}(\Omega)}. \tag{7}$$

This is proved by using Minkowski's inequality, the convexity of $t \mapsto t^{p_i}$ on $(0, +\infty)$ and estimate (6) in the following way

$$\begin{aligned}&\||\nabla(\rho_i * u)|\|_{L^{p_i}(\mathbb{R}^N)}^{p_i} = \int_{\mathbb{R}^N} \left(\sum_{j=1}^N \left(\rho_i * \frac{\partial u}{\partial x_j}\right)^2\right)^{\frac{p_i}{2}} dx \\ &\le \int_{\mathbb{R}^N} \left(\sum_{j=1}^N \left|\rho_i * \frac{\partial u}{\partial x_j}\right|\right)^{p_i} dx \le N^{p_i - 1} \sum_{j=1}^N \left\|\rho_i * \frac{\partial u}{\partial x_j}\right\|_{L^{p_i}(\mathbb{R}^N)}^{p_i} \\ &\le N^{p_i} \|\rho_i\|_{L^1(\mathbb{R}^N)}^{p_i} \||\nabla u|\|_{L^{p_i}(\Omega)}^{p_i},\end{aligned}$$

from which we get (7).

Since the operator $-\Delta_{p_i} - \mu_i \Delta_{q_i}$ is bounded on $W_0^{1,p_i}(\Omega)$ for $i = 1, 2$, and condition (H) holds, it is clear that the map A in (4) is bounded in the sense that it maps bounded sets into bounded sets.

Next we show the pseudomonotonicity of the nonlinear operator A in (4). To this end, let $\{(u_{1,n}, u_{2,n})\} \subset W$ be a sequence weakly converging to (u_1, u_2) in W that satisfies

$$\limsup_{n\to+\infty} \langle A(u_{1,n}, u_{2,n}), (u_{1,n} - u_1, u_{2,n} - u_2)\rangle \le 0. \tag{8}$$

We claim that for $i=1,2$ it holds

$$\lim_{n\to+\infty}\int_\Omega f_i(x,\rho_1*u_{1,n},\rho_2*u_{2,n},\nabla(\rho_1*u_{1,n}),\nabla(\rho_2*u_{2,n}))(u_{i,n}-u_i)dx=0. \tag{9}$$

Let us note that (5), (6), (7), Hölder's inequality and Rellich-Kondrachov compact embedding theorem imply the following convergence results as $n\to\infty$:

$$\int_\Omega|\sigma_i||u_{i,n}-u_i|dx\le\|\sigma_i\|_{L^{\gamma_i'}(\Omega)}\|u_{i,n}-u_i\|_{L^{\gamma_i}(\Omega)}\to 0,$$

$$\begin{aligned}&\int_\Omega|\rho_i*u_{i,n}|^{\alpha_i}|u_{i,n}-u_i|dx\\ &\le\|\rho_i\|^{\alpha_i}_{L^1(\mathbb{R}^N)}\|u_{i,n}\|^{\alpha_i}_{L^{p_i}(\Omega)}\|u_{i,n}-u_i\|_{L^{\frac{p_i}{p_i-\alpha_i}}(\Omega)}\to 0,\end{aligned}$$

$$\begin{aligned}&\int_\Omega|\nabla(\rho_i*u_{i,n})|^{\beta_i}|u_{i,n}-u_i|dx\\ &\le N^{p_i}\|\rho_i\|^{\beta_i}_{L^1(\mathbb{R}^N)}\|\nabla u_{i,n}\|^{\beta_i}_{L^{p_i}(\Omega)}\|u_{i,n}-u_i\|_{L^{\frac{p_i}{p_i-\beta_i}}(\Omega)}\to 0,\end{aligned}$$

$$\begin{aligned}&\int_\Omega|\rho_j*u_{j,n}|^{\frac{\alpha_ip_j}{p_i}}|u_{i,n}-u_i|dx\\ &\le\|\rho_j\|^{\frac{\alpha_ip_j}{p_i}}_{L^1(\mathbb{R}^N)}\|u_{j,n}\|^{\frac{\alpha_ip_j}{p_i}}_{L^{p_j}(\Omega)}\|u_{i,n}-u_i\|_{L^{\frac{p_i}{p_i-\alpha_i}}(\Omega)}\to 0,\quad i\ne j,\end{aligned}$$

$$\begin{aligned}&\int_\Omega|\nabla(\rho_j*u_{j,n})|^{\frac{\beta_ip_j}{p_i}}|u_{i,n}-u_i|dx\\ &\le N^{\frac{\beta_ip_j}{p_i}}\|\rho_j\|^{\frac{\beta_ip_j}{p_i}}_{L^1(\mathbb{R}^N)}\|\nabla u_{j,n}\|^{\frac{\beta_ip_j}{p_i}}_{L^{p_j}(\Omega)}\|u_{i,n}-u_i\|_{L^{\frac{p_i}{p_i-\beta_i}}(\Omega)}\to 0,\quad i\ne j.\end{aligned}$$

From these and hypothesis (H) we get readily (9).

Observe that combining (9) and (8) we obtain

$$\begin{aligned}\limsup_{n\to+\infty}[&\langle-\Delta_{p_1}u_{1,n}-\mu_1\Delta_{q_1}u_{1,n},u_{1,n}-u_1\rangle\\ &+\langle-\Delta_{p_2}u_{2,n}-\mu_2\Delta_{q_2}u_{2,n},u_{2,n}-u_2\rangle]\le 0.\end{aligned}\tag{10}$$

Then through a reasoning by contradiction (see [13]) we can prove that

$$\limsup_{n\to+\infty}\langle-\Delta_{p_i}u_{i,n}-\mu_i\Delta_{q_i}u_{i,n},u_{i,n}-u_i\rangle\le 0,\quad i=1,2. \tag{11}$$

By (11) and using the $(S)_+$-property of the map $-\Delta_{p_i}-\mu_i\Delta_{q_i}$ on $W_0^{1,p_i}(\Omega)$ (see [3,12]), it turns out the strong convergence $u_{i,n}\to u_i$ in $W_0^{1,p_i}(\Omega)$ for $i=1,2$. Consequently, for each $(v_1,v_2)\in W$ we have

$$\lim_{n\to+\infty}\langle A(u_{1,n},u_{2,n}),(u_{1,n}-v_1,u_{2,n}-v_2)\rangle=\langle A(u_1,u_2),(u_1-v_1,u_2-v_2)\rangle$$

because we have already shown that the operator A is continuous and bounded. Therefore the operator A in (4) is pseudomonotone.

We also need to check the coercivity of the nonlinear operator A, which means

$$\lim_{\|u\|\to+\infty} \frac{\langle Au, u\rangle}{\|u\|} = +\infty, \tag{12}$$

where we recall that $\|u\| = \||\nabla u_1|\|_{L^{p_1}(\Omega)} + \||\nabla u_2|\|_{L^{p_2}(\Omega)}$ for all $u = (u_1, u_2) \in W$.

By the definition of the operator A in (4) we have

$$\begin{aligned}
&\langle Au, u\rangle = \||\nabla u_1|\|^{p_1}_{L^{p_1}(\Omega)} + \||\nabla u_2|\|^{p_2}_{L^{p_2}(\Omega)} \\
&+\mu_1\||\nabla u_1|\|^{q_1}_{L^{q_1}(\Omega)} + \mu_2\||\nabla u_2|\|^{q_2}_{L^{q_2}(\Omega)} \\
&- \int_\Omega f_1(x, \rho_1 * u_1, \rho_2 * u_2, \nabla(\rho_1 * u_1), \nabla(\rho_2 * u_2))u_1 dx \\
&- \int_\Omega f_2(x, \rho_1 * u_1, \rho_2 * u_2, \nabla(\rho_1 * u_1), \nabla(\rho_2 * u_2))u_2 dx.
\end{aligned} \tag{13}$$

We note from assumption (H), Hölder's inequality, (5), (6) and (7) that the following estimate holds

$$\begin{aligned}
&\left|\int_\Omega f_i(x, \rho_1 * u_1, \rho_2 * u_2, \nabla(\rho_1 * u_1), \nabla(\rho_2 * u_2))u_i dx\right| \\
&\leq \|\sigma_i\|_{L^{\gamma_i'}(\Omega)}\|u_i\|_{L^{\gamma_i}(\Omega)} \\
&+ a_i\left(\|\rho_i\|^{\alpha_i}_{L^1(\mathbb{R}^N)}\|u_i\|^{\alpha_i}_{L^{p_i}(\Omega)} + \|\rho_j\|^{\frac{\alpha_i p_j}{p_i}}_{L^1(\mathbb{R}^N)}\|u_j\|^{\frac{\alpha_i p_j}{p_i}}_{L^{p_j}(\Omega)}\right)\|u_i\|_{L^{\frac{p_i}{p_i-\alpha_i}}(\Omega)} \\
&+ b_i\left(N^{p_i}\|\rho_i\|^{\beta_i}_{L^1(\mathbb{R}^N)}\||\nabla u_i|\|^{\beta_i}_{L^{p_i}(\Omega)} + N^{\frac{\beta_i p_j}{p_i}}\|\rho_j\|^{\frac{\beta_i p_j}{p_i}}_{L^1(\mathbb{R}^N)}\||\nabla u_j|\|^{\frac{\beta_i p_j}{p_i}}_{L^{p_j}(\Omega)}\right) \\
&\qquad\qquad \times \|u_i\|_{L^{\frac{p_i}{p_i-\beta_i}}(\Omega)}
\end{aligned}$$

for all $(u_1, u_2) \in W$, $i, j = 1, 2$, $i \neq j$. Here as before $u_i \in W^{1,p_i}_0(\Omega)$ is identified with $E_i(u_i)$, where $i = 1, 2$. Then, on the basis of Sobolev embedding theorem, we can find positive constants $\tilde{a}_i$ and $\tilde{b}_i$ such that

$$\begin{aligned}
&\left|\int_\Omega f_i(x, \rho_1 * u_1, \rho_2 * u_2, \nabla(\rho_1 * u_1), \nabla(\rho_2 * u_2))u_i dx\right| \\
&\leq \tilde{a}_i\left(\||\nabla u_i|\|_{L^{p_i}(\Omega)} + \||\nabla u_i|\|^{\alpha_i+1}_{L^{p_i}(\Omega)} + \||\nabla u_j|\|^{\frac{\alpha_i p_j}{p_i}}_{L^{p_j}(\Omega)}\||\nabla u_i|\|_{L^{p_i}(\Omega)}\right) \\
&+ \tilde{b}_i\left(\||\nabla u_i|\|^{\beta_i+1}_{L^{p_i}(\Omega)} + \||\nabla u_j|\|^{\frac{\beta_i p_j}{p_i}}_{L^{p_j}(\Omega)}\||\nabla u_i|\|_{L^{p_i}(\Omega)}\right)
\end{aligned}$$

for all $(u_1, u_2) \in W$, $i, j = 1, 2$, $i \neq j$. Through Young's inequality with $\varepsilon > 0$, the preceding inequality becomes

$$
\begin{aligned}
&\left| \int_\Omega f_i(x, \rho_1 * u_1, \rho_2 * u_2, \nabla(\rho_1 * u_1), \nabla(\rho_2 * u_2)) u_i dx \right| \qquad (14)\\
&\leq \tilde{a}_i \left(\||\nabla u_i|\|_{L^{p_i}(\Omega)} + \||\nabla u_i|\|_{L^{p_i}(\Omega)}^{\alpha_i+1} + \varepsilon \||\nabla u_i|\|_{L^{p_i}(\Omega)}^{p_i} + c(\varepsilon) \||\nabla u_j|\|_{L^{p_j}(\Omega)}^{\frac{\alpha_i p_j}{p_i - 1}} \right) \\
&+ \tilde{b}_i \left(\||\nabla u_i|\|_{L^{p_i}(\Omega)}^{\beta_i+1} + \varepsilon \||\nabla u_i|\|_{L^{p_i}(\Omega)}^{p_i} + c(\varepsilon) \||\nabla u_j|\|_{L^{p_j}(\Omega)}^{\frac{\beta_i p_j}{p_i - 1}} \right)
\end{aligned}
$$

for all $(u_1, u_2) \in W$, $i, j = 1, 2$, $i \neq j$, with a positive constant $c(\varepsilon)$ depending on ε. Let us insert this estimate in (13), which gives

$$
\begin{aligned}
&\langle Au, u \rangle \qquad (15)\\
&\geq (1 - \varepsilon \tilde{a}_1 - \varepsilon \tilde{b}_1) \||\nabla u_1|\|_{L^{p_1}(\Omega)}^{p_1} + (1 - \varepsilon \tilde{a}_2 - \varepsilon \tilde{b}_2) \||\nabla u_2|\|_{L^{p_2}(\Omega)}^{p_2} \\
&- \tilde{a}_1 \left(\||\nabla u_1|\|_{L^{p_1}(\Omega)} + \||\nabla u_1|\|_{L^{p_1}(\Omega)}^{\alpha_1+1} + c(\varepsilon) \||\nabla u_2|\|_{L^{p_2}(\Omega)}^{\frac{\alpha_1 p_2}{p_1 - 1}} \right) \\
&- \tilde{a}_2 \left(\||\nabla u_2|\|_{L^{p_2}(\Omega)} + \||\nabla u_2|\|_{L^{p_2}(\Omega)}^{\alpha_2+1} + c(\varepsilon) \||\nabla u_1|\|_{L^{p_1}(\Omega)}^{\frac{\alpha_2 p_1}{p_2 - 1}} \right) \\
&- \tilde{b}_1 \left(\||\nabla u_1|\|_{L^{p_1}(\Omega)}^{\beta_1+1} + c(\varepsilon) \||\nabla u_2|\|_{L^{p_2}(\Omega)}^{\frac{\beta_1 p_2}{p_1 - 1}} \right) \\
&- \tilde{b}_2 \left(\||\nabla u_2|\|_{L^{p_2}(\Omega)}^{\beta_2+1} + c(\varepsilon) \||\nabla u_1|\|_{L^{p_1}(\Omega)}^{\frac{\beta_2 p_1}{p_2 - 1}} \right).
\end{aligned}
$$

Notice from (1) that

$$\alpha_1, \beta_1 < p_1 - 1 \text{ and } \alpha_2, \beta_2 < p_2 - 1,$$

which forces

$$\frac{\alpha_2 p_1}{p_2 - 1}, \frac{\beta_2 p_1}{p_2 - 1} < p_1 \text{ and } \frac{\alpha_1 p_2}{p_1 - 1}, \frac{\beta_1 p_2}{p_1 - 1} < p_2.$$

Choose an $\varepsilon > 0$ sufficiently small to satisfy $1 - \varepsilon \tilde{a}_1 - \varepsilon \tilde{b}_1 > 0$ and $1 - \varepsilon \tilde{a}_2 - \varepsilon \tilde{b}_2 > 0$.

We note that given positive numbers $r < p$, for every $\tau > 0$ there is a constant $C(\tau) > 0$ depending on τ such that

$$t^r \leq \tau t^p + C(\tau), \ \forall t > 0. \qquad (16)$$

Apply repeatedly inequality (16) with p_1 and p_2 in place of p and with an r equal to the lower order exponents in (15), which correspond to the inequalities displayed below (15). Then, since $\tau > 0$ in (16) is arbitrary, we are able to find a $\sigma > 0$ with

$$\sigma < \min\{1 - \varepsilon \tilde{a}_1 - \varepsilon \tilde{b}_1, 1 - \varepsilon \tilde{a}_2 - \varepsilon \tilde{b}_2\}$$

(recall that $\varepsilon > 0$ was fixed) and for which (15) yields

$$\begin{aligned}&\langle Au, u\rangle\\ &\geq (1-\varepsilon\tilde{a}_1-\varepsilon\tilde{b}_1-\sigma)\||\nabla u_1|\|^{p_1}_{L^{p_1}(\Omega)}+(1-\varepsilon\tilde{a}_2-\varepsilon\tilde{b}_2-\sigma)\||\nabla u_2|\|^{p_2}_{L^{p_2}(\Omega)}\\ &-C(\varepsilon,\sigma)\\ &\geq C_0(\varepsilon,\sigma)\left(\||\nabla u_1|\|^{p_1}_{L^{p_1}(\Omega)}+\||\nabla u_2|\|^{p_2}_{L^{p_2}(\Omega)}\right)-C(\varepsilon,\sigma),\end{aligned}$$

where $C(\varepsilon,\sigma)$ is a positive constant depending on ε and σ, and

$$C_0(\varepsilon,\sigma)=\min\{1-\varepsilon\tilde{a}_2-\varepsilon\tilde{b}_2-\sigma, 1-\varepsilon\tilde{a}_2-\varepsilon\tilde{b}_2-\sigma)\}>0.$$

Assume without any loss of generality that $p_1 \geq p_2$. The above estimate and a well-known convexity inequality imply for all $u=(u_1,u_2)\in W$ that

$$\begin{aligned}&\langle Au, u\rangle \geq C_0(\varepsilon,\sigma)\left(\||\nabla u_1|\|^{p_1}_{L^{p_1}(\Omega)}+\||\nabla u_2|\|^{p_2}_{L^{p_2}(\Omega)}\right)-C(\varepsilon,\sigma)\\ &=C_0(\varepsilon,\sigma)\left(\||\nabla u_1|\|^{p_1}_{L^{p_1}(\Omega)}-\||\nabla u_1|\|^{p_2}_{L^{p_2}(\Omega)}\right)\\ &+C_0(\varepsilon,\sigma)\left(\||\nabla u_1|\|^{p_2}_{L^{p_2}(\Omega)}+\||\nabla u_2|\|^{p_2}_{L^{p_2}(\Omega)}\right)-C(\varepsilon,\sigma)\\ &\geq -C_0(\varepsilon,\sigma)+\frac{1}{2^{p_2-1}}C_0(\varepsilon,\sigma)\left(\||\nabla u_1|\|_{L^{p_2}(\Omega)}+\||\nabla u_2|\|_{L^{p_2}(\Omega)}\right)^{p_2}-C(\varepsilon,\sigma)\\ &=\frac{1}{2^{p_2-1}}C_0(\varepsilon,\sigma)\|u\|^{p_2}-C_0(\varepsilon,\sigma)-C(\varepsilon,\sigma).\end{aligned}$$

Since $p_2 > 1$, there results that the limit in (12) holds true.

Summarizing, we have shown that the operator $A : W \to W^*$ given by (4) is pseudomonotone, bounded, and coercive. So, by virtue of the main theorem on pseudomonotone operators (see [21, Theorem 27.A]), we know that there is $u=(u_1,u_2)\in W$ such that $Au = 0$, which is equivalent to the fact that $u=(u_1,u_2)$ is a (weak) solution of system $(P_{\mu_1,\mu_2,\rho_1,\rho_2})$. This completes the proof.

3 Uniqueness result

Now we focus on the uniqueness of solution to problem $(P_{\mu_1,\mu_2,\rho_1,\rho_2})$. Consider the vector field $f : \Omega\times\mathbb{R}^2\times(\mathbb{R}^N)^2\to\mathbb{R}^2$ given as

$$f(x,s,\xi)=(f_1(x,s,\xi), f_2(x,s,\xi)) \text{ a.e. } x\in\Omega,\ \forall s\in\mathbb{R}^2,\ \xi\in(\mathbb{R}^N)^2. \quad (17)$$

We suppose that f satisfies the following condition, which is slightly stronger than the one utilized in [13]:

(U) There are constants $a, b \geq 0$ and a function $\tau=(\tau_1,\tau_2)\in L^1(\Omega,\mathbb{R}^2)$ for which the map $f(x,s,\cdot)-\tau(x)$ is linear on $(\mathbb{R}^N)^2$ and the inequalities below hold

$$|f(x,s,\xi)-f(x,t,\xi)|\leq a|s-t| \text{ for a.e. } x\in\Omega,\ \forall s,t\in\mathbb{R}^2,\ \xi\in(\mathbb{R}^N)^2, \quad (18)$$

$$|f(x,s,\xi)-\tau(x)|\leq b|\xi| \text{ a.e. } x\in\Omega,\ \forall(s,\xi)\in\mathbb{R}^2\times(\mathbb{R}^N)^2. \quad (19)$$

By "$\cdot$" we will denote the standard scalar product in $\mathbb{R}^2$, while the notation "$|\cdot|$" means the Euclidean norm in $(\mathbb{R}^N)^2$ and in $\mathbb{R}^2$.

By λ_1 we denote the first eigenvalue of $-\Delta$ on $W_0^{1,2}(\Omega)$ whose variational characterization is

$$\lambda_1 = \inf_{u \in W_0^{1,2}(\Omega), u \neq 0} \frac{\| |\nabla u| \|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}.$$

Theorem 2 *Assume that conditions (H) and (U) hold true.*

(i) If $p_1 = p_2 = 2$ and

$$\sqrt{2}\lambda_1^{-1}(a + Nb\lambda_1^{\frac{1}{2}}) \max\{\|\rho_1\|_{L^1(\mathbb{R}^N)}, \|\rho_2\|_{L^1(\mathbb{R}^N)}\} < 1, \tag{20}$$

then the solution of problem $(P_{\mu_1,\mu_2,\rho_1,\rho_2})$ is unique for every $\mu_1, \mu_2 \geq 0$.

(ii) If $q_1 = q_2 = 2$ and the parameters μ_1 and μ_2 satisfy

$$\sqrt{2}\lambda_1^{-1}(a + Nb\lambda_1^{\frac{1}{2}}) \max\{\|\rho_1\|_{L^1(\mathbb{R}^N)}, \|\rho_2\|_{L^1(\mathbb{R}^N)}\} < \min\{\mu_1, \mu_2\},$$

then the solution of problem $(P_{\mu_1,\mu_2,\rho_1,\rho_2})$ is unique.

Proof By Theorem 1 we know that system $(P_{\mu_1,\mu_2,\rho_1,\rho_2})$ admits at least one (weak) solution $u = (u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$. Arguing indirectly let us admit that another solution $v = (v_1, v_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ to $(P_{\mu_1,\mu_2,\rho_1,\rho_2})$ exists.

Using the test functions $u_1 - v_1 \in W_0^{1,p_1}(\Omega)$ and $u_2 - v_2 \in W_0^{1,p_2}(\Omega)$ we get

$$\begin{aligned}
&\langle -\Delta_{p_1} u_1 + \Delta_{p_1} v_1, u_1 - v_1\rangle + \mu_1 \langle -\Delta_{q_1} u_1 + \Delta_{q_1} v_1, u_1 - v_1\rangle \\
&+\langle -\Delta_{p_2} u_2 + \Delta_{p_2} v_2, u_2 - v_2\rangle + \mu_2 \langle -\Delta_{q_2} u_2 + \Delta_{q_2} v_2, u_2 - v_2\rangle \\
&= \int_\Omega (f(x, \rho_1 * u_1, \rho_2 * u_2, \nabla(\rho_1 * u_1), \nabla(\rho_2 * u_2)) \\
&-f(x, \rho_1 * v_1, \rho_2 * v_2, \nabla(\rho_1 * u_1), \nabla(\rho_2 * u_2))) \cdot (u - v)\ dx \\
&+ \int_\Omega (f(x, \rho_1 * v_1, \rho_2 * v_2, \nabla(\rho_1 * u_1), \nabla(\rho_2 * u_2)) - \tau(x) \\
&-f(x, \rho_1 * v_1, \rho_2 * v_2, \nabla(\rho_1 * v_1), \nabla(\rho_2 * v_2)) + \tau(x)) \cdot (u - v)\ dx.
\end{aligned} \tag{21}$$

Next we distinguish two cases.

(i) Assume that $p_1 = p_2 = 2$. By (18), Cauchy-Schwarz and Minkowski's inequalities, estimate (5) and the variational characterization of λ_1, we get

$$\begin{aligned}
&\int_\Omega (f(x,\rho_1 * u_1, \rho_2 * u_2, \nabla(\rho_1 * u_1), \nabla(\rho_2 * u_2)) \qquad (22)\\
&-f(x,\rho_1 * v_1, \rho_2 * v_2, \nabla(\rho_1 * u_1), \nabla(\rho_2 * u_2))) \cdot (u-v)\ dx\\
&\le a \int_\Omega |(\rho_1 * (u_1 - v_1), \rho_2 * (u_2 - v_2))||u-v|\ dx\\
&\le a \left(\int_\Omega ((\rho_1 * (u_1 - v_1))^2 + (\rho_2 * (u_2 - v_2))^2)\ dx\right)^{\frac{1}{2}} \||u-v|\|_{L^2(\Omega)}\\
&\le a \left(\|\rho_1 * (u_1 - v_1)\|_{L^2(\Omega)} + \|\rho_2 * (u_2 - v_2)\|_{L^2(\Omega)}\right) \||u-v|\|_{L^2(\Omega)}\\
&\le a \max\left\{\|\rho_1\|_{L^1(\mathbb{R}^N}, \|\rho_2\|_{L^1(\mathbb{R}^N)}\right\} \left(\|u_1 - v_1\|_{L^2(\Omega)} + \|u_2 - v_2\|_{L^2(\Omega)}\right)\\
&\times \left(\|u_1 - v_1\|^2_{L^2(\Omega)} + \|u_2 - v_2\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}\\
&\le a\lambda_1^{-1} \max\left\{\|\rho_1\|_{L^1(\mathbb{R}^N}, \|\rho_2\|_{L^1(\mathbb{R}^N)}\right\}\\
&\times \left(\||\nabla(u_1 - v_1)|\|_{L^2(\Omega)} + \||\nabla(u_2 - v_2)|\|_{L^2(\Omega)}\right)\\
&\times \left(\||\nabla(u_1 - v_1)|\|^2_{L^2(\Omega)} + \||\nabla(u_2 - v_2)|\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}\\
&\le \sqrt{2} a\lambda_1^{-1} \max\left\{\|\rho_1\|_{L^1(\mathbb{R}^N}, \|\rho_2\|_{L^1(\mathbb{R}^N)}\right\}\\
&\times \left(\||\nabla(u_1 - v_1)|\|^2_{L^2(\Omega)} + \||\nabla(u_2 - v_2)|\|^2_{L^2(\Omega)}\right).
\end{aligned}$$

By the linearity of the map $f(x,s,\cdot) - \tau(x)$ on $(\mathbb{R}^N)^2$ as postulated in assumption (U), the linearity of the gradient and convolution in each variable, Cauchy-Schwarz and Minkowski's inequalities, (19), estimate (7), and the variational characterization of λ_1, we obtain

$$\begin{aligned}
&\int_\Omega (f(x,\rho_1 * v_1, \rho_2 * v_2, \nabla(\rho_1 * u_1), \nabla(\rho_2 * u_2)) - \tau(x) \qquad (23)\\
&-f(x,\rho_1 * v_1, \rho_2 * v_2, \nabla(\rho_1 * v_1), \nabla(\rho_2 * v_2)) + \tau(x)) \cdot (u-v)\ dx\\
&= \int_\Omega (f(x,\rho_1 * v_1, \rho_2 * v_2, \nabla(\rho_1 * (u_1 - v_1)), \nabla(\rho_2 * (u_2 - v_2))) - \tau(x))\\
&\cdot (u-v)\ dx\\
&\le \int_\Omega |f(x,\rho_1 * v_1, \rho_2 * v_2, \nabla(\rho_1 * (u_1 - v_1)), \nabla(\rho_2 * (u_2 - v_2))) - \tau(x)|\\
&\times |u-v|\ dx\\
&\le b \int_\Omega |(\nabla(\rho_1 * (u_1 - v_1)), \nabla(\rho_2 * (u_2 - v_2))||u-v|\ dx\\
&\le b \left(\int_\Omega (|\nabla(\rho_1 * (u_1 - v_1))|^2 + |\nabla(\rho_2 * (u_2 - v_2))|^2)\ dx\right)^{\frac{1}{2}} \||u-v|\|_{L^2(\Omega)}\\
&\le b \left(\||\nabla(\rho_1 * (u_1 - v_1)|\|_{L^2(\Omega)} + \||\nabla(\rho_2 * (u_2 - v_2))|\|_{L^2(\Omega)}\right) \||u-v|\|_{L^2(\Omega)}\\
&\le Nb \max\left\{\|\rho_1\|_{L^1(\mathbb{R}^N}, \|\rho_2\|_{L^1(\mathbb{R}^N)}\right\}\\
&\times (\||\nabla(u_1 - v_1)|\|_{L^2(\Omega)} + \||\nabla(u_2 - v_2)|\|_{L^2(\Omega)})
\end{aligned}$$

$$\times \left(\|u_1 - v_1\|^2_{L^2(\Omega)} + \|u_2 - v_2\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$$
$$\leq Nb\lambda_1^{-\frac{1}{2}} \max \left\{ \|\rho_1\|_{L^1(\mathbb{R}^N}, \|\rho_2\|_{L^1(\mathbb{R}^N)} \right\}$$
$$\times (\||\nabla(u_1 - v_1)|\|_{L^2(\Omega)} + \||\nabla(u_2 - v_2)|\|_{L^2(\Omega)})$$
$$\times \left(\|\nabla(u_1 - v_1)\|^2_{L^2(\Omega)} + \|\nabla(u_2 - v_2)\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$$
$$\leq \sqrt{2} Nb\lambda_1^{-\frac{1}{2}} \max \left\{ \|\rho_1\|_{L^1(\mathbb{R}^N}, \|\rho_2\|_{L^1(\mathbb{R}^N)} \right\}$$
$$\times \left(\||\nabla(u_1 - v_1)|\|^2_{L^2(\Omega)} + \||\nabla(u_2 - v_2)|\|^2_{L^2(\Omega)} \right).$$

By the monotonicity of $-\Delta_{q_i}$ for $i = 1, 2$, it follows that combining (21) in the case of $p_1 = p_2 = 2$ with (22) and (23) we arrive at

$$\||\nabla(u_1 - v_1)|\|^2_{L^2(\Omega)} + \||\nabla(u_2 - v_2)|\|^2_{L^2(\Omega)}$$
$$\leq \sqrt{2}\lambda_1^{-1}(a + Nb\lambda_1^{\frac{1}{2}}) \max \left\{ \|\rho_1\|_{L^1(\mathbb{R}^N}, \|\rho_2\|_{L^1(\mathbb{R}^N)} \right\}$$
$$\times \left(\||\nabla(u_1 - v_1)|\|^2_{L^2(\Omega)} + \||\nabla(u_2 - v_2)|\|^2_{L^2(\Omega)} \right).$$

In view of assumption (20), we conclude that $u_i = v_i$ for $i = 1, 2$.

(ii) Assume that $q_1 = q_2 = 2$ and choose (μ_1, μ_2) as required in part (ii). The computation performed for part (i) and the monotonicity of $-\Delta_{p_i}$ for $i = 1, 2$ entail

$$\mu_1 \||\nabla(u_1 - v_1)\|^2_{L^2(\Omega)} + \mu_2 \||\nabla(u_2 - v_2)|\|^2_{L^2(\Omega)}$$
$$\leq \sqrt{2}\lambda_1^{-1}(a + Nb\lambda_1^{\frac{1}{2}}) \max \left\{ \|\rho_1\|_{L^1(\mathbb{R}^N}, \|\rho_2\|_{L^1(\mathbb{R}^N)} \right\}$$
$$\times \left(\||\nabla(u_1 - v_1)|\|^2_{L^2(\Omega)} + \||\nabla(u_2 - v_2)|\|^2_{L^2(\Omega)} \right).$$

Then from the requirement on (μ_1, μ_2) we deduce that $u_i = v_i$ for $i = 1, 2$.

4 Dependence on $(\rho_1, \rho_2) \in L^1(\mathbb{R})^2$

We provide a result about the dependence of solution set on the parameter $(\rho_1, \rho_2) \in L^1(\mathbb{R})^2$ in problem $(P_{\mu_1,\mu_2,\rho_1,\rho_2})$. Recall that in [13] the considered problem did not contain convolutions. There were just parameters $(\mu_1, \mu_2) \in \mathbb{R}^2$ with respect to which asymptotic properties were established. Here we focus on the parameters $(\rho_1, \rho_2) \in L^1(\mathbb{R})^2$ with (μ_1, μ_2) fixed. As in Section 2, we set $W := W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$.

Theorem 3 *Assume that condition (H) holds. Then the multivalued map $\mathcal{S} : L^1(\mathbb{R})^2 \to 2^W$ assigning to every $(\rho_1, \rho_2) \in L^1(\mathbb{R})^2$ the solution set $\mathcal{S}(\rho_1, \rho_2)$ of problem $(P_{\mu_1,\mu_2,\rho_1,\rho_2})$ is upper semicontinuous.*

Proof Fix (μ_1, μ_2) with $\mu_1, \mu_2 \geq 0$. By Theorem 1 we know that $\mathcal{S}(\rho_1, \rho_2)$ is nonempty for every $(\rho_1, \rho_2) \in L^1(\Omega)^2$.

If the conclusion of the theorem were false there would exist $(\rho_{1,0}, \rho_{2,0}) \in L^1(\mathbb{R})^2$ such that the multivalued map $\mathcal{S} : L^1(\mathbb{R})^2 \to 2^W$ is not upper semicontinuous at the point $(\rho_{1,0}, \rho_{2,0})$. Thus a neighborhood V_0 of the solution set $\mathcal{S}(\rho_{1,0}, \rho_{2,0})$ in the space W can be found together with a strongly convergent sequence $(\rho_{1,n}, \rho_{2,n}) \to (\rho_{1,0}, \rho_{2,0})$ in $L^1(\mathbb{R})^2$ and a corresponding sequence $(u_{1,n}, u_{2,n}) \in W$ with $(u_{1,n}, u_{2,n}) \in \mathcal{S}(\rho_{1,n}, \rho_{2,n})$ and $(u_{1,n}, u_{2,n}) \notin V_0$ for every n.

The boundedness of the sequence $(\rho_{1,n}, \rho_{2,n})$ in $L^1(\mathbb{R})^2$ and an estimate likewise in (14) enable us to find constants $c > 0$ and $\delta \in (0,1)$ independent of $n \in \mathbb{N}$ such that

$$\left|\int_\Omega f_i(x, \rho_{1,n} * u_{1,n}, \rho_{2,n} * u_{2,n}, \nabla(\rho_{1,n} * u_{1,n}), \nabla(\rho_{2,n} * u_{1,n}))u_{i,n}dx\right| \tag{24}$$
$$\leq \delta \left(|||\nabla u_{1,n}|||^{p_1}_{L^{p_1}(\Omega)} + |||\nabla u_{2,n}|||^{p_2}_{L^{p_2}(\Omega)}\right) + c$$

for all $n \in \mathbb{N}$, $i = 1,2$. For the sake of clarity we prove this step in detail.

Assumption (H), Hölder's inequality, (5), (6) and (7) ensure the estimate

$$\begin{aligned}
&\left|\int_\Omega f_i(x, \rho_{1,n} * u_{1,n}, \rho_{2,n} * u_{2,n}, \nabla(\rho_{1,n} * u_{1,n}), \nabla(\rho_{2,n} * u_{1,n}))u_{i,n}dx\right| \\
&\leq \|\sigma_i\|_{L^{\gamma_i'}(\Omega)} \|u_{i,n}\|_{L^{\gamma_i}(\Omega)} \\
&+ a_i \left(\|\rho_{i,n}\|^{\alpha_i}_{L^1(\mathbb{R}^N)} \|u_{i,n}\|^{\alpha_i}_{L^{p_i}(\Omega)} + \|\rho_{j,n}\|^{\frac{\alpha_i p_j}{p_i}}_{L^1(\mathbb{R}^N)} \|u_{j,n}\|^{\frac{\alpha_i p_j}{p_i}}_{L^{p_j}(\Omega)}\right) \|u_{i,n}\|_{L^{\frac{p_i}{p_i-\alpha_i}}(\Omega)} \\
&+ b_i \left(N^{p_i} \|\rho_{i,n}\|^{\beta_i}_{L^1(\mathbb{R}^N)} |||\nabla u_{i,n}|||^{\beta_i}_{L^{p_i}(\Omega)} + N^{\frac{\beta_i p_j}{p_i}} \|\rho_{j,n}\|^{\frac{\beta_i p_j}{p_i}}_{L^1(\mathbb{R}^N)} |||\nabla u_{j,n}|||^{\frac{\beta_i p_j}{p_i}}_{L^{p_j}(\Omega)}\right) \\
&\qquad \times \|u_{i,n}\|_{L^{\frac{p_i}{p_i-\beta_i}}(\Omega)}
\end{aligned}$$

for all $n \in \mathbb{N}$, $i, j = 1,2$, $i \neq j$. Then through Sobolev embedding theorem and taking advantage that the sequence $(\rho_{i,n})$ is bounded in $L^1(\mathbb{R})^2$ for $i = 1,2$, there exist positive constants $\overline{a}_i$ and $\overline{b}_i$ such that

$$\begin{aligned}
&\left|\int_\Omega f_i(x, \rho_{1,n} * u_{1,n}, \rho_{2,n} * u_{2,n}, \nabla(\rho_{1,n} * u_{1,n}), \nabla(\rho_{2,n} * u_{1,n}))u_{i,n}dx\right| \\
&\leq \overline{a}_i \left(|||\nabla u_{i,n}|||_{L^{p_i}(\Omega)} + |||\nabla u_{i,n}|||^{\alpha_i+1}_{L^{p_i}(\Omega)} + |||\nabla u_{j,n}|||^{\frac{\alpha_i p_j}{p_i}}_{L^{p_j}(\Omega)} |||\nabla u_{i,n}|||_{L^{p_i}(\Omega)}\right) \\
&+ \overline{b}_i \left(|||\nabla u_{i,n}|||^{\beta_i+1}_{L^{p_i}(\Omega)} + |||\nabla u_{j,n}|||^{\frac{\beta_i p_j}{p_i}}_{L^{p_j}(\Omega)} |||\nabla u_{i,n}|||_{L^{p_i}(\Omega)}\right)
\end{aligned}$$

for all $n \in \mathbb{N}$, $i, j = 1, 2$, $i \neq j$. Now, similarly to (14), Young's inequality with $\varepsilon > 0$ yields

$$\left|\int_\Omega f_i(x, \rho_{1,n} * u_{1,n}, \rho_{2,n} * u_{2,n}, \nabla(\rho_{1,n} * u_{1,n}), \nabla(\rho_{2,n} * u_{1,n}))u_{i,n}dx\right| \tag{25}$$
$$\leq \overline{a}_i\Bigg(\||\nabla u_{i,n}\|_{L^{p_i}(\Omega)} + \||\nabla u_{i,n}|\|^{\alpha_i+1}_{L^{p_i}(\Omega)} + \varepsilon\||\nabla u_{i,n}|\|^{p_i}_{L^{p_i}(\Omega)}$$
$$+c(\varepsilon)\||\nabla u_{j,n}|\|^{\frac{\alpha_i p_j}{p_i-1}}_{L^{p_j}(\Omega)}\Bigg)$$
$$+\overline{b}_i\left(\||\nabla u_{i,n}|\|^{\beta_i+1}_{L^{p_i}(\Omega)} + \varepsilon\||\nabla u_{i,n}|\|^{p_i}_{L^{p_i}(\Omega)} + c(\varepsilon)\||\nabla u_{j,n}|\|^{\frac{\beta_i p_j}{p_i-1}}_{L^{p_j}(\Omega)}\right)$$

for all $n \in \mathbb{N}$, $i, j = 1, 2$, $i \neq j$, with a positive constant $c(\varepsilon)$ depending on ε. Fix an $\varepsilon \in (0, 1)$ in (25). Notice that we can apply (16) with

$$r \in \{1, \alpha_i + 1, \beta_i + 1\} \text{ and } p = p_i$$

as well as

$$r \in \left\{\frac{\alpha_i p_j}{p_i - 1}, \frac{\beta_i p_j}{p_i - 1}\right\} \text{ and } p = p_j$$

for $i, j = 1, 2$, $i \neq j$ (see assumption (H)). Apply it to the terms in (25) whose exponents match the indicated values of r. Since the constant $\tau > 0$ in (16) can be taken arbitrarily small, we infer from (25) that (24) holds true with a $\gamma \in (0, 1)$ and some constant $c > 0$.

The fact that $(u_{1,n}, u_{2,n}) \in \mathcal{S}(\rho_{1,n}, \rho_{2,n})$ reads as

$$\begin{cases} -\Delta_{p_1} u_{1,n} - \mu_1 \Delta_{q_1} u_{1,n} \\ \qquad = f_1(x, \rho_{1,n} * u_{1,n}, \rho_{2,n} * u_{2,n}, \nabla(\rho_{1,n} * u_{1,n}), \nabla(\rho_{2,n} * u_{2,n})) & \text{in } \Omega \\ -\Delta_{p_2} u_{2,n} - \mu_2 \Delta_{q_2} u_{2,n} \\ \qquad = f_2(x, \rho_{1,n} * u_{1,n}, \rho_{2,n} * u_{2,n}, \nabla(\rho_{1,n} * u_{1,n}), \nabla(\rho_{1,n} * u_{2,n})) & \text{in } \Omega \\ u_{1,n} = u_{2,n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Take $(u_{1,n}, u_{2,n})$ as test functions for the above equations. Then (24) shows that the sequence $(u_{1,n}, u_{2,n})$ is bounded in the space W. Thanks to the reflexivity of W, along a relabeled subsequence it holds $(u_{1,n}, u_{2,n}) \rightharpoonup (u_1, u_2)$ in W for some $(u_1, u_2) \in W$. Relying on Rellich-Kondrachov compact embedding theorem and on the boundedness of the sequence $(\rho_{1,n}, \rho_{2,n})$ in $L^1(\mathbb{R})^2$, by following the pattern of arguments in (9) we can readily derive

$$\lim_{n\to+\infty}\int_\Omega f_i(x, \rho_{1,n}*u_{1,n}, \rho_{2,n}*u_{2,n}, \nabla(\rho_{1,n}*u_{1,n}), \nabla(\rho_{2,n}*u_{2,n}))(u_{i,n}-u_i)dx = 0, \tag{26}$$

$i = 1, 2$. Act on the system $(P_{\mu_1,\mu_2,\rho_{1,n},\rho_{2,n}})$ with the test functions $(u_{1,n} - u_1, u_{2,n} - u_2)$. Taking into account (26), it results in the limit that

$$\lim_{n\to+\infty}[\langle -\Delta_{p_i} u_{i,n} - \mu_i \Delta_{q_i} u_{i,n}, u_{i,n} - u_i\rangle] = 0, \ i = 1, 2. \tag{27}$$

At this point the (S_+)-property of the operators $-\Delta_{p_i} - \mu\Delta_{q_i}$ (see [3,12]) ensures the strong convergence $(u_{1,n}, u_{2,n}) \to (u_1, u_2)$ in W. Therefore, for n sufficiently large, we must have $(u_{1,n}, u_{2,n}) \in V_0$. The obtained contradiction proves the result.

Remark 1 Under assumption (H), if $(\rho_{1,n}, \rho_{2,n}) \to (\rho_1, \rho_2)$ in $L^1(\mathbb{R})^2$ and $(u_{1,n}, u_{2,n}) \in W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$ is a weak solution of system $(P_{\mu_1,\mu_2,\rho_{1,n},\rho_{2,n}})$, then along a relabeled subsequence we have $(u_{1,n}, u_{2,n}) \to (u_1, u_2)$ in $W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$ for some weak solution $(u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ of system $(P_{\mu_1,\mu_2,\rho_1,\rho_2})$. This property is a consequence of Theorem 3.

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