Generalized Trigonometric Functions and Matrix Parameterization

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Abstract

The *Generalized Trigonometric Functions (GTF)* have been introduced using an appropriate redefinition of *Euler* type identities involving non-standard forms of imaginary numbers, realized by different types of matrices. In this paper we use the *GTF* to get parameterization of practical interest for non-singular matrices. The possibility of using this procedure to deal with applications in electron transport is also touched on.

Keywords: Trigonometric Functions, Matrix Algebra, Majorana Matrices, Cayley-Hamilton Theorem, Charged Beam Transport, Courant Snyder Theory

1. Introduction

According to refs. [1]-[5], the Generalized Trigonometric Functions *(GTF)* of order 2 $C(t)$, $S(t)$ are defined by means of the identity

$$e^{t\hat{M}} = C(t)\,\hat{1} + S(t)\,\hat{M} \tag{1}$$

where $\hat{M}$, $\hat{1}$ are respectively a 2×2 non-singular matrix and the unit, namely

$$\hat{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ \hat{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

From eq. (1) it also follows that

$$\begin{aligned} e^{t\,\lambda_+} &= C(t)\ + S(t)\,\lambda_+, \\ e^{t\,\lambda_-} &= C(t)\ + S(t)\,\lambda_- \end{aligned} \tag{3}$$

with $\lambda_\pm$ being the eigenvalues of $\hat{M}$, assumed to be non-singular, thus getting the explicit form of the *GTF*, namely

$$\begin{aligned} C(t) &= \frac{\lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t}}{\lambda_- - \lambda_+}, \\ S(t) &= \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \end{aligned} \tag{4}$$

The structure of eq. (1) is that of the *Euler-De Moivre* identity, with $\hat{M}$ playing the role of imaginary unit, on the other side eq. (3) represents the scalar counterpart of (1) and, accordingly, $\lambda_\pm$ are understood as conjugated imaginary units.

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The properties of the cos and sin like functions $C(t)$, $S(t)$ can be inferred from either eqs. (1)-(3), which yield for example (see also ref. [5])

$$\begin{aligned} & C^2 + \Delta_{\hat{M}} S^2 + Tr(\hat{M})\, CS = e^{Tr(\hat{M})\, t}, \\ & Tr(\hat{M}) = a + d, \\ & \Delta_{\hat{M}} = a\, d - b\, c \end{aligned} \tag{5}$$

and

$$\begin{aligned} & C(2\, t) = C^2 - \Delta_{\hat{M}} S^2, \\ & S(2\, t) = 2\, C(t)\, S(t) + Tr(\hat{M})\, S^2 \end{aligned} \tag{6}$$

The previous relationships are recognized as the fundamental trigonometric identity (eq. (5)) and as the duplication formulae (eq. (6)).

By keeping the derivative of both sides of eq. (1) with respect to the variable t, we find

$$\frac{d}{dt} e^{t\, \hat{M}} = \left(\frac{d}{dt} C(t)\right) \hat{1} + \left(\frac{d}{dt} S(t)\right) \hat{M} \tag{7}$$

being also

$$\frac{d}{dt} e^{t\, \hat{M}} = \hat{M} e^{t\, \hat{M}} = C(t) \hat{M} + S(t)\, \hat{M}^2 \tag{8}$$

and since

$$\hat{M}^2 = -\Delta_{\hat{M}} \hat{1} + Tr(\hat{M})\, \hat{M} \tag{9}$$

we end up, after combining eqs. (7)-(9) and equating "real" and "imaginary" parts, the following identities, specifying the properties under derivatives of the *GTF*

$$\begin{aligned} & \frac{d}{dt} C(t) = -\Delta_{\hat{M}} S(t), \\ & \frac{d}{dt} S(t) = Tr(\hat{M})\, S(t) + C(t) \end{aligned} \tag{10}$$

We can infer directly from eq. (4) that the second order *GTF's* exhibit, under variable reflection, the identities

$$\begin{aligned} & C(-t) = e^{-Tr(\hat{M})\, t} \left(-Tr(\hat{M})\, S(t) + C(t)\right) = e^{-Tr(\hat{M})\, t} \left(\frac{d}{dt} S(t)\right), \\ & S(-t) = -e^{-Tr(\hat{M})\, t} S(t) \end{aligned} \tag{11}$$

which underscore the significant difference with the ordinary *TF* (be they circular or hyperbolic) with definite even or odd parities.

Further properties can be argued by the use of other means; by keeping e.g. the freedom of treating $\hat{M}$ as an ordinary algebraic quantity we can formally derive integrals involving *GTF* thus finding e.g.

$$
\begin{aligned}
&\int^t dt' e^{t'\hat{M}} = {}_I C(t)\,\hat{1} + {}_I S(t)\,\hat{M},\\
&\int^t dt' e^{t'\hat{M}} = \frac{1}{\hat{M}} e^{t\hat{M}} = C(t)\,\hat{M}^{-1} + S(t)\,\hat{1},\\
&{}_I C(t) = \int^t dt' C(t'),\ {}_I S(t) = \int^t dt' S(t')
\end{aligned}
\tag{12}
$$

Moreover, since the following identity holds

$$
\begin{aligned}
\hat{M}^{-1} &= c_{-1}\,\hat{1} + s_{-1}\hat{M}\\
c_{-1} &= \frac{\lambda_-\lambda_+^{-1} - \lambda_+\lambda_-^{-1}}{\lambda_- - \lambda_+} = \frac{Tr(\hat{M})}{\Delta_{\hat{M}}},\\
s_{-1} &= \frac{\lambda_+^{-1} - \lambda_-^{-1}}{\lambda_+ - \lambda_-} = -\frac{1}{\Delta_{\hat{M}}}
\end{aligned}
\tag{13}
$$

we obtain the “primitives” of the *GTF’s*

$$
\begin{aligned}
{}_I C(t) &= \frac{Tr(\hat{M})}{\Delta_{\hat{M}}} C(t) + S(t),\\
{}_I S(t) &= -\frac{1}{\Delta_{\hat{M}}} C(t)
\end{aligned}
\tag{14}
$$

A straightforward consequence of the previous relationships is

$$
\begin{aligned}
\int_0^\infty dt' C(-t') &= \frac{Tr(\hat{M})}{\Delta_{\hat{M}}},\\
\int_0^\infty dt' S(-t') &= -\frac{1}{\Delta_{\hat{M}}}\hat{M}
\end{aligned}
\tag{15}
$$

which hold true only if the integrals are convergent, namely if $Re(\lambda_\pm)$ are both positive.

A further slightly more intriguing example is provided by the *Gaussian* integral

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} dt e^{-t^2\hat{M}} = \sqrt{\frac{\pi}{\hat{M}}} = \sqrt{\pi}\left(c_{-\frac{1}{2}}\,\hat{1} + s_{-\frac{1}{2}}\hat{M}\right),\\
&c_{-1/2} = \frac{\lambda_-\lambda_+^{-1/2} - \lambda_+\lambda_-^{-1/2}}{\lambda_- - \lambda_+},\\
&s_{-1/2} = \frac{\lambda_+^{-1/2} - \lambda_-^{-1/2}}{\lambda_+ - \lambda_-}
\end{aligned}
\tag{16}
$$

which yields the following generalizations of the *Fresnel* integrals, obtained by other means in ref. [5],

$$
\begin{aligned}
\int_{-\infty}^{+\infty} dt C(-t^2) &= \sqrt{\pi} c_{-\frac{1}{2}}\\
\int_{-\infty}^{+\infty} dt S(-t^2) &= \sqrt{\pi} s_{-\frac{1}{2}}
\end{aligned}
\tag{17}
$$

The convergence of these integrals depends on the eigenvalues $\lambda_{\pm}$, if convergence is ensured, eq. (17) provides the most general form of solution.

Iterating the procedure, leading to eqs. (10), namely by keeping successive derivatives with respect to t of both sides of (1) and by noting that

$$\hat{M}^n = c_n \hat{1} + s_n \hat{M} \tag{18}$$

we end up with

$$\begin{aligned}
\left(\frac{d}{dt}\right)^n C(t) &= c_n C(t) + c_{n+1} S(t) \\
\left(\frac{d}{dt}\right)^n S(t) &= s_n C(t) + s_{n+1} S(t)
\end{aligned} \tag{19}$$

It is evident that the coefficients $c_\nu,\ s_\nu$ are essentially *GTF* in which $e^{\lambda_{\pm} t}$ are replaced by $\lambda_{\pm}^{\nu}$. The relevant properties are discussed later in the paper.

The addition formulae too can be derived in terms of the $c_n,\ s_n$ coefficients as

$$\begin{aligned}
& C(t+t') = C(t)\,C(t') + c_2 S(t)\,S(t'), \\
& S(t+t') = (C(t)\ + s_2 S(t))\ S(t') + S(t)\,C(t'), \\
& s_2 = Tr(\hat{M}),\ c_2 = -\Delta_{\hat{M}}
\end{aligned} \tag{20}$$

In absence of the simple reflection properties of the ordinary circular functions, we can establish the subtraction formulae according to the expressions given below

$$\begin{aligned}
& C(t-t') = e^{-s_2 t'} \left[s_2 S(t') C(t) + C(t') C(t) - c_2 S(t)\,S(t') \right], \\
& S(t-t') = -e^{-s_2 t'} \left[C(t) S(t') - C(t') S(t) \right]
\end{aligned} \tag{21}$$

which, once combined with eq, (20), yields the following prosthaphaeresis like identities

$$\begin{aligned}
C(p) - e^{s_2 \frac{(p-q)}{2}} C(q) = &- s_2 S\left(\frac{p-q}{2}\right) C\left(\frac{p+q}{2}\right) + \\
&+ 2 c_2 S\left(\frac{p-q}{2}\right) S\left(\frac{p+q}{2}\right)
\end{aligned} \tag{22}$$

In the forthcoming section we will provide some examples aimed at providing the usefulness of this family of functions in applications.

2. GTF, Matrix parameterization and generalized complex forms

To proceed further, we remind that eq. (1) follows from the *Cayley-Hamilton* Theorem [6], which allows to write a given function (usually an exponential) of a matrix $\hat{\Sigma}$ in terms of its characteristic polynomial. We will now use the *GTF* to provide the reverse procedure, namely we write a given matrix $\hat{\Sigma}$ in exponential form, namely

$$\begin{aligned}
& \hat{\Sigma} = e^{\hat{T}}, \\
& \hat{\Sigma} = \begin{pmatrix} l & m \\ n & p \end{pmatrix},\ \hat{T} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}
\end{aligned} \tag{23}$$

The problem we are interested in is therefore that of finding the elements of the exponentiated matrix $\hat{T}$, once those of $\hat{\Sigma}$ are known. The use of eq. (1) yields

$$e^{\hat{T}} = C(1)\,\hat{1} + S(1)\,\hat{T} \tag{24}$$

where the *GTF* are expressed in terms of the eigenvalues of $\hat{T}$. It is therefore worth to remind that both $\hat{\Sigma}$, $\hat{T}$ are diagonalised through the same matrix $\hat{D}$ and therefore

$$\begin{aligned} &\hat{D}^{-1}\hat{\Sigma}\,\hat{D} = e^{\hat{D}^{-1}\hat{T}\,\hat{D}}, \\ &\hat{D}^{-1}\hat{\Sigma}\,\hat{D} = \begin{pmatrix} \sigma_+ & 0 \\ 0 & \sigma_- \end{pmatrix} = \begin{pmatrix} e^{\tau_+} & 0 \\ 0 & e^{\tau_-} \end{pmatrix} \end{aligned} \tag{25}$$

where $\sigma_\pm$, $\tau_\pm$ denote the eigenvalues of the $\hat{\Sigma}$ and $\hat{T}$ matrices respectively it is furthermore evident that

$$\tau_\pm = \ln(\sigma_\pm) \tag{26}$$

We can therefore write

$$\begin{aligned} &\hat{\Sigma} = C(1)\,\hat{1} + S(1)\,\hat{T}, \\ &C(1) = \frac{\ln(\sigma_-)\,\sigma_+ - \ln(\sigma_+)\,\sigma_-}{\ln(\sigma_-) - \ln(\sigma_+)}, \\ &S(1) = \frac{\sigma_+ - \sigma_-}{\ln(\sigma_+) - \ln(\sigma_-)} \end{aligned} \tag{27}$$

and

$$\hat{T} = \begin{pmatrix} \dfrac{l - C(1)}{S(1)} & \dfrac{m}{S(1)} \\ \dfrac{n}{S(1)} & \dfrac{p - C(1)}{S(1)} \end{pmatrix} \tag{28}$$

It is now worth noting that

$$\hat{\Sigma}^n = e^{n\,\hat{T}} = C(n)\,\hat{1} + S(n)\,\hat{T} \tag{29}$$

and it should be stressed that the arguments of the *GTF* in the elements of the matrix $\hat{T}$ in eq. (24) remains the unity.

The parameterization we have proposed is a generalized form of what is known in the Physics of charged beam transport as the *Courant-Snyder* parameterization, which is exploited to adapt the beam sizes to the characteristics of the transport device or in laser optics to transport an optical beam through ordinary lens systems [7].

In the following we will extend the method to matrices with larger dimensions, using higher order *GTF*. Before doing this, we take advantage from the present point of view to extend the notion of complex number, which will be defined as

$$\begin{aligned} &\zeta_+ = x + \lambda_+\, y, \\ &\zeta_- = x + \lambda_-\, y \end{aligned} \tag{30}$$

with "modulus"

$$\zeta_+\zeta_- = x^2 + Tr(\hat{M})\,x\,y + \Delta_{\hat{M}}y^2 \tag{31}$$

The relevant trigonometric form can be written as (λ may be either λ_+ or its conjugate form λ_-)

$$\begin{aligned} \zeta &= |A|\, e^{\lambda \vartheta}, \\ |A| &= \sqrt{\zeta_+ \zeta_-} e^{-Tr(\hat{M})\,\frac{\vartheta}{2}}, \\ \vartheta &= \frac{1}{\lambda_+ - \lambda_-} \ln\left[\frac{1 + \frac{y}{x}\lambda_+}{1 + \frac{y}{x}\lambda_-}\right] \end{aligned} \tag{32}$$

The conclusion, we may draw from this last result, is that the concept of imaginary number is more subtle than it might be thought, it is not necessarily associated with the roots of a negative number but can be constructed with any pair of numbers, solutions of a second degree algebraic equation [5].

We have tried to keep our treatment of *GTF* following in a close parallel with the ordinary circular trigonometry, it is therefore important to note that the geometrical image of the condition (5) is no more a circle but a more complicated curve not necessarily closed. Notwithstanding a "cos" and "sin" like interpretation of the *GTF* is still possible (see Figs. 1-2). It is however worth noting that *GTF* may be circular or hyperbolic like, according to whether $Im(\lambda)$ be $\neq 0$, or $= 0$. The argument of the *GTF* cannot be simply regarded as angles, notwithstanding, it is natural to ask whether there is any quantity playing the role of π, even though if e.g. $Tr(\hat{M}) \neq 0$ we are not dealing with periodic functions.

To clarify this point we try to keep advantage from the *Euler*-formula "$e^{i\frac{\pi}{2}} = i$" to define two distinct quantities $\pi_\pm$ such that

$$\begin{aligned} e^{\lambda_- \frac{\pi_-}{2}} &= \lambda_- \\ e^{\lambda_+ \frac{\pi_+}{2}} &= \lambda_+ \end{aligned} \tag{33}$$

yielding

$$\pi_\pm = \frac{2\ \ln(\lambda_\pm)}{\lambda_\pm} \tag{34}$$

It is furthermore worth noting the "funny" identities

$$\begin{aligned} e^{\lambda_\pm \pi_\pm} &= Tr(\hat{M})\lambda_\pm - \Delta_{\hat{M}} \\ \lambda_-^{\lambda_+} &= e^{\frac{\Delta_{\hat{M}}}{2}\pi_-}, \\ \lambda_+^{\lambda_-} &= e^{\frac{\Delta_{\hat{M}}}{2}\pi_+}, \\ e^{\lambda_-^2 \frac{\pi_-}{2}} &= e^{(Tr(\hat{M})\lambda_- \frac{\pi_-}{2}} e^{-\frac{\Delta_{\hat{M}}}{2}\pi_-} = \lambda_-^{\lambda_-} \end{aligned} \tag{35}$$

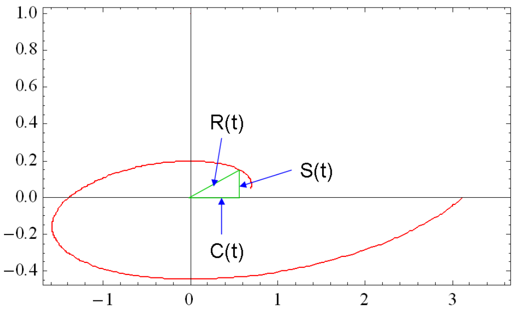
The last of which can also be reinterpreted as

$$\lambda_-^{\lambda_-} = \lambda_-^{Tr(\hat{M}) - \lambda_+} \tag{36}$$

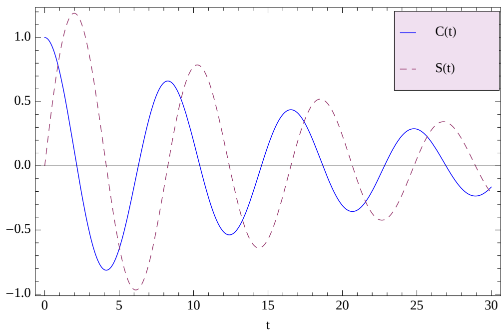
It is also evident that if $Im(\lambda_\pm) \neq 0$ the *GTF* functions exhibit infinite zeros on the real axis, which for C and S are, respectively, given by

$$\begin{aligned} c t_n^* &= \frac{1}{2(\lambda_- - \lambda_+)} \left[\lambda_- \pi_- - \lambda_+ \pi_+ - 4\, i\, n\, \pi\right], \\ s t_n^* &= \frac{2\, i n\, \pi}{(\lambda_- - \lambda_+)} \end{aligned} \tag{37}$$

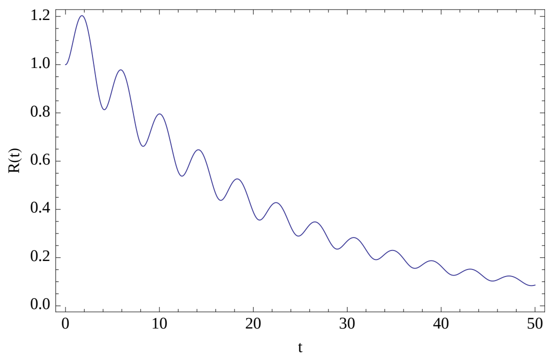
To appreciate the analogies and the differences as well, we have reported in Figs. 3 the function $C(t)$ and its counterpart $C(-t)$.



(a) Geometrical Images.

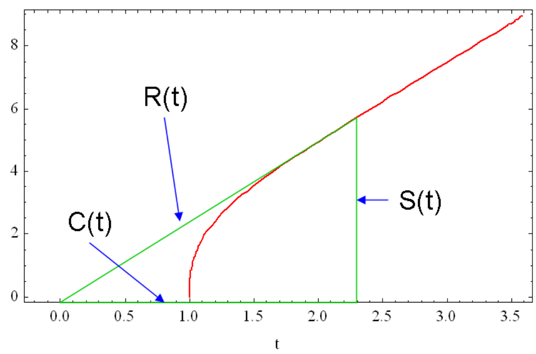


(b) Behavior vs. Argument.



(c) $R(t) = \sqrt{C(t)^2 + S(t)^2}$

Figure 1: Generalized Trigonometric Functions for $Im(\lambda) \neq 0$.



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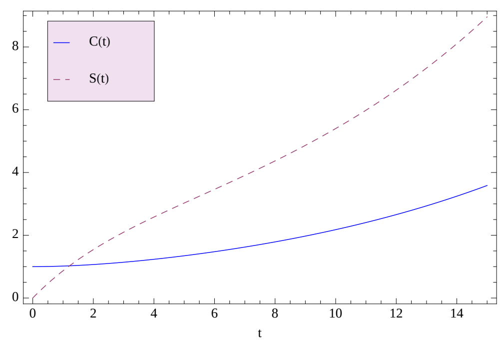
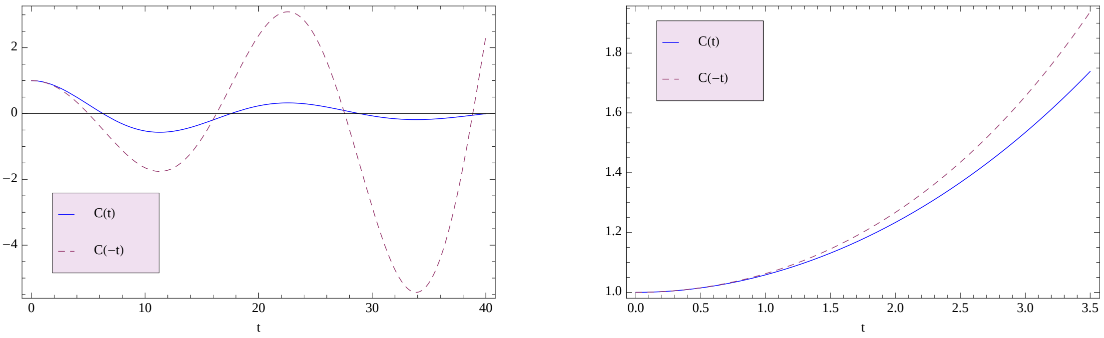


(b) Behavior vs. Argument.

Figure 2: Generalized Trigonometric Functions for $Im(\lambda) = 0$.



(a) $Im(\lambda) \neq 0$. **(b)** $Im(\lambda) = 0$.

Figure 3: Behavior of $C(t)$ and $C(-t)$ vs Argument.

3. Third and Higher Order GTF

According to terminology of ref. [5] the order of the *GTF* is associated with that of the corresponding generating matrix. If $\hat{M}$ is a 3×3 non-singular matrix with three distinct eigenvalues we have

$$e^{t\,\hat{M}} = C_0(t)\,\hat{1} + C_1(t)\,\hat{M} + C_2(t)\,\hat{M}^2 \tag{38}$$

We can introduce the third order *GTF*, $C_{0,1,2}(t)$

$$\begin{gathered}\begin{pmatrix} C_0(t) \\ C_1(t) \\ C_2(t) \end{pmatrix} = \left[\hat{V}(\lambda_1,\lambda_2,\lambda_3)\right]^{-1} \begin{pmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ e^{\lambda_3 t} \end{pmatrix} \\ \hat{V}(\lambda_1,\lambda_2,\lambda_3) = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix}\end{gathered} \tag{39}$$

where $\hat{V}(\lambda_1,\lambda_2,\lambda_3)$ is the *Vandermonde* matrix, constructed with the eigenvalues of $\hat{M}$. The inverse of $\hat{V}$ can be written as [8]

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{\lambda_2\lambda_3}{(\lambda_1-\lambda_2)\,(\lambda_1-\lambda_3)} & \dfrac{\lambda_1\lambda_3}{(\lambda_2-\lambda_1)\,(\lambda_2-\lambda_3)} & \dfrac{\lambda_1\lambda_2}{(\lambda_3-\lambda_1)\,(\lambda_3-\lambda_2)} \\ -\dfrac{\lambda_2+\lambda_3}{(\lambda_1-\lambda_2)\,(\lambda_1-\lambda_3)} & -\dfrac{\lambda_1+\lambda_3}{(\lambda_2-\lambda_1)\,(\lambda_2-\lambda_3)} & -\dfrac{\lambda_1+\lambda_2}{(\lambda_3-\lambda_1)\,(\lambda_3-\lambda_2)} \\ \dfrac{1}{(\lambda_1-\lambda_2)\,(\lambda_1-\lambda_3)} & \dfrac{1}{(\lambda_2-\lambda_1)\,(\lambda_2-\lambda_3)} & \dfrac{1}{(\lambda_3-\lambda_1)\,(\lambda_3-\lambda_2)} \end{pmatrix} \tag{40}$$

We can therefore write the third order *GTF* as

$$\begin{aligned} C_0(t) &= \frac{1}{\Delta(\lambda_1,\lambda_2,\lambda_3)} \sum_{i,j,k=1}^{3} \frac{\varepsilon_{i,j,k}}{2} \lambda_i\lambda_j(\lambda_j-\lambda_i)\,e^{\lambda_k t}, \\ C_1(t) &= \frac{1}{\Delta(\lambda_1,\lambda_2,\lambda_3)} \sum_{i,j,k=1}^{3} \frac{\varepsilon_{i,j,k}}{2} (\lambda_i^2-\lambda_j^2)\,e^{\lambda_k t}, \\ C_2(t) &= -\frac{1}{\Delta(\lambda_1,\lambda_2,\lambda_3)} \sum_{i,j,k=1}^{3} \frac{\varepsilon_{i,j,k}}{2} (\lambda_i-\lambda_j)\,e^{\lambda_k t} \end{aligned} \tag{41}$$

where

$$\Delta(\lambda_1,\,\lambda_2,\,\lambda_3) = (\lambda_2-\lambda_1)\,(\lambda_3-\lambda_1)\,(\lambda_3-\lambda_2) \tag{42}$$

Is the *Vandermonde* determinant and $\varepsilon_{i,j,k}$ is the *Levi-Civita* tensor.

By following the same procedure as before, we can extend to the third order the properties of the second order case. It is easily argued that they satisfy third order differential equations and that the relevant addition formulae read

$$\begin{aligned}
&C_0(t+t') = C_0(t)\,C_0(t') + {}_0c_3\,[C_1(t)\,C_2(t') + C_1(t')\,C_2(t)] + {}_0c_4 C_2(t)\,C_2(t'),\\
&C_1(t+t') = [C_0(t)\,C_1(t') + C_1(t)\,C_0(t')] + {}_1c_3\,[C_1(t)\,C_2(t') + C_1(t')\,C_2(t)] + {}_1c_4 C_2(t)\,C_2(t'),\\
&C_2(t+t') = [C_0(t)\,C_2(t') + C_1(t)\,C_1(t') + C_2(t)\,C_0(t')] + {}_2c_3\,[C_1(t)\,C_2(t') + C_1(t')\,C_2(t)] + {}_2c_4 C_2(t)\,C_2(t')
\end{aligned} \tag{43}$$

where ${}_\alpha c_n,\ \alpha = 0,\,1,\,2$ are the third order GTF with $e^{\lambda_\alpha t}$ replaced by λ_α^n.

More in general we also find that

$$C_\alpha(n\,t) = \sum_{\substack{n_1,n_2,n_3=0\\ n_1+n_2+n_3=n}}^{n} \binom{n}{n_1\ n_2\ n_3} {}_\alpha c_{n-n_1} C_0^{n_1} C_1^{n_2} C_2^{n_3} \tag{44}$$

with $\binom{n}{n_1\ n_2\ n_3}$ being the multinomial coefficient.

It is also easily understood that the analogous of eqs. (13), (15) for the third order *GTF* read

$$\begin{aligned}
&{}_I C_0(t) = {}_0c_{-1} C_0(t) + C_1(t),\\
&{}_I C_1(t) = C_2(t) + {}_1c_{-1} C_2\\
&{}_I C_2(t) = {}_2c_{-1} C_0(t),\\
&\int_0^\infty dt C_\alpha(-t) = {}_\alpha c_{-1},\\
&\int_{-\infty}^\infty dt C_\alpha(-t^2) = \sqrt{\pi}{}_\alpha c_{-\frac{1}{2}},\\
&\alpha = 0,\,1,\,2
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
{}_0c_\nu &= \frac{1}{\Delta(\lambda_1,\lambda_2,\lambda_3)} \sum_{i,j,k=1}^{3} \frac{\varepsilon_{i,j,k}}{2} \lambda_i\lambda_j(\lambda_j-\lambda_i)\,\lambda_k^\nu,\\
{}_1c_\nu &= \frac{1}{\Delta(\lambda_1,\lambda_2,\lambda_3)} \sum_{i,j,k=1}^{3} \frac{\varepsilon_{i,j,k}}{2} (\lambda_i^2-\lambda_j^2)\,\lambda_k^\nu,\\
{}_2c_\nu &= -\frac{1}{\Delta(\lambda_1,\lambda_2,\lambda_3)} \sum_{i,j,k=1}^{3} \frac{\varepsilon_{i,j,k}}{2} (\lambda_i-\lambda_j)\,\lambda_k^\nu
\end{aligned} \tag{46}$$

It is now worth stressing that the following identities hold true in the case of third order matrices expressed in terms of *GTF*, namely

$$\hat{M}^n = {}_0c_n\hat{1} + {}_1c_n\hat{M} + {}_2c_n\hat{M}^2 \tag{47}$$

Let us now consider the possibility of extending the *Courant-Snyder* parameterization to third order matrices. To this aim we set

$$\hat{\Sigma} = e^{\hat{T}} \tag{48}$$

The explicit form of the matrix $\hat{T}$ can be obtained by setting

$$\hat{\Sigma} = C_0(1)\,\hat{1} + C_1(1)\,\hat{T} + C_2(1)\,\hat{T}^2 \tag{49}$$

where ${}_{\alpha}C(1)\ C_{\alpha}(1)$ are written in terms of the eigenvalues of the matrix $\hat{T}$ according to the prescription discussed in sec. 2, furthermore, since

$$\hat{\Sigma}^{-1} = C_0(-1)\,\hat{1} + C_1(-1)\,\hat{T} + C_2(-1)\,\hat{T}^2 \tag{50}$$

the matrix $\hat{T}$can be obtained as

$$\hat{T} = \frac{C_2(-1)\,\hat{\Sigma} - C_2(1)\,\hat{\Sigma}^{-1} + [C_2(-1)\,C_0(1) - C_2(1)\,C_0(-1)]\ \hat{1}}{C_2(-1)\,C_1(1) - C_1(-1)\,C_2(1)} \tag{51}$$

The use in applications of these last results in applications will be discussed elsewhere.

It is evident that the results we have obtained so far can be extended to an arbitrary $n \times n$ matrix, it is however instructive to consider more specific examples involving particular cases as e.g. a 5×5 anti-symmetric matrix $\hat{F}$, which can be exponentiated as it follows [9]

$$e^{t\,\hat{F}} = \hat{1} + \frac{1}{\sqrt{\Gamma}}\left[f_1(t)\,\hat{F} + f_2(t)\,\hat{F}^2 + f_3(t)\hat{F}^3 + f_4(t)\,\hat{F}^4\,\right] \tag{52}$$

where

$$\begin{aligned}
&\Gamma = Tr(\hat{F}^4) - \frac{1}{4}\left[Tr(\hat{F}^2)\right]^2,\\
&\theta_{\pm}^2 = -\frac{1}{4}Tr(\hat{F}^2) \pm \frac{1}{2}\sqrt{\Gamma},\\
&f_1(t) = \left(\frac{\sin(\theta_- t)}{\theta_-}\theta_+^2 - \frac{\sin(\theta_+ t)}{\theta_+}\theta_-^2\right)\\
&f_2(t) = \left(\frac{1-\cos(\theta_- t)}{\theta_-^2}\theta_+^2 - \frac{1-\cos(\theta_+ t)}{\theta_+^2}\theta_-^2\right)\\
&f_3(t) = \left(\frac{\sin(\theta_- t)}{\theta_-} - \frac{\sin(\theta_+ t)}{\theta_+}\right),\\
&f_4(t) = \left(\frac{1-\cos(\theta_- t)}{\theta_-^2} - \frac{1-\cos(\theta_+ t)}{\theta_+^2}\right)
\end{aligned} \tag{53}$$

We can provide the identification of the f functions with the $fifth$ order *GTF*

$$\begin{aligned}
&C_0(t) = 1,\\
&C_{\alpha}(t) = \frac{1}{\sqrt{\Gamma}}f_{\alpha}(t),\\
&\alpha = 1, ..., 4
\end{aligned} \tag{54}$$

It is also worth noting that, from the previous identities, the following relationships are easily inferred

$$
\begin{aligned}
\hat{F}^{2n+1} &= \frac{1}{\sqrt{\Gamma}}\left[{}_1f_{2n+1}\,\hat{F} + {}_3f_{2n+1}\hat{F}^3\right],\\
\hat{F}^{2n} &= \hat{1} + \frac{1}{\sqrt{\Gamma}}\left[{}_2f_{2n}\,\hat{F}^2 + {}_4f_{2n}\hat{F}^4\right]
\end{aligned}
\tag{55}
$$

where the coefficients

$$
\begin{aligned}
{}_1f_{2n+1} &= \left(\frac{\theta_-^{2n+1}}{\theta_-}\theta_-^2 - \frac{\theta_+^{2n+1}}{\theta_+}\theta_+^2\right),\\
{}_3f_{2n+1} &= \left(\frac{\theta_-^{2n+1}}{\theta_-} - \frac{\theta_+^{2n+1}}{\theta_+}\right),\\
{}_2f_n &= \left(\frac{\theta_-^{2n}}{\theta_-^2}\theta_+^2 - \frac{\theta_+^{2n}}{\theta_+^2}\theta_-^2\right),\\
{}_4f_n &= \left(\frac{\theta_-^{2n}}{\theta_-^2} - \frac{\theta_+^{2n}}{\theta_+^2}\right)
\end{aligned}
\tag{56}
$$

play a role analogous to that of ${}_\alpha c_n$ introduced in the previous sections.

4. Final Comments

In the previous sections we have introduced the properties of the auxiliary coefficients c_n and s_n, their role is fairly important within the present context and warrants further analysis.

To this aim we note that they satisfy the following recurrences

$$
\begin{aligned}
&\begin{pmatrix} c_{n+1} \\ s_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -\Delta_{\hat{M}} \\ 1 & Tr(\hat{M}) \end{pmatrix}\begin{pmatrix} c_n \\ s_n \end{pmatrix},\\
&\begin{pmatrix} c_0 \\ s_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{aligned}
\tag{57}
$$

which follow from the identities

$$
\begin{aligned}
\hat{M}^{n+1} &= c_{n+1}\,\hat{1} + s_{n+1}\hat{M},\\
\hat{M}^{n+1} &= -\Delta_M\, s_n\hat{1} + \left[c_n + Tr(\hat{M})\, s_n\right]\,\hat{M}
\end{aligned}
\tag{58}
$$

The above recurrences can be cast in the decoupled form

$$
c_{n+2} - Tr(\hat{M})\, c_{n+1} + \Delta_{\hat{M}}\, c_n = 0 \tag{59}
$$

and for s_n we find an analogous expression.

The solution of the difference equation (59) can be obtained by the use of the *Binet* method [10], after setting $c_n = r^n$ we find indeed

$$
c_n = \alpha_1 r_+^n + \alpha_2 r_-^n \tag{60}
$$

with $r_\pm$ being solutions of the auxiliary equation

$$r^2 - Tr(\hat{M})r + \Delta_{\hat{M}} = 0 \tag{61}$$

and $\alpha_{1,2}$ being defined through the "initial constants" $c_{0,1}$.
Accordingly we obtain

$$c_n = \frac{1}{r_- - r_+}\left[c_0(r_- r_+^n - r_+ r_-^n) + c_1\,(r_-^n - r_+^n)\right] \tag{62}$$

It is also interesting to note that, by rescaling $n = m - 2$, eq. (59) writes

$$c_m = -\Delta_{\hat{M}}\, c_{m-2} + Tr(\hat{M})\, c_{m-1} \tag{63}$$

Eq. (63), for $Tr(\hat{M}) = 1$ and $\Delta_{\hat{M}} = -1$ (e.g. the eigenvalues of $\hat{M}$ are the golden ratio and the opposite of the golden ratio conjugate), reduces to the *Fibonacci* sequence. The link between the previous coefficient and the *Fibonacci* trigonometry will be discussed elsewhere.

These coefficients play a more general role when extended to the case of higher order matrices and the systematic study of their properties may simplify the calculations of problems where exponentiation of matrices are involved.

In the past, different generalizations of the trigonometric functions have been proposed, in addition to those quoted in this paper different avenues have been explored along this direction. The tool exploited within such a framework can be comprised into three different branches:

a) Use of matrix methods and generalization of the Euler exponential rule.

b) Extension of the trigonometric fundamental identity, providing a thread with elliptic functions [11].

c) Generalized forms of the series expansion, providing a link with integer order *Mittag-Leffler* function [12, 13, 14].

This last point of view provides a significant step forward in the theory of special functions, yielding a tool for applications in the field of classical and quantum optics [15, 16, 17].

Preliminary attempts to merge the points of view *a*) and *c*) have been put forward in refs. [18, 19].

Even though the matter presented in this paper may sound abstract there are important applications in beam transport optics as illustrated below.

The use of 4×4 matrices is currently employed to deal with transverse coupling in charged beam transport [20]. *Baumgarten* [21] has proposed the use of real *Dirac* matrices [22] to construct a generalization of the one dimensional *Courant-Snyder* theory of beam transport.

Within such a context the beam transport through a solenoid can be written as

$$\frac{d}{ds}\begin{pmatrix} x \\ \frac{x'}{K} \\ y \\ \frac{y'}{K} \end{pmatrix} = K \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ \frac{x'}{K} \\ y \\ \frac{y'}{K} \end{pmatrix} \tag{64}$$

where K is the solenoid strength and the column vector is represented by the position an velocity for the transverse coordinates (x, y), finally $s$is the propagation coordinate, playing the role of time.

The solution of the previous system of differential equation can be written as

$$\begin{aligned}
&\underline{Z} = \hat{U}(s)\underline{Z}_0, \\
&\hat{U}(s) = e^{Ks\hat{T}} \\
&\hat{T} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}
\end{aligned} \tag{65}$$

The use of the techniques outlined in the previous section yields for the evolution operator

$$\hat{U}(s) = \hat{1} + \frac{\sin(2Ks)}{2}\hat{T} + \frac{1-\cos(2Ks)}{4}\hat{T}^2 - \frac{\sin(2Ks)}{16}\hat{T}^3 \tag{66}$$

however the above expression simplifies since $\hat{T}^3 = -4\,\hat{T}$.

In this case the *GTF* are simple combinations of the ordinary circular functions.

The method proposed is however fairly important because the (sixteen) real *Majorana* matrices provide a basis for the 4×4 matrices and it could be interesting to develop a systematic study within the context of *GTF* viewed as the associated auxiliary functions. The relevant applications might be interesting for a plethora of problems including e.g. four level systems interacting with external radiation.

We conclude this paper by noting that, in terms of the *Majorana* matrices, the solenoid transport matrix reads

$$\begin{aligned}
&\hat{T} = \gamma_0 - \gamma_9, \\
&\gamma_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \gamma_9 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{67}$$

Since γ_0, γ_9 are commuting quantities, we can also write

$$\begin{aligned}
&e^{\hat{T}\,\xi} = e^{\hat{\gamma}_0\xi}e^{-\hat{\gamma}_9\xi}, \\
&e^{\hat{\gamma}_{0,9}\xi} = \cos(\xi)\,\hat{1} + \sin(\xi)\,\hat{\gamma}_{0,9}
\end{aligned} \tag{68}$$

and easily recover the result reported in eq. (66).

A more systematic analysis will be presented elsewhere.

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