This manuscript is the AAM version of the DOI 10.1007/978-3-030-51945-2_29, published in Bastos, M.A., Castro, L., Karlovich, A.Y. (eds) Operator Theory, Functional Analysis and Applications. Operator Theory: Advances and Applications, vol 282 (2021).

Riesz-Fischer maps, semi-frames and frames in rigged Hilbert spaces

Francesco Tschinke

Abstract. In this note we present a review, some considerations and new results about maps with values in a distribution space and domain in a σ-finite measure space X. Namely, this is a survey about Bessel maps, frames and bases (in particular Riesz and Gel'fand bases) in a distribution space. In this setting, the Riesz-Fischer maps and semi-frames are defined and new results about them are obtained. Some examples in tempered distributions space are examined.

Mathematics Subject Classification (2010). Primary 42C15; Secondary 47A70, 46F05.

Keywords. Frames, Bases, Distributions, Rigged Hilbert space.

1. Introduction

Given a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot|\cdot\rangle$ and norm $\|\cdot\|$, a frame is a sequence of vectors $\{f_n\}$ in $\mathcal{H}$ if there exist $A, B > 0$ such that:

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f|f_n\rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

As known, this notion generalizes orthonormal bases, and has reached an increasing level of popularity in many fields of interests, such as signal theory, image processing, etc., but it is also an important tool in pure mathematics: in fact it plays key roles in wavelet theory, time-frequency analysis, the theory of shift-invariant spaces, sampling theory and many other areas (see [10, 11, 18, 20, 24]).

A generalization of frame, the *continuous* frame, was proposed by Kaiser [24] and by Alí, Antoine, Gazeau [1, 2]: if (X,μ) is a measure space with μ as σ-finite positive measure, a function $F : x \in X \mapsto F_x \in \mathcal{H}$ is a continuous frame with respect to (X,μ) if:

i) F is weakly measurable, i.e. the map $x \in X \mapsto \langle f|F_x\rangle \in \mathbb{C}$ is μ-measurable for all $f \in \mathcal{H}$;

ii) there exist $A, B > 0$ such that, for all $f \in \mathcal{H}$:

$$A\|f\|^2 \leq \int_X |\langle f|F_x\rangle|^2 d\mu \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

Today, the notion of continuous frames in Hilbert spaces and their link with the theory of coherent states is well-known in the literature.

With the collaboration of C. Trapani and T. Triolo [29], the author introduced bases and frames in distributional spaces. To illustrate the motivations for this study, we have to consider the *rigged Hilbert space* (or Gel'fand triple) [16, 17]: that is, if $\mathcal{H}$ is a Hilbert space, the triple:

$$\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times],$$

where $\mathcal{D}[t]$ is a dense subspace of $\mathcal{H}$ endowed with a locally convex topology t stronger than the Hilbert norm and $\mathcal{D}^\times[t^\times]$ is the conjugate dual space of $\mathcal{D}$ endowed with the strong dual topology $t^\times$. If $\mathcal{D}$ is reflexive, the inclusions are dense and continuous.

In this setting, let us consider the *generalized eigenvectors* of an operator, i.e. eigenvectors that do not belong to $\mathcal{H}$. More precisely: if A is an essentially self-adjoint operator in $\mathcal{D}$ which maps $\mathcal{D}[t]$ into $\mathcal{D}[t]$ continuously, then A has a continuous extension $\hat{A}$ given by its adjoint, (i.e. $\hat{A} = A^\dagger$) from $\mathcal{D}^\times$ into itself. A *generalized eigenvector* of A, with eigenvalue $\lambda \in \mathbb{C}$, is an eigenvector of $\hat{A}$; that is, a conjugate linear functional $\omega_\lambda \in \mathcal{D}^\times$ such that:

$$\langle Af|\omega_\lambda\rangle = \lambda \langle f|\omega_\lambda\rangle, \quad \forall f \in \mathcal{D}.$$

The above equality can be read as $\hat{A}\omega_\lambda = A^\dagger\omega_\lambda = \lambda\omega_\lambda$.

A simple and explicative example is given by the derivative operator: $A := i\frac{d}{dx} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ where $\mathcal{S}(\mathbb{R})$ is the Schwartz space (i.e. infinitely differentiable rapidly decreasing functions). The rigged Hilbert space is:

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}^\times(\mathbb{R}), \tag{1.1}$$

where the set $\mathcal{S}^\times(\mathbb{R})$ is known as space of *tempered distributions*. Then $\omega_\lambda(x) = \frac{1}{\sqrt{2\pi}}e^{-i\lambda x}$ -that does not belong to $L^2(\mathbb{R})$- is a generalized eigenvector of A with λ as eigenvalue.

Each function ω_λ can be viewed as a regular distribution of $\mathcal{S}^\times(\mathbb{R})$ through the following integral representation:

$$\langle\phi|\omega_\lambda\rangle = \int_{\mathbb{R}} \phi(x)\overline{\omega_\lambda(x)}dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \phi(x)e^{i\lambda x}dx := \check{\phi}(\lambda)$$

and the linear functional $\phi \mapsto \check{\phi}(\lambda)$ defined on $\mathcal{S}(\mathbb{R})$ is continuous. Furthermore by the Fourier-Plancherel theorem, one has: $\|\check{\phi}\|_2^2 = \int_{\mathbb{R}} |\langle\phi|\omega_\lambda\rangle|^2 dx = \|\phi\|_2^2$.

With a limiting procedure, the Fourier transform can be extended to $L^2(\mathbb{R})$. Since a function $f \in L^2(\mathbb{R})$ defines a regular tempered distribution, we have, for all $\phi \in \mathcal{S}(\mathbb{R})$:

$$\begin{aligned}\langle \phi | f \rangle := \int_{\mathbb{R}} f(x)\phi(x)dx &= \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\lambda) e^{i\lambda x} d\lambda \right) \phi(x) dx = \\ &= \int_{\mathbb{R}} \hat{f}(\lambda) \check{\phi}(\lambda) d\lambda = \int_{\mathbb{R}} \hat{f}(\lambda) \langle \phi | \omega_\lambda \rangle \, d\lambda.\end{aligned}$$

That is:

$$f = \int_{\mathbb{R}} \hat{f}(\lambda) \omega_\lambda d\lambda. \tag{1.2}$$

in weak sense. The family $\{\omega_\lambda;\, \lambda \in \mathbb{R}\}$ of the previous example can be considered as the range of a weakly measurable function $\omega : \mathbb{R} \to \mathcal{S}^\times(\mathbb{R})$ which allows a representation as in (1.2) of any $f \in L^2(\mathbb{R})$ in terms of generalized eigenvectors of A. This is an example of a *distribution basis.* More precisely, since the Fourier-Plancherel theorem corresponds to the Parseval identity, this is an example of *Gel'fand distribution basis* [29, Subsec. 3.4], that plays, in $\mathcal{S}^\times(\mathbb{R})$, the role of an orthonormal basis in a Hilbert space.

The example above is a particular case of the Gel'fand-Maurin theorem (see [17, 19] for details), which states that, if $\mathcal{D}$ is a domain in a Hilbert space $\mathcal{H}$ which is a nuclear space under a certain topology τ, and A is an essentially self-adjoint operator on $\mathcal{D}$ which maps $\mathcal{D}[t]$ into $\mathcal{D}[t]$ continuously, then A admits a *complete set of generalized eigenvectors.*

If $\sigma(\overline{A})$ is the spectrum of the closure of the operator A, the completeness of the set $\{\omega_\lambda;\, \lambda \in \sigma(\overline{A})\}$ is understood in the sense that the Parseval identity holds, that is:

$$\|f\| = \left(\int_{\sigma(\overline{A})} |\, \langle f | \omega_\lambda \rangle \,|^2 d\lambda \right)^{1/2}, \quad \forall f \in \mathcal{D}. \tag{1.3}$$

To each λ there corresponds the subspace $\mathcal{D}^\times_\lambda \subset \mathcal{D}^\times$ of all generalized eigenvectors whose eigenvalue is λ. For all $f \in \mathcal{D}$ it is possible to define a linear functional $\tilde{f}_\lambda$ on $\mathcal{D}^\times_\lambda$ by $\tilde{f}_\lambda(\omega_\lambda) := \langle \omega_\lambda | f \rangle$ for all $\omega_\lambda \in \mathcal{D}^\times_\lambda$. Following [16, 17], the correspondence $\mathcal{D} \to \mathcal{D}^{\times\times}_\lambda$ defined by $f \mapsto \tilde{f}_\lambda$ is called the *spectral decomposition of the element f corresponding to A.* If $\tilde{f}_\lambda \equiv 0$ implies $f = 0$ (i.e. the map $f \mapsto \tilde{f}_\lambda$ is injective) then A is said to have *a complete system of generalized eigenvectors.*

The completeness and condition (1.3) may be interpreted as a kind of *orthonormality* of the ω_λ's: the family $\{\omega_\lambda\}_{\lambda \in \sigma(\overline{A})}$ in [29] is called a *Gel'fand basis.*

Another meaningful situation comes from quantum mechanics. Let us consider the rigged Hilbert space (1.1) corresponding to the one-dimensional case. The Hamiltonian operator H is an essentially self-adjoint operator on $\mathcal{S}(\mathbb{R})$, with self-adjoint extension $\overline{H}$ on the domain $\mathcal{D}(\overline{H})$, dense in $L^2(\mathbb{R})$. According to the *spectral expansion theorem* in the case of non-degenerate spectrum, for all $f \in L^2(\mathbb{R})$, the following decomposition holds:

$$f = \sum_{n \in J} c_n u_n + \int_{\sigma_c} c(\alpha) u_\alpha d\mu(\alpha).$$

The set $\{u_n\}_{n \in J}$, $J \subset \mathbb{N}$, is an orthonormal system of eigenvectors of H; the measure μ is a continuous measure on $\sigma_c \subset \mathbb{R}$ and $\{u_\alpha\}_{\alpha \in \sigma_c}$ are *generalized eigenvectors* of H in $\mathcal{S}^\times(\mathbb{R})$. This decomposition is *unique*. Furthermore:

$$\|f\|^2 = \sum_{n \in J} |c_n|^2 + \int_{\sigma_c} |c(\alpha)|^2 d\mu(\alpha).$$

The subset σ_c, corresponding to the continuous spectrum, is a union of intervals of $\mathbb{R}$, i.e. the index α is continuous. The generalized eigenvectors u_α are distributions: they do not belong to $L^2(\mathbb{R})$, therefore the "orthonormality" between generalized eigenvectors is not defined. Nevertheless, it is often denoted by the physicists with the Dirac delta: " $\langle u_\alpha | u_{\alpha'} \rangle$"$=\delta_{\alpha - \alpha'}$.

Frames, semi-frames, Riesz bases, etc. are families of sequences that generalize orthonormal bases in Hilbert space maintaining the possibility to *reconstruct* vectors of the space as superposition of more 'elementary' vectors renouncing often to the uniqueness of the representation, but gaining in versatility.

In this sense, they have been considered in literature in various spaces of functions and distributions: see for example the following (not exhaustive) list: [5, 8, 12, 14, 15, 25, 26].

It is remarkable that in a separable Hilbert space, orthonormal bases and Riesz bases are countable and notions corresponding to Riesz basis have been formulated in the continuous setting, but it is known that they exist only if the space given by the index set is discrete [7, 22, 23]. On the other hand, in the distributions and rigged Hilbert space setting the corresponding objects can be continuous.

Revisiting some results of [29] about Bessel maps, frames and (Gel'fand and Riesz) bases in distribution set-up, in this paper the notions of Riesz-Fischer map and of semi-frames in a space of distributions are proposed.

After some preliminaries and notations (Section 2), in Section 3 distribution Bessel maps are considered and the notion of distribution Riesz-Fischer maps is proposed, showing some new results about them (such as bounds and duality properties). Since distribution Bessel maps are not, in general, bounded by a Hilbert norm, we consider appropriate to define in Section 4 the *distribution semi-frames*, notion already introduced in a Hilbert space by J.-P. Antoine and P. Balasz [4]. Finally, distribution frames, distribution bases, Gel'fand and Riesz bases, considered in [29], are revisited in Section 5 with some additional examples.

2. Preliminary definitions and facts

2.1. Rigged Hilbert space

Let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{H}$ endowed with a locally convex topology t finer than the topology induced by a Hilbert norm. Denote as $\mathcal{D}^\times$ the vector space of all continuous conjugate linear functionals on $\mathcal{D}[t]$, i.e., the conjugate dual of $\mathcal{D}[t]$, endowed with the *strong dual topology* $t^\times = \beta(\mathcal{D}^\times, \mathcal{D})$, which can be defined by the seminorms:

$$q_{\mathcal{M}}(F) = \sup_{g\in\mathcal{M}} |\langle F|g\rangle|, \quad F \in \mathcal{D}^\times, \tag{2.1}$$

where $\mathcal{M}$ is a bounded subset of $\mathcal{D}[t]$. In this way, a *rigged Hilbert space* is defined in a standard fashion:

$$\mathcal{D}[t] \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}^\times[t^\times], \tag{2.2}$$

where $\hookrightarrow$ denotes a continuous injection. Since the Hilbert space $\mathcal{H}$ can be identified with a subspace of $\mathcal{D}^\times[t^\times]$, we will systematically read (2.2) as a chain of topological inclusions: $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$. These identifications imply that the sesquilinear form $B(\cdot,\cdot)$ which puts $\mathcal{D}$ and $\mathcal{D}^\times$ in duality is an extension of the inner product of $\mathcal{H}$; i.e. $B(\xi,\eta) = \langle\xi|\eta\rangle$, for every $\xi,\eta \in \mathcal{D}$ (to simplify notations we adopt the symbol $\langle\cdot|\cdot\rangle$ for both of them) and also the embedding map $I_{\mathcal{D},\mathcal{D}^\times} : \mathcal{D} \to \mathcal{D}^\times$ can be taken to act on $\mathcal{D}$ as $I_{\mathcal{D},\mathcal{D}^\times} f = f$ for every $f \in \mathcal{D}$. For more insights, besides to [16, 17], see also [21]. In this paper, if is not otherwise specified, we will work with a rigged Hilbert space $\mathcal{D}[t] \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}^\times[t^\times]$ with $\mathcal{D}[t]$ reflexive, in this way the embedding $\hookrightarrow$ is continuous and dense.

2.2. The space $\mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$

If $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$ is a rigged Hilbert space, let us denote by $\mathcal{L}(\mathcal{D},\mathcal{D}^\times)$ the vector space of all continuous linear maps from $\mathcal{D}[t]$ into $\mathcal{D}^\times[t^\times]$. If $\mathcal{D}[t]$ is barreled (e.g., reflexive), an involution $X \mapsto X^\dagger$ can be introduced in $\mathcal{L}(\mathcal{D},\mathcal{D}^\times)$ by the identity:

$$\left\langle X^\dagger\eta|\xi\right\rangle = \overline{\langle X\xi|\eta\rangle}, \quad \forall\xi,\eta \in \mathcal{D}.$$

Hence, in this case, $\mathcal{L}(\mathcal{D},\mathcal{D}^\times)$ is a †-invariant vector space. We also denote by $\mathcal{L}(\mathcal{D})$ the algebra of all continuous linear operators $Y : \mathcal{D}[t] \to \mathcal{D}[t]$ and by $\mathcal{L}(\mathcal{D}^\times)$ the algebra of all continuous linear operators $Z : \mathcal{D}^\times[t^\times] \to \mathcal{D}^\times[t^\times]$. If $\mathcal{D}[t]$ is reflexive, for every $Y \in \mathcal{L}(\mathcal{D})$ there exists a unique operator $Y^\times \in \mathcal{L}(\mathcal{D}^\times)$, the adjoint of Y, such that

$$\langle\Phi|Yg\rangle = \left\langle Y^\times\Phi|g\right\rangle, \quad \forall\Phi \in \mathcal{D}^\times, g \in \mathcal{D}.$$

In similar way an operator $Z \in \mathcal{L}(\mathcal{D}^\times)$ has an adjoint $Z^\times \in \mathcal{L}(\mathcal{D})$ such that $(Z^\times)^\times = Z$. In the monograph [3] the topic is treated more deeply.

2.3. Weakly measurable maps

In this paper a *weakly measurable map* is considered as a subset of $\mathcal{D}^\times$: if (X,μ) is a measure space with μ a σ-finite positive measure, $\omega : x \in X \mapsto \omega_x \in \mathcal{D}^\times$ is a weakly measurable map if, for every $f \in \mathcal{D}$, the complex valued function $x \mapsto \langle f|\omega_x\rangle$ is μ-measurable. Since the form which puts $\mathcal{D}$ and $\mathcal{D}^\times$ in conjugate duality is an extension of the inner product of $\mathcal{H}$, we write $\langle f|\omega_x\rangle$ for $\overline{\langle \omega_x|f\rangle}$, $f \in \mathcal{D}$. If not otherwise specified, throughout the paper we will work with a measure space (X,μ) above described.

Definition 2.1. Let $\mathcal{D}[t]$ be a locally convex space, $\mathcal{D}^\times$ its conjugate dual and $\omega : x \in X \to \omega_x \in \mathcal{D}^\times$ a weakly measurable map, then:

i) ω is *total* if, $f \in \mathcal{D}$ and $\langle f|\omega_x\rangle = 0$ μ-a.e. $x \in X$ implies $f = 0$;
ii) ω is *μ-independent* if the unique measurable function ξ on (X,μ) such that, if $\int_X \xi(x)\,\langle g|\omega_x\rangle\, d\mu = 0$ for every $g \in \mathcal{D}$, then $\xi(x) = 0$ μ-a.e.

3. Bessel and Riesz-Fischer distribution maps

3.1. Bessel distribution maps

Definition 3.1. Let $\mathcal{D}[t]$ be a locally convex space. A weakly measurable map ω is a *Bessel distribution map* (briefly: Bessel map) if for every $f \in \mathcal{D}$, $\int_X |\langle f|\omega_x\rangle|^2 d\mu < \infty$.

The following Proposition is the analogue of Proposition 2 and Theorem 3 in [30, Section 2, Chapter 4].

Proposition 3.2 ([29, Proposition 3.1])**.** *If $\mathcal{D}[t]$ a Fréchet space, and $\omega : x \in X \mapsto \omega_x \in \mathcal{D}^\times$ a weakly measurable map. The following statements are equivalent:*

(i) *ω is a Bessel map;*
(ii) *there exists a continuous seminorm p on $\mathcal{D}[t]$ such that:*

$$\left(\int_X |\langle f|\omega_x\rangle|^2 d\mu\right)^{1/2} \leq p(f), \quad \forall f \in \mathcal{D}; \tag{3.1}$$

(iii) *for every bounded subset $\mathcal{M}$ of $\mathcal{D}$ there exists $C_\mathcal{M} > 0$ such that:*

$$\sup_{f\in\mathcal{M}} \left| \int_X \xi(x)\,\langle \omega_x|f\rangle\, d\mu \right| \leq C_\mathcal{M} \|\xi\|_2, \quad \forall \xi \in L^2(X,\mu). \tag{3.2}$$

We have also the following:

Lemma 3.3 ([29, Lemma 3.4])**.** *If $\mathcal{D}$ is a Fréchet space and ω a Bessel distribution map, then:*

$$\int_X \langle f|\omega_x\rangle\, \omega_x d\mu$$

converges for every $f \in \mathcal{D}$ to an element of $\mathcal{D}^\times$. Moreover, the map $\mathcal{D} \ni f \mapsto \int_X \langle f|\omega_x\rangle\, \omega_x d\mu \in \mathcal{D}^\times$ is continuous.

Let ω be a Bessel map: the previous lemma allows to define on $\mathcal{D} \times \mathcal{D}$ the sesquilinear form Ω:

$$\Omega(f,g) = \int_X \langle f|\omega_x\rangle \langle \omega_x|g\rangle \, d\mu. \tag{3.3}$$

By Proposition 3.2, one has:

$$|\Omega(f,g)| = \left| \int_X \langle f|\omega_x\rangle \langle g|\omega_x\rangle \, d\mu \right| \leq \| \langle f|\omega_x\rangle \|_2 \, \|\langle g|\omega_x\rangle\|_2 \leq p(f)p(g), \forall f,g \in \mathcal{D}.$$

This means that Ω is jointly continuous on $\mathcal{D}[t]$. Hence there exists an operator $S_\omega \in \mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$, with $S_\omega = S_\omega^\dagger$, $S_\omega \geq 0$, such that:

$$\Omega(f,g) = \langle S_\omega f|g\rangle = \int_X \langle f|\omega_x\rangle \langle \omega_x|g\rangle \, d\mu, \quad \forall f,g \in \mathcal{D} \tag{3.4}$$

that is,

$$S_\omega f = \int_X \langle f|\omega_x\rangle \, \omega_x d\mu, \quad \forall f \in \mathcal{D}.$$

Since ω is a Bessel distribution map and $\xi \in L^2(X,\mu)$, we put for all $g \in \mathcal{D}$:

$$\Lambda_\omega^\xi(g) := \int_X \xi(x) \langle \omega_x|g\rangle \, d\mu. \tag{3.5}$$

Then Λ_ω^ξ is a continuous conjugate linear functional on $\mathcal{D}$, i.e. $\Lambda_\omega^\xi \in \mathcal{D}^\times$. We write:

$$\Lambda_\omega^\xi := \int_X \xi(x)\omega_x d\mu$$

in weak sense. Therefore we can define a linear map $T_\omega : L^2(X,\mu) \to \mathcal{D}^\times[t^\times]$, which will be called the *synthesis operator*, by:

$$T_\omega : \xi \mapsto \Lambda_\omega^\xi.$$

By (3.2), it follows that T_ω is continuous from $L^2(X,\mu)$, endowed with its natural norm, into $\mathcal{D}^\times[t^\times]$. Hence, it possesses a continuous adjoint $T_\omega^\times : \mathcal{D}[t] \to L^2(X,\mu)$, which is called the *analysis operator*, acting as follows:

$$T_\omega^\times : f \in \mathcal{D}[t] \mapsto \xi_f \in L^2(X,\mu), \text{ where } \xi_f(x) = \langle f|\omega_x\rangle, \; x \in X.$$

One has that $S_\omega = T_\omega T_\omega^\times$.

3.2. Riesz-Fischer distribution map

Definition 3.4. Let $\mathcal{D}[t]$ be a locally convex space. A weakly measurable map $\omega : x \in X \mapsto \omega_x \in \mathcal{D}^\times$ is called a *Riesz-Fischer distribution map* (briefly: Riesz-Fischer map) if, for every $h \in L^2(X,\mu)$, there exists $f \in \mathcal{D}$ such that:

$$\langle f|\omega_x\rangle = h(x) \quad \mu\text{-a.e.} \tag{3.6}$$

In this case, we say that f is a solution of equation $\langle f|\omega_x\rangle = h(x)$.

Clearly, if f_1 and f_2 are solutions of (3.6), then $f_1 - f_2 \in \omega^\perp := \{g \in \mathcal{D} : \langle g|\omega_x\rangle = 0, \quad \mu - a.e.\}$. If ω is total, the solution is unique. We prove the following:

Lemma 3.5. *Let $\mathcal{D}$ be a reflexive locally convex space, $h(x)$ be a measurable function and $x \in X \mapsto \omega_x \in \mathcal{D}^\times[t^\times]$ a weakly measurable map. Then the equation:*

$$\langle f|\omega_x\rangle = h(x) \tag{3.7}$$

admits a solution $f \in \mathcal{D}$ if, and only if, there exists a bounded subset $\mathcal{M}$ of $\mathcal{D}$ such that $|h(x)| \leq \sup_{f\in\mathcal{M}} |\langle f|\omega_x\rangle|$ μ-a.e.

Proof. Necessity is obvious. Conversely, let $x \in X$ be a point such that $\langle f|\omega_x\rangle = h(x) \neq 0$. Let us consider the subspace V_x of $\mathcal{D}^\times$ given by $V_x := \{\alpha\omega_x\}_{\alpha\in\mathbb{C}}$, and let us define the functional μ on V_x by: $\mu(\alpha\omega_x) := \alpha h(x)$. We have that $|\mu(\alpha\omega_x)| = |\alpha h(x)| \leq |\alpha| \sup_{f\in\mathcal{M}} |\langle f|\omega_x\rangle| = \sup_{f\in\mathcal{M}} |\langle f|\alpha\omega_x\rangle|$, in other words: $|\mu(F_x)| \leq \sup_{f\in\mathcal{M}} |\langle f|F_x\rangle|$ for all $F_x \in V_x$. By the Hahn-Banach theorem, there exists an extension $\tilde{\mu}$ to $\mathcal{D}^\times$ such that $|\tilde{\mu}(F)| \leq \sup_{f\in\mathcal{M}} |\langle f|F\rangle|$, for every $F \in \mathcal{D}^\times$. Since $\mathcal{D}$ is reflexive, there exists $\bar{f} \in \mathcal{D}$ such that $\tilde{\mu}(F) = \langle \bar{f}|F\rangle$. The statement follows from the fact that $\mu(\omega_x) = h(x)$. □

If M is a subspace of $\mathcal{D}$ and the topology of $\mathcal{D}$ is generated by the family of seminorms $\{p_\alpha\}_{\alpha\in I}$, then the topology on the quotient space $\mathcal{D}/M$ is defined, as usual, by the seminorms $\{\tilde{p}_\alpha\}_{\alpha\in I}$, where $\tilde{p}_\alpha(\tilde{f}) := \inf\{p_\alpha(g) : g \in f + M\}$. The following proposition can be compared to the case of Riesz-Fischer sequences: see [30, Chapter 4, Section 2, Proposition 2].

Proposition 3.6. *Assume that $\mathcal{D}[t]$ is a Fréchet space. If the map $\omega : x \in X \to \omega_x \in \mathcal{D}^\times$ is a Riesz-Fischer map, then for every continuous seminorm p on $\mathcal{D}$, there exists a constant $C > 0$ such that, for every solution f of* (3.6)*,*

$$\tilde{p}(\tilde{f}) := \inf\{p(g) : g \in f + \omega^\perp\} \leq C\| \langle f|\omega_x\rangle \|_2.$$

Proof. Since $\omega^\perp$ is closed, it follows that the quotient $\mathcal{D}/\omega^\perp := \mathcal{D}_{\omega^\perp}$ is a Fréchet space. If $f \in \mathcal{D}$, we put $\tilde{f} := f + \omega^\perp$. Let $h \in L^2(X,\mu)$ and f a solution of (3.6) corresponding to h; then, we can define an operator $S : L^2(X,\mu) \to \mathcal{D}_{\omega^\perp}$ by $h \mapsto \tilde{f}$. Let us consider a sequence $h_n \in L^2(X,\mu)$ such that $h_n \to 0$ and, for each $n \in \mathbb{N}$, let f_n be a corresponding solution of (3.6). One has that $\int_X |h_n(x)|^2 d\mu \to 0$, i.e. $\int_X |\langle f_n|\omega_x\rangle|^2 d\mu \to 0$. This implies that $\langle f_n|\omega_x\rangle \to 0$ in measure, so there exists a subsequence such that $\langle f_{n_k}|\omega_x\rangle \to 0$ a.e. (see [13]). On the other hand, if $Sh_n = \tilde{f}_n$ is a sequence convergent to $\tilde{f}$ in $\mathcal{D}_{\omega^\perp}$ w.r. to the quotient topology defined by the seminorms $\tilde{p}(\cdot)$, it follows that the sequence is convergent in the *weak* topology of $\mathcal{D}_{\omega^\perp}$, i.e.:

$$\left\langle \tilde{f}_n|\tilde{F}\right\rangle \to \left\langle \tilde{f}|\tilde{F}\right\rangle \quad \forall \tilde{F} \in \mathcal{D}^\times_{\omega^\perp}.$$

Let us consider the canonical surjection $\rho : \mathcal{D} \to \mathcal{D}_{\omega^\perp}$, $\rho : f \mapsto \tilde{f} = f + \omega^\perp$. Its transpose map (adjoint) $\rho^\dagger : \mathcal{D}^\times_{\omega^\perp} \to \mathcal{D}^\times$ is injective (see [21], p. 263) and $\rho^\dagger[\mathcal{D}^\times_{\omega^\perp}] = \omega^{\perp\perp}$. Then $\rho^\dagger : \mathcal{D}^\times_{\omega^\perp} \to \omega^{\perp\perp}$ is invertible. Hence,

$$\left\langle \tilde{f}_n|\tilde{F}\right\rangle = \langle \rho(f_n)|(\rho^\dagger)^{-1}(F)\rangle = \langle f_n|\rho^\dagger((\rho^\dagger)^{-1}(F))\rangle = \langle f_n|F\rangle, \quad \forall F \in \omega^{\perp\perp}.$$

Thus, if $\tilde{f}_n \to \tilde{f}$ in the topology of $\mathcal{D}_{\omega^\perp}$, then $\langle f_n|F\rangle \to \langle f|F\rangle$, for all $F \in \omega^{\perp\perp}$, and, in particular, since $\omega \subset \omega^{\perp\perp}$, one has $\langle f_n|\omega_x\rangle \to \langle f|\omega_x\rangle$. Since $\langle f_n|\omega_x\rangle$ has a subsequence convergent to 0, one has $f \in \omega^\perp$. From the closed graph theorem, it follows that the map S is continuous, i.e. for all continuous seminorms $\tilde{p}$ on $\mathcal{D}_{\omega^\perp}$ there exists $C > 0$ such that: $\tilde{p}(Sh) \leq C\|h\|_2$, for all $h \in L^2(X,\mu)$. The statement follows from the definition of Riesz-Fischer map. □

Corollary 3.7. *Assume that $\mathcal{D}[t]$ is a Fréchet space. If the map $\omega : x \in X \to \omega_x \in \mathcal{D}^\times$ is a total Riesz-Fischer map, then for every continuous seminorm p on $\mathcal{D}$, there exists a constant $C > 0$ such that, for the solution f of* (3.6),

$$p(f) \leq C\| \langle f|\omega_x\rangle \|_2.$$

Remark 3.8. For an arbitrary weakly measurable map ω, we define the subset of $\mathcal{D}[t]$: $D(V_\omega) := \{f \in \mathcal{D} : \langle f|\omega_x\rangle \in L^2(X,\mu)\}$ and the analysis operator $V_\omega : f \in D(V_\omega) \mapsto \langle f|\omega_x\rangle \in L^2(X,\mu)$. Clearly, ω is a Riesz-Fischer map if and only if $V_\omega : D(V_\omega) \to L^2(X,\mu)$ is surjective. If ω is total, it is injective too, so V_ω is invertible. A consequence of Corollary 3.7 is that $V_\omega^{-1} : L^2(X,\mu) \to D(V_\omega)$ is continuous.

3.3. Duality

Definition 3.9. Let $\mathcal{D} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}^\times$ be a rigged Hilbert space and ω a weakly measurable map. We call *dual map of* ω, if it exists, a weakly measurable map θ such that for all $f, g \in \mathcal{D}$:

$$\left| \int_X \langle f|\theta_x\rangle \langle \omega_x|g\rangle \, d\mu \right| < \infty$$

and

$$\langle f|g\rangle = \int_X \langle f|\theta_x\rangle \langle \omega_x|g\rangle \, d\mu, \quad \forall f, g \in \mathcal{D}.$$

Proposition 3.10. *Suppose that ω is a Riesz-Fischer map. Then the map θ is a Bessel map.*

Proof. For all $h \in L^2(X,\mu)$ there exists $\bar{f} \in \mathcal{D}$ such that $\langle \bar{f}|\omega_x\rangle = h(x)$ μ-a.e. Since θ is a dual map, one has that: $|\int_X h(x) \langle \theta_x|g\rangle \, d\mu| < \infty$ for all $h \in L^2(X,\mu)$. It follows that $\langle \theta_x|g\rangle \in L^2(X,\mu)$ (see [28, Chapter 6, Exercise 4]). □

Proposition 3.11. *Let $\mathcal{D}$ be reflexive and let ω be a μ-independent Bessel map. Furthermore, suppose that for all $h \in L^2(X,\mu)$ there exists a bounded subset $\mathcal{M} \subset \mathcal{D}$ such that:*

$$\left| \int_X h(x) \langle \omega_x|g\rangle \, d\mu \right| \leq \sup_{f \in \mathcal{M}} | \langle f|g\rangle |, \quad \forall g \in \mathcal{D},$$

then the dual map θ is a Riesz-Fischer map.

Proof. Since $h \in L^2(X,\mu)$, and since ω is a Bessel map, one has: $|\int_X h(x)\langle\omega_x|g\rangle\, d\mu| < \infty$. Let us consider $g = \int_X \langle g|\omega_x\rangle\, \theta_x d\mu$ as element of $\mathcal{D}^\times$. We define the following functional on $\mathcal{D}$ (as subspace of $\mathcal{D}^\times$): $\mu(g) := \int_X h(x)\langle\omega_x|g\rangle\, d\mu$. By hypothesis, one has:

$$|\mu(g)| \leq \sup_{f\in\mathcal{M}} |\langle f|g\rangle|.$$

By the Hahn-Banach theorem, there exists an extension $\tilde{\mu}$ to $\mathcal{D}^\times$ such that:

$$|\tilde{\mu}(G)| \leq \sup_{f\in\mathcal{M}} |\langle f|G\rangle|, \quad \forall G \in \mathcal{D}^\times.$$

Since $\mathcal{D}$ is reflexive, there exists $\tilde{f} \in \mathcal{D}^{\times\times} = \mathcal{D}$ such that $\tilde{\mu}(G) = \left\langle \tilde{f}|G\right\rangle$. In particular $\left\langle \tilde{f}|g\right\rangle = \int_X h(x)\langle\omega_x|g\rangle\, d\mu$. Since θ is dual of ω, we have too: $\left\langle \tilde{f}|g\right\rangle = \int_X \left\langle \tilde{f}|\theta_x\right\rangle \langle\omega_x|g\rangle\, d\mu$. But ω is μ-independent, then it follows that $h(x) = \left\langle \tilde{f}|\theta_x\right\rangle$ μ-a.e. □

4. Semi-frames and frames

4.1. Distribution semi-frames

Definition 4.1. Given a rigged Hilbert space $\mathcal{D} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}^\times$, a Bessel map ω is a *distribution upper semi-frame* if it is complete (total) and if there exists $B > 0$:

$$0 < \int_X |\langle f|\omega_x\rangle|^2 d\mu \leq B\|f\|^2,\ \forall f \in \mathcal{D}, f \neq 0. \tag{4.1}$$

Since the injection $\mathcal{D} \hookrightarrow \mathcal{H}$ is continuous, it follows that there exists a continuous seminorm p on $\mathcal{D}$ such that $\|f\| \leq p(f)$ for all $f \in \mathcal{D}$. If $\xi \in L^2(X,\mu)$, then the continuous conjugate functional Λ^ξ_ω on $\mathcal{D}$ defined in (3.5) is bounded in $\mathcal{D}[\|\cdot\|]$; it follows that it has a bounded extension $\tilde{\Lambda}^\xi_\omega$ to $\mathcal{H}$, defined, as usual, by a limiting procedure. Therefore, there exists a unique vector $h_\xi \in \mathcal{H}$ such that:

$$\tilde{\Lambda}^\xi_\omega(g) = \langle h_\xi|g\rangle, \quad \forall g \in \mathcal{H}.$$

This implies that the synthesis operator T_ω takes values in $\mathcal{H}$, it is bounded and $\|T_\omega\| \leq B^{1/2}$; its hilbertian adjoint $C_\omega := T^*_\omega$ extends the analysis operator $T^\times_\omega$.

The action of C_ω can be easily described: if $g \in \mathcal{H}$ and $\{g_n\}$ is a sequence of elements of $\mathcal{D}$, norm converging to g, then the sequence $\{\eta_n\}$, where $\eta_n(x) = \langle g_n|\omega_x\rangle$, is convergent in $L^2(X,\mu)$. Put $\eta = \lim_{n\to\infty}\eta_n$. Then,

$$\langle T_\omega\xi|g\rangle = \lim_{n\to\infty}\int_X \xi(x)\langle\omega_x|g_n\rangle\, d\mu = \int_X \xi(x)\overline{\eta(x)}d\mu. \tag{4.2}$$

Hence $T^*_\omega g = \eta$.

The function $\eta \in L^2(X,\mu)$ depends linearly on g, for each $x \in X$. Thus we can define a linear functional $\check{\omega}_x$ by

$$\langle g|\check{\omega}_x\rangle = \lim_{n\to\infty} \langle g_n|\omega_x\rangle\,, \quad g \in \mathcal{H};\, g_n \to g. \tag{4.3}$$

Of course, for each $x \in X$, $\check{\omega}_x$ extends ω_x; however $\check{\omega}_x$ need not be continuous, as a functional on $\mathcal{H}$. We conclude that:

$$T^*_\omega : g \mapsto \langle g|\check{\omega}_x\rangle \in L^2(X,\mu).$$

Moreover, in this case, the sesquilinear form Ω in (3.4), which is well defined on $\mathcal{D}\times\mathcal{D}$, is bounded with respect to $\|\cdot\|$ and possesses a bounded extension $\hat{\Omega}$ to $\mathcal{H}$. Hence there exists a bounded operator $\hat{S}_\omega$ in $\mathcal{H}$, such that:

$$\hat{\Omega}(f,g) = \left\langle \hat{S}_\omega f|g\right\rangle, \quad \forall f,g \in \mathcal{H}. \tag{4.4}$$

Since

$$\left\langle \hat{S}_\omega f|g\right\rangle = \int_X \langle f|\omega_x\rangle \langle \omega_x|g\rangle\, d\mu, \quad \forall f,g \in \mathcal{D}, \tag{4.5}$$

$\hat{S}_\omega$ extends the frame operator S_ω and $S_\omega : \mathcal{D} \to \mathcal{H}$. It is easily seen that $\hat{S}_\omega = \hat{S}^*_\omega$ and $\hat{S}_\omega = T_\omega T^*_\omega$. By definition, we have:

$$0 < \|\hat{S}_\omega f\| \le B\|f\|,\ \forall f \in \mathcal{H}, \quad f \neq 0.$$

Then $\hat{S}_\omega$ is bounded, self-adjoint and injective too. This means that $\mathsf{Ran}\, S_\omega$ is dense in $\mathcal{H}$, and $\hat{S}_\omega^{-1}$ is densely defined. If ω is not a frame, $\hat{S}_\omega^{-1}$ is an unbounded, self-adjoint operator (see [4]).

Remark 4.2. If $\{\omega_x\}_{x\in X}$ is an upper semi-frame, then there exists a continuous seminorm p on $\mathcal{D}$ such that $\|\langle f|\omega_x\rangle\|_2 \le p(f)$ for all $f \in \mathcal{D}$. In fact, the injection $\mathcal{D} \hookrightarrow \mathcal{H}$ is continuous, i.e. $\|f\| \le p(f)$ for all $f \in \mathcal{D}$. The converse is not true: let us consider the rigged Hilbert space $\mathcal{S}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}) \hookrightarrow \mathcal{S}^\times(\mathbb{R})$; the system of derivative of Dirac's deltas $\{\delta'_x\}_{x\in\mathbb{R}}$ is total. Since $\mathcal{S}(\mathbb{R})$ is a Fréchet space, *(ii)* of Proposition 3.2 it holds. However $\{\delta'_x\}_{x\in\mathbb{R}}$ is not a distribution upper semi-frame; in fact:

$$\int_\mathbb{R} |\langle \phi|\delta'_x\rangle|^2 dx = \|\phi'\|_2^2 \quad \forall \phi \in \mathcal{S}(\mathbb{R}),$$

but the derivative operator $\frac{d}{dx} : \mathcal{S}(\mathbb{R}) \to L^2(\mathbb{R})$ is unbounded (clearly with respect to the topology of the Hilbert norm).

Remark 4.3. In [29] it is defined the notion of *bounded Bessel map*, that is a Bessel map in rigged Hilbert space such that: $\int_X |\langle f|\omega_x\rangle|^2 d\mu \le B\|f\|^2$, $\forall f \in \mathcal{D}$. It is a more general notion than upper bounded semi-frame. In fact, we can consider, as example, the distribution $\omega_x := \eta_K(x)\delta_x$ where $\eta_K(x)$ is a C^∞-function with compact support K and $M := \max_{x\in K}|\eta_K(x)|$:

$$\int_\mathbb{R} |\langle \phi|\omega_x\rangle|^2 dx = \int_\mathbb{R} |\langle \phi|\eta_K(x)\delta_x\rangle|^2 dx =$$
$$= \int_\mathbb{R} |\overline{\eta_K(x)}\phi(x)|^2 dx \le M^2 \int_K |\phi(x)|^2 dx \le M^2\|\phi\|_2^2.$$

Therefore ω is a bounded Bessel map, but it is not total, then it is not an upper semi-frame.

Definition 4.4. Given a rigged Hilbert space $\mathcal{D} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}^\times$, a Bessel map ω is a *distribution lower semi-frame* if there exists $A > 0$ such that:

$$A\|f\|^2 \leq \int_X |\langle f|\omega_x\rangle|^2 d\mu, \ \forall f \in \mathcal{D}. \tag{4.6}$$

By definition, it follows that ω is total. If $\mathcal{D}$ is a Fréchet space, by Proposition 3.2 one has $S_\omega \in \mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$ and, if ω is not a frame, S_ω is unbounded. Furthermore, S_ω is injective, and S_ω^{-1} is bounded.

Example. Let us consider the space $\mathcal{O}_M$, known (see [27]) as the set of infinitely differentiable functions on $\mathbb{R}$ that are polynomially bounded together with their derivatives. Let us consider $g(x) \in \mathcal{O}_M$ such that $0 < m < |g(x)|$. If we define $\omega_x := g(x)\delta_x$, then $\{\omega_x\}_{x\in\mathbb{R}}$ is a distribution lower semi-frame with $A = m^2$.

The proof of the following lemma is analogous to that of Lemma 2.5 in [4]:

Lemma 4.5. *Let ω be an upper semi-frame with upper frame bound M and θ a total family dual to ω. Then θ is a lower semi-frame, with lower frame bound M^{-1}.*

4.2. Distribution Frames

This section is devoted to distribution frames, with main results already shown in [29].

Definition 4.6 ([29, Definition 3.6])**.** Let $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$ be a rigged Hilbert space, with $\mathcal{D}[t]$ a reflexive space and ω a Bessel map. We say that ω is a *distribution frame* if there exist $A, B > 0$ such that:

$$A\|f\|^2 \leq \int_X |\langle f|\omega_x\rangle|^2 d\mu \leq B\|f\|^2, \quad \forall f \in \mathcal{D}. \tag{4.7}$$

A distribution frame ω is clearly, in particular, an upper bounded semi-frame. Thus, we can consider the operator $\hat{S}_\omega$ defined in (4.4). It is easily seen that, in this case,

$$A\|f\| \leq \|\hat{S}_\omega f\| \leq B\|f\|, \quad \forall f \in \mathcal{H}.$$

This inequality, together with the fact that $\hat{S}_\omega$ is symmetric, implies that $\hat{S}_\omega$ has a bounded inverse $\hat{S}_\omega^{-1}$ everywhere defined in $\mathcal{H}$.

Remark 4.7. It is worth noticing that the fact that ω and S_ω extend to $\mathcal{H}$ does not mean that ω a frame in the Hilbert space $\mathcal{H}$, because we do not know if the extension of S_ω has the form of (3.4) with $f, g \in \mathcal{H}$.

To conclude this section, we recall a list of properties of frames proved in [29].

Lemma 4.8 ([29, Lemma 3.8]). *Let ω be a distribution frame. Then, there exists $R_\omega \in \mathcal{L}(\mathcal{D})$ such that $S_\omega R_\omega f = R_\omega^\times S_\omega f = f$, for every $f \in \mathcal{D}$.*

As a consequence, the reconstruction formulas for distribution frames hold for all $f \in \mathcal{D}$:

$$f = R_\omega^\times S_\omega f = \int_X \langle f|\omega_x\rangle R_\omega^\times \omega_x d\mu;$$

$$f = S_\omega R_\omega f = \int_X \langle R_\omega f|\omega_x\rangle \omega_x d\mu.$$

These representations have to be interpreted in the weak sense.

Remark 4.9. The operator R_ω acts as an inverse of S_ω. On the other hand the operator $\hat{S}_\omega$ has a bounded inverse $\hat{S}_\omega^{-1}$ everywhere defined in $\mathcal{H}$. It results that [29, Remark 3.7]: $\hat{S}_\omega^{-1}\mathcal{D} \subset \mathcal{D}$ and $R_\omega = \hat{S}_\omega^{-1} \upharpoonright_{\mathcal{D}}$.

There exists the dual frame:

Proposition 4.10 ([29, Lemma 3.10]). *Let ω be a distribution frame. Then there exists a weakly measurable function θ such that:*

$$\langle f|g\rangle = \int_X \langle f|\theta_x\rangle \langle \omega_x|g\rangle d\mu, \quad \forall f, g \in \mathcal{D}.$$

Where $\theta_x := R_\omega^\times \omega_x$. The frame operator S_θ for θ is well defined and we have: $S_\theta = I_{\mathcal{D},\mathcal{D}^\times} R_\omega$.

The distribution function θ, constructed in Proposition 4.10, is also a distribution frame, called the *canonical dual frame* of ω. Indeed, it results that [29]:

$$B^{-1}\|f\|^2 \leq \langle S_\theta f|f\rangle \leq A^{-1}\|f\|^2, \quad \forall f \in \mathcal{D}.$$

4.3. Parseval distribution frames

Definition 4.11. If ω is a distribution frame, then we say that:

a) ω is a *tight* distribution frame if we can choose $A = B$ as frame bounds. In this case, we usually refer to A as a frame bound for ω;
b) ω is a *Parseval* distribution frame if $A = B = 1$ are frame bounds.

More explicitly a weakly measurable distribution function ω is called a *Parseval distribution frame* if [29, Definition 3.13]:

$$\int_X |\langle f|\omega_x\rangle|^2 d\mu = \|f\|^2, \quad f \in \mathcal{D}.$$

It is clear that a Parseval distribution frame is a frame in the sense of Definition 4.6 with $S_\omega = I_{\mathcal{D}}$, the identity operator of $\mathcal{D}$.

Lemma 4.12 ([29, Lemma 3.14]). *Let $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times$ be a rigged Hilbert space and $\omega : x \in X \mapsto \omega_x \in \mathcal{D}^\times$ a weakly measurable map. The following statements are equivalent.*

(i) *ω is a Parseval distribution frame;*
(ii) $\langle f|g\rangle = \int_X \langle f|\omega_x\rangle \langle \omega_x|g\rangle d\mu, \quad \forall f, g \in \mathcal{D};$

(iii) $f = \int_X \langle f|\omega_x\rangle \omega_x d\mu$, *the integral on the r.h.s. is understood as a continuous conjugate linear functional on* $\mathcal{D}$, *that is an element of* $\mathcal{D}^\times$.

The representation in (iii) of Lemma 4.12 is not necessarily unique.

5. Distribution basis

Definition 5.1 **(**[29, Definition 2.3]**).** Let $\mathcal{D}[t]$ be a locally convex space, $\mathcal{D}^\times$ its conjugate dual and $\omega : x \in X \mapsto \omega_x \in \mathcal{D}^\times$ a weakly measurable map. Then ω is a *distribution basis* for $\mathcal{D}$ if, for every $f \in \mathcal{D}$, there exists a *unique* measurable function ξ_f such that:

$$\langle f|g\rangle = \int_X \xi_f(x) \langle \omega_x|g\rangle \, d\mu, \quad \forall f, g \in \mathcal{D}$$

and, for every $x \in X$, the linear functional $f \in \mathcal{D} \to \xi_f(x) \in \mathbb{C}$ is continuous in $\mathcal{D}[t]$.

The above formula can be represented by:

$$f = \int_X \xi_f(x)\omega_x d\mu$$

in weak sense.

Remark 5.2. Clearly, if ω is a distribution basis, then it is μ-independent. Furthermore, since $f \in \mathcal{D} \mapsto \xi_f(x) \in \mathbb{C}$ continuously, there exists a unique weakly μ-measurable map $\theta : X \to \mathcal{D}^\times$ such that: $\xi_f(x) = \langle f|\theta_x\rangle$ for every $f \in \mathcal{D}$. We call θ *dual* map of ω. If θ is μ-independent, then it is a distribution basis too.

5.1. Gel'fand distribution basis

The Gel'fand distribution basis, introduced in [29], is a good substitute for the notion of an *orthonormal basis* which is meaningless in the present framework.

Definition 5.3. A weakly measurable map ζ is *Gel'fand distribution basis* if it is a μ-independent Parseval distribution frame.

By definition and Lemma 4.12, this means that, for every $f \in \mathcal{D}$ there exists a unique function $\xi_f \in L^2(X, \mu)$ such that:

$$f = \int_X \xi_f(x)\zeta_x d\mu \tag{5.1}$$

with $\xi_f(x) = \langle f|\zeta_x\rangle$ μ-a.e. Furthermore $\|f\|^2 = \int_X |\langle f|\zeta_x\rangle|^2 d\mu$ and ζ is total too.

For every $x \in X$, the map $f \in \mathcal{H} \mapsto \xi_f(x) \in \mathbb{C}$ defines as in (4.3) a linear functional $\check{\zeta}_x$ on $\mathcal{H}$, then for all $f \in \mathcal{H}$:

$$f = \int_X \langle f|\check{\zeta}_x\rangle \zeta_x d\mu.$$

We have the following characterization result [29]:

Proposition 5.4 ([29, Proposition 3.15]). *Let $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times$ be a rigged Hilbert space and let $\zeta : x \in X \mapsto \zeta_x \in \mathcal{D}^\times$ be a Bessel distribution map. Then the following statements are equivalent.*

(a) *ζ is a Gel'fand distribution basis.*
(b) *The synthesis operator T_ζ is an isometry of $L^2(X,\mu)$ onto $\mathcal{H}$.*

Example ([29, Example 3.17]). Given the rigged Hilbert space:

$$\mathcal{S}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}) \hookrightarrow \mathcal{S}^\times(\mathbb{R}),$$

for $x \in \mathbb{R}$ the function $\zeta_x(y) = \frac{1}{\sqrt{2\pi}} e^{-ixy}$, defines a (regular) tempered distribution: in fact, denoting as usual by $\hat{g}$, $\check{g}$, respectively, the Fourier transform and the inverse Fourier transform of $g \in L^2(\mathbb{R})$, one has that $\mathcal{S}(\mathbb{R}) \ni \phi \mapsto \langle \phi | \zeta_x \rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi(y) e^{-ixy} dy = \hat{\phi}(x) \in \mathbb{C}$. For all $x \in \mathbb{R}$ the set of functions $\zeta := \{\zeta_x(y)\}_{x \in \mathbb{R}}$ is a Gel'fand distribution basis, because the synthesis operator $T_\zeta : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by:

$$(T_\zeta \xi)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \xi(y) e^{-ixy} dy = \hat{\xi}(x), \quad \forall \xi \in L^2(\mathbb{R})$$

is an isometry onto $L^2(\mathbb{R})$ by Plancherel theorem. The analysis operator is: $T_\zeta^* f = \check{f}$, for all $f \in L^2(\mathbb{R})$.

Example ([29, Example 3.18]). Let us consider again $\mathcal{S}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}) \hookrightarrow \mathcal{S}^\times(\mathbb{R})$. For $x \in \mathbb{R}$, let us consider the Dirac delta $\delta_x : \mathcal{S}(\mathbb{R}) \to \mathbb{C}, \quad \phi \mapsto \langle \phi | \delta_x \rangle := \phi(x)$. The set of Dirac deltas $\delta := \{\delta_x\}_{x \in \mathbb{R}}$ is a Gel'fand distribution basis. In fact, the Parseval identity holds:

$$\int_{\mathbb{R}} |\langle \delta_x | \phi \rangle|^2 dx = \int_{\mathbb{R}} |\phi(x)|^2 dx = \|\phi\|_2^2, \quad \forall \phi \in \mathcal{S}(\mathbb{R}).$$

The synthesis operator: $T_\delta : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is:

$$\langle T_\delta \xi | \phi \rangle = \int_{\mathbb{R}} \xi(x) \langle \delta_x | \phi \rangle \, dx = \int_{\mathbb{R}} \xi(x) \overline{\phi(x)} dx = \langle \xi | \phi \rangle, \quad \forall \phi \in \mathcal{S}(\mathbb{R}).$$

Then $T_\delta \xi = \xi$ for all $\xi \in L^2(\mathbb{R})$. Since T_δ is an identity, it is an isometry onto $L^2(\mathbb{R})$.

5.2. Riesz distribution basis

Proposition 5.4 and (5.1) suggest a more general class of bases that will play the same role as Riesz bases in the ordinary Hilbert space framework.

Definition 5.5. Let $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times$ be a rigged Hilbert space. A weakly measurable map $\omega : x \in X \mapsto \omega_x \in \mathcal{D}^\times$ is a *Riesz distribution basis* if ω is a μ-independent distribution frame.

One has the following:

Proposition 5.6 ([29, Proposition 3.19]). *Let $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times$ be a rigged Hilbert space and let $\omega : x \in X \mapsto \omega_x \in \mathcal{D}^\times$ be a Bessel distribution map. Then the following statements are equivalent:*

(a) *ω is a Riesz distribution basis;*
(b) *If ζ is a Gel'fand distribution basis, then the operator W defined, for $f \in \mathcal{H}$, by:*

$$f = \int_X \xi_f(x)\zeta_x d\mu \mapsto Wf = \int_X \xi_f(x)\omega_x d\mu$$

is continuous and has bounded inverse;
(c) *the synthesis operator T_ω is a topological isomorphism of $L^2(X,\mu)$ onto $\mathcal{H}$.*

Proposition 5.7. *If ω is a Riesz distribution basis then ω possesses a unique dual frame θ which is also a Riesz distribution basis.*

Example. Let us consider $f \in C^\infty(\mathbb{R})$: $0 < m < |f(x)| < M$. Let us define $\omega_x := f(x)\delta_x$: then $\{\omega_x\}_{x\in\mathbb{R}}$ is a distribution frame, in fact:

$$\int_{\mathbb{R}} |\langle \omega_x|\phi\rangle|^2 dx = \int_{\mathbb{R}} |\overline{f(x)}\phi(x)|^2 dx \le M^2\|\phi\|_2^2, \quad \forall \phi \in \mathcal{S}(\mathbb{R}),$$

and

$$m^2\|\phi\|_2^2 \le \int_{\mathbb{R}} |\overline{f(x)}\phi(x)|^2 dx \le M^2\|\phi\|_2^2, \quad \forall \phi \in \mathcal{S}(\mathbb{R}).$$

Furthermore, $\{\omega_x\}_{x\in\mathbb{R}}$ is μ-independent. In fact, putting:

$$\int_{\mathbb{R}} \xi(x)\langle \omega_x|g\rangle\, dx = 0, \quad \forall g \in \mathcal{S}(\mathbb{R}),$$

one has:

$$\int_{\mathbb{R}} \xi(x)\langle \omega_x|g\rangle\, dx = \int_{\mathbb{R}} \xi(x)\overline{f(x)}\langle \delta_x|g\rangle dx = 0, \quad \forall g \in \mathcal{S}(\mathbb{R}).$$

Since $\{\delta_x\}_{x\in\mathbb{R}}$ is μ-independent, it follows that $\xi(x)\overline{f(x)} = 0$ a.e., then $\xi(x) = 0$ a.e.. By definition, $\{\omega_x\}_{x\in\mathbb{R}}$ is a Riesz distribution basis.

Concluding remarks

In a Hilbert space, frames, semi-frames, Bessel, Riesz-Fischer sequences, and Riesz bases are related through the action of a linear operator on elements of an orthonormal basis (see also [29, Remark 3.22]). On the other hand, in literature some studies on bounds (upper and lower) of these sequences have been already considered and their links with the linear operators related to them have been studied (see [6, 9, 4]). For that, it is desirable to continue an analogous study in rigged Hilbert spaces by considering linear operators in $\mathcal{L}(\mathcal{D},\mathcal{D}^\times)$.

Acknowledgement

The author would like to thank R. Corso for some valuable comments and suggestions.

References

1. S. T. Ali, J. P. Antoine, J. P. Gazeau, *Coherent states, Wavelets and their Generalizations*, Springer, II ed. 2014.
2. S. T. Ali, J. P. Antoine, J. P. Gazeau, *Continuous frames in Hilbert spaces*, Annals of Physics, **222**, (1993), 1-37.
3. J.-P. Antoine, A. Inoue, C. Trapani, *Partial *-Algebras and their Operator Realizations*, Kluwer, Dordrecht, 2002.
4. J.-P. Antoine, P. Balazs, *Frames and semi-frames*, J. Phys. A: Math. Theor. 44, 205201, (2011).
5. J.-P. Antoine, C. Trapani, *Reproducing pairs of measurable functions and partial inner product spaces*, Advances in Operator Theory, **2**, (2017), 126-146.
6. P. Balazs, D. T. Stoeva, J.-P. Antoine, *Classification of general sequences by frame-related operators*, Sampl. Theory Signal Image Process., **10**, (2), (2011), 151-170.
7. P. Balasz, M. Speckbacher, *Frames, their relatives and reproducing kernel Hilbert spaces*, J. Phys. A: Math. Theor., **53**, (2020), 015204.
8. G. Bellomonte, C.Trapani, *Riesz-like bases in Rigged Hilbert Spaces*, Zeitschr. Anal. Anwen., **35**, (2016), 243-265.
9. P. Casazza, O. Christensen, S. Li, A. Lindner, *Riesz-Fischer Sequences and Lower Frame Bounds*, Zeitschr. Anal. Anwen., **21** (2002), No. 2, 305-314.
10. O. Christensen, *Frames and Bases-An Introductory Course*, Birkhäuser, 2008.
11. O. Christensen, *An Introduction to Frames and Riesz Bases*, Boston, Birkhauser, 2016.
12. E. Cordero, H. Feichtinger, F. Luef, *Banach Gelfand Triples for Gabor Analysis*, Pseudodifferential operators, Lecture Notes in Math., 1949: 133, Springer, Berlin, 2008.
13. G. de Barra, *Measure theory and integration*, New Age International (P) limited Publishers, 1981.
14. H.G. Feichtinger, K. Gröchenig *Gabor Frames and Time-Frequency Analysis of Distributions*, J. Functional Anal., **146**, (1997), 464-495.
15. H.G. Feichtinger, G. Zimmermann, *A Banach space of test functions for Gabor analysis* in H.G. Feichtinger and T. Strohmer, editors, *Gabor analysis and Algorithms - Theory and Applications*, Birkähäuser Boston, 1998.
16. I.M. Gel'fand, G.E. Shilov, E. Saletan, *Generalized Functions*, Vol. III, Academic Press, New York, 1967.
17. I. M. Gel'fand, N. Ya. Vilenkin, *Generalized Functions*, Vol. IV, Academic Press, New York, 1964.
18. K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, Basel, 2001.
19. G.G. Gould, *The Spectral Representation of Normal Operators on a Rigged Hilbert Space*, J. London Math. Soc. **43** (1968), 745-754.
20. C. Heil, *A Basis Theory Primer*, Expanded Edition, Birkhuser/Springer, New York, 2011.
21. J. Horvath, *Topological Vector Spaces and Distributions*, Addison-Wesley, 1966.

22. H. Hosseini Giv, M. Radjabalipour, *On the structure and properties of lower bounded analytic frames*, Iran. J. Sci. Technol., 37A3, (2013), 227-230.
23. M. S. Jakobsen, J. Lemvig, *Density and duality theorems for regular Gabor frames* J. Functional Anal., **270**,1, (2016), 229-263.
24. G. Kaiser, *A friendly guide to wavelets*, Birkhäuser, Boston, 1994.
25. G. Kyriazis, P. Petrushev, *On the construction of frames for spaces of distributions*, J. Functional Anal., **257**, (2009), 2159-2187.
26. S. Pilipovic, D. T. Stoeva, *Fréchet frames, general definition and expansion*, Analysis and Appl., **12**, No. 02, pp. 195-208, (2014).
27. M. Reed, B. Simon, *Methods of modern mathematical physics*, Vol.I, II, Academic Press, New York, 1980.
28. W. Rudin, *Real and Complex analysis*, McGraw-Hill, 1987.
29. C. Trapani, S. Triolo, F. Tschinke, *Distribution Frames and Bases*, J. Fourier Anal. and Appl., **25**, (2019), pp. 2109-2140.
30. R. M. Young, *An Introduction to Non-Harmonic Fourier Series*, Revised Edition, 2nd Edition, Academic Press, (2001).

Francesco Tschinke
Università di Palermo,
I-90123 Palermo (Italy)
e-mail: `francesco.tschinke@unipa.it`