Mediterr. J. Math. (2025) 22:68
https://doi.org/10.1007/s00009-025-02816-3
1660-5446/25/030001-20
*published online* April 10, 2025
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Mediterranean Journal of Mathematics

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# Further Results on Two Stronger Variants of Weyl's Theorem

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**Abstract.** In this paper, we give some results on two new variants of the classical Weyl type theorems for operators defined on Banach spaces. We investigate the relationship between these variants, and we also give some perturbation results. In the last part the theory is applied to Toeplitz operators on Hardy spaces with continuous symbol.

**Mathematics Subject Classification.** Primary 47A10, 47A11; Secondary 47A53, 47A55.

**Keywords.** $S$-Weyl's theorem, property $(t)$, property $(gaz)$, Toeplitz operators on Hardy spaces.

## 1. Introduction

A well known result, due to Weyl [18], concerning the spectral structure of self-adjoint operators $T$ on a Hilbert space, states that the spectral points $\lambda$ of $T$ for which $\lambda I - T$ is a Fredholm operator having index 0, are exactly all isolated points of the spectrum that are eigenvalues having finite multiplicity. A bounded linear operator $T \in L(X)$ on an infinite dimensional Banach space $X$, whose spectrum has the structure above mentioned for self-adjoint operators, is said, in the modern literature, to satisfy Weyl's theorem. The classical Weyl's theorem admits some stronger versions, the most important are the $a$-Weyl's theorem and property $(w)$, studied in several papers by different authors, see the recent monograph [2]. In this paper, we consider other two variants of Weyl's theorem, the first one, called *$S$-Weyl's theorem* , has been introduced in [4], the second one, called *property* $(t)$, has been introduced in [16]. Property $(t)$ entails both $a$-Weyl theorem and property $(w)$, while $S$-Weyl's theorem entails property $(t)$, but the converse of these implications are, in general, not true. For instance every finite-dimensional operator $T$ satisfies property $(t)$, but $S$-Weyl's theorem fails for $T$, see next Theorem 4.10. In this paper, we also establish some perturbation results concerning the permanence of $S$-Weyl's theorem and property $(t)$, under commuting

Riesz operators, or under commuting operators $K$ for which $K^n$ is finite-dimensional for some $n \in \mathbb{N}$. We also consider compact perturbations (not necessarily commuting) in the case where $T$ is a Hilbert space operator. Our theory is, in the last part, applied to some Toeplitz operators $T_\phi$ on Hardy spaces with continuous symbol $\phi$.

## 2. Definitions and Preliminary Results

A Banach space operator $T \in L(X)$, is *semi-Fredholm* if it is either upper Fredholm or lower Fredholm, where $T$ is *upper Fredholm*, $T \in \Phi_+(X)$, if $T(X)$ is closed and $\alpha(T) := \dim \ker(T) < \infty$, and $T$ is *lower Fredholm*, $T \in \Phi_-(X)$, if $\beta(T) = \dim(X/T(X)) < \infty$. $T$ is said to be *Fredholm*, $T \in \Phi(X)$, if $T \in \Phi_+(X) \cap \Phi_-(X)$. The *semi-Fredholm index* of $T$ is the (finite or infinite) integer $\operatorname{ind}(T) := \alpha(T) - \beta(T)$. The class of *Weyl operators* is defined by $W(X) := \{T \in \Phi(X) : \operatorname{ind} T = 0\}$, the class of *upper semi-Weyl operators* is defined by $W_+(X) := \{T \in \Phi_+(X) : \operatorname{ind} T \leq 0\}$, while the class of *lower semi-Weyl operators* is defined by $W_-(X) := \{T \in \Phi_-(X) : \operatorname{ind} T \geq 0\}$. Clearly, $W(X) = W_+(X) \cap W_-(X)$. These classes of operators generate the following spectra: the *Weyl spectrum* defined by $\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\}$, and analogously are defined the *upper semi-Weyl spectrum* $\sigma_{\mathrm{uw}}(T)$, and the *lower semi-Weyl spectrum* $\sigma_{\mathrm{lw}}(T)$.

The ascent of $T$, $p(T)$, is the least nonnegative integer $n$ such that $\ker T^n = \ker T^{n+1}$ while the descent of $T$, $q(T)$, is the least nonnegative integer $n$ such that $T^n(X) = T^{n+1}(X)$. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$. Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ if and only if $\lambda$ is a pole of the resolvent, see [14, Proposition 50.2].

The class of all *Browder operators* is defined as the set $B(X) := \{T \in \Phi(X) : p(T), q(T) < \infty\}$, the class of all *upper semi-Browder operators* is defined as $B_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\}$. Obviously, $B(X) \subseteq W(X)$ and $B_+(X) \subseteq W_+(X)$. The class of Browder operators generates the *Browder spectrum*, defined by $\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B(X)\}$, and analogously is defined the *upper semi-Browder spectrum* $\sigma_{\mathrm{ub}}(T)$. Clearly, $\sigma_{\mathrm{uw}}(T) \subseteq \sigma_{\mathrm{ub}}(T)$ and $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$.

Recall that $R \in L(X)$ is said to be *Riesz* if $\lambda I - R \in \Phi(X)$ for every $\lambda \neq 0$, or equivalently, $\lambda I - R \in W_+(X)$ for every $\lambda \neq 0$, see [2, Theorem 3.2]. The spectrum of a Riesz operator is finite, or consists of a sequence of eigenvalues (of finite multiplicity) which clusters at 0. It is well known that the Weyl spectra and the Browder spectra are stable under Riesz commuting perturbations, see [2, Chapter 3].

In the sequel we denote by $\sigma_{\mathrm{ap}}(T)$ the classical *approximate point spectrum*, and by $\sigma_{\mathrm{s}}(T)$ the *surjectivity spectrum*. Evidently, $\sigma_{\mathrm{uw}}(T) \subseteq \sigma_{\mathrm{ap}}(T)$ and $\sigma_{\mathrm{ap}}(T^*) = \sigma_{\mathrm{s}}(T)$. An useful tool for studying Weyl type theorems is given by *the single valued extension property* at a point $\lambda_0 \in \mathbb{C}$, (abbreviated SVEP at $\lambda_0$). Recall that $T \in L(X)$ has SVEP at $\lambda_0$ if for every open disk $U$ of $\lambda_0$, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$. An operator $T \in L(X)$

is said to have SVEP if $T$ has SVEP at every point $\lambda \in \mathbb{C}$. Evidently, an operator $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbf{C}\backslash\sigma(T)$, and both $T$ and $T^*$ have SVEP at the isolated points of the spectrum. In [6] it shown that the localized SVEP is stable under Riesz commuting perturbation $R$, i.e., $T$ has SVEP at $\lambda \Leftrightarrow\ T + R$ has SVEP at $\lambda$. Furthermore, if $T$ has SVEP then $\sigma(T) = \sigma_{\rm s}(T)$, and dually if $T^*$ has SVEP then $\sigma(T) = \sigma_{\rm ap}(T)$, see [2, Theorem 2.68]. Note that

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda,$$

and dually

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda.$$

These implications are equivalences if $\lambda I - T \in \Phi_{\pm}(X)$, see [2, Chapter 2].

Semi-Fredholm operators have been generalized by Berkani ([10], and [11]) in the following way: for every $T \in L(X)$ and a nonnegative integer $n$ let us denote by $T_{[n]}$ the restriction of $T$ to $T^n(X)$, viewed as a map from the space $T^n(X)$ into itself (we set $T_{[0]} = T$). $T \in L(X)$ is said to be *semi B-Fredholm*, (respectively. *B-Fredholm, upper semi B-Fredholm*, *lower semi B-Fredholm,*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). In this case $T_{[m]}$ is a semi-Fredholm operator for all $m \geq n$ with the same index of $T_{[n]}$. This enables one to define the index of a semi B-Fredholm as $\text{ind } T = \text{ ind } T_{[n]}$.

A bounded operator $T \in L(X)$ is said to be *B-Weyl* (respectively, *upper semi B-Weyl*, *lower semi B-Weyl*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is Weyl (respectively, upper semi-Weyl. lower semi-Weyl). The *B-Weyl spectrum* is defined by $\sigma_{\rm bw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Weyl}\}$, and analogously are defined the *upper semi B-Weyl spectrum* $\sigma_{\rm ubw}(T)$ and the *lower semi B-Weyl spectrum* $\sigma_{\rm lbw}(T)$.

The concept of Drazin invertibility has been introduced in a more abstract setting than operator theory. In the case of the Banach algebra $L(X)$, $T \in L(X)$ is said to be *Drazin invertible* (with a finite index), $T \in D(X)$, if $p(T) = q(T) < \infty$. Clearly, $\lambda I - T \in L(X)$ is Drazin invertible if and only if $\lambda I - T$ is invertible or $\lambda$ is a pole of the resolvent. Drazin invertibility for bounded operators suggests the following definitions:

**Definition 2.1.** An operator $T \in L(X)$, is said to be *left Drazin invertible*, $T \in LD(X)$, if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed. $T \in L(X)$, is said to be *right Drazin invertible*, $T \in RD(X)$, if $q := q(T) < \infty$ and $T^q(X)$ is closed. If $\lambda I - T$ is left Drazin invertible and $\lambda \in \sigma_{\rm ap}(T)$ then $\lambda$ is said to be a *left pole*. A left pole $\lambda$ is said to have *finite rank* if $\alpha(\lambda I - T) < \infty$. If $\lambda I - T$ is right Drazin invertible and $\lambda \in \sigma_{\rm s}(T)$ then $\lambda$ is said to be a *right pole*. A right pole $\lambda$ is said to have finite rank if $\beta(\lambda I - T) < \infty$.

Evidently, $T \in L(X)$ is Drazin invertible if and only if $T$ is both left Drazin invertible and right Drazin invertible. The *Drazin spectrum* is defined as $\sigma_{\rm d}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin D(X)\}$. and analogously, are defined the *left Drazin spectrum* $\sigma_{\rm ld}(T)$, and the *right Drazin spectrum* $\sigma_{\rm rd}(T)$. We have $\sigma_{\rm ld}(T) \cup \sigma_{\rm rd}(T) = \sigma_{\rm d}(T)$. Every left Drazin invertible operator is upper semi

B-Weyl and every right Drazin invertible operator is lower semi B-Weyl, see [2, Theorem 3.47], so we have $\sigma_{\rm bw}(T) \subseteq \sigma_{\rm d}(T)$ and

$$\sigma_{\rm ubw}(T) \subseteq \sigma_{\rm ld}(T), \quad \sigma_{\rm lbw}(T) \subseteq \sigma_{\rm rd}(T).$$

The following theorem has been proved in [2, Theorem 3.78].

**Theorem 2.2.** *Suppose that $T, K \in L(X)$ commutes and that $K^n$ is finite-dimensional for some $n \in \mathbb{N}$. Then*

$$\sigma_{\rm ld}(T+K) = \sigma_{\rm ld}(T) \quad \textit{and} \quad \sigma_{\rm d}(T+K) = \sigma_{\rm d}(T). \tag{1}$$

Denote by $\Pi(T)$ and $\Pi_a(T)$ the set of all poles and the set of left poles of $T$, respectively. Clearly, $\Pi(T) = \sigma(T)\backslash\sigma_{\rm d}(T)$ and $\Pi_a(T) = \sigma_{\rm ap}(T)\backslash\sigma_{\rm ld}(T)$. If $\operatorname{iso} A$ denotes the set of all isolated points of a subset $A \subseteq \mathbb{C}$, we have $\Pi(T) \subseteq \operatorname{iso}\sigma(T)$ and $\Pi_a(T) \subseteq \operatorname{iso}\sigma_{\rm ap}(T)$, see [2, Theorem 4.3]. The next lemma has been proved in [9, Lemma 2.4].

**Lemma 2.3.** *Let $T \in L(X)$. Then*

$$\sigma_{\rm ld}(T) = \sigma_{\rm d}(T) \Leftrightarrow \sigma_{\rm ap}(T) = \sigma(T).$$

*Analogously,*

$$\sigma_{\rm rd}(T) = \sigma_{\rm d}(T) \Leftrightarrow \sigma_{\rm s}(T) = \sigma(T).$$

## 3. Preliminary Results on the Upper Semi-Weyl Spectrum $\sigma_{\rm uw}(T)$

In the theory of Weyl type theorems the upper semi-Weyl spectrum $\sigma_{\rm uw}(T)$ has a prominent role. In this section we shall fix some preliminary properties of this spectrum that will be useful in the sequel. Let $\rho(T) := \mathbb{C}\setminus\sigma(T)$ denote the resolvent of $T$ and consider the *semi-Fredholm region* of $T$, defined:

$$\rho_{\rm sf}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi_{\pm}(X)\}.$$

The sets $\rho_{\rm sf}(T)$, $\rho_{\rm w}(T) := \mathbb{C}\backslash\sigma_{\rm w}(T)$, $\rho_{\rm uw}(T) := \mathbb{C}\backslash\sigma_{\rm uw}(T)$, and $\rho_{\rm ap}(T) := \mathbb{C}\backslash\sigma_{\rm ap}(T)$ are nonempty open subsets of $\mathbb{C}$, so they can be decomposed into components, i.e., maximal open, connected, pairwise disjoint non-empty subsets of $\mathbb{C}$. If $T \in L(X)$, set $p_{00}(T) := \sigma(T)\backslash\sigma_{\rm b}(T)$, $p_{00}^a(T) := \sigma_{\rm ap}(T)\backslash\sigma_{\rm ub}(T)$, $\rho_{\rm ap}(T) := \mathbb{C}\backslash\sigma_{\rm ap}(T)$ and $\rho_{\rm uw}(T) := \mathbb{C}\backslash\sigma_{\rm uw}(T)$.

Recall that given a compact set $\sigma \subset \mathbb{C}$, a *hole* of $\sigma$ is a bounded component of the complement $\mathbb{C}\backslash\sigma$. Since $\mathbb{C}\backslash\sigma$ has always an unbounded component, then $\mathbb{C}\setminus\sigma$ is connected precisely when $\sigma$ has no hole.

**Theorem 3.1.** *If $\sigma_{\rm uw}(T)$ has no hole then $\sigma_{\rm uw}(T) = \sigma_{\rm w}(T)$. Furthermore,*

$$\rho_{\rm uw}(T) = \rho_{\rm ap}(T) \cup p_{00}^a(T) = \rho(T) \cup p_{00}(T).$$

*Proof.* If $\sigma_{\rm uw}(T)$ has no hole then $\rho_{\rm uw}(T)$ is connected, so, by [7, Lemma 3.5], we have $\sigma_{\rm uw}(T) = \sigma_{\rm w}(T)$. The equality $\rho_{\rm uw}(T) = \rho_{\rm ap}(T) \cup p_{00}^a(T)$ has been shown in [7, Lemma 3.2]. From $\rho_{\rm uw}(T) = \rho_{\rm w}(T)$ it follows that $\rho_{\rm w}(T)$ is connected, hence, again from [7, Lemma 3.2], we have $\rho_{\rm uw}(T) = \rho(T) \cup p_{00}(T)$ □

**Example 3.2.** The assumption that $\sigma_{\rm w}(T)$ has no hole does not imply that $\sigma_{\rm uw}(T) = \sigma_{\rm w}(T)$. For instance, let $T \in \ell^p(\mathbb{N})$ be the unilateral right shift, defined by

$$Tx := \sum_{n=1}^{\infty} x_n e_{n+1} \quad \text{for all } x = (x_n) \in \ell^p(\mathbb{N}).$$

The hyper-range $T^\infty(X) := \bigcap_{n=1}^{\infty} T^n(X)$ is $\{0\}$, so, by [2, Theorem 4.83], $\sigma(T) = \sigma_{\rm w}(T) = \mathbf{D}(0,1)$, the unit disk, while $\sigma_{\rm ap}(T)$ is the unit circle $\mathbf{\Gamma}$. From the inclusion $\sigma_{\rm uw}(T) \subseteq \sigma_{\rm ap}(T)$, we see that $\sigma_{\rm uw}(T) \subseteq \mathbf{\Gamma}$. Let $\lambda \in \mathbf{\Gamma}$, and suppose that $\lambda \notin \sigma_{\rm uw}(T)$. It is well-known that every unilateral right shift has SVEP, so, being $\lambda I - T \in W_+(X)$, we have $p(\lambda I - T) < \infty$ and hence $\lambda \in B_+(X)$, so $\lambda I - T$ is left Drazin invertible. Since $\lambda \in \sigma_{\rm ap}(T)$ then $\lambda$ is a left pole of $T$, consequently an isolated point of $\sigma_{\rm ap}(T) = \mathbf{\Gamma}$, see [2, Theorem 4.3], and this is impossible. Therefore, $\mathbf{\Gamma} = \sigma_{\rm uw}(T) \neq \sigma_{\rm w}(T)$.

An operator $T \in L(X)$ is said to satisfy *Browder's theorem* if $\sigma_{\rm w}(T) = \sigma_{\rm b}(T)$, or equivalently $T$ has SVEP at every $\lambda \notin \sigma_{\rm w}(T)$, and $T \in L(X)$ is said to satisfy *a-Browder's theorem* if $\sigma_{\rm uw}(T) = \sigma_{\rm ub}(T)$, or equivalently $T$ has SVEP at every $\lambda \notin \sigma_{\rm uw}(T)$, see [2, Chapter 5]. It is known that $a$-Browder's theorem entails Browder's theorem. An operator $T \in L(X)$ is said to satisfy the *generalized Browder's theorem* if $\sigma_{\rm bw}(T) = \sigma_{\rm d}(T)$, while $T \in L(X)$ is said to satisfy the *generalized a-Browder's theorem* if $\sigma_{\rm ubw}(T) = \sigma_{\rm ld}(T)$. Browder's theorem and generalized Browder's theorem are equivalent, and analogously $a$-Browder's theorem and generalized $a$-Browder's theorem are equivalent, see [2, Chapter 5]. In the sequel we shall need the following result, see [7, Theorem 3.4].

**Theorem 3.3.** *Let $T \in L(X)$. Then we have:*

(i) *$\sigma_{\rm w}(T)$ has no hole if and only if $\sigma(T)$ has no hole and $T$ satisfies Browder's theorem.*
(ii) *$\sigma_{\rm uw}(T)$ has no hole if and only if $\sigma_{\rm ap}(T)$ has no hole and $T$ satisfies a-Browder's theorem.*

**Theorem 3.4.** *Suppose that $\sigma_{\rm uw}(T)$ has no hole. Then $\sigma_{\rm ap}(T) = \sigma(T)$, or equivalently $\sigma_{\rm ld}(T) = \sigma_{\rm d}(T)$. Dually, if $\sigma_{\rm lw}(T)$ has no hole then $\sigma_{\rm s}(T) = \sigma(T)$, or equivalently $\sigma_{\rm rd}(T) = \sigma_{\rm d}(T)$.*

*Proof.* If $\sigma_{\rm uw}(T)$ has no hole then, by part (ii) of Theorem 3.3, $T$ satisfies $a$-Browder's theorem and $\sigma_{\rm ap}(T)$ has no hole. By Theorem 3.1 we also have $\sigma_{\rm uw}(T) = \sigma_{\rm w}(T)$, so $\sigma_{\rm w}(T)$ has no hole, and hence, from part (i) of Theorem 3.3, $\sigma(T)$ has no hole, i.e., the resolvent $\rho(T)$ is connected. Let $\Omega$ denote the unique unbounded connected component of $\rho(T)$. Evidently, since $\rho(T) \subseteq \rho_{\rm ap}(T)$, we have

$$\Omega \subseteq \rho_{\rm ap}(T) \subseteq \rho_{\rm sf}(T).$$

Trivially, $\Omega$ is contained in the unique unbounded component $\Lambda$ of $\rho_{\rm ap}(T)$. Now, if $\lambda \notin \sigma_{\rm ap}(T)$ then $\lambda \in \Lambda$. Since for $\mu \in \Lambda$ sufficiently large, $T^*$ has SVEP at $\mu$ then $T^*$ has SVEP at every point of $\Lambda$, see [1, Theorem 3.36], and

in particular, $q(\mu I - T) < \infty$ for every $\mu \in \Lambda$. In particular, $q(\lambda I - T) < \infty$ and this implies that

$$\beta(\lambda I - T) \leq \alpha(\lambda I - T) = 0,$$

see [2, Theorem 1.22]. Therefore, $\lambda \notin \sigma(T)$, and hence $\sigma_{\rm ap}(T) = \sigma(T)$. The last statement follows from duality, if $\sigma_{\rm lw}(T) = \sigma_{\rm uw}(T^*)$ has no hole then $\sigma(T^*) = \sigma_{\rm ap}(T^*)$ and hence $\sigma(T) = \sigma(T^*) = \sigma_{\rm s}(T)$. □

The isolated points of the Weyl spectra are related as follows, see [2, Theorem 3.58]:

**Lemma 3.5.** *If $T \in L(X)$, then we have:*

$$iso\ \sigma_{\rm w}(T) \subseteq iso\ \sigma_{\rm uw}(T) \cap iso\ \sigma_{\rm lw}(T) \subseteq iso\ \sigma_{\rm sf}(T).$$

An operator $T \in L(X)$ is said to be *isoloid* if every $\lambda \in {\rm iso}\,\sigma(T)$ is an eigenvalue. $T \in L(X)$ is said to be *finite-isoloid* if every $\lambda \in {\rm iso}\,\sigma(T)$ is an eigenvalue having finite multiplicity.

**Lemma 3.6.** *Let $T \in L(X)$ and suppose that $iso\,\sigma_{\rm uw}(T) = \emptyset$. Then $T$ is finite-isoloid and $iso\ \sigma_{\rm ap}(T) \subseteq \rho_{\rm sf}(T)$.*

*Proof.* Let $\lambda_0 \in {\rm iso}\,\sigma(T)$ be arbitrary. Then there is $\varepsilon > 0$ such that $\lambda I - T$ is invertible for all $0 < |\lambda - \lambda_0| < \varepsilon$, in particular $\lambda I - T \in W_+(X)$ for all $0 < |\lambda - \lambda_0| < \varepsilon$. Now, we have either $\lambda_0 \in \sigma_{\rm uw}(T)$ or $\lambda_0 \notin \sigma_{\rm uw}(T)$. If were $\lambda_0 \in \sigma_{\rm uw}(T)$ then we would have $\lambda_0 \in {\rm iso}\,\sigma_{\rm uw}(T) \neq \emptyset$, a contradiction. Therefore, $\lambda_0 \notin \sigma_{\rm uw}(T)$, i.e., $\lambda_0 I - T \in W_+(X)$. We have $\lambda_0 \in \sigma_{\rm ap}(T)$, since every isolated point of the spectrum belongs to $\sigma_{\rm ap}(T)$. Since $T$ has SVEP at every isolated point of $\sigma_{\rm ap}(T)$, see [2, Chapter 2], and $\lambda_0 I - T \in W_+(X)$, it then follows that $p(\lambda_0 I - T) < \infty$, see [2, Theorem 2.97], hence $\lambda_0 I - T \in B_+(X)$. But $(\lambda_0 I - T)(X)$ is closed, so the condition $\lambda_0 \in \sigma_{\rm ap}(T)$ entails that $\alpha(\lambda_0 I - T) > 0$, thus $\lambda$ is an eigenvalue of $T$. We also have $\alpha(\lambda_0 I - T) < \infty$, since $\lambda_0 I - T \in B_+(X)$, so $T$ is finite-isoloid.

To show the inclusion ${\rm iso}\ \sigma_{\rm ap}(T) \subseteq \rho_{\rm sf}(T)$, let $\lambda_0 \in {\rm iso}\ \sigma_{\rm ap}(T)$. Then there exists $\delta > 0$ such that $\lambda I - T$ is bounded below for all $0 < |\lambda - \lambda_0| < \delta$, in particular $\lambda I - T \in W_+(X)$ for all $0 < |\lambda - \lambda_0| < \delta$. Suppose that $\lambda_0 \notin \rho_{\rm sf}(T)$. Then $\lambda_0 \in \sigma_{\rm sf}(T) \subseteq \sigma_{\rm uw}(T)$, hence $\lambda_0 \in {\rm iso}\,\sigma_{\rm uw}(T)$, and this is impossible. □

**Remark 3.7.** The result of Lemma 3.6 cannot be reversed, i.e., $T$ finite-isoloid does not imply that ${\rm iso}\,\sigma_{\rm uw}(T) = \emptyset$. For instance, a Riesz operator $T$ having infinite spectrum is obviously finite-isoloid. Indeed, $\sigma_{\rm uw}(T) = \{0\}$, since $\lambda I - T \in W_+(X)$ for all $\lambda \neq 0$, and obviously ${\rm iso}\,\sigma_{\rm uw}(T) \neq \emptyset$.

Let us now consider an analytic function defined on an open disk containing the spectrum $\sigma(T)$. Then $f(T)$ is defined by the classical functional calculus. The set of all analytic functions defined on an open disk containing the spectrum $\sigma(T)$ will be denoted by $\mathcal{H}(\sigma(T))$. Recall that the spectral mapping theorem holds for $\sigma_{\rm ap}(T)$ and $\sigma_{\rm s}(T)$, i.e., for every $f \in \mathcal{H}(\sigma(T))$ we have

$$\sigma_{\rm ap}(f(T)) = f(\sigma_{\rm ap}(T)) \quad \text{and} \quad \sigma_{\rm s}(f(T)) = f(\sigma_{\rm s}(T)).$$

In general the spectral mapping theorem does not hold for the Weyl spectra and the $B$-Weyl spectra. However, if either $T$ or $T^*$ has SVEP, then the spectral mapping theorem holds for the Weyl and the $B$-Weyl spectra, see [2, Corollary 3.120 and Corollary 3.123].

## 4. S-Weyl's Theorem and Property $(t)$

Define $\Delta_a^g(T) := \sigma_{\rm ap}(T)\backslash\sigma_{\rm ubw}(T)$ and $\Delta^g(T) := \sigma(T)\backslash\sigma_{\rm ubw}(T)$. Since $\sigma_{\rm ubw}(T) \subseteq \sigma_{\rm ld}(T)$, we have

$$\Pi_a(T) = \sigma_{\rm ap}(T)\backslash\sigma_{\rm ld}(T) \subseteq \Delta_a^g(T) \subseteq \Delta^g(T).$$

It is easy to check that $p_{00}(T) \subseteq p_{00}^a(T)$ for all $T \in L(X)$. Obviously every point of $p_{00}(T)$ is an isolated point of $\sigma(T)$, and hence an isolated point of $\sigma_{\rm ap}(T)$. Define

$$\pi_{00}(T) := \{\lambda \in \text{ iso } \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\},$$

and

$$\pi_{00}^a(T) := \{\lambda \in \text{ iso } \sigma_{\rm ap}(T) : 0 < \alpha(\lambda I - T) < \infty\}.$$

It is easily seen that

$$p_{00}(T) \subseteq \pi_{00}(T) \subseteq \pi_{00}^a(T) \quad \text{and} \quad p_{00}^a(T) \subseteq \pi_{00}^a(T). \tag{2}$$

**Definition 4.1.** An operator $T \in L(X)$ is said to verify *property (gaz)* if $\Delta^g(T) = \Pi_a(T)$.

Property $(gaz)$ entails remarkable spectral equalities:

**Theorem 4.2.** *Let $T \in L(X)$. Then the following statements are equivalent:*

(i) *$T$ has property $(gaz)$;*
(ii) *$a$-Browder's theorem holds and $\sigma_{\rm ap}(T) = \sigma(T)$, or equivalently $\sigma_{\rm ld}(T) = \sigma_{\rm d}(T)$;*
(iii) $\sigma(T) \setminus \sigma_{\rm uw}(T) = p_{00}^a(T)$.

*Furthermore, if $T$ has property $(gaz)$ then*

$$\sigma_{\rm uw}(T) = \sigma_{\rm w}(T) = \sigma_{\rm ub}(T) = \sigma_{\rm b}(T), \tag{3}$$

*and*

$$\sigma_{\rm ubw}(T) = \sigma_{\rm bw}(T) = \sigma_{\rm ld}(T) = \sigma_{\rm d}(T). \tag{4}$$

*Consequently, $\Pi(T) = \Pi_a(T)$ and $p_{00}(T) = p_{00}^a(T)$.*

*Proof.* The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) have been shown in [9]. The equalities (3) and (4) have been proved in [4, Theorem 3.2]. □

An easy consequence of Theorem 4.2 is that, if $T^*$ has SVEP then property $(gaz)$ holds for $T$. Indeed, as observed before, the SVEP for $T^*$ entails that $T$ satisfies $a$-Browder's theorem and that $\sigma_{\rm ap}(T) = \sigma(T)$. Analogously, if $T$ has SVEP then property $(gaz)$ holds for $T^*$.

The classical Weyl type theorems are defined as follows, see [2, Chapter 6] for details.

**Definition 4.3.** A bounded operator $T \in L(X)$ is said to satisfy *Weyl's theorem* if $\sigma(T)\backslash\sigma_{\rm w}(T) = \pi_{00}(T)$, $T$ is said to satisfy *a-Weyl's theorem* if $\sigma_{\rm ap}(T)\backslash\sigma_{\rm uw}(T) = \pi^a_{00}(T)$, $T$ is said to satisfy property $(\omega)$ if $\sigma_{\rm ap}(T)\backslash\sigma_{\rm uw}(T) = \pi_{00}(T)$.

Both $a$-Weyl's theorem and property $(\omega)$ entails Weyl's theorem. We now define other two stronger variants of Weyl's theorem, the first one introduced in [4], the second one introduced in [16].

**Definition 4.4.** A bounded operator $T \in L(X)$ is said to satisfy the *S-Weyl's theorem*, abbreviated $(SW)$, if $\sigma(T)\backslash\sigma_{\rm ubw}(T) = \pi_{00}(T)$. $T \in L(X)$ is said to satisfy *property* $(t)$ if $\sigma(T)\backslash\sigma_{\rm uw}(T) = \pi_{00}(T)$.

**Remark 4.5.** Property $(t)$ entails that $\sigma_{\rm ap}(T) = \sigma(T)$. Indeed, let $\lambda \notin \sigma_{\rm ap}(T)$. Clearly, $\lambda \notin \sigma_{\rm uw}(T)$. Suppose that $\lambda \in \sigma(T)$. Then $\lambda \in \sigma(T)\backslash\sigma_{\rm uw}(T) = \pi_{00}(T)$, so $\alpha(\lambda I - T) < \infty$. Since $\lambda \in \pi_{00}(T)$ then $\lambda \in {\rm iso}\,\sigma(T)$, so both $T$ and $T^*$ have SVEP at $\lambda$. Since $\lambda I - T \in W_+(X)$ it then follows that $\lambda I - T$ is Browder, see [2, Theorem 2.100]. In particular, $\lambda$ is a pole of the resolvent, and hence an eigenvalue of $T$. This contradicts our assumption that $\lambda \notin \sigma_{\rm ap}(T)$. Therefore, $\sigma(T) \subseteq \sigma_{\rm ap}(T)$, and since the opposite inclusion is obvious, it then follows that $\sigma_{\rm ap}(T) = \sigma(T)$.

Examples of operators for which $S$-Weyl's theorem holds are Riesz operators having infinite spectrum. Other important examples are given by the dual $T^*$ of a non-invertible symmetry $T$ which is not quasi-nilpotent, or by every unilateral weighted left shift on $\ell^p(\mathbb{N})$, see [4].

**Theorem 4.6.** *Suppose that* ${\rm iso}\,\sigma_{\rm uw}(T) = \emptyset$. *Then S-Weyl's theorem holds for* $T$ *if and only if* $T$ *has property* $(t)$.

*Proof.* Note first that the equality

$$\sigma_{\rm uw}(T) = \sigma_{\rm ubw}(T) \cup {\rm iso}\,\sigma_{\rm uw}(T) \tag{5}$$

holds for every operator. Indeed, let $\lambda_0 \in \sigma_{\rm uw}(T)\backslash\sigma_{\rm ubw}(T)$. Since $\lambda_0 I - T$ is upper semi-$B$-Weyl then there exists an open disk $\mathbb{D}$ centered at $\lambda_0$ such that $\lambda I - T$ is upper-Weyl for every $\lambda \in \mathbb{D}\backslash\{\lambda_0\}$, see [2, Theorem 1.117]. Since $\lambda_0 \in \sigma_{\rm uw}(T)$ it then follows that $\lambda_0 \in {\rm iso}\,\sigma_{\rm uw}(T)$. Hence the equality (5) holds. The condition ${\rm iso}\,\sigma_{\rm uw}(T) = \emptyset$ then implies that $\sigma_{\rm uw}(T) = \sigma_{\rm ubw}(T)$, thus $S$-Weyl's theorem for $T$ and property $(t)$ are equivalent. □

**Theorem 4.7.** *If* $T$ *has property* $(t)$ *then property* $(gaz)$ *holds for* $T$.

*Proof.* Trivially, $p^a_{00}(T) \subseteq \sigma_{\rm ap}(T)\backslash\sigma_{\rm uw}(T) \subseteq \sigma(T)\backslash\sigma_{\rm uw}(T)$, so we need only to prove, by Theorem 4.2, the inclusion $\sigma(T)\backslash\sigma_{\rm uw}(T) \subseteq p^a_{00}(T)$. Let $\lambda \in \sigma(T)\backslash\sigma_{\rm uw}(T)$. Since $T$ has property $(t)$ then $\lambda \in \pi_{00}(T)$, so $\lambda \in {\rm iso}\,\sigma(T)$ and hence $T$ has SVEP at $\lambda$. This implies that $p(\lambda I - T) < \infty$, so $\lambda I - T \in B_+(X)$ and hence $\lambda \notin p^a_{00}(T)$. □

**Remark 4.8.** The property $(t)$ entails the equalities $\sigma_{\rm ap}(T) = \sigma(T)$ and $\pi^a_{00}(T) = \pi_{00}(T)$, so we have

$$\pi^a_{00}(T) = \pi_{00}(T) = \sigma_{\rm ap}(T)\backslash\sigma_{\rm uw}(T),$$

from which we conclude that property $(t)$ for $T$ entails that property $(\omega)$ and $a$-Weyl's theorem hold for $T$.

The next theorem clarifies the relationships between $S$-Weyl's theorem and the properties $(t)$ and $(gaz)$.

**Theorem 4.9.** *If $T \in L(X)$, let us consider the following statements:*

(i) *$T$ satisfies $S$-Weyl's theorem;*
(ii) *$T$ satisfies property $(gaz)$ and $\Pi_a(T) = \pi_{00}(T)$;*
(iii) *$T$ has property $(t)$.*

*Then*

$$\text{(i)} \Leftrightarrow \text{(ii)} \Rightarrow \text{(iii)}.$$

*Furthermore, if $T$ is finite-isoloid then* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

*Proof.* (i) $\Rightarrow$ (ii) Suppose that $T$ satisfies $S$-Weyl's theorem, i.e., $\sigma(T)\backslash\sigma_{\mathrm{ubw}}(T) = \pi_{00}(T)$. We show first that $\sigma_{\mathrm{ap}}(T) = \sigma(T)$. Let $\lambda \notin \sigma_{\mathrm{ap}}(T)$. Then $\lambda I - T$ is bounded below, in particular $\lambda I - T \in W_+(X)$, hence $\lambda \notin \sigma_{\mathrm{ubw}}(T)$. If were $\lambda \in \sigma(T)$ then $S$-Weyl's theorem entails that $\lambda \in \pi_{00}(T)$, so $\alpha(\lambda I - T) > 0$, a contradiction since $\lambda I - T$ is bounded below. Hence $\sigma_{\mathrm{ap}}(T) = \sigma(T)$.

To prove that $T$ has property $(gaz)$ it suffices, by Theorem 4.2, to show that $T$ satisfies $a$-Browder's theorem, or equivalently generalized $a$-Browder's theorem, i.e., that $T$ has SVEP at every $\lambda \notin \sigma_{\mathrm{ubw}}(T)$.

Let $\lambda \notin \sigma_{\mathrm{ubw}}(T)$ be arbitrarily given. Then $\lambda \notin \sigma(T)$ or $\lambda \in \sigma(T)$. If $\lambda \notin \sigma(T)$ then, trivially, $T$ has SVEP at $\lambda$. Suppose that $\lambda \in \sigma(T)$. Then $\lambda \in \sigma(T)\backslash\sigma_{\mathrm{ubw}}(T) = \pi_{00}(T)$, so $\lambda$ is an isolated point of $\sigma(T)$, thus $T$ has SVEP at $\lambda$. The equality $\Pi_a(T) = \pi_{00}(T)$ is immediate, from the definition of $S$-Weyl's theorem.

(ii) $\Rightarrow$ (i) Assume that $T$ satisfies property $(gaz)$ and $\Pi_a(T) = \pi_{00}(T)$. Then, by Theorem 4.2, we have

$$\sigma(T)\backslash\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{ap}}(T)\backslash\sigma_{\mathrm{ld}}(T) = \Pi_a(T) = \pi_{00}(T),$$

thus $T$ satisfies $S$-Weyl's theorem.

(ii) $\Rightarrow$ (iii) Suppose that $T$ has property $(gaz)$ and $\Pi_a(T) = \pi_{00}(T)$. Property $(gaz)$ entails that $\sigma_{\mathrm{ap}}(T) = \sigma(T)$, thus $\pi_{00}(T) = \pi^a_{00}(T)$. By Lemma 2.3 we have $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{d}}(T)$. We show that $p^a_{00}(T) = \pi_{00}(T)$. From (2) we know that $p^a_{00}(T) \subseteq \pi^a_{00}(T) = \pi_{00}(T)$. To show the reverse inclusion, let $\lambda \in \pi_{00}(T)$. Then, by Lemma 2.3, $\lambda \in \Pi_a(T) = \sigma_{\mathrm{ap}}(T)\backslash\sigma_{\mathrm{ld}}(T) = \sigma(T)\backslash\sigma_{\mathrm{d}}(T)$, so $\lambda \notin \sigma_{\mathrm{d}}(T)$ and hence $p(\lambda I - T) = q(\lambda I - T) < \infty$. This implies, by [2, Theorem 122], that $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$, hence $\lambda I - T$ is Browder, in particular $\lambda \notin \sigma_{\mathrm{ub}}(T)$. Since $\lambda$ is a spectral point it then follows that $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{ub}}(T) = p^a_{00}(T)$. Therefore, $\pi_{00}(T) = p^a_{00}(T)$.

Now, by Theorem 4.2, property $(gaz)$ entails $a$-Browder's theorem, i.e., $\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{uw}}(T)$, so

$$\pi_{00}(T) = p^a_{00}(T) = \sigma(T)\backslash\sigma_{\mathrm{uw}}(T),$$

and this proves property $(t)$.

Suppose now that $T$ is finite-isoloid and $T$ has property $(t)$, i.e., $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$. We show that (ii) holds. From Theorem 4.7 we know that

$T$ has property $(gaz)$, so by the equivalence (i) $\Leftrightarrow$ (ii) it suffices to prove $\Pi_a(T) = \pi_{00}(T)$. From Theorem 4.2 we know that $\Pi_a(T) = \Pi(T)$, so every $\lambda \in \Pi_a(T)$ is a pole and hence an isolated point of $\sigma(T)$. Since $T$ is finite-isoloid is then follows that $\alpha(\lambda I - T) < \infty$. Moreover, being $\lambda$ an eigenvalue, we have $0 < \alpha(\lambda I - T)$, hence $\lambda \in \pi_{00}(T)$. This proves that $\Pi_a(T) \subseteq \pi_{00}(T)$. Conversely, if $\lambda \in \pi_{00}(T) = \sigma(T)\backslash\sigma_{\rm uw}(T)$, then $\lambda \notin \sigma_{\rm uw}(T)$ and from (3) we get $\lambda \notin \sigma_{\rm b}(T)$. Therefore, $\lambda I - T$ is Browder and hence $\lambda \in \Pi(T) = \Pi_a(T)$. □

Part (iii) improves Theorem 4.6, since, as noted in Remark 3.7, the condition to be finite-isoloid is weaker than the condition $\operatorname{iso}\sigma_{\rm uw}(T) = \emptyset$.

Recall that an operator $K \in L(X)$ is said to be *algebraic* if there exists a complex polynomial $h$ such that $h(K) = 0$. Examples of algebraic operators are nilpotent operators and operators $K$ for which $K^n$ is a finite-dimensional operator for some $n \in \mathbb{N}$. Every algebraic operator has a finite spectrum, say $\{\lambda_1, \dots, \lambda_n\}$. Moreover, $\sigma_{\rm ld}(K) = \emptyset$, i.e., every $\lambda_i \in \sigma(K)$ is a left pole, see [2, Theorem 3.93], hence $\Pi_a(T) = \sigma(T)$. The spectrum of $K^*$ is finite, so $K^*$ has SVEP and hence $\sigma_{\rm ap}(K) = \sigma(K)$ so, $\Pi_a(K) = \sigma_{\rm ap}(T)$.

**Theorem 4.10.** *Every finite dimensional operator on an infinite dimensional Banach space satisfies property $(t)$ but does not satisfy S-Weyl's theorem.*

*Proof.* Let $K$ be a finite-dimensional operator on an infinite-dimensional Banach space $X$. Evidently, $0 \in \sigma(K)$, since $K(X) \neq X$. Hence $0 < p := p(K) = q(K) < \infty$, since 0 is a pole. If were $\alpha(K) < \infty$, from the decomposition $X = \ker K^p \oplus K^p(X)$, see [2, Theorem 1.35], where $\ker K^p$ and $K^p(X)$ are both finite-dimensional, we would have that $X$ is finite-dimensional. Thus, $\alpha(K) = \infty$, and hence $0 \notin \pi_{00}(K)$. On the other hand, 0 is a left pole, so $0 \in \Pi_a(K)$, from which we conclude that $K$ does not satisfies S-Weyl's theorem.

To show that property $(t)$ holds for $K$, write $\sigma(K) = \{0, \lambda_1, \lambda_2, \dots \lambda_n\}$, where $\lambda_i \neq 0$. A finite-dimensional operator is Riesz, so $\lambda_i I - K \in W_+(X)$ for every $\lambda_i$ and hence $\sigma_{\rm uw}(K) = \{0\}$ (the last spectrum never is empty), hence

$$\sigma(K)\backslash\sigma_{\rm uw}(K) = \{\lambda_1, \lambda_2, \dots \lambda_n\}.$$

Since every $\lambda_i$ is an eigenvalue we have $0 < \alpha(\lambda_i I - K) < \infty$, so also $\pi_{00}(K) = \{\lambda_1, \lambda_2, \dots \lambda_n\}$, and hence property $(t)$ holds for $K$. □

**Theorem 4.11.** *Let $T \in L(X)$ finite-isoloid.*

(i) *If $T$ satisfies property $(gaz)$, then*

$$\sigma_{\rm ubw}(T) = \sigma_{\rm uw}(T) \quad \textit{and} \quad \sigma_{\rm bw}(T) = \sigma_{\rm w}(T). \tag{6}$$

(ii) *If $T$ satisfies property $(t)$ then $\Pi(T) = \pi_{00}(T)$.*

*Proof.* (i) Suppose that $T$ ha property $(gaz)$. Since $\sigma_{\rm ubw}(T) \subseteq \sigma_{\rm uw}(T)$, in order to prove the first equality in (6) it suffices to show the reverse inclusion.

Let $\lambda \notin \sigma_{\rm ubw}(T)$. Since $T$ has property $(gaz)$ then $T$ satisfies $a$-Browder's theorem, by Theorem 4.2, or equivalently generalized $a$-Browder's theorem, so $\sigma_{\rm ubw}(T) = \sigma_{\rm ld}(T)$. Property $(gaz)$ entails also that $\sigma_{\rm ubw}(T) = \sigma_{\rm d}(T)$,

again by Theorem 4.2. Hence $\lambda \notin \sigma_{\mathrm{d}}(T)$, so $\lambda I - T$ is Drazin invertible, thus $\lambda \in \operatorname{iso}\sigma(T)$. Since $T$ is finite-isoloid we then have $\alpha(\lambda I - T) < \infty$, so, from [2, Theorem 1.22], $\lambda I - T$ is Weyl, and hence $\lambda \notin \sigma_{\mathrm{uw}}(T)$. Therefore, $\sigma_{\mathrm{uw}}(T) \subseteq \sigma_{\mathrm{ubw}}(T)$, from which we conclude that $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ubw}}(T)$. By Theorem 4.2, we have $\Pi(T) = \Pi_a(T) = \sigma_{\mathrm{ap}}(T)\backslash\sigma_{\mathrm{ld}}(T)$, and since $a$-Browder's theorem holds we then have $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{uw}}(T)$. Hence

$$\Pi(T) = \sigma_{\mathrm{ap}}(T)\backslash\sigma_{\mathrm{uw}}(T) = \pi_{00}(T),$$

since property $(t)$ entails that $T$ satisfies property $(\omega)$. □

## 5. Perturbation Results

In this section we give some perturbation results concerning property $(t)$ and $S$-Weyl's theorem. First we note that if $\operatorname{iso}\sigma_{\mathrm{ap}}(T) = \emptyset$ then $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{ap}}(T + K)$ for every $K \in L(X)$ that commutes with $T$ and such that $K^n$ is finite-dimensional for some $n \in \mathbb{N}$, see [2, Theorem 3.29 ]. The condition $T^*$ has SVEP does not ensure that $T$ satisfies $S$-Weyl' s theorem. For instance if $T$ is finite-dimensional then $T^*$ has SVEP, while $S$-Weyl's theorem fails for $T$, by Theorem 4.10.

**Theorem 5.1.** *Let $T, K \in L(X)$ be such that $K^n$ is finite dimensional for some $n \in \mathbb{N}$ and $KT = TK$.*

(i) *If $T^*$ has SVEP and $\operatorname{iso}\sigma_{\mathrm{ap}}(T) = \emptyset$ then $T + K$ satisfies S-Weyl's theorem.*

(ii) *If $T$ has SVEP and $\operatorname{iso}\sigma_{\mathrm{s}}(T) = \emptyset$ then $T^* + K^*$ satisfies S-Weyl's theorem.*

*Proof.* (i) Since $K$ is a Riesz operator then $K^*$ is still a Riesz operator, see [2, Theorem 3.7]. The SVEP for $T^*$ entails the SVEP for $T^* + K^*$, hence $\sigma(T + K) = \sigma_{\mathrm{ap}}(T + K)$, or equivalently, by Lemma 2.3, $\sigma_{\mathrm{d}}(T + K) = \sigma_{\mathrm{ld}}(T + K)$. The SVEP for $T^* + K^*$ also entails that $a$-Browder's theorem holds for $T + K$, see [2, Chapter 5]. By Theorem 4.2 then $T + K$ satisfies property $(gaz)$. Now, since $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{ap}}(T + K)$, we have $\operatorname{iso}\sigma_{\mathrm{ap}}(T + K) = \emptyset$ and this entails that $\operatorname{iso}\sigma(T + K) = \emptyset$, since the isolated point of the spectrum belongs to the approximate point spectrum for every operator. Hence $\Pi(T + K) = \emptyset$ and $\pi_{00}(T + K) = \emptyset$. On the other hand,

$$\begin{aligned}\Pi_a(T + K) &= \sigma_{\mathrm{ap}}(T + K) \setminus \sigma_{\mathrm{ld}}(T + K) = \sigma(T + K) \setminus \sigma_{\mathrm{d}}(T + K)\\ &= \Pi(T + K) = \emptyset,\end{aligned}$$

hence $\pi_{00}(T + K) = \Pi_a(T + K)$, so $S$-Weyl's theorem holds for $T + K$, by Theorem 4.9.

(ii) The proof of the second statement is similar: if $T$ has SVEP then $T + K$ has SVEP, and this implies that $\sigma_{\mathrm{ap}}(T^* + K^*) = \sigma(T^* + K^*)$. Moreover, $a$-Browder's theorem holds for $T^* + K^*$, so $T^* + K^*$ has property $(gaz)$, By Theorem 4.2 we then have $\Pi_a(T^* + K^*) = \Pi(T^* + K^*)$. Again by [2, Theorem 3.29], the assumption

$$\emptyset = \operatorname{iso}\sigma_{\mathrm{s}}(T) = \operatorname{iso}\sigma_{\mathrm{ap}}(T^*),$$

implies $\sigma_{\rm ap}(T^*) = \sigma_{\rm ap}(T^* + K^*)$, and hence $\operatorname{iso}\sigma(T^* + K^*) = \emptyset$. Therefore,

$$\Pi(T^* + K^*) = \pi_{00}(T^* + K^*) = \emptyset.$$

By Lemma 2.3 we also have $\sigma_{\rm d}(T^* + K^*) = \sigma_{\rm ld}(T^* + K^*)$, from which we obtain

$$\Pi_a(T^* + K^*) = \Pi(T^* + K^*) = \emptyset.$$

Hence, $S$-Weyl's theorem holds for $T^* + K^*$. □

**Example 5.2.** Theorem 5.1 applies to unilateral weighted right shifts. Recall that $R \in L(\ell^p(\mathbb{N}))$ is said to be an *unilateral weighted right shifts*, $1 \leq p \leq \infty$, if

$$Rx := \sum_{n=1}^{\infty} \omega_n x_n e_{n+1} \quad \text{for all } x = (x_n) \in \ell^p(\mathbb{N}),$$

where $(e_n)$ is the standard basis of $\ell^p(\mathbb{N})$ and $(\omega_n)$ any bounded sequence of strictly positive real numbers. Suppose that $R$ is not quasi-nilpotent, i.e., the spectral radius $r(R) > 0$. For these operators the spectrum is the closed disk $\mathbf{D}(0, r(R))$, see [2, Theorem 4.83], hence $\operatorname{iso}\sigma(R)$ is empty. Every unilateral weighted right shift $R$ has SVEP, so Theorem 5.1 applies to $R^*$. The dual of $R$ is given by the *unilateral weighted left shift* defined by

$$Lx := \sum_{n=1}^{\infty} \omega_{n+1} x_{n+1}\, e_n \quad \text{for all } x := (x_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N}).$$

Hence $L + K$ satisfies $S$-Weyl's theorem for every commuting $K$ for which $K^n$ is finite dimensional.

In general $S$-Weyl's theorem is not preserved under commuting finite-dimensional perturbations. A trivial example is provided setting $T = 0$ and $K$ any finite-dimensional operator. Evidently, $T = 0$ satisfies $S$-Weyl's theorem while, according to Theorem 4.10, $T + K = K$ fails $S$-Weyl's theorem.

**Theorem 5.3.** *Let $T, K \in L(X)$, suppose that $K^n$ is finite dimensional for some $n \in \mathbb{N}$ and $TK = KT$. Then we have:*

(i) *If $T$ has property $(gaz)$ then $T + K$ has property $(gaz)$.*
(ii) *If $T$ is isoloid and $T$ has property $(t)$ then $T + K$ has property $(t)$.*
(iii) *If $T$ is finite-isoloid and satisfies $S$-Weyl's theorem then $T + K$ satisfies $S$-Weyl's theorem.*

*Proof.* (i) Note first that $K^n$ is a Riesz operator, and from [2, Theorem 3.4] it then follows that $K$ is a Riesz operator. By Theorem 4.2 then it suffices to prove that $a$-Browder's theorem holds for $T+K$ and $\sigma_{\rm ld}(T+K) = \sigma_{\rm d}(T+K)$. Property $(gaz)$ for $T$ entails that $\sigma_{\rm ld}(T) = \sigma_{\rm d}(T)$ and hence $\sigma_{\rm ld}(T + K) = \sigma_{\rm d}(T + K)$. Furthermore, property $(gaz)$ entails $a$-Browder' theorem for $T$, by Theorem 4.2, or equivalently $T$ has SVEP at every $\lambda \notin \sigma_{\rm uw}(T)$. From the stability of the localized SVEP under Riesz commuting perturbation we then obtain that $T + K$ has SVEP at every $\lambda \notin \sigma_{\rm uw}(T) = \sigma_{\rm uw}(T + K)$, so $a$-Browder's theorem holds for $T + K$. Therefore, $T + K$ has property $(gaz)$.

The assertion (ii) has been proved in [17, Theorem 3.5].

(iii) Suppose that $T$ satisfies $S$-Weyl's theorem and that $T$ is finite-isoloid. Then $T$ satisfies property $(gaz)$ and from part (ii) we know that $T+K$ satisfies property $(gaz)$. To show $S$-Weyl theorem it then suffices to prove that $\Pi_a(T+K) = \pi_{00}(T+K)$. Now, since $T+K$ satisfies property $(gaz)$, by Theorem 4.2 we have that $\Pi_a(T+K) = \Pi(T+K)$. We show that $\Pi(T+K) = \pi_{00}(T+K)$. Let $\lambda \in \Pi(T+K) = \sigma(T+K)\backslash\sigma_{\rm d}(T+K)$. Then $\lambda$ is a pole of $T+K$, hence an eigenvalue of $\sigma(T+K)$, so $0 < \alpha(\lambda I - (T+K))$. Now, if $\lambda \notin \sigma_{\rm d}(T+K) = \sigma_{\rm d}(T)$, then $\lambda I - T$ is Drazin invertible. There are two possibilities: $\lambda \notin \sigma(T)$ or $\lambda \in \sigma(T)$. If $\lambda \notin \sigma(T)$ then $\lambda I - T$ is invertible, hence a Browder operator, so $\lambda I - (T+K)$ is Browder, hence $\alpha(\lambda I - (T+K)) < \infty$. If $\lambda \in \sigma(T)$ then $\lambda$ is a pole of $T$, hence an isolated point of $\sigma(T)$, and being $T$ finite-isoloid we then have $\alpha(\lambda I - T) < \infty$. But $p(\lambda I - T) = q(\lambda I - T) < \infty$, so, by [2, Theorem 1.22], $\lambda I - T)$ is Browder, and hence also $\lambda I - (T+K)$ is Browder, thus $\alpha(\lambda I - (T+K)) < \infty$ also in this case. Therefore, $\Pi(T+K) \subseteq \pi_{00}(T+K)$.

To show the reverse inclusion, suppose that $\lambda \in \pi_{00}(T+K)$. There are two possibilities: $\lambda \notin \sigma(T)$ or $\lambda \in \sigma(T)$. If $\lambda I - T$ is invertible then $\lambda I - T$ is Browder, so also $\lambda I - (T+K)$ is Browder, in particular is Drazin invertible. Hence $\lambda \in \sigma(T+K)\backslash\sigma_{\rm d}(T+K) = \Pi(T+K)$. Consider the other case $\lambda \in \sigma(T) = \sigma_{\rm ap}(T)$. Then

$$\lambda \in \pi_{00}(T+K) \cap \sigma_{\rm ap}(T),$$

and hence, see [15, Lemma 2.3] or [2, Theorem 6.11, part (ii)], $\lambda \in {\rm iso}\sigma(T)$. Since $T$ is finite-isoloid then $0 < \alpha(\lambda I - T) < \infty$, so $\lambda \in \pi_{00}(T)$. But $T$ satisfies $S$-Weyl's theorem, hence Weyl's theorem, and this implies that $\pi_{00}(T) \cap \sigma_{\rm w}(T) = \emptyset$, i.e., $\lambda \notin \sigma_{\rm w}(T)$. Weyl's theorem entails Browder's theorem, hence $\lambda \notin \sigma_{\rm w}(T) = \sigma_{\rm b}(T)$, i.e., $\lambda I - T$ is Browder. It then follows that $\lambda I - (T+K)$ is Browder, in particular Drazin invertible. Therefore, $\lambda \in \sigma(T+K) \setminus \sigma_{\rm d}(T+K) = \Pi(T+K)$. Consequently,

$$\pi_{00}(T+K) = \Pi(T+K) = \Pi_a(T+K);$$

so $T+K$ satisfies $S$-Weyl's theorem. □

Recall that the Weyl spectra are stable under (not necessarily commuting) compact perturbations.

**Theorem 5.4.** *Let $T \in L(X)$ such that $\sigma_{\rm uw}(T)$ has no hole and ${\rm iso}\,\sigma_{\rm uw}(T) = \emptyset$. Then we have:*

(i) *$T$ satisfies $S$-Weyl's theorem.*
(ii) *If $R \in L(X)$ is a Riesz operator that commutes with $T$, then $T+R$ satisfies $S$-Weyl's theorem.*
(iii) *$S$-Weyl's theorem holds for $T+K$ for every compact operator $K \in L(X)$.*

*Proof.* (i) By assumption, $\sigma_{\rm uw}(T)$ has no hole, so $\sigma(T) = \sigma_{\rm ap}(T)$ and hence $\pi_{00}(T) = \pi^a_{00}(T)$. We claim that $\pi_{00}(T) = p_{00}(T)$. The inclusion $p_{00}(T) \subseteq \pi_{00}(T)$ is clear, so it suffices to prove the reverse inclusion. Let $\lambda \in \pi_{00}(T) = \pi^a_{00}(T)$. Then $\lambda \in {\rm iso}\,\sigma_{\rm ap}(T)$. Since ${\rm iso}\,\sigma_{\rm uw}(T) = \emptyset$ it then follows that $\lambda I -$

$T \in \Phi_{\pm}(X)$, by Lemma 3.6, and hence $\lambda I - T \in W_+(X)$, since $\alpha(\lambda I - T) < \infty$. Then $\lambda I - T \in B(X)$, see [2, Theorem 2.100], and hence $\lambda \in p_{00}(T)$. Therefore

$$\pi_{00}(T) = p_{00}(T) \subseteq \sigma(T) \backslash \sigma_{\mathrm{ubw}}(T).$$

Conversely, since $\operatorname{iso} \sigma_{\mathrm{uw}}(T) = \emptyset$ entails that $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ubw}}(T)$, from Theorem 3.1 we get

$$\sigma(T) \backslash \sigma_{\mathrm{ubw}}(T) = \sigma(T) \backslash \sigma_{\mathrm{uw}}(T) \subseteq \rho(T) \cup p_{00}(T),$$

and this easily implies $\sigma(T) \backslash \sigma_{\mathrm{ubw}}(T) \subseteq p_{00}(T) = \pi_{00}(T)$. Hence $S$-Weyl's theorem holds for $T$.

The statement (ii) is clear from part (i), since $\sigma_{\mathrm{uw}}(T + R) = \sigma_{\mathrm{uw}}(T)$ has no hole and $\operatorname{iso} \sigma_{\mathrm{uw}}(T + R) = \operatorname{iso} \sigma_{\mathrm{uw}}(T) = \emptyset$.

(iii) According part (i) the assertion follows from the equality $\operatorname{iso} \sigma_{\mathrm{uw}}(T) = \operatorname{iso} \sigma_{\mathrm{uw}}(T + K) = \emptyset$ and since $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{uw}}(T + K)$ has no hole. □

**Example 5.5.** Without the assumption $\operatorname{iso} \sigma_{\mathrm{uw}}(T) = \emptyset$ the result of the previous theorem is not true. To see this, let $Q$ denote an injective quasi-nilpotent operator on $\ell_2(\mathbb{N})$ and consider the 1-dimensional operator:

$$U(x_1, x_2, \dots) := \left(-\frac{1}{2}x_1, 0, 0, \cdots\right) \quad \text{for every } (x_n) \in \ell_2(\mathbb{N}).$$

Then $QU = UQ$. Let $T := \frac{1}{2}I \oplus Q$ and $K := U \oplus 0$. Then $\sigma(T) = \sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{uw}}(T) = \{0, \frac{1}{2}\}$, and it is easily seen that $\pi_{00}(T) = \emptyset$. Moreover, $T^*$ has SVEP, since $\sigma(T^*)$ is finite, and this implies that $T$ has property $(gaz)$. Evidently,

$$\sigma_{\mathrm{d}}(T) = \sigma_{\mathrm{d}}\left(\frac{1}{2}I\right) \cup \sigma_{\mathrm{d}}(Q) = \left\{0, \frac{1}{2}\right\},$$

and by Lemma 2.3, we have $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{d}}(T)$, so $\Pi_a(T) = \Pi(T) = \sigma(T) \backslash \sigma_{\mathrm{d}}(T)$
$= \emptyset$. Hence, $T$ satisfies $S$-Weyl's theorem, by Theorem 4.9. On the other hand,

$$\sigma(T + K) = \sigma_{\mathrm{uw}}(T + K) = \left\{0, \frac{1}{2}\right\},$$

and $\pi_{00}(T + K) = \{0\}$, so $\sigma(T + K) \backslash \sigma_{\mathrm{uw}}(T + K) = \emptyset \neq \pi_{00}(T + K)$, hence $T + K$ does not satisfies property $(t)$, so $S$-Weyl's theorem does not hold for $T + K$.

By Lemma 3.5 $\operatorname{iso} \sigma_{\mathrm{uw}}(T) = \emptyset$ entails $\operatorname{iso} \sigma_{\mathrm{w}}(T) = \emptyset$. The next example shows that in Theorem 5.4 the conditions $\operatorname{iso} \sigma_{\mathrm{uw}}(T) = \emptyset$ and $\sigma_{\mathrm{uw}}(T)$ has no holes, cannot be changed into the weaker conditions $\operatorname{iso} \sigma_{\mathrm{w}}(T) = \emptyset$ and $\sigma_{\mathrm{w}}(T)$ has no hole.

**Example 5.6.** Let $T := R \oplus S$ where $R$ is the unilateral right shift on $\ell_2(\mathbb{N})$ and

$$S(x_1, x_2, x_3, \cdots) := \left(\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \cdots\right) \quad \text{for all } (x_k) \in \ell_2(\mathbb{N}).$$

Then $\sigma(T) = \sigma_{\rm w}(T) = \mathbf{D}(0,1)$, the closed disk in $\mathbb{C}$, has no hole and has no isolated points. It is easily seen that

$$\sigma_{\rm ap}(T) = \sigma_{\rm uw}(T) = \mathbf{\Gamma} \cup \{0\},$$

where $\mathbf{\Gamma}$ is the unit circle, so $T$ does not have property $(gaz)$, since $\sigma_{\rm ap}(T) \neq \sigma(T)$, and hence $S$-Weyl's theorem fails for $T$.

Evidently, if $T$ is finite-isoloid then $E(T) = \pi_{00}(T)$. Furthermore, if $T$ has also property $(gaz)$ then $\sigma(T) = \sigma_{\rm ap}(T)$, so $E_a(T) = \pi^a_{00}(T)$. Both $a$-Weyl's theorem and property $(\omega)$, admit a generalized version. $T \in L(X)$ is said to satisfy *generalized $a$-Weyl's theorem* if $\sigma_{\rm ap}(T) \setminus \sigma_{\rm ubw}(T) = E_a(T)$, where

$$E_a(T) := \{\lambda \in \operatorname{iso}\sigma_{\rm ap}(T) : 0 < \alpha(\lambda I - T)\},$$

while $T$ is said to satisfy *generalized property* $(\omega)$ if $\sigma_{\rm ap}(T)\backslash\sigma_{\rm ubw}(T) = E(T)$, where $E(T) = \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(\lambda I - T)\}$. Generalized $a$-Weyl's theorem entails $a$-Weyl's theorem, and analogously generalized property $(\omega)$ entails property $(\omega)$.

**Theorem 5.7.** *Let $T \in L(X)$ be finite-isoloid. Then both generalized $a$-Weyl's theorem and generalized property $(\omega)$ hold for $T$.*

*Proof.* $T$ satisfies $S$-Weyl's theorem by Theorem 5.4, or equivalently property $(t)$. We have $E(T) = \pi_{00}(T)$ and property $(t)$ entails $\sigma(T) = \sigma_{\rm ap}(T)$, so $E(T) = E_a(T) = \pi^a_{00}(T) = \pi_{00}(T)$. Since, as observed before, $T$ satisfies $a$-Weyl's theorem and property $(\omega)$ we then have, by Theorem 4.11,

$$E(T) = E_a(T) = \pi^a_{00}(T) = \sigma_{\rm ap}(T)\backslash\sigma_{\rm uw}(T) = \sigma_{\rm ap}(T)\backslash\sigma_{\rm ubw}(T),$$

hence $T$ satisfies generalized $a$-Weyl's theorem and generalized property $(\omega)$. □

**Theorem 5.8.** *Let $T \in L(X)$ such that $\sigma(T)$ has no hole and $\operatorname{iso}\sigma(T) = \emptyset$. If $T^*$ has SVEP then $T+K$ satisfies $S$-Weyl's theorem for every Riesz operator $K$ that commutes with $T$.*

*Proof.* The SVEP for $T^*$ implies that $\sigma_{\rm ap}(T) = \sigma(T)$, hence $\sigma_{\rm ap}(T)$ has no hole. Moreover, by [2, Theorem 5.27] $a$-Browder's theorem holds for $T$, i.e. $\sigma_{\rm uw}(T) = \sigma_{\rm ub}(T)$. From part (ii) of Theorem 3.3 it then follows that $\sigma_{\rm uw}(T)$ has no hole. Furthermore,

$$\sigma_{\rm ap}(T)\backslash\sigma_{\rm uw}(T) = p^a_{00}(T) \subseteq \operatorname{iso}\sigma_{\rm ap}(T) = \operatorname{iso}\sigma(T) = \emptyset,$$

from which we conclude that $\sigma_{\rm uw}(T) = \sigma_{\rm ap}(T) = \sigma(T)$ has no isolated points. Theorem 5.4 then applies. □

Evidently Theorem 5.8 applies to every weighted left shifts $L$, since the dual $L^*$ is a weighted right shift, and every weighted right shift has SVEP. Furthermore the spectrum of $L$ is the closed unit disk of $\mathbb{C}$.

**Theorem 5.9.** *Let $T \in L(H)$, where $H$ is a complex separable infinite dimensional Hilbert space. Then*

(i) *Suppose that* $\sigma_{\rm sf}(T)$ *has no hole and the interior part of* $\sigma_{\rm sf}(T)$ *is empty. Then property* $(gaz)$ *holds for* $T^*+K^*$ *for every compact operator* $K \in L(H)$.
(ii) $T+K$ *satisfies S-Weyl's theorem for all compact operators* $K \in L(H)$ *if and only if* $\sigma_{\rm uw}(T)$ *has no hole and* $\operatorname{iso}\sigma_{\rm uw}(T)=\emptyset$.

*Proof.* (i) This is an immediate consequence of a result of Zhu and Li, see [21, Theorem 3.1]. Indeed the conditions on $\sigma_{\rm sf}(T)$ entails that $T+K$ has SVEP for every compact operator $K \in L(H)$. Hence property $(gaz)$ holds for $T^*+K^*$.
(ii) By Theorem 5.4 we have only to show the implication $(\Rightarrow)$. Suppose that $T+K$ satisfies $S$-Weyl's theorem for all compact operators $K \in L(H)$. Then $T+K$ satisfies property $(t)$ and hence, by a result of S. Zhu and C. Guang Li see [20, Corollary 2.7], $\sigma_{\rm uw}(T)$ has no hole and $\operatorname{iso}\sigma_{\rm w}(T)=\emptyset$. The condition that $\sigma_{\rm uw}(T)$ has no hole entails, by Theorem 3.1, that $\sigma_{\rm w}(T)=\sigma_{\rm uw}(T)$, so $\operatorname{iso}\sigma_{\rm uw}(T)=\emptyset$. □

## 6. Analytic Toeplitz operators on Hardy Spaces

Let $\mu$ be the normalized Lebesque measure on $\mathbf{T}$, and $L^2(\mathbf{T})$ the classical Hilbert space defined with respect to $\mu$. The set of functions $\{\chi_n\}_{n\in\mathbb{Z}}$, where the functions $\chi_n$ are defined on $\mathbf{T}$ by $\chi_n(e^{it}) := e^{int}$ for every $n \in \mathbb{Z}$, is a orthogonal basis of $L^2(\mathbf{T})$. The *Hardy space* $H^2(\mathbf{T})$ is defined as the closed subspace of all $f \in L^2(\mathbf{T})$ for which

$$\frac{1}{2\pi}\int_0^{2\pi} f\chi_n \mathrm{dt} = 0 \quad \text{for } n = 1, 2, \dots.$$

Let $P$ denote the projection of $L^2(\mathbf{T})$ onto $H^2(\mathbf{T})$. The *Toeplitz operator* $T_\phi$ *on* $H^2(\mathbf{T})$, *with symbol* $\phi$, is defined by

$$T_\phi f := P(\phi f) \quad \text{for } f \in H^2(\mathbf{T}).$$

The Toeplitz operators with continuous symbols are particularly amenable to study. Indeed, let us consider the case that the symbol $\phi$ is a continuous function on $\mathbf{T}$. The essential spectrum $\sigma_{\rm e}(T_\phi)$ of $T_\phi$, defined by $\sigma_{\rm e}(T_\phi) := \{\lambda \in \mathbb{C} : \lambda I - T_\phi \text{ is not Fredholm}\}$, coincides with $\phi(\mathbf{T})$, precisely:

$$\phi(\mathbf{T}) = \sigma_{\rm e}(T) \subseteq \sigma_{\rm ap}(T_\phi) \cap \sigma_{\rm s}(T_\phi),$$

while

$$\sigma_{\rm uw}(T) = \sigma_{\rm ap}(T) \quad \text{and} \quad \sigma_{\rm w}(T) = \sigma(T),$$

see [3, Theorem 33]. If $\lambda \notin \phi(\mathbf{T})$ denote by $i(\phi,\lambda)$ the *winding number* of $\mathbf{\Gamma}$ with respect to $\lambda$ (recall that the winding number $i(\phi,\lambda)$ of a closed curve $\mathbf{\Gamma}$ in the plane around a given point $\lambda$ is an integer representing the total number of times that curve travels counterclockwise around the point). In this case

$$\operatorname{ind}(\lambda I - T_\phi) = -i(\phi,\lambda),$$

s [19].

Now, suppose that the orientation of the curve $\phi(\mathbf{T})$ of the curve $\phi(\mathbf{T})$ traced out by $\phi$ is clockwise. Then, for a spectral point $\lambda \notin \phi(\mathbf{T})$, we have $i(\phi, \lambda) < 0$ and hence $\operatorname{ind}(\lambda I - T_\phi) = -i(\phi, \lambda) > 0$, so $\lambda \in \sigma_{\mathrm{uw}}(T_\phi)$, hence $\sigma_{\mathrm{uw}}(T_\phi)$ is the closed subset of the plane delimited by the curve $\phi(\mathbf{T})$, so $\rho_{\mathrm{uw}}(T_\phi)$ is connected. Furthermore, if $\phi$ is nonconstant

$$\operatorname{iso}\sigma_{\mathrm{w}}(T_\phi) = \operatorname{iso}\sigma_{\mathrm{uw}}(T_\phi) = \emptyset,$$

see [3, Theorem 35].

**Theorem 6.1.** *Let $\phi$ be continuous and nonconstant such that the orientation of the curve $\phi(\mathbf{T})$ traced out by $\phi$ is clockwise. Then $T_\phi$ and $T_\phi + K$ satisfies S-Weyl's theorem for every Riesz operator $K$ that commutes with $T_\phi$. Moreover, $T'_\phi$ satisfies S-Weyl's theorem. If $K$ is compact, the commutativity $T_\phi K = KT_\phi$ is not required.*

*Proof.* $S$-Weyl's theorem for $T_\phi$ and $T_\phi + K$ is a direct consequence of Theorem 5.4. The second assertion, concerning $T'_\phi$, has been proved in [4, Theorem 4.4 ]. The assertion concerning the case where $K$ is compact follows from Theorem 5.9. □

**Remark 6.2.** The Riesz operator $K$ in the previous theorem is not necessarily a Toeplitz operator. Note that a compact Toeplitz operator $K$ is 0. If $\sigma(T_\phi)$ has planar Lebesgue measure zero and $K$ is compact, the commutativity of $T_\phi$ with $K$ is not required, Indeed, in this case $\operatorname{int}\sigma_{\mathrm{uw}}(T_\phi) = \operatorname{int}\sigma_{\mathrm{uw}}(T_\phi + K) = \emptyset$, so $T_\phi + K$ satisfies property $(gaz)$ by Theorem 5.9. Being $T_\phi + K$ finite-isoloid it then follows that $T_\phi + K$ satisfies $S$-Weyl's theorem, by Theorem 4.9.

Actually, we can say much more. Indeed, if the orientation of the curve $\phi(\mathbf{T})$ traced out by $\phi$ is clockwise then $T'_\phi$ has SVEP, see [8, Theorem 4.9], so the spectral mapping theorem holds for the upper semi $B$-Weyl spectrum [2]. From [5, Theorem 2.3 and Theorem 3.3] it then follows that $f(T_\phi)$ satisfies $S$-Weyl's theorem for every $f \in \mathcal{H}(\sigma(T_\phi))$.

**Theorem 6.3.** *Let $\phi$ be continuous and nonconstant such that the orientation of the curve $\phi(\mathbf{T})$ traced out by $\phi$ is counterclockwise. Then $T'_\phi$ and $T'_\phi + K'$ satisfies S-Weyl's theorem for every Riesz operator $K$ that commutes with $T_\phi$. Moreover, $T_\phi$ satisfies S-Weyl' theorems. If $K$ is compact the commutativity is not required.*

*Proof.* If $\Omega$ denote the open part of the plane inside the curve $\phi(\mathbf{T})$, then

$$\sigma_{\mathrm{lw}}(T_\phi) = \sigma_{\mathrm{uw}}(T'_\phi) = \Omega \cup \phi(\mathbf{T}),$$

so $\rho_{\mathrm{uw}}(T'_\phi)$ is connected and $\sigma_{\mathrm{uw}}(T'_\phi)$ has no hole. Again, by [3, Theorem 35], if $\phi$ is nonconstant

$$\operatorname{iso}\sigma_{\mathrm{lw}}(T_\phi) = \operatorname{iso}\sigma_{\mathrm{uw}}(T'_\phi) = \emptyset,$$

thus Theorem 5.4 applies to $T'_\phi$ and $T'_\phi + K'$, being the adjoint of a Riesz operator also Riesz. If $K$ is compact the result follows from Theorem 5.9. □

Also in this case we can say much more. Indeed in this case $T_\phi$ has SVEP, see [8, Theorem 4.9], so the spectral mapping theorem holds for the upper semi $B$-Weyl spectrum. Again from [5, Theorem 2.3 and Theorem 3.3] it then follows that $f(T'_\phi)$ satisfies $S$-Weyl's for every $f \in \mathcal{H}(\sigma(T_\phi))$.

**Remark 6.4.** If the orientation of the curve $\phi(\mathbf{T})$ traced out by $\phi$ changes and $\sigma(T_\phi)$ has no hole, the operators $T_\phi$ and $T'_\phi$ satisfy $S$-Weyl's theorem, but this result cannot be extended to $f(T)$, see [3, Example 39 ].

Now, suppose that $\phi$ is continuous and $f \in \mathcal{H}(\sigma(T_\phi))$. Then $T_{f\circ\phi}$ is still a Toeplitz operator with continuous symbol. In general, the operator $f(T_\phi)$ is not a Toeplitz operator and $T_{f\circ\phi} \neq f(T_\phi)$, see [3, Example 39].

**Theorem 6.5.** *If $\sigma_{\mathrm{ap}}(T_{f\circ\phi})$ has no holes then $S$-Weyl's theorem holds for $T_{f\circ\phi}$ and $f(T_\phi) + K$ for all $f \in \mathcal{H}(\sigma(T_\phi))$ and every compact operator $K \in L(H^2(\mathbf{T})$. In this case*

$$\sigma_{\mathrm{ap}}(T_{f\circ\phi}) = f(\sigma_{\mathrm{ap}}(T_\phi))$$

*and*

$$\sigma(T_{f\circ\phi}) = f(\sigma(T_\phi))$$

*Proof.* For every $f \in \mathcal{H}(\sigma(T_\phi))$, $T_{f\circ\phi}$ is a Toeplitz operator, so $\sigma_{\mathrm{uw}}(T_{f\circ\phi}) = \sigma_{\mathrm{ap}}(T_{f\circ\phi})$ has no holes. Furthermore, since $f \circ \phi$ is nonconstant then, as observed above, $\sigma_{\mathrm{uw}}(T_{f\circ\phi})$ has no isolated points. Therefore, $T_{f\circ\phi}$ satisfies $S$-Weyl's theorem by Theorem 5.4.

Let $K$ be any compact operator on $H^2(\mathbf{T})$. In [13] it is shown that for every continuous function $\phi$ there exists a compact operator $K_1$ on $H^2(\mathbf{T})$ such that

$$f(T_\phi) = T_{f\circ\phi} + K_1.$$

But the Weyl spectra are invariant under compact (not necessarily commuting) operators, so

$$\sigma_{\mathrm{uw}}(f(T) + K) = \sigma_{\mathrm{uw}}(T_{f\circ\phi} + K_1 + K)$$

has no holes and does not have isolated points, and again by Theorem 5.4 this implies that $f(T) + K$ satisfies $S$-Weyl's theorem.

The last assertion follows from [3, Theorem 45 and Corollary 46], since $f(T_\phi)$ satisfies $a$-Weyl's theorem and property $(\omega)$. □

**Acknowledgements**

S:T. acknowledge partial financial support from Palermo University, UNIPA D26-PREMIO-GRUPPI RIC-2023 FFR2023 and via FFR and G.N.A.M.P.A. of the INdAM. The authors thank the referee/s for his/her useful comments and suggestions.

**Author contributions** All the authors give the same contribution.

**Funding** Open access funding provided by Università degli Studi di Palermo within the CRUI-CARE Agreement.

**Data Availability** No datasets were generated or analyzed during the current study.

**Declarations**

**Conflict of interest** The authors declare no competing interests.

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Received: December 11, 2024.

Revised: December 11, 2024.

Accepted: February 10, 2025.