SUPERLINEAR ROBIN PROBLEMS WITH INDEFINITE LINEAR PART

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Abstract. We consider a semilinear Robin problem with an indefinite linear part and a superlinear reaction term, which does not satisfy the usual in such cases AR-condition. Using variational methods, together with truncation-perturbation techniques and Morse theory (critical groups), we establish the existence of three nontrivial solutions. Our result extends in different ways the multiplicity theorem of Wang.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following semilinear Robin problem

$$
\begin{cases} -\Delta u(z) + \xi(z)u(z) = f(z, u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}
$$

In this problem $\xi \in L^s(\Omega)$, $s > N$, is an indefinite potential function. So, the linear part of problem (1) is indefinite. The reaction term $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and, for a.a. $z \in \Omega$, $x \to f(z, x)$ is continuous), which exhibits superlinear growth near $\pm\infty$ but without satisfying the usual in such cases Ambrosetti-Rabinowitz condition (the AR-condition for short). In the boundary condition, $\dfrac{\partial u}{\partial n}$ denotes the usual normal derivative on $\partial\Omega$ defined by extension of the map

$$
u \to \frac{\partial u}{\partial n} = (\nabla u, n)_{\mathbb{R}^N} \quad \text{for all } u \in C^1(\overline{\Omega}),
$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The boundary coefficient β belongs in $W^{1,\infty}(\partial\Omega)$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$. If $\beta \equiv 0$, then we recover the Neumann problem.

Our aim in this paper, is to extend the well-known multiplicity result (three solutions theorem) of Wang [21]. In Wang [21] the problem is Dirichlet, there is no potential term (that is, $\xi \equiv 0$ and so the linear part of the equation in [21] is coercive) and the reaction term is autonomous (that is, $f(z, x) = f(x)$), $f \in C^1(\mathbb{R}, \mathbb{R})$, $f(0) = f'(0) = 0$ and satisfies the AR-condition. Here, we weaken significantly all these requirements. We mention, that the lack of smoothness for the function $x \to f(z, x)$ (in our case this function is only continuous and not necessarily C^1), makes the use of Morse theory problematic. It is well-known that the most powerful tools of Morse theory, are available if the energy (Euler) functional of the problem is C^2, which is not possible if $f(z, \cdot)$ is only continuous.

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We mention that superlinear problems with coercive left hand side, were studied by Miyagaki-Souto [13] (parametric semilinear Dirichlet problems) and by Aizicovici-Papageorgiou-Staicu [2] (Neumann problems), [3] (Dirichlet problems), Fang-Liu [8], Li-Yang [11], Liu [12], Sun [19] (all four works prove only existence theorems) for equations driven by the p-Laplacian. Finally we mention that multiplicity theorems for different classes of semilinear problems with an indefinite potential term, were proved by Papageorgiou-Papalini [15] (Dirichlet problems). Papageorgiou-Smyrlis [18] (Neumann problems) and Papageorgiou-Rădulescu [17] (Robin problems).

Our approach uses variational methods based on the critical point theory and truncation-perturbation techniques coupled with Morse theory (critical groups).

2. Mathematical Background

Let X be a Banach space and let X^* denote its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X,\mathbb{R})$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following property holds:

"Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and $(1+\|u_n\|)\varphi'(u_n) \to 0$ in X^* as $n \to +\infty$, admits a strongly convergent subsequence."

This is a compactness-type condition on the functional φ and it leads to a deformation theorem from which one can derive the minimax theory for the critical values of φ. Prominent in that theory is the celebrated "mountain pass theorem" of Ambrosetti-Rabinowitz [4]. Here we state this result in a slightly more general form (see Gasiński-Papageorgiou [9]).

Theorem 1. *If $\varphi \in C^1(X,\mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > \rho > 0$, $\max\{\varphi(u_0), \varphi(u_1)\} < \inf[\varphi(u) : \|u - u_0\| = \rho] = m_\rho$ and $c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$, then $c \geq m_\rho$ and c is a critical value of φ.*

The spaces which we will use in the study of problem (1) are the Sobolev space $H^1(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the "boundary" spaces $L^p(\partial\Omega)$ $(1 \leq p \leq \infty)$.

The space $H^1(\Omega)$ is a Hilbert space with inner product

$$(u, v) = \int_\Omega uv dz + \int_\Omega (\nabla u, \nabla v)_{\mathbb{R}^N} dz \quad \text{for all } u, v \in H^1(\Omega)$$

and corresponding norm

$$\|u\| = \left[\|u\|_2^2 + \|\nabla u\|_2^2\right]^{1/2} \quad \text{for all } u \in H^1(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is an ordered Banach space with positive cone given by

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior which contains the set

$$D_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Then using $\sigma(\cdot)$ we can define in the usual way the boundary Lebesgue spaces $L^p(\partial\Omega)$ $(1 \leq p \leq \infty)$.

From the theory of Sobolev spaces, we know that there exists a unique continuous linear map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$, known as the trace map such that

$$\gamma_0(u) = u\big|_{\partial\Omega} \quad \text{for all } u \in H^1(\Omega) \cap C(\overline{\Omega}).$$

So, we understand the trace map as an expression of the "boundary values" of a Sobolev function. We know that the linear map γ_0 is compact into $L^p(\Omega)$ with $p \in \left[1, \frac{2(N-1)}{N-2}\right)$ if $N \geq 3$ and into $L^p(\Omega)$ for all $p \geq 1$ if $N = 1, 2$. In what follows, for the sake of notational simplicity, we drop the use of the trace map γ_0. All restrictions of Sobolev functions on $\partial\Omega$, are understood in the sense of traces.

Also, we will need the principal eigenvalue in the spectrum of $u \to -\Delta u + \xi(z)u$ with Robin boundary condition. So, we consider the following linear eigenvalue problem:

$$\begin{cases} -\Delta u(z) + \xi(z)u(z) = \widehat{\lambda} u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Suppose that

- $\xi \in L^{N/2}(\Omega)$ if $N \geq 3$, $\xi \in L^p(\Omega)$ with $p > 1$ if $N = 2$ and $\xi \in L^1(\Omega)$ if $N = 1$;
- $\beta \in W^{1,\infty}(\partial\Omega)$, $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

Consider the C^1-functional $\eta : H^1(\Omega) \to \mathbb{R}$ defined by

$$\eta(u) = \|\nabla u\|_2^2 + \int_\Omega \xi(z)u^2 dz + \int_{\partial\Omega} \beta(z)u^2 d\sigma \quad \text{for all } u \in H^1(\Omega).$$

From D'Aguì-Marano-Papageorgiou [6], we know that there exists $\mu > 0$ such that

$$\eta(u) + \mu\|u\|_2^2 \geq c_0\|u\|^2 \quad \text{for all } u \in H^1(\Omega), \text{ some } c_0 > 0. \tag{3}$$

Using (3) and the spectral theorem for compact self-adjoint operators on a Hilbert space, we can have a complete description of the spectrum of (2). So, we obtain a strictly increasing sequence $\{\widehat{\lambda}_k\}_{k\geq 1}$ of distinct eigenvalues such that $\widehat{\lambda}_k \to +\infty$ as $k \to +\infty$. If by $E(\widehat{\lambda}_k)$, $k \in \mathbb{N}$, we denote the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_k$, then we have the following orthogonal direct sum decomposition

$$H^1(\Omega) = \overline{\bigoplus_{k\geq 1} E(\widehat{\lambda}_k)}.$$

For the first eigenvalue $\widehat{\lambda}_1$, we have the following properties

$$\widehat{\lambda}_1 = \inf \left[\frac{\eta(u)}{\|u\|_2^2} : u \in H^1(\Omega), u \neq 0 \right]; \tag{4}$$

$$\widehat{\lambda}_1 \text{ is simple (that is, } \dim E(\widehat{\lambda}_1) = 1). \tag{5}$$

The infimum in (4) is realized on $E(\widehat{\lambda}_1)$. This in conjunction with (5) imply that the elements of $E(\widehat{\lambda}_1)$ have fixed sign. Let $\widehat{u}_1$ denote the L^2-normalized (that is, $\|\widehat{u}_1\|_2 = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1$, If $\xi \in L^s(\Omega)$ $s > N$, then using the regularity theory of Wang [20], we have $\widehat{u}_1 \in C_+ \setminus \{0\}$, In addition Harnack's inequality (see, for example, Motreanu-Motreanu-Papageorgiou [14] (p. 212)) implies that $\widehat{u}_1(z) > 0$ for all $z \in \Omega$. Finally if $\xi^+ \in L^\infty(\Omega)$, then Hopf's theorem (see, for example, Gasiński-Papageorgiou [9] (p. 738)) implies that $\widehat{u}_1 \in D_+$. Finally, we mention that the eigenfunctions corresponding to an eigenvalue $\widehat{\lambda}_k$, $k \in \mathbb{N}$, $k \neq 1$, are nodal (that is, sign changing) and the eigenspace $E(\widehat{\lambda}_k)$ has the "Unique Continuation Property", that is, if $u \in E(\widehat{\lambda}_k)$ and u vanishes on a set of positive measure, then $u = 0$.

For details, see D'Aguì-Marano-Papageorgiou [6] and Papageorgiou-Smyrlis [18].

Let X be a Banach space, $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We introduce the following sets

$$\begin{aligned} \varphi^c &= \{u \in X : \varphi(u) \leq c\}, \\ K_\varphi &= \{u \in X : \varphi'(u) = 0\}, \\ K_\varphi^c &= \{u \in K_\varphi : \varphi(u) = c\}. \end{aligned}$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every $k \in \mathbb{N}_0$, by $H_k(Y_1, Y_2)$ we denote the k^{th}-relative singular homology group with integer coefficients for the pair (Y_1, Y_2). Recall that if $k \in -\mathbb{N}$, then $H_k(Y_1, Y_2) = 0$. Suppose that $u_0 \in K_\varphi^c$ is isolated. Then the critical groups of φ at u_0 are defined by

$$C_k(\varphi, u_0) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u_0\}) \quad \text{for all } k \in \mathbb{N}_0,$$

with U being a neighborhood of u_0 such that $K_\varphi \cap \varphi^c \cap U = \{u_0\}$. The excision property of singular homology theory implies that this definition is independent of the choice of the neighborhood U.

Suppose that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition and $-\infty < \inf \varphi(K_\varphi)$. Let $c < \inf \varphi(K_\varphi)$. Then the critical groups of φ at infinity, are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0.$$

From the second deformation theorem (see Gasiński-Papageorgiou [9], (p. 628)), we know that if $c, c' < \inf \varphi(K_\varphi)$, then the sets φ^c and $\varphi^{c'}$ are homotopy equivalent and so

$$H_k(X, \varphi^c) = H_k(X, \varphi^{c'}) \quad \text{for all } k \in \mathbb{N}_0$$

(see Motreanu-Motreanu-Papageorgiou [14] (p. 143)). So, it follows that the above definition of critical groups at infinity is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose that the critical set K_φ is finite. We introduce the following polynomials in $t \in \mathbb{R}$:

$$\begin{aligned} M(t, u) &= \sum_{k \in \mathbb{N}_0} \text{rank } C_k(\varphi, u) t^k \quad \text{for all } t \in \mathbb{R}, \text{ all } u \in K_\varphi, \\ P(t, \infty) &= \sum_{k \in \mathbb{N}_0} \text{rank } C_k(\varphi, \infty) t^k \quad \text{for all } t \in \mathbb{R}. \end{aligned}$$

The "Morse relation" says that

$$\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t) Q(t) \quad \text{for all } t \in \mathbb{R},$$

with $Q(t) = \sum_{k \in \mathbb{N}_0} \beta_k t^k$ being a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients β_k.

In what follows $A \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ is the continuous linear operator defined by

$$\langle A(u), v \rangle = \int_\Omega (\nabla u, \nabla v)_{\mathbb{R}^N} dz \quad \text{for all } u, v \in H^1(\Omega).$$

Also, we say that a Banach space X has the "Kadec-Klee property", if the following implication holds:

$$u_n \xrightarrow{w} u \text{ in } X \text{ and } \|u_n\| \to \|u\| \Rightarrow u_n \to u \text{ in } X.$$

We know that locally uniformly convex Banach spaces (in particular Hilbert spaces), have the Kadec-Klee property (see Gasiński-Papageorgiou [9] (p. 901)).

Finally we introduce some basic notation which will be used in the sequel. So, if $x \in \mathbb{R}$, then we set $x^{\pm} = \max\{\pm x, 0\}$. For $u \in W^{1,p}(\Omega)$ we define

$$u^{\pm}(\cdot) = u(\cdot)^{\pm}.$$

We know that

$$u = u^+ - u^-,\ |u| = u^+ + u^-,\ u^{\pm} \in W^{1,p}(\Omega).$$

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$ and by 2^* the critical Sobolev exponent defined by

$$2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N \geq 3, \\ +\infty & \text{if } N = 1, 2. \end{cases}$$

3. Three Solutions Theorem

The hypotheses on the data of problem (1) are the following:

$H(\xi)$: $\xi \in L^s(\Omega)$ with $s > N$ and $\xi^+ \in L^\infty(\Omega)$.

$H(\beta)$: $\beta \in W^{1,\infty}(\partial\Omega)$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

Remark 1. If $\beta \equiv 0$, then we have the Neumann problem.

H: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z, x)| \leq a(z)(1 + |x|^{r-1})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $2 < r < 2^*$;

(ii) if $F(z, x) = \int_0^x f(z, s)ds$, then $\lim\limits_{x \to \pm\infty} \frac{F(z,x)}{x^2} = +\infty$ uniformly for a.a. $z \in \Omega$;

(iii) if $e(z, x) = f(z, x)x - 2F(z, x)$, then there exists $d \in L^1(\Omega)$ such that

$$e(z, x) \leq e(z, y) + d(z) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq y \text{ or } y \leq x \leq 0;$$

(iv) there exist $\vartheta_0 > 0$ and a function $\vartheta \in L^\infty(\Omega)$ such that

$$\vartheta(z) \leq \widehat{\lambda}_1 \text{ for a.a. } z \in \Omega,\ \vartheta \not\equiv \widehat{\lambda}_1,$$

$$-\vartheta_0 \leq \liminf_{x \to 0} \frac{f(z, x)}{x} \leq \limsup_{x \to 0} \frac{f(z, x)}{x} \leq \vartheta(z) \text{ uniformly for a.a. } z \in \Omega.$$

Remark 2. Evidently hypotheses $H(ii), (iii)$ imply that

$$\lim_{x \to \pm\infty} \frac{f(z, x)}{x} = +\infty \quad \text{uniformly for a.a. } z \in \Omega.$$

So the reaction term of problem (1) is superlinear in the x-variable. However, we point out that the superlinearity of $f(z, \cdot)$ is not formulated using the common for such problems AR-condition. We recall that the AR-condition says that there exist $q > 2$ and $M > 0$ such that

$$0 < qF(z, x) \leq f(z, x)x \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M, \tag{6}$$

$$0 < \operatorname{essinf}_\Omega F(\cdot, \pm M). \tag{7}$$

Integrating (6) and using (7), we obtain the following weaker condition

$$c_1|x|^q \leq F(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M, \text{ some } c_1 > 0. \tag{8}$$

From (8) and (6) we see that the AR-condition implies that $f(z,\cdot)$ has at least $(q-1)$-polynomial growth near $\pm\infty$. This excludes from consideration superlinear nonlinearities with "slower" growth near $\pm\infty$ (see the examples below). Here, instead of the AR-condition (see (6), (7)), we employ a quasimonotonicity condition on the function $e(z,\cdot)$ (see hypothesis $H(iii)$). This condition is a little more general than the one used by Li-Yang [11]. If there exists $M > 0$ such that

$$x \to \frac{f(z,x)}{x} \text{ is nondecreasing on } [M,+\infty),$$
$$x \to \frac{f(z,x)}{x} \text{ is nonincreasing on } (-\infty,-M],$$

then hypothesis $H(iii)$ is satisfied. At zero we have nonuniform nonresonance with respect to the principal eigenvalue $\widehat{\lambda}_1$ (see hypothesis $H(iii)$).

Example 1. The following functions satisfy hypotheses H. For the sake of simplicity we drop the z-dependence.

$$f_1(x) = \begin{cases} cx & \text{if } |x| \leq 1, \\ |x|^{r-2}x + c - 1 & \text{if } 1 < |x|, \end{cases} \text{ with } c < \widehat{\lambda}_1,\ 2 < r < 2^*,$$

$$f_2(x) = \begin{cases} c(x - |x|^{\tau-2}x) & \text{if } |x| \leq 1, \\ |x|\ln|x| & \text{if } 1 < |x|, \end{cases} \text{ with } c < \widehat{\lambda}_1,\ 2 < \tau.$$

Note that f_2 does not satisfy the AR-condition.

We start by producing two nontrivial constant sign smooth solutions.

Proposition 1. *If hypotheses $H(\xi)$, $H(\beta)$, H hold, then problem* (1) *has at least two nontrivial constant sign solutions $u_0 \in D_+$ and $v_0 \in -D_+$.*

Proof. Let $\mu > 0$ be as in (3) and consider the Carathéodory function $g_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$g_+(z,x) = \begin{cases} 0 & \text{if } x \leq 0, \\ f(z,x) + \mu x & \text{if } 0 < x. \end{cases} \tag{9}$$

We set $G_+(z,x) = \int_0^x g_+(z,s)ds$ and consider the C^1-functional $\psi_+ : H^1(\Omega) \to \mathbb{R}$ defined by

$$\psi_+(u) = \frac{1}{2}\eta(u) + \frac{\mu}{2}\|u\|_2^2 - \int_\Omega G_+(z,u)dz \quad \text{for all } u \in H^1(\Omega).$$

Claim 1: The functional ψ_+ satisfies the C-condition.

Let $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ be a sequence such that

$$|\psi_+(u_n)| \leq M_1 \quad \text{for some } M_1 > 0, \text{ all } n \in \mathbb{N}, \tag{10}$$

$$(1 + \|u_n\|)\psi_+'(u_n) \to 0 \quad \text{in } H^1(\Omega)^*. \tag{11}$$

From (11) we have

$$\left|\langle A(u_n), h\rangle + \int_\Omega (\xi(z)+\mu)u_n h dz + \int_{\partial\Omega} \beta(z)u_n h d\sigma - \int_\Omega g_+(z,u_n)hdz\right| \leq \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \tag{12}$$
$$\text{for all } h\in H^1(\Omega), \text{ with } \varepsilon_n\to 0^+.$$

In (12) we choose $h=-u_n^-\in H^1(\Omega)$. Then

$$\eta(u_n^-)+\mu\|u_n^-\|_2^2\leq\varepsilon_n \quad \text{for all } n\in\mathbb{N} \text{ (see (9))},$$
$$\Rightarrow \quad c_0\|u_n^-\|^2\leq\varepsilon_n \quad \text{for all } n\in\mathbb{N} \text{ (see (3))},$$
$$\Rightarrow \quad u_n^-\to 0 \text{ in } H^1(\Omega). \tag{13}$$

From (10) and (13), we have

$$\|\nabla u_n^+\|_2^2 + \int_\Omega \xi(z)(u_n^+)^2dz + \int_{\partial\Omega}\beta(z)(u_n^+)^2d\sigma - \int_\Omega 2F(z,u_n^+)dz \leq M_2 \tag{14}$$
$$\text{for some } M_2>0, \text{ all } n\in\mathbb{N} \text{ (see (9))}.$$

On the other hand, if in (12) we choose $h=u_n^+\in H^1(\Omega)$, then

$$-\|\nabla u_n^+\|_2^2 - \int_\Omega \xi(z)(u_n^+)^2dz - \int_{\partial\Omega}\beta(z)(u_n^+)^2d\sigma + \int_\Omega f(z,u_n^+)u_n^+dz \leq \varepsilon_n \quad \text{for all } n\in\mathbb{N}. \tag{15}$$

We add (14) and (15) and obtain

$$\int_\Omega e(z,u_n^+)dz \leq M_3 \quad \text{for some } M_3>0, \text{ all } n\in\mathbb{N}. \tag{16}$$

We show that $\{u_n^+\}_{n\geq1}\subseteq H^1(\Omega)$ is bounded. Arguing by contradiction, suppose that by passing to a subsequence if necessary, we have

$$\|u_n^+\|\to+\infty. \tag{17}$$

Let $y_n=\dfrac{u_n^+}{\|u_n^+\|}$, $n\in\mathbb{N}$. Then $\|y_n\|=1$, $y_n\geq 0$ for all $n\in\mathbb{N}$ and so we may assume that

$$y_n\xrightarrow{w} y \text{ in } H^1(\Omega) \text{ and } y_n\to y \text{ in } L^\tau(\Omega) \text{ and in } L^2(\partial\Omega),\ y\geq 0, \tag{18}$$
$$\text{with } \tau=\max\left\{\frac{2s}{s-1}, r\right\} \text{ (note that } \tau<2^*).$$

First suppose that $y\neq 0$ and let $S=\{z\in\Omega : y(z)>0\}$. Then $|S|_N>0$ and we have

$$u_n^+(z)\to+\infty \quad \text{for a.a. } z\in S.$$

Using hypothesis $H(ii)$, we have

$$\frac{F(z,u_n^+)}{\|u_n^+\|^2} = \frac{F(z,u_n^+)}{(u_n^+)^2}y_n^2\to+\infty \quad \text{for a.a. } z\in S. \tag{19}$$

Using (19) and Fatou's lemma (note that hypothesis $H(ii)$ permits its use), we have

$$\int_S \frac{F(z,u_n^+)}{\|u_n^+\|^2}dz\to+\infty \quad \text{as } n\to+\infty. \tag{20}$$

Hypothesis $H(ii)$ implies that we can find $M_4>0$ such that

$$F(z,x)\geq 0 \quad \text{for a.a. } z\in\Omega, \text{ all } |x|\geq M_4. \tag{21}$$

Also, from (17) we see that without any loss of generality we may assume that

$$\|u_n^+\| \geq 1 \quad \text{for all } n \in \mathbb{N}. \tag{22}$$

We have

$$\int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^2}dz = \int_S \frac{F(z,u_n^+)}{\|u_n^+\|^2}dz + \int_{\Omega\setminus S} \frac{F(z,u_n^+)}{\|u_n^+\|^2}dz \quad \text{for all } n \in \mathbb{N}. \tag{23}$$

We estimate the second integral in the right hand side of (23). Then

$$\begin{aligned}\int_{\Omega\setminus S} \frac{F(z,u_n^+)}{\|u_n^+\|^2}dz &= \int_{(\Omega\setminus S)\cap\{u_n^+\geq M_4\}} \frac{F(z,u_n^+)}{\|u_n^+\|^2}dz + \int_{(\Omega\setminus S)\cap\{u_n^+< M_4\}} \frac{F(z,u_n^+)}{\|u_n^+\|^2}dz \\ &\geq -M_5 \quad \text{for some } M_5>0, \text{ all } n \in \mathbb{N} \\ &\qquad \text{(see (21), (22) and hypothesis } H(i)).\end{aligned} \tag{24}$$

Returning to (23) and using (24), we obtain

$$\begin{aligned}&\int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^2}dz \geq \int_S \frac{F(z,u_n^+)}{\|u_n^+\|^2}dz - M_5 \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow\quad &\lim_{n\to+\infty} \int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^2}dz = +\infty \quad \text{(see (20))}.\end{aligned} \tag{25}$$

From (10) and (13), we have

$$\begin{aligned}\int_\Omega 2F(z,u_n^+)dz &\leq M_6 + \eta(u_n^+) \quad \text{for some } M_6>0, \text{ all } n \in \mathbb{N} \\ &\leq M_6 + \|\nabla u_n^+\|_2^2 + \|\xi^+\|_\infty \|u_n^+\|_2^2 + \int_{\partial\Omega} \beta(z)(u_n^+)^2 d\sigma \\ &\qquad \text{for all } n \in \mathbb{N} \text{ (see hypothesis } H(\xi)),\end{aligned}$$

$$\begin{aligned}\Rightarrow\quad \int_\Omega \frac{2F(z,u_n^+)}{\|u_n^+\|^2}dz &\leq \frac{M_6}{\|u_n^+\|^2} + \|\nabla y_n\|_2^2 + \|\xi^+\|_\infty \|y_n\|_2^2 + \int_{\partial\Omega} \beta(z) y_n^2 d\sigma \\ &\leq c_2(1+\|y_n\|^2) = 2c_2 \\ &\qquad \text{for some } c_2>0, \text{ all } n \in \mathbb{N} \text{ (recall that } \|y_n\|_1 = 1 \text{ for all } n \in \mathbb{N}).\end{aligned} \tag{26}$$

Comparing (25) and (26) we have a contradiction.

Next suppose that $y = 0$. Let $k > 0$ and set $v_n = (2k)^{1/2} y_n \in H^1(\Omega)$, $n \in \mathbb{N}$. Then

$$\begin{aligned}&v_n \to 0 \text{ in } L^r(\Omega) \text{ (see (18) and recall that } y=0), \\ \Rightarrow\quad &\int_\Omega F(z,v_n)dz \to 0 \quad \text{(see hypothesis } H(i)).\end{aligned} \tag{27}$$

From (17) we see that we can find $n_0 \in \mathbb{N}$ such that

$$(2k)^{1/2} \frac{1}{\|u_n^+\|} \leq 1 \quad \text{for all } n \geq n_0. \tag{28}$$

Let $t_n \in [0,1]$ be such that

$$\psi_+(t_n u_n^+) = \max[\psi_+(t u_n^+) : 0 \leq t \leq 1] \quad \text{for all } n \in \mathbb{N}. \tag{29}$$

From (28) and (29) it follows that

$$\psi_+(t_n u_n^+) \geq \psi_+(v_n)$$

$$
\begin{aligned}
&= k[\eta(y_n) + \mu\|y_n\|_2^2] - \int_\Omega F(z, v_n)dz - \frac{\mu}{2}\|v_n\|_2^2 \quad \text{(see (9))} \\
&\geq kc_0 - \int_\Omega F(z, v_n)dz - \frac{\mu}{2}\|v_n\|_2^2 \quad \text{for all } n \geq n_0 \\
&\quad \text{(see (3) and recall that } \|y_n\| = 1 \text{ for all } n \in \mathbb{N}).
\end{aligned}
$$

Since $v_n \to 0$ in $L^2(\Omega)$ and using also (27), we see that we can find $n_1 \geq n_0$ such that

$$\psi_+(t_n u_n^+) \geq \frac{kc_0}{2} \quad \text{for all } n \geq n_1.$$

But recall that $k > 0$ is arbitrary. So, we infer that

$$\psi_+(t_n u_n^+) \to +\infty \text{ as } n \to +\infty. \tag{30}$$

We know that

$$\psi_+(0) = 0 \text{ and } \psi_+(u_n^+) \leq M_7 \quad \text{for some } M_7 > 0, \text{ all } n \in \mathbb{N} \text{ (see (10) and (13))}.$$

Therefore (30) implies that we can find $n_2 \in \mathbb{N}$ such that

$$t_n \in (0, 1) \quad \text{for all } n \geq n_2. \tag{31}$$

From (29) and (31), we have

$$
\begin{aligned}
& \frac{d}{dt}\psi_+(tu_n^+)\Big|_{t=t_n} = 0 \quad \text{for all } n \geq n_2, \\
\Rightarrow \quad & \langle \psi_+'(t_n u_n^+), u_n^+ \rangle = 0 \quad \text{for all } n \geq n_2 \text{ (by the chain rule)}, \\
\Rightarrow \quad & \langle \psi_+'(t_n u_n^+), t_n u_n^+ \rangle = 0 \quad \text{for all } n \geq n_2, \\
\Rightarrow \quad & \eta(t_n u_n^+) = \int_\Omega f(z, t_n u_n^+)(t_n u_n^+)dz \quad \text{for all } n \geq n_2 \text{ (see (9))}. \qquad (32)
\end{aligned}
$$

Hypothesis $H(iii)$ and (31) imply that

$$
\begin{aligned}
& \int_\Omega e(z, t_n u_n^+)dz \leq \int_\Omega e(z, u_n^+)dz + \|d\|_1 \quad \text{for all } n \geq n_2, \\
\Rightarrow \quad & \int_\Omega e(z, t_n u_n^+)dz \leq M_8 \quad \text{for some } M_8 > 0, \text{ all } n \geq n_2 \text{ (see (16))}, \\
\Rightarrow \quad & \int_\Omega f(z, t_n u_n^+)(t_n u_n^+)dz \leq M_8 + \int_\Omega 2F(z, t_n u_n^+)dz \quad \text{for all } n \geq n_2.
\end{aligned}
$$

We use this inequality in (32) and obtain that

$$2\psi_+(t_n u_n^+) \leq M_8 \quad \text{for all } n \geq n_2 \text{ (see (9))}. \tag{33}$$

Comparing (30) and (33) we have a contradiction.

So, we have proved that

$$
\begin{aligned}
& \{u_n^+\}_{n\geq 1} \subseteq H^1(\Omega) \text{ is bounded}, \\
\Rightarrow \quad & \{u_n\}_{n\geq 1} \subseteq H^1(\Omega) \text{ is bounded (see (13))}.
\end{aligned}
$$

Hence we may assume that

$$u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \text{ and } u_n \to u \text{ in } L^\tau(\Omega) \text{ and in } L^2(\partial\Omega). \tag{34}$$

In (12) we choose $h = u_n - u \in H^1(\Omega)$, pass to the limit as $n \to +\infty$ and use (34). Then

$$\begin{aligned} & \lim_{n\to+\infty} \langle A(u_n), u_n - u\rangle = 0, \\ \Rightarrow \quad & \|\nabla u_n\|_2 \to \|\nabla u\|_2, \\ \Rightarrow \quad & u_n \to u \text{ in } H^1(\Omega) \text{ by the Kadec-Klee property (see (34)).} \end{aligned}$$

This proves Claim 1.

<u>Claim 2</u>: $u = 0$ is a local minimizer of the functional ψ_+.

Hypotheses $H(i), (iv)$ imply that given $\varepsilon > 0$, we can find $c_3 = c_3(\varepsilon) > 0$ such that

$$F(z,x) \leq \frac{1}{2}(\vartheta(z)+\varepsilon)x^2 + c_3|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{35}$$

Then for every $u \in H^1(\Omega)$ we have

$$\begin{aligned} \psi_+(u) &= \frac{1}{2}\eta(u^-) + \frac{\mu}{2}\|u^-\|_2^2 + \frac{1}{2}\eta(u^+) + \frac{\mu}{2}\|u^+\|_2^2 - \int_\Omega G_+(z,u^+)dz \\ &\geq \frac{c_0}{2}\|u^-\|^2 + \frac{1}{2}\eta(u^+) - \frac{1}{2}\int_\Omega \vartheta(z)(u^+)^2 dz - \frac{\varepsilon}{2}\|u^+\|^2 - c_4\|u\|^r \\ &\qquad \text{for some } c_4 > 0 \text{ (see (3), (9) and (35))} \\ &\geq \frac{c_0}{2}\|u^-\|^2 + \frac{c_5-\varepsilon}{2}\|u^+\|^2 - c_4\|u\|^r \\ &\qquad \text{for some } c_5 > 0 \text{ (see D'Aguì-Marano-Papageorgiou [6], Lemma 2.2).} \end{aligned}$$

Choosing $\varepsilon \in (0, c_5)$ we infer that

$$\psi_+(u) \geq c_6\|u\|^2 - c_4\|u\|^r \quad \text{for some } c_6 > 0, \text{ all } u \in H^1(\Omega). \tag{36}$$

Since $r > 2$, from (36) it follows that

$$u = 0 \text{ is a local minimizer of } \psi_+.$$

This proves Claim 2.

<u>Claim 3</u>: $K_{\psi_+} \setminus \{0\} \subseteq D_+$.

Let $u \in K_{\psi_+}$, $u \neq 0$. Then we have

$$\langle A(u), h\rangle + \int_\Omega (\xi(z)+\mu)uh dz + \int_{\partial\Omega} \beta(z)uh d\sigma = \int_\Omega g_+(z,u)h dz \quad \text{for all } h \in H^1(\Omega). \tag{37}$$

In (37) we choose $h = -u^- \in H^1(\Omega)$. Then

$$\begin{aligned} & \eta(u^-) + \mu\|u^-\|_2^2 = 0 \text{ (see (9))} \\ \Rightarrow \quad & c_0\|u^-\|^2 \leq 0 \text{ (see (3))}, \\ \Rightarrow \quad & u \geq 0, \ u \neq 0. \end{aligned}$$

Then using (9), we see that (37) becomes

$$\begin{aligned} & \langle A(u), h\rangle + \int_\Omega \xi(z)uh dz + \int_{\partial\Omega} \beta(z)uh d\sigma = \int_\Omega f(z,u)h dz \quad \text{for all } h \in H^1(\Omega), \\ \Rightarrow \quad & -\Delta u(z) + \xi(z)u(z) = f(z,u(z)) \quad \text{for a.a. } z \in \Omega, \\ & \frac{\partial u}{\partial n} + \beta(z)u = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{38}$$

(see Papageorgiou-Rădulescu [16]).

Hypotheses $H(i),(iv)$ imply that

$$|f(z,x)| \le c_7(|x|+|x|^{r-1}) \quad \text{for a.a. } z\in\Omega, \text{ all } x\in\mathbb{R}, \text{ some } c_7>0. \tag{39}$$

Then from (38) we have

$$-\Delta u(z) = \left[\frac{f(z,u(z))}{u(z)} - \xi(z)\right] u(z) \quad \text{for a.a. } z\in\Omega. \tag{40}$$

Let $a_0(z) = \dfrac{f(z,u(z))}{u(z)} - \xi(z)$. We have

$$\begin{aligned} |a_0(z)| &\le \frac{|f(z,u(z))|}{|u(z)|} + |\xi(z)| \\ &\le c_7(1+|u(z)|^{r-2}) + |\xi(z)| \quad \text{for a.a. } z\in\Omega \text{ (see (39))}. \end{aligned}$$

Since $u\in H^1(\Omega)$, using the Sobolev embedding theorem, we have

$$|u(\cdot)|^{r-2} \in L^{\frac{2^*}{r-2}}(\Omega).$$

By hypothesis $r<2^*$ and so we have

$$\frac{N}{2} < \frac{2^*}{r-2}.$$

Therefore

$$a_0\in L^\lambda(\Omega) \text{ with } \lambda > \frac{N}{2} \text{ (see hypothesis } H(\xi)).$$

Then from (40) and Lemma 5.1 of Wang [20] we have $u\in L^\infty(\Omega)$.

So, using hypotheses $H(i)$ and $H(\xi)$, we see that

$$z\to f(z,u(z)) - \xi(z)u(z) \text{ belongs in } L^s(\Omega),\ s>N.$$

Invoking Lemma 5.2 of Wang [20] (the Calderon-Zygmund estimates) we infer that $u\in C_+\setminus\{0\}$.

Hypotheses $H(i),(iv)$ imply that if $\rho=\|u\|_\infty$, then we can find $\widetilde{\xi}_\rho>0$ such that

$$f(z,x)x + \widetilde{\xi}_\rho x^2 \ge 0 \quad \text{for a.a. } z\in\Omega, \text{ all } |x|\le\rho. \tag{41}$$

From (38) and (41) it follows that

$$\begin{aligned} &\Delta u(z) \le (\|\xi^+\|_\infty + \widetilde{\xi}_\rho)u(z) \quad \text{for a.a. } z\in\Omega \text{ (see hypothesis } H(\xi)), \\ \Rightarrow\ & u\in D_+ \text{ (by the strong maximum principle)}. \end{aligned}$$

This proves Claim 3.

On account of Claim 3, we may assume that K_{ψ_+} is finite (otherwise we already have an infinity of distinct positive smooth solutions of problem (1) (see (9)) and so we are done). Claim 2 implies that we can find $\rho\in(0,1)$ small such that

$$0=\psi_+(0) < \inf[\psi_+(u) : \|u\|=\rho] = m_\rho^+ \tag{42}$$

(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 29).

Also, hypothesis $H(ii)$ implies that for every $u\in D_+$, we have

$$\psi_+(tu)\to -\infty \text{ as } t\to+\infty. \tag{43}$$

From (42), (43) and Claim 1, we see that we can apply Theorem 1 (the mountain pass theorem) and find $u_0 \in H^1(\Omega)$ such that

$$u_0 \in K_{\psi_+} \text{ and } m_\rho^+ \leq \psi_+(u_0).$$

It follows that $u_0 \in D_+$ is a positive solution of problem (1) (see Claim 3, (42) and (9)).

For the negative solution we argue in a similar fashion. So, we introduce the Carathéodory function $g_- : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$g_-(z,x) = \begin{cases} f(z,x) + \mu x & \text{if } x < 0, \\ 0 & \text{if } 0 \leq x. \end{cases}$$

We set $G_-(z,x) = \int_0^x g_-(z,s)ds$ and consider the C^1-functional $\psi_- : H^1(\Omega) \to \mathbb{R}$ defined by

$$\psi_-(u) = \frac{1}{2}\eta(u) + \frac{\mu}{2}\|u\|_2^2 - \int_\Omega G_-(z,u)dz \quad \text{for all } u \in H^1(\Omega).$$

Using ψ_- and reasoning as we did for ψ_+, we produce a negative solution $v_0 \in -D_+$. □

To produce a third nontrivial smooth solution, we will use tools from Morse theory (critical groups). The fact that ψ_+ are not C^2-functionals complicates things.

Proposition 2. *If hypotheses $H(\xi)$, $H(\beta)$, H hold, then $C_k(\psi_\pm, \infty) = 0$ for all $k \in N_0$.*

Proof. We will do the proof for the functional ψ_+, the proof for ψ_- being similar. Let $\partial B_1^+ = \{u \in H^1(\Omega) : \|u\| = 1, u^+ \neq 0\}$. We consider the deformation $h_+ : [0,1] \times \partial B_1^+ \to \partial B_1^+$ defined by

$$h_+(t,u) = \frac{(1-t)u + t\widehat{u}_1}{\|(1-t)u + t\widehat{u}_1\|} \quad \text{for all } (t,u) \in [0,1] \times \partial B_1^+.$$

We have

$$h_+(0,\cdot) = \mathrm{id}\big|_{\partial B_1^+} \text{ and } h_+(1,\cdot) = \frac{\widehat{u}_1}{\|\widehat{u}_1\|} \in \partial B_1^+ \quad (\text{recall that } \widehat{u}_1 \in D_+).$$

So, it follows that

$$\partial B_1^+ \text{ is contractible in itself.} \tag{44}$$

Hypotheses $H(i), (ii)$ imply that given any $k > 0$, we can find $c_8 = c_8(k) > 0$ such that

$$F(z,x) \geq \frac{k}{2}x^2 - c_8 \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{45}$$

Then for $u \in \partial B_1^+$ and $t > 0$, we have

$$\begin{aligned} \psi_+(tu) &= \frac{t^2}{2}\eta(u) + \frac{\mu}{2}t^2\|u\|_2^2 - \int_\Omega G_+(z,tu)dz \\ &\leq \frac{t^2}{2}\eta(u) + \frac{\mu}{2}t^2\|u\|_2^2 - \frac{k}{2}t^2\|u^+\|_2^2 - \frac{\mu}{2}t^2\|u^+\|_2^2 + c_8|\Omega|_N \text{ (see (9) and (45))} \\ &\leq \frac{t^2}{2}[\eta(u^-) + \mu\|u^-\|_2^2] + \frac{t^2}{2}[\eta(u^+) - k\|u^+\|_2^2 + c_8|\Omega|_N \end{aligned}$$

$$\leq \frac{t^2}{2}c_9 + \frac{t^2}{2}[c_{10} - k\|u^+\|_2^2] + c_8|\Omega|_N \quad \text{for some } c_9 > 0,\ c_{10} > 0$$
$$\text{(see hypotheses } H(\xi),\ H(\beta) \text{ and recall that } \|u\| = 1)$$
$$= \frac{t^2}{2}[c_{11} - k\|u^+\|_2^2] + c_8|\Omega|_N \quad \text{with } c_{11} = c_9 + c_{10} > 0.$$

Recall that $k > 0$ is arbitrary. We choose $k > \dfrac{c_{11}}{\|u^+\|_2^2}$. It follows that

$$\psi_+(tu) \to -\infty \text{ as } t \to +\infty \text{ for all } u \in \partial B_1^+. \tag{46}$$

For $u \in \partial B_1^+$ and $t > 0$, we have

$$\frac{d}{dt}\psi_+(tu) = \langle \psi_+'(tu), u\rangle \text{ (by the chain rule)}$$
$$= \frac{1}{t}\langle \psi_+'(tu), tu\rangle$$
$$= \frac{1}{t}\left[\langle A(tu), tu\rangle + \int_\Omega (\xi(z)+\mu)(tu)^2 dz + \int_{\partial\Omega} \beta(z)(tu)^2 d\sigma - \int_\Omega g_+(z,tu)(tu)dz\right]$$
$$= \frac{1}{t}\left[\eta(tu) + \mu\|tu\|_2^2 - \int_\Omega g_+(z,tu)(tu)dz\right]$$
$$\leq \frac{1}{t}\left[\eta(tu) + \mu\|tu\|_2^2 - \int_\Omega (2F(z,tu^+) + \mu(tu^+))dz + \|d\|_1\right]$$
(from hypothesis $H(iii)$ we have $0 \leq e(z,x) + d(z)$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$)
$$\leq \frac{1}{t}[2\psi_+(tu) + \|d\|_1]. \tag{47}$$

From (46) and (47) we see that

$$\psi_+(tu) < -\frac{\|d\|_1}{2} \text{ and } \frac{d}{dt}\psi_+(tu) < 0 \text{ for all } t > 0 \text{ big.} \tag{48}$$

We choose

$$m < \min\left\{-\frac{\|d\|_1}{2}, \inf_{\overline{B}_1} \psi_+\right\}. \tag{49}$$

From (48) and the implicit function theorem, we know that there exists $\zeta \in C(\partial B_1^+)$ with $\zeta \geq 1$ (see (49)) such that

$$\psi_+(tu) \begin{cases} > m & \text{if } t \in [0, \zeta(u)), \\ = m & \text{if } t = \zeta(u), \\ < m & \text{if } \zeta(u) < t. \end{cases} \tag{50}$$

From (49) and (50) it follows that

$$\psi_+^m = \{tu : u \in \partial B_1^+, t \geq \zeta(u)\}. \tag{51}$$

We set

$$E_+ = \{tu : u \in \partial B_1^+, t \geq 1\}.$$

Evidently we have

$$\psi_+^m \subseteq E_+ \text{(see (51))}.$$

We consider the deformation $\widetilde{h}_+ : [0,1] \times E_+ \to E_+$ defined by

$$\widetilde{h}_+(s, tu) = \begin{cases} (1-s)tu + s\zeta(u)u & \text{if } t \in [1, \zeta(u)), \\ tu & \text{if } \zeta(u) \leq t, \end{cases} \tag{52}$$

(recall that $\zeta \geq 1$). We have

$$\begin{aligned} &\widetilde{h}_+(s,\cdot)\big|_{\psi_+^m} = \mathrm{id}\big|_{\psi_+^m} \text{ for all } s \in [0,1] \text{ (see (51), (52))}, \\ &\widetilde{h}_+(1, E_+) \subseteq \psi_+^m \text{ (see (51)), (52))}. \end{aligned}$$

This means that

$$\begin{aligned} &\psi_+^m \text{ is a strong deformation retract of } E_+ \\ \Rightarrow\ & H_k(H^1(\Omega), \psi_+^m) = H_k(H^1(\Omega), E_+) \quad \text{for all } k \in \mathbb{N}_0 \\ & \text{(see Motreanu-Motreanu-Papageorgiou [14] (p. 143))}. \end{aligned} \tag{53}$$

Also, we consider the deformation $\overline{h}_+ : [0,1] \times E_+ \to E_+$ defined by

$$\overline{h}_+(s, tu) = (1-s)tu + s\frac{tu}{\|tu\|} \quad \text{for all } s \in [0,1], \text{ all } tu \in E_+.$$

We have

$$\begin{aligned} &\overline{h}_+(1, E_+) \subseteq \partial B_1^+, \\ &\overline{h}_+(1,\cdot)\big|_{\partial B_1^+} = \mathrm{id}\big|_{\partial B_1^+}. \end{aligned}$$

So, E_+ is deformable into ∂B_1^+ and the latter is a retract of E_+. So, from Theorem 6.5, p. 325 of Dugundji [7], we have

$$\begin{aligned} &\partial B_1^+ \text{ is a deformation retract of } E_+ \\ \Rightarrow\ & H_k(H^1(\Omega), \partial B_1^+) = H_k(H^1(\Omega), E_+) \quad \text{for all } k \in \mathbb{N}_0 \\ & \text{(see Motreanu-Motreanu-Papageorgiou [14] (p. 143))}, \\ \Rightarrow\ & H_k(H^1(\Omega), \partial B_1^+) = H_k(H^1(\Omega), \psi_+^m) \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (53))}. \end{aligned} \tag{54}$$

From (44) and Motreanu-Motreanu-Papageorgiou [14] (p. 147), we have

$$\begin{aligned} & H_k(H^1(\Omega), \partial B_1^+) = 0 \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow\ & H_k(H^1(\Omega), \psi_+^m) = 0 \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (54))}. \end{aligned} \tag{55}$$

Choosing $m < 0$ even more negative if necessary (see (49)), from (55) we infer that

$$C_k(\psi_+, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0 \text{ (recall } K_{\psi_+} \text{ is finite)}.$$

In a similar fashion, we show that

$$C_k(\psi_-, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

□

Let $\varphi : H^1(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem (1) defined by

$$\varphi(u) = \frac{1}{2}\eta(u) - \int_\Omega F(z,u)dz \quad \text{for all } u \in H^1(\Omega).$$

Evidently $\varphi \in C^1(H^1(\Omega))$.

Proposition 3. *If hypotheses $H(\xi)$, $H(\beta)$, H hold, then the functional φ satisfies the C-condition.*

Proof. Consider a sequence $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ such that

$$|\varphi(u_n)| \leq M_9 \quad \text{for some } M_9 > 0, \text{ all } n \in \mathbb{N}, \tag{56}$$

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \quad \text{in } H^1(\Omega)^* \text{ as } n \to +\infty. \tag{57}$$

From (57) we have

$$\left| \langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n h dz + \int_{\partial\Omega} \beta(z) u_n h d\sigma - \int_\Omega f(z, u_n) h dz \right| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \tag{58}$$
$$\text{for all } h \in H^1(\Omega), \text{ with } \varepsilon_n \to 0^+.$$

In (58) we choose $h = u_n \in H^1(\Omega)$. Then

$$-\eta(u_n) + \int_\Omega f(z, u_n) u_n dz \leq \varepsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{59}$$

Also, from (56) we have

$$\eta(u_n) - \int_\Omega 2F(z, u_n) dz \leq 2M_9 \quad \text{for all } n \in \mathbb{N}. \tag{60}$$

We add (59) and (60) and obtain that

$$\int_\Omega e(z, u_n) dz \leq M_{10} \quad \text{for some } M_{10} > 0, \text{ all } n \in \mathbb{N}. \tag{61}$$

As in the proof of Proposition 1, using (61) we will show the boundedness of $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$. We argue indirectly. So, suppose that

$$\|u_n\| \to +\infty. \tag{62}$$

Let $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. We have $\|y_n\| = 1$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \text{ and } y_n \to y \text{ in } L^\tau(\Omega) \text{ and in } L^2(\partial\Omega) \tag{63}$$
$$\left(\text{recall that } \tau = \max\left\{\frac{2s}{s-1}, r\right\}\right).$$

If $y \neq 0$, then as in the proof of Proposition 1, we reach a contradiction.
If $y = 0$, then for $k \geq 1$ we set $v_n = (2k)^{1/2} y_n$, $n \in \mathbb{N}$. We have

$$v_n \to 0 \text{ in } L^r(\Omega) \text{ (see (63))},$$
$$\Rightarrow \int_\Omega F(z, v_n) dz \to 0 \quad \text{(see hypothesis } H(i)). \tag{64}$$

From (62) we see that we can find $n_0 \in \mathbb{N}$ such that

$$(2k)^{1/2} \frac{1}{\|u_n\|} \leq 1 \quad \text{for all } n \geq n_0. \tag{65}$$

Let $t_n \in [0, 1]$ be such that

$$\varphi(t_n u_n) = \max[\varphi(t u_n) : 0 \leq t \leq 1], \quad \text{for all } n \in \mathbb{N}. \tag{66}$$

Then (65) and (66) imply that

$$\varphi(t_n u_n) \geq \varphi(v_n)$$

$$
\begin{aligned}
&= k\eta(y_n) - \int_\Omega F(z,v_n)dz \\
&= k[\eta(y_n) + \mu\|y_n\|_2^2] - \int_\Omega F(z,v_n)dz - \mu k\|y_n\|_2^2 \\
&\geq kc_0 - \int_\Omega F(z,v_n)dz - k\mu\|y_n\|_2^2 \\
&\qquad \text{(see (3) and recall that } \|y_n\| = 1,\ n \in \mathbb{N}) \\
&\geq k(c_0 - \mu\|y_n\|_2^2) - k\left|\int_\Omega F(z,v_n)dz\right| \quad \text{for all } n \geq n_0 \qquad (67)\\
&\qquad \text{(recall that } k \geq 1).
\end{aligned}
$$

From (63) (recall that $y = 0$) and (64), we see that we can find $n_1 \in \mathbb{N}$, $n_1 \geq n_0$ such that

$$
\|y_n\|_2^2 < \frac{c_0}{3\mu} \text{ and } \left|\int_\Omega F(z,v_n)dz\right| < \frac{c_0}{3\mu} \quad \text{for all } n \geq n_1. \tag{68}
$$

From (67) and (68) it follows that

$$
\varphi(t_n u_n) \geq k\frac{c_0}{3\mu} \quad \text{for all } n \geq n_1.
$$

But $k \geq 1$ is arbitrary. So, we infer that

$$
\varphi(t_n u_n) \to -\infty \text{ as } n \to +\infty. \tag{69}
$$

Using (69) and reasoning as in the proof of Proposition 1 (see Claim 1, the part of the proof after (30)), we reach again a contradiction. So, we conclude that φ satisfies the C-condition. □

We assume that K_φ is finite. Otherwise, we already have an infinity of distinct (smooth by the regularity theory of Wang [20]) solutions of problem (1) and so we are done. The finiteness of K_φ together with Proposition 3 permit the computation of the critical groups of φ at infinity.

Proposition 4. *If hypotheses $H(\xi)$, $H(\beta)$, H hold, then $C_k(\varphi,\infty) = 0$ for all $k \in N_0$.*

Proof. As in the proof of Proposition 2, using (45) we show that

$$
\varphi(tu) \to -\infty \text{ as } t \to +\infty \text{ for all } u \in \partial B_1 = \{v \in H^1(\Omega) : \|v\| = 1\}. \tag{70}
$$

For $u \in \partial B_1$ and $t > 0$, we have

$$
\begin{aligned}
\frac{d}{dt}\varphi(tu) &= \langle \varphi'(tu), u\rangle \text{ (by the chain rule)} \\
&= \frac{1}{t}\langle \varphi'(tu), tu\rangle \\
&= \frac{1}{t}\left[\eta(tu) - \int_\Omega f(z,tu)(tu)dz\right] \\
&\leq \frac{1}{t}\left[\eta(tu) - \int_\Omega 2F(z,tu)dz + \|d\|_1\right] \text{ (see hypothesis } H(iii)) \\
&= \frac{1}{t}[2\varphi(tu) + \|d\|_1]. \qquad (71)
\end{aligned}
$$

From (70) and (71) it follows that

$$\frac{d}{dt}\varphi(tu) < 0 \quad \text{for all } t > 0 \text{ big.}$$

As in the proof of Proposition 2, invoking the implicit function theorem we can find $\gamma \in C(\partial B_1)$ such that

$$\gamma > 0 \text{ and } \varphi(\gamma(u)u) = \rho_0 < -\frac{\|d\|_1}{2}.$$

We extend $\gamma(\cdot)$ on all of $H^1(\Omega) \setminus \{0\}$ by setting

$$\widehat{\gamma}(u) = \frac{1}{\|u\|}\gamma\left(\frac{u}{\|u\|}\right) \quad \text{for all } u \in H^1(\Omega) \setminus \{0\}.$$

Evidently $\widehat{\gamma} \in C(H^1(\Omega) \setminus \{0\})$ and $\varphi(\widehat{\gamma}(u)u) = \rho_0$. Moreover, we have

(72) $$\varphi(u) = \rho_0 \quad \Rightarrow \quad \widehat{\gamma}(u) = 1.$$

So, if we set

(73) $$\gamma_0(u) = \begin{cases} 1 & \text{if } \varphi(u) \leq \rho_0, \\ \widehat{\gamma}(u) & \text{if } \rho_0 < \varphi(u), \end{cases}$$

then we infer that

$$\gamma_0 \in C(H^1(\Omega) \setminus \{0\}) \text{ (see (72)).}$$

We consider the deformation $h : [0,1] \times (H^1(\Omega) \setminus \{0\}) \to H^1(\Omega) \setminus \{0\}$ defined by

$$h(t,u) = (1-t)u + t\gamma_0(u)u \quad \text{for all } (t,u) \in [0,1] \times (H^1(\Omega) \setminus \{0\}).$$

We have

$$\begin{aligned} &h(0,u) = u; \\ &h(1,u) = \gamma_0(u)u \in \varphi^{\rho_0} \text{ (see (72), (73));} \\ &h(t,\cdot)\big|_{\varphi^{\rho_0}} = \mathrm{id}\big|_{\varphi^{\rho_0}} \text{ (see (73)).} \end{aligned}$$

These properties imply that

(74) $$\varphi^{\rho_0} \text{ is a strong deformation retract of } H^1(\Omega) \setminus \{0\}.$$

Let $\widehat{r} : H^1(\Omega) \setminus \{0\} \to \partial B_1$ be the radial retraction map defined by

$$\widehat{r}(u) = \frac{u}{\|u\|} \quad \text{for all } u \in H^1(\Omega) \setminus \{0\}.$$

Let $h_0 : [0,1] \times (H^1(\Omega) \setminus \{0\}) \to H^1(\Omega) \setminus \{0\}$ be the deformation defined by

$$h_0(t,u) = (1-t)u + t\widehat{r}(u) \quad \text{for all } (t,u) \in [0,1] \times (H^1(\Omega) \setminus \{0\}).$$

This deformation shows that

(75) $$H^1(\Omega) \setminus \{0\} \text{ is deformable into } \partial B_1.$$

Moreover, the map $\widehat{r}(\cdot)$ shows that

(76) $$\partial B_1 \text{ is a retract of } H^1(\Omega) \setminus \{0\}.$$

From (75), (76) and Theorem 6.5, p. 325 of Dugundji [7], we infer that

(77) $$\partial B_1 \text{ is a deformation retract of } H^1(\Omega) \setminus \{0\}.$$

From (74) and (77) it follows that

$$
\begin{aligned}
&\varphi^{\rho_0} \text{ and } \partial B_1 \text{ are homotopy equivalent,}\\
\Rightarrow\quad & H_k(H^1(\Omega),\varphi^{\rho_0}) = H_k(H^1(\Omega),\partial B_1) \quad \text{for all } k\in\mathbb{N}_0 \qquad (78)\\
&\qquad\qquad \text{(see Motreanu-Motreanu-Papageorgiou [14] (p. 143)).}
\end{aligned}
$$

The space $H^1(\Omega)$ is infinite dimensional and so it follows that ∂B_1 is contractible (see Gasiński-Papageorgiou [10], Problems 4.154, 4.159). So, we have

$$
\begin{aligned}
& H_k(H^1(\Omega),\partial B_1) = 0 \quad \text{for all } k\in\mathbb{N}_0,\\
&\qquad\qquad \text{(see Motreanu-Motreanu-Papageorgiou [14] (p. 147)),}\\
\Rightarrow\quad & H_k(H^1(\Omega),\varphi^{\rho_0}) = 0 \quad \text{for all } k\in\mathbb{N}_0 \text{ (see (78)).}
\end{aligned}
$$

Choosing $\rho_0 < -\frac{\|d\|_1}{2}$ even more negative if necessary, we conclude that

$$
C_k(\varphi,\infty) = 0 \quad \text{for all } k\in\mathbb{N}_0.
$$

□

From the proof of Proposition 1 we know that $u_0\in D_+$, the positive solution of problem (1), is a critical point of ψ_+ of mountain pass type. So, we have

$$
C_1(\psi_+,u_0) \neq 0 \qquad (79)
$$

(see Motreanu-Motreanu-Papageorgiou [14] (p. 168)). From (9) it is clear that

$$
\psi_+\Big|_{C_+} = \varphi\Big|_{C_+}.
$$

Since $u_0\in D_+$ and $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$, from Chang [5] (p. 14), we infer that

$$
\begin{aligned}
& C_k(\varphi,u_0) = C_k(\psi_+,u_0) \quad \text{for all } k\in\mathbb{N}_0, \qquad (80)\\
\Rightarrow\quad & C_1(\varphi,u_0)\neq 0 \text{ (see (79)).}
\end{aligned}
$$

Because φ is not a C^2-functional (recall that $f(z,\cdot)$ is only continuous), in general we can not say that

$$
C_k(\varphi,u_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k\in\mathbb{N}_0.
$$

Similarly for the negative solution $v_0\in -D_+$ (see Proposition 1).

The next proposition, shows that although φ lacks smoothness, we can still have a precise computation of the critical groups of φ at $u_0\in D_+$ and at $v_0\in -D_+$.

Proposition 5. *If hypotheses $H(\xi)$, $H(\beta)$, H hold and $u_0\in D_+$, $v_0\in -D_+$, from Proposition 1, are the only nontrivial constant sign solutions of problem* (1), *then $C_k(\varphi,u_0) = C_k(\varphi,v_0) = \delta_{k,1}\mathbb{Z}$ for all $k\in\mathbb{N}_0$.*

Proof. Recall that

$$
K_{\psi_+} \subseteq D_+\cup\{0\}
$$

(see the proof of Proposition 1, Claim 3). We know that

$$
\varphi'\Big|_{C_+} = \psi_+'\Big|_{C_+} \text{ (see (9)).}
$$

The critical set K_φ is the set of positive solutions of (1) and by hypothesis $u_0\in D_+$ is the only nontrivial positive solution of problem (1). So, we have

$$
K_{\psi_+} = \{0,u_0\}. \qquad (81)
$$

We choose $\lambda < 0 < s < \psi_+(u_0) = \varphi(u_0)$ (see the proof of Proposition 1). We consider the following triple of sets

$$\psi_+^\lambda \subseteq \psi_+^s \subseteq H^1(\Omega).$$

For this triple of sets, we consider the corresponding long exact sequence of singular homology groups (for notational simplicity, we set $H = H^1(\Omega)$).

$$\cdots \to H_k(H, \psi_+^\lambda) \xrightarrow{i_*} H_k(H, \psi_+^s) \xrightarrow{\widehat{\partial}_*} H_{k-1}(\psi_+^s, \psi_+^\lambda) \to \cdots \tag{82}$$

(see Motreanu-Motreanu-Papageorgiou [14] (pp. 143-144)). In (82) i_* is the homomorphism induced by the inclusion

$$(H, \psi_+^\lambda) \hookrightarrow^i (H, \psi_+^s),$$

while $\widehat{\partial}_*$ is the composed boundary homomorphism (see Motreanu-Motreanu-Papageorgiou [14] (p. 144)).

Since $\lambda < 0 = \psi_+(0)$, from (81) it follows that

$$H_k(H, \psi_+^\lambda) = C_k(\psi_+, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0 \text{ (see Proposition 2).} \tag{83}$$

Also, since $s \in (0, \psi_+(u_0))$, from (81) and Motreanu-Motreanu-Papageorgiou [14] (p. 157), we have

$$H_{k-1}(\psi_+^s, \psi_+^\lambda) = C_{k-1}(\psi_+, 0) = \delta_{k-1,0}\mathbb{Z} = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \tag{84}$$

(see the proof of Proposition 2, Claim 2).

Similarly, we have

$$H_k(H, \psi_+^s) = C_k(\psi_+, u_0) \quad \text{for all } k \in \mathbb{N}_0. \tag{85}$$

Taking into account (83), (84), (85), we see that in (82) only the tail (that is, $k = 1$) of the long exact sequence, is nontrivial.

From the rank theorem, we have

$$\begin{aligned}
\text{rank } C_1(\psi_+, u_0) &= \text{rank ker } \widehat{\partial}_* + \text{rank im } \widehat{\partial}_* \text{ (see (85))} \\
&= \text{rank im } i_* + \text{rank im } \widehat{\partial}_* \text{ (because (82) is exact)} \\
&= \text{rank im } \widehat{\partial}_* \text{ (see (83))} \\
&\leq 1 \text{ (see (84))} \\
&\Rightarrow \quad C_k(\psi_+, u_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \\
&\qquad \text{(see (79) and recall that only the tail of (82) is nontrivial)} \\
&\Rightarrow \quad C_k(\varphi, u_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (80)).}
\end{aligned}$$

Similarly we show that

$$C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

□

Now we are ready for the multiplicity theorem (three solutions theorem).

Theorem 2. *If hypotheses $H(\xi)$, $H(\beta)$, H hold, then problem* (1) *has at least three nontrivial smooth solutions $u_0 \in D_+$, $v_0 \in -D_+$, $y_0 \in C^1(\overline{\Omega}) \setminus \{0\}$.*

Proof. From Proposition 1, we already have two nontrivial constant sign smooth solutions

$$u_0 \in D_+ \text{ and } v_0 \in -D_+.$$

We assume that

$$K_\varphi = \{0, u_0, v_0\}. \tag{86}$$

Otherwise we already have a third nontrivial solution which is in $C^1(\overline{\Omega})$ (by the regularity theory of Wang [20]). From (86) and Proposition 5, we have

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{87}$$

As in the proof of Proposition 1, Claim 2, we show that

$$u = 0 \text{ is a local minimizer of } \varphi,$$
$$\Rightarrow \quad C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{88}$$

From Proposition 4 we have

$$C_k(\varphi, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{89}$$

From (86), (87), (88), (89) and the Morse relation with $t = -1$, we have

$$2(-1)^1 + (-1)^0 = 0,$$

a contradiction. So, we can find $y_0 \in H^1(\Omega)$ such that

$$y_0 \in K_\varphi \text{ and } y_0 \notin \{0, u_0, v_0\}.$$

This is the third nontrivial solution of problem (1) and the regularity theory of Wang [20] implies that $y_0 \in C^1(\overline{\Omega}) \setminus \{0\}$. ☐

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