TRACE CODIMENSIONS OF ALGEBRAS AND THEIR EXPONENTIAL GROWTH

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Abstract. The trace codimensions give a quantitative description of the identities satisfied by an algebra with trace. Here we study the asymptotic behaviour of the sequence of trace codimensions $c_n^{tr}(A)$ and of pure trace codimensions $c_n^{ptr}(A)$ of a finite dimensional algebra A over a field of characteristic zero. We find an upper and lower bound of both codimensions and as a consequence we get that the limits $\lim_{n\to\infty}\sqrt[n]{c_n^{tr}(A)}$ and $\lim_{n\to\infty}\sqrt[n]{c_n^{ptr}(A)}$ always exist and are integers. This result gives a positive answer to a conjecture of Amitsur in this setting. Finally we characterize the varieties of algebras whose exponential growth is bounded by 2.

1. Introduction

This paper is concerned with finite dimensional algebras over a field F of characteristic zero endowed with a trace function. Recall that a trace function on an algebra A over F is an F-linear map $\mathrm{tr}: A \to Z(A)$, where $Z(A)$ is the center of A, satisfying $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ and $\mathrm{tr}(\mathrm{tr}(a)b) = \mathrm{tr}(a)\mathrm{tr}(b)$, for all $a, b \in A$. A typical example is $M_k(F)$, the algebra of $k \times k$ matrices over F with the usual trace.

One defines in a natural way $F\langle X, \mathrm{Tr}\rangle$, the free algebra with (formal) trace Tr over F on a countable set X. Its elements are called trace polynomials and the elements of the commutative subalgebra generated by the elements $\mathrm{Tr}(M)$, where M is any monomial in the elements of X, are called pure trace polynomials. Whenever any such polynomial f vanishes in a given algebra with trace A, we say that f is a trace identity or a pure trace identity of A, respectively.

For instance since the characteristic of F is zero, the coefficients of a Cayley-Hamilton polynomial $CH_k(a)$ of $a \in M_k(F)$, can be expressed as pure trace polynomials evaluated in a ([1]). Hence $CH_k(x)$ is an example of a trace identity of $M_k(F)$.

The set of trace identities of a given algebra A is a T-ideal $\mathrm{Id}^{tr}(A)$ of $F\langle X, Tr\rangle$, i.e., an ideal invariant under all endomorphisms of the free algebra with trace and this is the main object of our study. Probably the most significant result obtained in this area is due to Procesi ([15]) and Razmyslov ([18]) who showed that the trace identities of $M_k(F)$ are consequences of the Cayley-Hamilton polynomial $CH_k = CH_k(x)$; in other words CH_k generates $\mathrm{Id}^{tr}(M_k(F))$ as a T-ideal. Another important and inspiring result is due to Procesi ([16]) who proved that if an algebra with trace A satisfies a formal Cayley-Hamilton polynomial of degree k, then it can be embedded in $k \times k$ matrices over a commutative algebra and the trace on A coincides with the usual trace of matrices.

Here we want to study T-ideals of trace identities through some growth functions that can the attached to them. This point of view comes from the theory of polynomial identities. Recall that an algebra A over F is a PI-algebra if it satisfies a non-trivial polynomial identity, i.e., a polynomial of the free algebra $F\langle X\rangle$ without trace.

The polynomial identities of A form a T-ideal in $F\langle X\rangle$ denoted $\mathrm{Id}(A)$. It is known that over a field of characteristic zero, every proper T-ideal is finitely generated as a T-ideal ([14]). Nevertheless to exhibit such generators is a difficult problem. For instance the generators of $\mathrm{Id}(M_k(F))$ are known only for $k \le 2$ ([5, 17]). Then one introduces a growth function that can be attached to any T-ideal ([19]) as follows. Because of the hypothesis on the characteristic, one studies only multilinear polynomials and let P_n be the space of multilinear polynomials in the variables $x_1, \ldots, x_n$. Let A be any PI-algebra and let $c_n(A)$ be the dimension of P_n modulo the identities of A, i.e., $c_n(A) = \dim_F \frac{P_n}{P_n \cap \mathrm{Id}(A)}$. It was proved in [19] that the sequence $c_n(A)$, $n = 1, 2, \ldots$, called the sequence of codimensions of A, is exponentially bounded. Later in [6] and [7] it was also proved that for some constants $C_1 > 0, C_2, t, s, d$ we have that

$$C_1 n^t d^n \le c_n(A) \le C_2 n^s d^n,$$

where d is an integer. As a consequence we get that $d = \exp(A) = \lim_{n\to\infty}\sqrt[n]{c_n(A)}$ exists. It is called the PI-exponent of A. This answers in the positive a conjecture of Amitsur. It should be also mentioned that in [2, 4] (see also [10]) it was proved that in the above inequalities $t = s$ is a half integer and if $1 \in A$, $C_1 = C_2$.

Here we want to extend the above result to trace identities. As in the ordinary case since the trace identities are completely determined by their multilinear polynomials, we consider MT_n, the space of multilinear trace polynomials in the first n variables $x_1, \ldots, x_n$ and PT_n, the space of pure trace polynomials in the same variables. Then we define

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two codimensions

$$c_n^{tr}(A) = \dim_F \frac{MT_n}{MT_n \cap \mathrm{Id}^{tr}(A)} \qquad \text{and} \qquad c_n^{ptr}(A) = \dim_F \frac{PT_n}{PT_n \cap \mathrm{Id}^{ptr}(A)},$$

called the n-th trace codimension and the n-th pure trace codimension of A, respectively.

The asymptotic behaviour of the trace codimensions of the matrix algebra $M_k(F)$ was studied by Regev ([21]) who proved that the codimensions and the trace codimensions of $M_k(F)$ are asymptotically equal.

In general even if a PI-algebra has a trace function, the trace codimensions could have an overexponential behavior. Nevertheless recently Berele in [3] proved a trace analogue of Regev's theorem by showing that a PI-algebra has exponentially bounded trace codimensions if and only if it satisfies a special trace identity.

In [11] the authors characterized algebras with trace with polynomially bounded trace codimensions. An analogous characterization concerning the pure trace codimensions will be proved here (see Theorem 21).

The main results of this paper are as follows. We shall consider unitary finite dimensional algebras with trace tr over a field F of characteristic zero. We shall assume that $\mathrm{tr}(J) = 0$, where J is the Jacobson radical of the algebra A. In analogy with the ordinary case, we shall prove that $c_n^{ptr}(A)$ and $c_n^{tr}(A)$ are bounded from above and below, up to a polynomial factor, by d^n and d'^n, respectively, where d and d' are two integers that are the dimensions of suitable semisimple subalgebras called trace admissible. As a consequence we shall obtain that such codimensions either are polynomially bounded or grow exponentially. Moreover $\lim_{n\to\infty} \sqrt[n]{c_n^{tr}(A)}$ and $\lim_{n\to\infty} \sqrt[n]{c_n^{ptr}(A)}$ exist and are integers. We refer to such limits as the trace exponent $\exp^{tr}(A)$ and the pure trace exponent $\exp^{ptr}(A)$ of A, respectively.

In the last section we shall characterize the identities of finite dimensional algebras with trace having trace exponent and pure trace exponent bounded by 2. As a consequence, we obtain new results concerning the varieties of trace algebras of minimal (pure) trace exponent.

2. The general setting

Let F be a field of characteristic zero and A a unitary finite dimensional associative F-algebra with trace tr (a trace algebra). Recall that a trace tr on A is an F-linear map $\mathrm{tr}\colon A \to A$ satisfying, for any $a, b \in A$, the following axioms:

1. $\mathrm{tr}(a)b = b\mathrm{tr}(a)$,
2. $\mathrm{tr}(ab) = \mathrm{tr}(ba)$,
3. $\mathrm{tr}(\mathrm{tr}(a)b) = \mathrm{tr}(a)\mathrm{tr}(b)$.

Notice that, for any $a \in A$, $\mathrm{tr}(a) \in Z(A)$, where $Z(A)$ denotes the center of the algebra A.

In what follows we shall consider only trace algebras whose trace takes values in $F \equiv F \cdot 1$, where 1 is the unit of the algebra.

In order to talk about polynomial identities in the setting of algebras with trace, the first step is to introduce $F\langle X, \mathrm{Tr}\rangle$, the free algebra with trace on a countable set $X = \{x_1, x_2, \ldots\}$, where Tr is a formal trace. Let $\mathcal{M}$ denote the set of monomials in the elements of X. Then $F\langle X, \mathrm{Tr}\rangle$ is the algebra generated by the free algebra $F\langle X\rangle$ together with the set of central (commuting) elements $\mathrm{Tr}(M)$, $M \in \mathcal{M}$, subject to the conditions $\mathrm{Tr}(MN) = \mathrm{Tr}(NM)$ and $\mathrm{Tr}(\mathrm{Tr}(M)N) = \mathrm{Tr}(M)\mathrm{Tr}(N)$, for all M, $N \in \mathcal{M}$. In other words,

$$F\langle X, \mathrm{Tr}\rangle \cong F\langle X\rangle \otimes F[\mathrm{Tr}(M) : M \in \mathcal{M}].$$

The elements of the free algebra with trace are called trace polynomials or pure trace polynomials in case all the variables appear inside a trace.

An element $f = f(x_1, \ldots, x_n, \mathrm{Tr}) \in F\langle X, \mathrm{Tr}\rangle$ is a trace identity for A if, after substituting the variables x_i with arbitrary elements $a_i \in A$ and Tr with the trace tr, we obtain 0. A pure trace polynomial is called a pure trace identity. We denote by $\mathrm{Id}^{tr}(A)$ (resp. $\mathrm{Id}^{ptr}(A)$) the set of trace identities (resp. pure trace identities) of A, which is a trace T-ideal (T^{tr}-ideal) of the free algebra with trace, i.e., an ideal invariant under all endomorphisms of $F\langle X, \mathrm{Tr}\rangle$.

As in the ordinary case, $\mathrm{Id}^{tr}(A)$ and $\mathrm{Id}^{ptr}(A)$ are completely determined by their multilinear polynomials. We denote by MT_n the vector space of multilinear trace polynomials in the first n variables $x_1, \ldots, x_n$. Its elements are linear combinations of expressions of the type

$$\mathrm{Tr}(x_{i_1} \cdots x_{i_a}) \cdots \mathrm{Tr}(x_{j_1} \cdots x_{j_b}) x_{l_1} \cdots x_{l_c},$$

where $\{i_1, \ldots, i_a, \ldots, j_1, \ldots, j_b, l_1, \ldots, l_c\} = \{1, \ldots, n\}$.

The non-negative integer

$$c_n^{tr}(A) = \dim_F \frac{MT_n}{MT_n \cap \mathrm{Id}^{tr}(A)}$$

is called the n-th trace codimension of A.

Also we denote by PT_n the vector space of multilinear pure trace polynomials:

$$PT_n = \mathrm{span}_F \left\{ \mathrm{Tr}(x_{i_1} \cdots x_{i_a}) \cdots \mathrm{Tr}(x_{j_1} \cdots x_{j_b}) : \{i_1, \ldots, j_b\} = \{1, \ldots, n\} \right\}.$$

The n-th pure trace codimension of A is defined as

$$c_n^{ptr}(A) = \dim_F \frac{PT_n}{PT_n \cap \mathrm{Id}^{tr}(A)}.$$

The vector spaces MT_n and PT_{n+1} are isomorphic and we have that

$$\dim_F MT_n = \dim_F PT_{n+1} = (n+1)!$$

(see [1, Proposition 2.3.15]).

For any non-zero $\alpha \in F$, let $\varphi_\alpha \colon F\langle X, \mathrm{Tr}\rangle \to F\langle X, \mathrm{Tr}\rangle$ be the linear map sending a monomial m into $\alpha^s m$, where s is the number of traces appearing in m. Notice that:

1. $\varphi_\alpha(fg) = \varphi_\alpha(f)\varphi_\alpha(g)$, for any $f, g \in F\langle X, \mathrm{Tr}\rangle$,
2. $\varphi_{\alpha^{-1}}(\varphi_\alpha(f)) = \varphi_\alpha(\varphi_{\alpha^{-1}}(f)) = f$, for any $f \in F\langle X, \mathrm{Tr}\rangle$.

For any $f \in F\langle X, \mathrm{Tr}\rangle$ we shall denote by $f^\alpha = \varphi_\alpha(f)$.

Given any algebra (A, t) with trace t, for any non-zero $\alpha \in F$, we denote by t_α the corresponding proportional trace to t, i.e., $t_\alpha = \alpha t$. The following result holds.

Lemma 1. *Let $f \in F\langle X, Tr\rangle$. Then $f \in Id^{tr}((A,t))$ if and only if $\varphi_{\alpha^{-1}}(f) = f^{\alpha^{-1}} \in Id^{tr}((A, t_\alpha))$.*

Proof. The result follows since, for any $a_1, \ldots, a_k \in A$, $f(a_1, \ldots, a_k, t) = f^{\alpha^{-1}}(a_1, \ldots, a_k, t_\alpha)$. □

Now we are in a position to prove the following theorem.

Theorem 2. *Let (A,t) be an algebra with trace and assume that each trace identity of (A,t) is a consequence of some trace polynomials $f_1, \ldots, f_n$. For any non-zero $\alpha \in F$, if $f = f(x_1, \ldots, x_k) \in Id^{tr}((A, t_\alpha))$, then f is a consequence of $f_1^{\alpha^{-1}}, \ldots, f_n^{\alpha^{-1}}$.*

Proof. By Lemma 1, we have that $f \in \mathrm{Id}^{tr}((A,t_\alpha))$ if and only if $\varphi_\alpha(f) = f^\alpha \in \mathrm{Id}^{tr}((A,t))$. By hypothesis, f^α is a consequence of $f_1, \ldots, f_n$. It follows that

$$f^\alpha = \sum_{i=1}^{n} \left(\sum_j k_i^{(j)} f_i \left(l_{i_1}^{(j)}, \ldots, l_{i_t}^{(j)} \right) h_i^{(j)} \right),$$

where the $k_i^{(j)}$'s, the $h_i^{(j)}$'s and the $l_{i_p}^{(j)}$'s are suitable trace polynomials. Now, as a consequence of the properties of the map φ_α, we get

$$\begin{aligned} f &= \varphi_{\alpha^{-1}}(\varphi_\alpha(f)) = \varphi_{\alpha^{-1}}(f^\alpha) \\ &= \varphi_{\alpha^{-1}} \left(\sum_{i=1}^{n} \left(\sum_j k_i^{(j)} f_i \left(l_{i_1}^{(j)}, \ldots, l_{i_t}^{(j)} \right) h_i^{(j)} \right) \right) \\ &= \sum_{i=1}^{n} \left(\sum_j \varphi_{\alpha^{-1}} \left(k_i^{(j)} \right) f_i^{\alpha^{-1}} \left(l_{i_1}^{(j)}, \ldots, l_{i_t}^{(j)} \right) \varphi_{\alpha^{-1}} \left(h_i^{(j)} \right) \right). \end{aligned}$$

In conclusion, f is a consequence of $f_1^{\alpha^{-1}}, \ldots, f_n^{\alpha^{-1}}$ and the proof is complete. □

Now let us consider the algebra $M_k(F)$ of $k \times k$ matrices over F endowed with the usual trace t, defined, for each matrix, as the sum of all the elements in the main diagonal. We remark that every trace on $M_k(F)$ is proportional to t, i.e, if tr is a trace on $M_n(F)$, then there exists $\alpha \in F$ such that $\mathrm{tr} = \alpha t$. We shall use the notation $M_k^{t_\alpha}$ to indicate the algebra $M_k(F)$ endowed with the trace t_α. In particular $M_k^{t_1}$ will denote the algebra $M_k(F)$ with the usual trace.

Given a permutation $\sigma = (i_1 \cdots i_{r_1})(j_1 \cdots j_{r_2}) \cdots (l_1 \cdots l_{r_t}) \in S_{k+1}$ written as the product of disjoint cycles, including one-cycles with $r_1 \geq r_2 \geq \cdots \geq r_t$, we associate to it the pure trace monomial:

$$ptr_\sigma = \mathrm{Tr}\left(x_{i_1} \cdots x_{i_{r_1}}\right) \mathrm{Tr}\left(x_{j_1} \cdots x_{j_{r_2}}\right) \cdots \mathrm{Tr}\left(x_{l_1} \cdots x_{l_{r_t}}\right).$$

In the following theorem we recall a celebrated result proved independently by Procesi and Razmyslov ([15, 18]).

Theorem 3.

a) $\mathrm{Id}^{ptr}(M_k^{t_1})$ is generated by the single pure trace polynomial $g = \sum_{\sigma \in S_{k+1}} \mathrm{sgn}(\sigma) ptr_\sigma$.

b) $\mathrm{Id}^{tr}(M_k^{t_1})$ is generated by the k-th Cayley-Hamilton polynomial CH_k.

As a consequence of the previous two theorems we get the following.

Corollary 4. *For any non-zero $\alpha \in F$, $\mathrm{Id}^{ptr}(M_k^{t_\alpha})$ is generated by the pure trace polynomial $g^{\alpha^{-1}}$ and $\mathrm{Id}^{tr}(M_k^{t_\alpha})$ is generated by the trace polynomial $CH_k^{\alpha^{-1}}$. Moreover*

$$c_n^{ptr}(M_k^{t_\alpha}) = c_n^{ptr}(M_k^{t_1}) \quad \text{and} \quad c_n^{tr}(M_k^{t_\alpha}) = c_n^{tr}(M_k^{t_1}).$$

Recall that two functions $f(x), g(x)$ of a real variable are asymptotically equal, and we write $f(x) \simeq g(x)$, if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. We have the following.

Theorem 5. *For any non-zero $\alpha \in F$, $c_n^{ptr}(M_k^{t_\alpha}) \simeq Cn^{-\frac{k^2-1}{2}}(k^2)^n$, where C is a constant.*

Proof. By the previous corollary $c_n^{ptr}(M_k^{t_\alpha}) = c_n^{ptr}(M_k^{t_1})$. Now the result follows from [20]. □

3. A candidate for the trace exponent and the pure trace exponent

In what follows A will always denote a unitary finite dimensional trace algebra over an algebraically closed field F of characteristic zero. By the Wedderburn-Malcev theorem for algebras with trace ([11, Theorem 26]) we can write

$$A = \bar{A} + J,$$

where $\bar{A}$ is a maximal semisimple subalgebra of A and $J = J(A)$ is the Jacobson radical of A. Moreover, since F is algebraically closed we have that

$$\bar{A} = A_1 \oplus \cdots \oplus A_m = M_{n_1}(F) \oplus \cdots \oplus M_{n_m}(F), \tag{1}$$

where $M_{n_i}(F)$ is the simple algebra of $n_i \times n_i$ matrices over F, $i = 1, \ldots, m$. Clearly $\bar{A}$ is a trace-subalgebra, i.e., it is stable under the trace.

From now on we shall assume that the trace on J is zero.

We should remark that each simple component in (1) is not in general a trace-subalgebra. Nevertheless for every simple component A_i we consider

$$\mathrm{tr} : A_i \to F \cdot 1_{\bar{A}} \to F \cdot 1_{A_i},$$

where the second map is the projection onto the i-th component. In this way we may think of A_i as a simple algebra with induced trace. A similar remark applies to a direct sum of simple components.

Now, for any $a = a_1 + \cdots + a_m + j \in A$, $a_i \in M_{n_i}(F)$, $j \in J$,

$$\mathrm{tr}(a) = \mathrm{tr}(a_1) + \cdots + \mathrm{tr}(a_m),$$

since $\mathrm{tr}(j) = 0$. Let us denote by $e_{ij}^{(k)}$ the matrix units of the full matrix algebra $M_{n_k}(F)$. By taking into account that $\mathrm{tr}(ab) = \mathrm{tr}(ba)$, for any $a, b \in A$, it easily follows that $\mathrm{tr}(e_{ij}^{(k)}) = 0$ if $i \neq j$ and $\mathrm{tr}(e_{ii}^{(k)}) = \mathrm{tr}(e_{jj}^{(k)})$, fo any i, j. Hence, for any matrix $a_k \in M_{n_k}(F)$, $a_k = \sum_{i,j} a_{ij} e_{ij}^{(k)}$, we get that

$$\mathrm{tr}(a_k) = \mathrm{tr}\Big(\sum_{i,j} a_{ij} e_{ij}^{(k)}\Big) = \sum_{j=1}^{n_k} a_{jj} \mathrm{tr}\Big(e_{jj}^{(k)}\Big) = \mathrm{tr}\Big(e_{11}^{(k)}\Big) t_1^k(a),$$

where t_1^k is the usual trace on the matrix algebra $M_{n_k}(F)$. In conclusion, if we write $\alpha_k = \mathrm{tr}\Big(e_{11}^{(k)}\Big) \in F$ and $t_{\alpha_k} = \alpha_k t_1^k$, $k = 1, \ldots, m$, we get that

$$\mathrm{tr}(a) = \mathrm{tr}(a_1 + \cdots + a_m + j) = \sum_{k=1}^{m} t_{\alpha_k}(a_k).$$

One of the aims of this paper is to determine the exponential rate of growth of the sequences of the trace and pure trace codimensions of A. We make the following.

Definition 6. *Let $C_1, \ldots, C_k$ be distinct subalgebras of A from the set $\{A_1, \ldots, A_m\}$. The algebra $C = C_1 \oplus \cdots \oplus C_k$ is called*

- *admissible if $C_1 J \cdots J C_{k-1} J C_k \neq 0$,*
- *pure trace admissible if $tr(C_1) \cdots tr(C_k) \neq 0$,*
- *trace admissible if $tr(C_1) \cdots tr(C_l) C_{l+1} J \cdots J C_{l+m-1} J C_{l+m} \neq 0$, for some $l, m \geq 0, l + m = k$.*

Notice that an admissible or pure trace admissible algebra is trace admissible. Clearly a pure trace admissible algebra takes into account only the simple components with non-zero trace. The algebra $C_1 \oplus \cdots \oplus C_k + J$ is called a reduced algebra in the first case, a pure trace reduced algebra in the second case and a trace reduced algebra in the last case.

We then define the following three integers:

$$d_1 = d_1(A) = \max\ (\dim C),\ \text{ where } C \text{ runs over all admissible subalgebras of } \bar{A}, \tag{2}$$

$$d_2 = d_2(A) = \max\ (\dim C),\ \text{ where } C \text{ runs over all pure trace admissible subalgebras of } \bar{A}, \tag{3}$$

$$d = d(A) = \max\ (\dim C),\ \text{ where } C \text{ runs over all trace admissible subalgebras of } \bar{A}. \tag{4}$$

It is well-known that d_1 equals $\lim_{n\to\infty} \sqrt[n]{c_n(A)}$ and is the PI-exponent $\exp(A)$ of the algebra A ([6, 7]).

Here we shall prove that

$$d_2 = \lim_{n\to\infty} \sqrt[n]{c_n^{ptr}(A)} \quad \text{and} \quad d = \lim_{n\to\infty} \sqrt[n]{c_n^{tr}(A)}.$$

In order to reach our goal we shall show that $c_n^{ptr}(A)$ and $c_n^{tr}(A)$ are bounded from above and below, up to a polynomial factor, by d_2^n and d^n, respectively.

4. A combinatorial approach to the lower bound

Let $A = A_1 \oplus \cdots \oplus A_m + J$ be a unitary finite dimensional algebra with trace tr over an algebraically closed field F of characteristic zero, $\mathrm{tr}(J) = 0$. The goal of this section is to find a lower bound for $c_n^{ptr}(A)$ and $c_n^{tr}(A)$.

We start with the following remark.

Remark 7. *For $n \geq 1$, let $f_1, \dots, f_t \in PT_n$ be linearly independent modulo $\mathrm{Id}^{ptr}(A)$. If $g_1, \dots, g_t \in PT_{n+1}$ are obtained from the f_i's by substituting a fixed variable, say x_1, with x_1x_{n+1}, then $g_1, \dots, g_t$ are still linearly independent modulo $\mathrm{Id}^{ptr}(A)$. In particular $c_n^{ptr}(A) \leq c_{n+1}^{ptr}(A)$ and the pure trace codimensions are non decreasing.*

Proof. Suppose that $\sum_{i=1}^n \alpha_i g_i \in \mathrm{Id}^{ptr}(A)$, for some scalars α_i's. By evaluating the variable x_{n+1} into the unit element of A we get $\sum_{i=1}^n \alpha_i f_i \in \mathrm{Id}^{ptr}(A)$. Since the f_i's are linearly independent mod $\mathrm{Id}^{ptr}(A)$, then $\alpha_i = 0$, for all i. □

Notice that the result is still true if we take $f_1, \dots, f_t \in MT_n$. Hence $c_n^{tr}(A) \leq c_{n+1}^{tr}(A)$ and the trace codimensions are non decreasing.

Now for some $k \in \{1, \dots, m\}$, assume that

$$\mathrm{tr}(A_1) \cdots \mathrm{tr}(A_k) \neq 0.$$

Take any $n \geq 1$ and write $n = n_1 + \cdots + n_k$ as a sum of non-negative integers. Divide the variables $x_1, \dots, x_n$ into k disjoint sets of order $n_1, \dots, n_k$, respectively. There are $\binom{n}{n_1,\dots,n_k}$ ways of doing so and we consider one of them

$$\{x_1, \dots, x_n\} = I_{n_1} \cup \cdots \cup I_{n_k}.$$

We add to each set $I_{n_1}, \dots, I_{n_k}$ one extra variable and call them $y_1, \dots, y_k$, respectively. Consider the spaces $PT_{n_1+1}, \dots, PT_{n_k+1}$ of multilinear pure trace polynomials in the corresponding sets of variables. If $n_i \neq 0$ consider a basis of PT_{n_i} modulo $\mathrm{Id}^{ptr}(A_i)$ and by the above remark construct the corresponding polynomials $g_1^{(i)}, \dots, g_{r_i}^{(i)} \in PT_{n_i+1}$ that are linearly independent modulo $\mathrm{Id}^{ptr}(A_i)$. If $n_i = 0$, we set $g_1^{(i)} = \mathrm{Tr}(y_i) \in PT_{n_i+1}$. Hence

$$g_1^{(1)}, \dots, g_{r_1}^{(1)} \in PT_{n_1+1} \text{ are linearly independent modulo } PT_{n_1+1} \cap \mathrm{Id}^{ptr}(A_1),$$

$$\vdots$$

$$g_1^{(k)}, \dots, g_{r_k}^{(k)} \in PT_{n_k+1} \text{ are linearly independent modulo } PT_{n_k+1} \cap \mathrm{Id}^{tr}(A_k).$$

Lemma 8. *The set of polynomials*

$$\mathcal{F}_{I_{n_1},\dots,I_{n_k}} = \{g_{i_1}^{(1)} g_{i_2}^{(2)} \cdots g_{i_k}^{(k)} \mid 1 \leq i_1 \leq r_1, \dots, 1 \leq i_k \leq r_k\} \subseteq PT_{n+k}$$

is linearly independent modulo $\mathrm{Id}^{ptr}(A)$.

Proof. The proof is by induction on k, the case $k = 1$ being trivial.

Suppose that there is a non-zero linear combination

$$f = \sum_{i_1,\dots,i_k} \alpha_{i_1,\cdots,i_k} g_{i_1}^{(1)} \cdots g_{i_k}^{(k)} \in \mathrm{Id}^{ptr}(A).$$

Pick a non-zero coefficient which we may assume for simplicity to be $\alpha_{1,j_2,\cdots,j_k}$ and write $f = \sum_{s=1}^{r_1} \beta_s g_s^{(1)}$, where

$$\beta_s = \sum_{i_2,\dots,i_k} \alpha_{s,i_2,\cdots,i_k} g_{i_2}^{(2)} \cdots g_{i_k}^{(k)}.$$

Since $\alpha_{1,j_2,\cdots,j_k} \neq 0$, then β_1 is a non-zero linear combination. Now, by induction the set of polynomials

$$\{g_{i_2}^{(2)} \cdots g_{i_k}^{(k)} \mid 1 \leq i_2 \leq r_2, \dots, 1 \leq i_k \leq r_k\} \subseteq PT_{n_2+\cdots+n_k+k-1}$$

is linearly independent modulo $\mathrm{Id}^{ptr}(A)$. Hence there is a non-zero evaluation $\varphi : F\langle X\rangle \to A$ such that $\varphi(\beta_1) \neq 0$. Then we get $\sum_{s=1}^{r_1} \varphi(\beta_s) g_s^{(1)} \in \mathrm{Id}^{ptr}(A_1)$. Since $g_1^{(1)}, \dots, g_s^{(1)}$ are linearly independent modulo $\mathrm{Id}^{ptr}(A_1)$, we get that $\varphi(\beta_s) = 0$, for all $1 \leq s \leq r_1$. In particular $\varphi(\beta_1) = 0$, and this is a contradiction. □

Now for any decomposition $n = n_1 + \cdots + n_k$ into a sum of non-negative integers, and any fixed distribution of the variables $\{x_1, \dots, x_n\} = I_{n_1} \cup \cdots \cup I_{n_k}$, we write

$$\mathcal{P}_{I_{n_1},\dots,I_{n_k}} = \mathrm{span}_F \mathcal{F}_{I_{n_1},\dots,I_{n_k}}.$$

As a consequence of the previous lemma we have the following.

Corollary 9. *We have that*

$$\dim \frac{\mathcal{P}_{I_{n_1},\dots,I_{n_k}}}{\mathcal{P}_{I_{n_1},\dots,I_{n_k}} \cap \mathrm{Id}^{ptr}(A)} = c^{ptr}_{n_1}(A_1)\cdots c^{ptr}_{n_k}(A_k),$$

where we set $c^{tr}_{n_i}(A_i) = 1$ if $n_i = 0$.

In the previous discussion and lemma we have added to each set I_i the extra variable y_i for convenience since we are going to use it in the next lemma. Nevertheless we have the following.

Remark 10. *The conclusion of Lemma 8 still holds without adding the extra variables to the sets I_{n_i} or, which is the same, if we evaluate the variables y_i into the unit element of the corresponding simple algebra A_i.*

Next for a decomposition $n = n_1 + \cdots + n_k$ we consider all possible distributions of the variables $\{x_1,\dots,x_n\}$ and we write $\mathcal{G}_{n_1,\dots,n_k}$ for the set of polynomials which is the union of the $\binom{n}{n_1,\dots,n_k}$ sets of polynomials corresponding to all distributions of the variables (they are of the type $\mathcal{F}_{I_{n_1},\dots,I_{n_k}}$). We have the following.

Lemma 11. *The set* $\bigcup_{n_1+\cdots+n_k=n} \mathcal{G}_{n_1,\dots,n_k} \subseteq PT_{n+k}$ *is linearly independent modulo* $\mathrm{Id}^{ptr}(A)$.

Proof. Suppose that a non-zero linear combination f of all these polynomials is a trace identity of A and assume that a polynomial $g = g^{(1)}_{i_1}\cdots g^{(k)}_{i_k}$ in one of the sets $\mathcal{F}_{I_{n_1},\dots,I_{n_k}}$ has a non zero coefficient.

Recall that each set I_{n_i}, $1 \le i \le k$, contains an extra variable called y_i.

Let $\varphi : F\langle X\rangle \to A$ be an evaluation of f such that $\varphi(g) \neq 0$, where the variables y_i are evaluated in the unit element of the corresponding simple algebra. Such evaluation exists by Remark 10.

Now let $h = h^{(1)}_{j_1}\cdots h^{(k)}_{j_k}$ be any polynomial in $\mathcal{F}_{I_{n'_1},\dots,I_{n'_k}}$, for some $n'_1 + \cdots + n'_k = n$. Notice that even for this decomposition each set $I_{n'_i}$, $1 \le i \le k$, contains the extra variable y_i.

Clearly if a variable of the set I_{n_i} appears in a set $I_{n'_j}$ with $j \neq i$, then $\varphi(h) = 0$ since $I_{n'_j}$ contains the variable y_j that has been evaluated in the unit element of A_j and $A_iA_j = A_jA_i = 0$. Hence if $\varphi(h) \neq 0$ we must have that $I_{n_i} = I_{n'_i}$, for all i. Thus we have obtained a non-trivial linear combination of polynomials in $\mathcal{F}_{I_{n_1},\dots,I_{n_k}}$ that vanishes under any evaluation φ in A where the variables y_i have been evaluated in the unit element of the corresponding simple algebra.

By Remark 10 we have that the polynomials of $\mathcal{F}_{I_{n_1},\dots,I_{n_k}}$ where we have evaluated the variables y_i into the unit elements of the corresponding simple algebra A_i, are linearly independent modulo $\mathrm{Id}^{ptr}(A)$. This completes the proof. □

We can now compute the lower bound of the pure trace codimensions.

Lemma 12. *For the unitary finite dimensional trace algebra A we have that*

$$c^{ptr}_n(A) \ge Cn^t d_2^n,$$

for some constants $C > 0, t$, where $d_2 = d_2(A)$ is the integer defined in (3).

Proof. Let d_2 be the maximal dimension of a pure trace admissible subalgebra C of A. We assume that $C = A_1 \oplus \cdots \oplus A_k$, and, so, $\mathrm{tr}(A_1)\cdots\mathrm{tr}(A_k) \neq 0$. Take any $N > k$ and write $N = n + k$. Recall that by Theorem 5 $c^{ptr}_{n_i}(A_i) \simeq C_i n_i^{t_i}(\dim A_i)^{n_i}$, for some constants $C_i > 0, t_i$, and then by Lemma 11 we have:

$$\begin{aligned}
c^{ptr}_N(A) &\ge \left| \bigcup_{n_1+\cdots+n_k=n} \mathcal{G}_{n_1,\dots,n_k} \right| = \sum_{\substack{I_{n_1},\dots,I_{n_k}\\ n_1+\cdots+n_k=n}} \dim \frac{\mathcal{P}_{I_{n_1},\dots,I_{n_k}}}{\mathcal{P}_{I_{n_1},\dots,I_{n_k}} \cap \mathrm{Id}^{ptr}(A)} \\
&= \sum_{n_1+\cdots+n_k=n} \binom{n}{n_1,\dots,n_k} c^{ptr}_{n_1}(A_1)\cdots c^{ptr}_{n_k}(A_k) \\
&\ge Cn^t \sum_{n_1+\cdots+n_k=n} \binom{n}{n_1,\dots,n_k} (\dim A_1)^{n_1}\cdots(\dim A_k)^{n_k} \\
&= Cn^t(\dim A_1 + \cdots + \dim A_k)^n = Cn^t d_2^n \ge C'N^t d_2^N.
\end{aligned}$$

for some constants $C' > 0, t$. □

Next we shall determine a lower bound for the trace codimensions sequence $c^{tr}_n(A)$.

Suppose that

$$\mathrm{tr}(A_1)\cdots\mathrm{tr}(A_l)A_{l+1}JA_{l+2}J\cdots JA_{l+m} \neq 0,$$

for some $l, m > 0$. Take any $n \ge 1$ and write $n = n_l + n_m$ as a sum of two non-negative integers. Let

$$e_{l+1}j_1e_{l+2}j_2\cdots j_{m-1}e_{l+m} \neq 0,$$

where e_i is the unit element of A_i, and consider the subalgebra B of A generated by the simple algebras $B_1 = A_{l+1}, \dots, B_m = A_{l+m}$ and by the elements

$$E_{1,2} = e_{l+1}j_1e_{l+2},\ E_{2,3} = e_{l+2}j_2e_{l+3},\ \dots,\ E_{m-1,m} = e_{l+m-1}j_{m-1}e_{l+m}.$$

It is well known that the algebra B is isomorphic to the algebra of upper block triangular matrices $UT(B_1, \dots, B_m) = UT(r_1, \dots, r_m)$ where $r_i = \sqrt{\dim B_i}$. Also, for some constants $C > 0, s$,

$$c_n(UT(r_1, \dots, r_m)) \simeq Cn^s(r_1^2 + \cdots + r_m^2)^n = Cn^s(\dim A_{l+1} + \cdots + \dim A_{l+m})^n \tag{5}$$

(see [9, Theorem 8.6.1]).

Now recall that $n = n_l + n_m$ and divide the variables $x_1, \dots, x_n$ into two disjoint sets of order n_l and n_m, respectively. Clearly there are $\binom{n}{n_l}$ ways of doing so and we consider one of them $\{x_1, \dots, x_n\} = I_{n_l} \cup I_{n_m}$.

Consider the spaces PT_{n_l} and P_{n_m} of multilinear pure trace polynomials and ordinary polynomials in the corresponding sets of variables, respectively. Let $f_1, \dots, f_a \in PT_{n_l+l}$ be the polynomials containing the extra variables $y_1, \dots, y_l$ constructed in Lemma 11, that are linearly independent modulo $\mathrm{Id}^{ptr}(A)$. Let also $g_1, \dots, g_b$ be a basis of P_{n_m} modulo $\mathrm{Id}(B)$.

Remark 13. *The set of polynomials*

$$\mathcal{F}_{I_{n_l}, I_{n_m}} = \{f_ig_j \mid 1 \le i \le a, 1 \le j \le b\} \in MT_{n+l}$$

is linearly independent modulo $\mathrm{Id}^{tr}(A)$.

Proof. Suppose we have a linear combination $h = \sum_{i,j} \alpha_{ij} f_i g_j \in \mathrm{Id}^{tr}(A)$, and say, $\alpha_{11} \neq 0$. Write

$$h = \sum_j \gamma_j g_j \in \mathrm{Id}^{tr}(A), \tag{6}$$

where $\gamma_j = \sum_i \alpha_{ij} f_i$. Then $\gamma_1 = \sum_i \alpha_{i1} f_i$ is a non-zero linear combination and since the polynomials $f_1, \dots, f_a$ are linearly independent modulo $\mathrm{Id}^{tr}(A)$, we get that there is a non-zero evaluation $\varphi : F\langle X\rangle \to A$ such that $\varphi(\gamma_1) \neq 0$.

Then from (6) we get $\sum_{j=1}^b \varphi(\gamma_j) g_j \in \mathrm{Id}(A) \subseteq \mathrm{Id}(B)$. Since $g_1, \dots, g_b$ are linearly independent modulo $\mathrm{Id}(B)$, we get that $\varphi(\gamma_j) = 0$, for all $1 \le j \le b$. In particular $\varphi(\gamma_1) = 0$, and this is a contradiction. □

Now we write $\mathcal{P}_{I_{n_l}, I_{n_m}} = \mathrm{span}_F \mathcal{F}_{I_{n_l}, I_{n_m}}$. Then by the proof of Lemma 12 and (5) we have that

$$\dim \frac{\mathcal{P}_{I_{n_l}, I_{n_m}}}{\mathcal{P}_{I_{n_l}, I_{n_m}} \cap \mathrm{Id}^{tr}(A)} \ge Cn^u(\dim A_1 + \cdots + \dim A_l)^{n_l}(\dim A_{l+1} + \cdots + \dim A_{l+m})^{n_m}, \tag{7}$$

for some constants $C > 0, u$.

As above, for any fixed decomposition $n = n_l + n_m$ we consider all possible distributions of the variables between two sets of order n_l and n_m, respectively. We write $\mathcal{G}_{n_l, n_m}$ for the set of polynomials which is the union of the $\binom{n}{n_l}$ sets of polynomials in the variables belonging to each distribution.

We claim that the set $\cup_{n_l+n_m=n} \mathcal{G}_{n_l, n_m}$ is linearly independent modulo $\mathrm{Id}^{tr}(A)$.

In fact suppose that a non-zero linear combination of all these polynomials

$$\sum \alpha_{ij} f_i g_j \in \mathrm{Id}^{tr}(A) \tag{8}$$

is a trace identity of A and assume that a polynomial, say f_1g_1, in one of the sets $\mathcal{F}_{I_{n_l}, I_{n_m}}$ has a non zero coefficient α_{11}.

Let $\varphi : F\langle X\rangle \to A$ be an evaluation such that $\varphi(f_1g_1) \neq 0$ and the extra variables y_t, $t = 1, \dots, l$, of the pure trace polynomial f_1 are evaluated in the unit element of the corresponding simple algebra.

Let f_ig_j be any polynomial in $\mathcal{F}_{I_{n'_l}, I_{n'_m}}$, for some $n'_l + n'_m = n$. Notice that even for this decomposition f_i contains the extra variables $y_1, \dots, y_l$. If one of the variables of the set I_{n_m} appears in the set $I_{n'_l}$, then $\varphi(f_ig_j) = 0$ since the polynomial f_i contains the extra variables $y_1, \dots, y_l$. Hence $I_{n_m} \subseteq I_{n'_m}$. If $I_{n_m} \neq I_{n'_m}$, then there is a variable x of the set I_{n_l} such that $x \in I_{n'_m} \supseteq I_{n_m}$ and, so, $\varphi(f_ig_j) = 0$ since $\varphi(g_j) = 0$. Thus $I_{n_l} = I_{n'_l}$ and $I_{n_m} = I_{n'_m}$.

We have proved that if φ is any non zero evaluation of the polynomial f_1g_1, then by (8) we have that $\varphi(\sum \alpha_{ij} f_i g_j) = 0$ where we have restricted the summation to all $f_ig_j \in \mathcal{F}_{I_{n_l}, I_{n_m}}$. Since by Remark 13 the polynomials in $\mathcal{F}_{I_{n_l}, I_{n_m}}$ are linearly independent modulo $\mathrm{Id}^{tr}(A)$, we get a contradiction and the claim is established.

Now we are in a position to prove the following result.

Lemma 14. *For the unitary finite dimensional trace algebra A we have that*

$$c_n^{tr}(A) \ge Cn^t d^n,$$

for some constants $C > 0, t$, where $d = d(A)$ is the integer defined in (4).

Proof. Let $A_1 \oplus \cdots \oplus A_{l+m}$ be a trace admissible subalgebra of A of maximal dimension d and suppose

$$\operatorname{tr}(A_1)\cdots\operatorname{tr}(A_l)A_{l+1}JA_{l+2}J\cdots JA_{l+m} \neq 0.$$

If $l=0$ or $m=0$ the result follows from [6, 7] and Lemma 12, since $c_n^{tr}(A) \geq c_n^{ptr}(A), c_n(A)$, for all $n \geq 1$. Now assume that $l>0$ and $m>0$. Let $N>l$ any integer and write $N=n+l$. Then by the previous discussion and (7) we have that

$$\begin{aligned} c_N^{tr}(A) &\geq \left|\bigcup_{n_l+n_m=n} \mathcal{G}_{n_l,n_m}\right| = \sum_{\substack{I_{n_l},I_{n_m}\\ n_l+n_m=n}} \dim \frac{\mathcal{P}_{I_{n_l},I_{n_m}}}{\mathcal{P}_{I_{n_l},I_{n_m}} \cap \operatorname{Id}^{tr}(A)} \\ &\geq Cn^t \sum_{n_l+n_m=n} \binom{n}{n_l} (\dim A_1 + \cdots + \dim A_l)^{n_l} (\dim A_{l+1} + \cdots + \dim A_{l+m})^{n_m} \\ &= Cn^t(\dim A_1 + \cdots + \dim A_{l+m})^n = Cn^t d^n \geq C'N^t d^n, \end{aligned}$$

for some constants $C'>0, t$. □

5. The exponential growth is integral

We recall our general setting: $A = A_1 \oplus \cdots \oplus A_m + J$ is a unitary finite dimensional associative algebra with trace tr over an algebraically closed field F of characteristic zero, $s \geq 0$ is the smallest integer such that $J^{s+1}=0$, $A_i = M_{d_i}(F)$, $1 \leq i \leq m$.

Fix a basis $\{u_1,\dots,u_l\}$ of A which is the union of the standard bases of the A_i's and of a basis of J.

For $1 \leq i \leq n$ consider the generic elements

$$\xi_i = \sum_{j=1}^{l} \xi_{i,j} \otimes u_j \in F[\xi_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq l] \otimes_F A, \tag{9}$$

where the elements $\xi_{i,j}$ are commutative variables.

Let $\mathcal{U} = F\{\xi_1,\dots,\xi_n\}$ be the algebra over F generated by the ξ_i's, $i=1,\dots,n$.

It is easy to check that $\mathcal{U}$ can be endowed with a trace function t by defining

$$t(\xi_{i,j} \otimes u_i) = \xi_{i,j} \otimes \operatorname{tr}(u_i).$$

Since a multilinear trace polynomial is a trace identity of A if and only if it vanishes on the generic elements ξ_i, it follows that

$$c_n^{tr}(A) = \dim_F \operatorname{span}\{f(\xi_{\sigma(1)},\dots,\xi_{\sigma(k)})\xi_{\sigma(k+1)}\cdots\xi_{\sigma(n)} \mid \sigma \in S_n,\ 0 \leq k \leq n\}$$

and

$$c_n^{ptr}(A) = \dim_F \operatorname{span}\{f(\xi_{\sigma(1)},\dots,\xi_{\sigma(n)}) \mid \sigma \in S_n\},$$

where $f(x_1,\dots,x_l) \in PT_l$ is a pure trace monomial.

Lemma 15. *For the unitary finite dimensional trace algebra A we have that*

$$c_n^{tr}(A) \leq Cn^k d^n \quad \text{and} \quad c_n^{ptr}(A) \leq C_1 n^t d_2^n,$$

for some constants C, C_1, k, t, where d and d_2 are the integers defined in (4) and (3), respectively.

Proof. We start by computing an upper bound of $c_n^{tr}(A)$.

By (9) we replace each ξ_i and we get

$$\begin{aligned} f(\xi_{\sigma(1)},\dots,\xi_{\sigma(k)})\xi_{\sigma(k+1)}\cdots\xi_{\sigma(n)} &= \sum_{j_1,\dots,j_n=1}^{l} \xi_{\sigma(1),j_1}\cdots\xi_{\sigma(n),j_n} \otimes f(u_{j_1},\dots,u_{j_k})u_{j_{k+1}}\cdots u_{j_n} \\ &= \sum_{h=1}^{l}\sum_{j_1,\dots,j_n=1}^{l} \alpha_{j_1,\dots j_n,h}\xi_{\sigma(1),j_1}\cdots\xi_{\sigma(n),j_n} \otimes u_h, \end{aligned} \tag{10}$$

where the above equalities follow from the fact that tr takes values in F and, so, $f(u_{j_1},\dots,u_{j_k})u_{j_{k+1}},\dots,u_{j_n}$ can be written as a linear combination of basis elements. Hence

$$\begin{aligned} c_n^{tr}(A) &\leq \dim_F \operatorname{span}\{\xi_{\sigma(1),j_1}\cdots\xi_{\sigma(n),j_n} \otimes u_h \mid \sigma \in S_n,\ 1 \leq j_1,\dots,j_n,\ h \leq l\} \\ &\leq l \cdot \left|\{\xi_{\sigma(1),j_1}\cdots\xi_{\sigma(n),j_n} \mid f(u_{j_1},\dots,u_{j_k})u_{j_{k+1}}\cdots u_{j_n} \neq 0,\ \sigma \in S_n,\ 1 \leq j_1,\dots,j_n \leq l\}\right|. \end{aligned} \tag{11}$$

Thus in order to find an upper bound of $c_n^{tr}(A)$, we have to count (or to find an upper bound of the number of) the monomials $\xi_{\sigma(1),j_1}\cdots\xi_{\sigma(n),j_n}$ such that $f(u_{j_1},\dots,u_{j_k})u_{j_{k+1}}\cdots u_{j_n}$ is non-zero.

Now, this product is non-zero only if $t \leq s$ basis elements u_{j_i} come from J. How many monomials can we write with t basis elements from J? Since the trace is zero on any radical element, only $u_{j_{k+1}},\dots,u_{j_n}$ can be elements of J. Hence there are $n-k \leq n$ possible positions for t elements of J, so there are in all $\leq \binom{n}{t}$ monomials, but in each position we can write any element of a basis of J, so in all we have $\leq \binom{n}{t}(\dim J)^t \leq \binom{n}{s}(\dim J)^s$ monomials.

Now that we have taken care of the u_{j_i}'s that come from J, all the other basis elements come from the semisimple part. But $f(u_{j_1},\dots,u_{j_k})u_{j_{k+1}}\cdots u_{j_n}\neq 0$ only if this product comes from a configuration of the type

$$\mathrm{tr}(A_{l_1})\cdots\mathrm{tr}(A_{l_r})A_{l_{r+1}}J\cdots JA_{l_k}\neq 0,$$

i.e., only if the semisimple basis elements come from a trace admissible subalgebra.

We fix a trace admissible subalgebra $B=A_{l_1}\oplus\cdots\oplus A_{l_k}$. How many possible monomials $u_{j_1}\cdots u_{j_n}$ with the semisimple variables coming from B can we write? If there are t radical elements, such number is $\binom{n}{t}(\dim J)^t(\dim B)^{n-t}$.

Thus if the semisimple variables come from the trace admissible subalgebra B, we can write at most

$$\sum_{t=0}^{s}\binom{n}{t}(\dim J)^t(\dim B)^{n-t}\leq(\dim J)^s\sum_{t=0}^{s}\binom{n}{t}(\dim B)^n\leq Cn^s(\dim B)^n$$

monomials, for some constant C. Notice that we do not know how many basis elements of J come in a product, so we have to count all possibilities $0,1,\dots,s$.

Now, the number of trace admissible subalgebras is finite, say r, hence the number of possible non-zero monomials is $\leq Cn^srd^n$, where d is the maximal dimension of a trace admissible subalgebra. It follows that

$$c_n^{tr}(A)\leq C_1n^sd^n,$$

as claimed.

Finally, recalling that no radical elements can appear inside a non-zero trace, it is clear that each $f(u_{j_1},\dots,u_{j_n})$ involves only semisimple elements. If d_2 is the maximal dimension of a pure trace admissible subalgebra, we get that $c_n^{ptr}(A)\leq Cd_2^n$, for some constant C and the proof is complete.

□

We are in a position to prove the main result of this paper.

Theorem 16. *Let $A=\bar{A}+J$ be a unitary finite dimensional algebra with trace tr over a field F of characteristic zero and let $tr(J)=0$. Then there exist constants $a,a'>0$ and b,b',t,t',r,r' such that*

$$an^td^n\leq c_n^{tr}(A)\leq bn^rd^n\qquad\text{and}\qquad a'n^{t'}d'^n\leq c_n^{ptr}(A)\leq b'n^{r'}d'^n,$$

where d and d' equal the dimensions of some subalgebras of A.

Hence the trace exponent of A $\exp^{tr}(A)=\lim_{n\to\infty}\sqrt[n]{c_n^{tr}(A)}$ and the pure trace exponent of A $\exp^{ptr}(A)=\lim_{n\to\infty}\sqrt[n]{c_n^{ptr}(A)}$ exist and are integers.

Proof. Since the trace and the pure trace codimensions do not change by extending the ground field, we may assume that the field is algebraically closed. Now the result follows by putting together Lemmas 12, 14 and 15. □

As an immediate consequence of the above theorem we get the following.

Corollary 17. *Under the hypotheses of Theorem 16 both sequences $c_n^{tr}(A)$ and $c_n^{ptr}(A)$, $n=1,2,\dots$ either are polynomially bounded or grow exponetially.*

Let A be a finite dimensional algebra as above. Clearly $\exp^{ptr}(A)=\sum_{tr(A_i)\neq 0}\dim A_i$.

We remark that if $A_1\oplus\cdots\oplus A_{l+m}$ is a trace admissible subalgebra of A and $\mathrm{tr}(A_1)\cdots\mathrm{tr}(A_l)A_{l+1}J\cdots JA_{l+m}\neq 0$, then, by constructing the algebra B given in (5), it follows that A contains a subalgebra isomorphic to

$$D=A_1\oplus\cdots\oplus A_l\oplus UT(A_{l+1},\dots,A_{l+m})\cong\begin{pmatrix} A_1 & & 0 & 0 & & 0\\ & \ddots & & & & \\ 0 & & A_l & 0 & & 0\\ 0 & & 0 & A_{l+1} & & *\\ & & & & \ddots & \\ 0 & & 0 & 0 & & A_{l+m}\end{pmatrix}.$$

If in D we assume that $\mathrm{tr}(A_{l+1})=\cdots=\mathrm{tr}(A_{l+m})=0$, then $\exp^{ptr}(A)=\dim(A_1\oplus\cdots\oplus A_l)$ and $\exp(A)=\dim(A_{l+1}\oplus\cdots\oplus A_{l+m})$. This shows that in general $\exp^{ptr}(A)$ and $\exp(A)$ are not comparable. Nevertheless the following relations hold: $\exp(A),\exp^{ptr}(A)\leq\exp^{tr}(A)$ and $\exp^{tr}(A)\leq\exp(A)+\exp^{ptr}(A)$.

6. Polynomially bounded trace codimensions

In this section we shall present two results concerning algebras with trace having the trace codimension sequence and the pure trace codimension sequence bounded by a polynomial. We recall that $A=\bar{A}+J=A_1\oplus\cdots\oplus A_m+J$ is a unitary finite dimensional algebra with trace which is zero on J.

We start by introducing some algebras with trace.

We denote by $UT_2 = UT_2(F)$ the algebra of 2×2 upper-triangular matrices endowed with zero trace and by $D_2 = D_2(F)$ the commutative algebra of 2×2 diagonal matrices. For any $\alpha, \beta \in F$, it is possible to define on D_2 the following trace function:

$$t_{\alpha,\beta}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \alpha a + \beta b.$$

We denote by $D_2^{t_{\alpha,\beta}}$ the algebra D_2 endowed with the trace $t_{\alpha,\beta}$. Such trace algebras have been extensively studied in [11, 12, 13] where it was proved that their trace codimensions grow exponentially. Here we remark that, up to isomorphism, we can define on D_2 only the following trace functions:

- $t_{\alpha,0}$, for any $\alpha \in F$,
- $t_{\beta,\beta}$, for any non-zero $\beta \in F$,
- $t_{\gamma,\delta}$, for any distinct non-zero $\gamma, \delta \in F$.

The following result is a particular case of a more general result [11, Theorem 30] without any restriction on the value of the trace on J.

Recall that, if A is an algebra with trace, then $\mathrm{var}^{tr}(A)$ (the variety generated by A) is the class of all algebras with trace satisfying all the trace identities satified by A.

Theorem 18. *Let $A = \bar{A} + J$ be a unitary finite dimensional algebra with trace tr over a field F of characteristic zero, $tr(J) = 0$. Then the sequence $c_n^{tr}(A)$, $n = 1, 2, \ldots$, is polynomially bounded if and only if UT_2, $D_2^{t_{\alpha,0}}$, $D_2^{t_{\beta,\beta}}$, $D_2^{t_{\gamma,\delta}} \notin var^{tr}(A)$, for any choice of $\alpha, \beta, \gamma, \delta \in F \setminus \{0\}$, $\gamma \neq \delta$.*

As a consequence we recover the result of Corollary 17.

Our next goal is to prove the pure trace analogue of the above theorem. The list of trace algebras to be excluded from the variety will be smaller. In fact, since $\mathrm{Tr}(x) \equiv 0$ is a pure trace identity of UT_2 then $c_n^{ptr}(UT_2) = 0$, for all $n \geq 1$. Moreover, $D_2^{t_{\alpha,0}}$ satisfying the pure trace identity $\mathrm{Tr}(x_1)\mathrm{Tr}(x_2) - \alpha \mathrm{Tr}(x_1x_2) \equiv 0$ has pure trace codimensions $c_n^{ptr}(D_2^{t_{\alpha,0}}) = 1$, for $\alpha \neq 0$.

Next we collect some results about the pure trace identities and pure trace codimensions of the algebras $D_2^{t_{\beta,\beta}}$ and $D_2^{t_{\gamma,\delta}}$. We omit the proofs since they are easily obtained following the ones given in [11] and the relation between pure trace codimensions and trace codimensions, in the non-degenerate case.

Proposition 19. *Let $\beta \in F \setminus \{0\}$. The pure trace T-ideal $\mathrm{Id}^{ptr}(D_2^{t_{\beta,\beta}})$ is generated by the polynomials:*

- $f_1 = Tr(x_1[x_2, x_3])$,
- $f_2 = Tr(x_1)Tr(x_2)Tr(x_3) + \beta^2 Tr(x_1x_2x_3) + \beta^2 Tr(x_2x_1x_3) - \beta Tr(x_1x_2)Tr(x_3) - \beta Tr(x_1x_3)Tr(x_2) - \beta Tr(x_2x_3)Tr(x_1)$.

Moreover

$$c_n^{ptr}(D_2^{t_{\beta,\beta}}) = 2^{n-1}.$$

Proposition 20. *Let $\gamma, \delta \in F \setminus \{0\}$, $\gamma \neq \delta$. The pure trace T-ideal $\mathrm{Id}^{ptr}(D_2^{t_{\gamma,\delta}})$ is generated by the polynomials:*

- $f_1 = Tr(x_1[x_2, x_3])$,
- $f_3 = Tr(x_1)Tr(x_2)Tr(x_3)Tr(x_4) - (\gamma\delta^2 + \gamma^2\delta)Tr(x_1x_2x_3x_4) + \gamma\delta Tr(x_1x_2x_4)Tr(x_3) + \gamma\delta Tr(x_1x_3x_4)Tr(x_2) + \gamma\delta Tr(x_2x_3x_4)Tr(x_1) - (\gamma+\delta)Tr(x_1x_4)Tr(x_2)Tr(x_3) + (\gamma^2+\gamma\delta+\delta^2)Tr(x_1x_4)Tr(x_2x_3) - \gamma\delta Tr(x_2x_4)Tr(x_1x_3) - \gamma\delta Tr(x_3x_4)Tr(x_1x_2) + \gamma\delta Tr(x_1x_2x_3)Tr(x_4) - (\gamma+\delta)Tr(x_1)Tr(x_2x_3)Tr(x_4)$,
- $f_4 = Tr(x_1)Tr(x_2)Tr(x_3x_4) - Tr(x_1x_4)Tr(x_2)Tr(x_3) + Tr(x_1x_2)Tr(x_3)Tr(x_4) - Tr(x_2x_3)Tr(x_1)Tr(x_4) + (\gamma+\delta)[Tr(x_1x_4)Tr(x_2x_3) - Tr(x_1x_2)Tr(x_3x_4)]$.

Moreover

$$c_n^{ptr}(D_2^{t_{\gamma,\delta}}) = 2^n - n.$$

Now, we shall prove the following result. We denote by $\mathrm{var}^{ptr}(A)$ the class of all algebras with trace satisfying the pure trace identities of A.

Theorem 21. *Let $A = \bar{A} + J$ be a unitary finite dimensional algebra with trace tr over a field F of characteristic zero, $tr(J) = 0$. Then the sequence $c_n^{ptr}(A)$, $n = 1, 2, \ldots$, is polynomially bounded if and only if $D_2^{t_{\beta,\beta}}, D_2^{t_{\gamma,\delta}} \notin var^{ptr}(A)$, for any choice of $\beta, \gamma, \delta \in F \setminus \{0\}$, $\gamma \neq \delta$.*

Proof. By Propositions 19 and 20, the pure trace codimensions of the algebras $D_2^{t_{\beta,\beta}}$ and $D_2^{t_{\gamma,\delta}}$ grow exponentially. Hence, if $c_n^{ptr}(A)$ is polynomially bounded, then $D_2^{t_{\beta,\beta}}$, $D_2^{t_{\gamma,\delta}} \notin \mathrm{var}^{ptr}(A)$, for any β, γ, $\delta \in F \setminus \{0\}$, $\gamma \neq \delta$.

Conversely suppose that $D_2^{t_{\beta,\beta}}$, $D_2^{t_{\gamma,\delta}} \notin \mathrm{var}^{ptr}(A)$, for any β, γ, $\delta \in F \setminus \{0\}$, $\gamma \neq \delta$. Since we are dealing with codimensions, and these do not change under extensions of the base field, we may assume that the field F is algebraically closed. By the Wedderburn-Malcev decomposition for trace algebras, we get that

$$A = M_{n_1}(F) \oplus \cdots \oplus M_{n_m}(F) + J, \quad m \geq 1,$$

and there exist constants α_i such that, for $a_i \in M_{n_i}(F)$, we have

$$tr(a_1, \ldots, a_k) = \sum_{i=1}^{m} t_{\alpha_i}(a_i).$$

Since $D_2^{t_{\beta,\beta}} \notin \text{var}^{tr}(A)$, by [11, Lemma 28] we get that $n_i = 1$, for every $i = 1, \ldots, m$. Hence

$$A = A_1 \oplus \cdots \oplus A_m + J$$

where for every $i = 1, \ldots, m$, $A_i \cong F$ and the trace on it is t_{α_i}. Now, if $\alpha_i = 0$ for any i, we are dealing with an algebra A with zero trace. Hence $c_n^{ptr}(A) = 0$ and we are finished in this case. On the other hand, if there exist i and j such that $\alpha_i, \alpha_j \neq 0$, as in [11, Theorem 30], we get that $D_2^{t_{\alpha_i,\alpha_j}} \in \text{var}^{tr}(A)$, a contradiction.

Hence we must have $A = A_1 \oplus \cdots \oplus A_m + J$, where for every $i = 1, \ldots, m$, $A_i \cong F$ and there exists just one $\alpha_i \neq 0$. Since the pure trace polynomials $\text{Tr}(x_1)\text{Tr}(x_2) - \alpha_i \text{Tr}(x_1 x_2) \equiv 0$ and $\text{Tr}(x_1[x_2, x_3]) \equiv 0$ are identities of A, it follows that $c_n^{ptr}(A) = 1$ and the proof is complete. □

As a consequence we have the following.

Corollary 22. *Under the hypotheses of the above theorem, if $c_n^{ptr}(A)$ is polynomially bounded then either $c_n^{ptr}(A) = 0$ or $c_n^{ptr}(A) = 1$, for all $n \geq 1$.*

7. Exponents bounded by 2

In this section we shall characterize the varieties of algebras with trace having trace exponent and pure trace exponent ≤ 2.

We start by introducing some algebras with trace (or pure trace) exponent equal to 3.

Let $UT_3 = UT_3(F)$ be the algebra of 3×3 upper-triangular matrices. For any $\alpha, \beta, \gamma \in F$, it is possible to define on UT_3 the following trace:

$$t_{\alpha,\beta,\gamma}\left(\begin{pmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{pmatrix}\right) = \alpha a + \beta b + \gamma c.$$

We denote by $UT_3^{t_{\alpha,\beta,\gamma}}$ the algebra UT_3 with such a trace.
Now consider the following trace subalgebras of $UT_3^{t_{\alpha,\beta,\gamma}}$:

$D_3^{t_{\alpha,\beta,\gamma}}$, the commutative algebra of 3×3 diagonal matrices with induced trace $t_{\alpha,\beta,\gamma}$;
$L_3^{t_{\alpha,\beta,\gamma}} = Fe_{11} + Fe_{22} + Fe_{33} + Fe_{13}$, with induced trace $t_{\alpha,\beta,\gamma}$.

Clearly $D_3^{t_{\alpha,\beta,\gamma}} \subseteq L_3^{t_{\alpha,\beta,\gamma}}$ is a trace subalgebra.

Remark 23. *The following facts are easily proved.*

1. *For any $\alpha, \beta, \gamma \in F$, $\exp^{tr}\left(UT_3^{t_{\alpha,\beta,\gamma}}\right) = 3$.*
2. *For any $\alpha, \beta, \gamma \in F$, $\beta \neq 0$, $\exp^{tr}\left(L_3^{t_{\alpha,\beta,\gamma}}\right) = 3$.*
3. *For any $\alpha, \beta, \gamma \in F$, $\alpha, \beta \neq 0$, $\exp^{tr}\left(D_3^{t_{\alpha,\beta,\gamma}}\right) = 3$.*
4. *For any $\alpha, \beta, \gamma \in F \setminus \{0\}$, $\exp^{ptr}\left(D_3^{t_{\alpha,\beta,\gamma}}\right) = 3$.*
5. *For any $\alpha \in F$, $\exp^{tr}\left(M_2^{t_\alpha}\right) = 4$. In case $\alpha \neq 0$, $\exp^{ptr}\left(M_2^{t_\alpha}\right) = 4$.*

Now we are in a position to characterize algebras with trace A with $\exp^{tr}(A) > 2$.

Theorem 24. *Let $A = \bar{A} + J$ be a unitary finite dimensional algebra with trace tr over a field F of characteristic zero with $tr(J) = 0$. Then $\exp^{tr}(A) > 2$ if and only if $B \in var^{tr}(A)$, for some $B \in \left\{UT_3^{t_{0,0,0}}, L_3^{t_{0,\xi,0}}, D_3^{t_{\alpha,\beta,\gamma}}, D_3^{t_{\delta,\epsilon,0}}, M_2^{t_\kappa}\right\}$, where $\alpha, \beta, \gamma, \delta, \epsilon, \xi \in F \setminus \{0\}$, $\kappa \in F$.*

Proof. Suppose that $\exp^{tr}(A) > 2$. Without loss of generality we may assume that the field F is algebraically closed. Hence we can write $\bar{A} = A_1 \oplus \cdots \oplus A_m = M_{n_1}(F) \oplus \cdots \oplus M_{n_1}(F)$ and, for any $i = 1, \ldots, m$, A_i can be viewed as a "trace subalgebra" of A with trace $t_{\alpha_i} = \alpha_i t_1^i$ (t_1^i is the usual trace on $M_{k_i}(F)$). If for some $j \in \{1, \ldots, m\}$ $\dim_F A_j \geq 4$ then, by [11, Lemma 28], $M_2^{t_{\alpha_j}} \in \text{var}^{tr}(A)$. Then suppose that $\dim_F A_j < 4$, for every $j = 1, \ldots, m$. It follows that $A_j \cong F$, for any $j \in \{1, \ldots, m\}$ and, since $\exp^{tr}(A) > 2$, there exist distinct A_i, A_j, A_k such that one of the following conditions occurs:

1. $\text{tr}(A_i)\text{tr}(A_j)\text{tr}(A_k) \neq 0$;
2. $\text{tr}(A_i)\text{tr}(A_j)A_k \neq 0$ and $\text{tr}(A_k) = 0$;
3. $\text{tr}(A_i)A_j J A_k \neq 0$ and $\text{tr}(A_j) = \text{tr}(A_k) = 0$;
4. $A_i J A_j J A_k \neq 0$ and $\text{tr}(A_i) = \text{tr}(A_i) = \text{tr}(A_k) = 0$.

Let e_1, e_2, e_3 be the unit elements of A_i, A_j, A_k, respectively and let $\mathrm{tr}(e_1) = \alpha$, $\mathrm{tr}(e_2) = \beta$ and $\mathrm{tr}(e_3) = \gamma$, for some $\alpha, \beta, \gamma \in F$.

Suppose first that $\mathrm{tr}(A_i)\mathrm{tr}(A_j)\mathrm{tr}(A_k) \neq 0$. Hence $\mathrm{tr}(e_1)\mathrm{tr}(e_2)\mathrm{tr}(e_3) \neq 0$. If we consider the subalgebra with trace U generated by the elements e_1, e_2, e_3, then the linear map $\varphi\colon U \to D_3^{t_{\alpha,\beta,\gamma}}$ defined by

$$\varphi(e_1) = e_{11}, \quad \varphi(e_2) = e_{22}, \quad \varphi(e_3) = e_{33},$$

is an isomorphism of algebras with trace. Hence $D_3^{t_{\alpha,\beta,\gamma}} \in \mathrm{var}^{tr}(A)$, $\alpha, \beta, \gamma \neq 0$ and we are done in this case.

Now suppose that $\mathrm{tr}(A_i)\mathrm{tr}(A_j)A_k \neq 0$ and $\mathrm{tr}(A_k) = 0$. As in the proof of case 1., it is possible to get that $D_3^{t_{\alpha,\beta,0}} \in \mathrm{var}^{tr}(A), \alpha, \beta \neq 0$.

At this point assume that 3. holds: $\mathrm{tr}(A_i)A_jJA_k \neq 0$ and $\mathrm{tr}(A_j) = \mathrm{tr}(A_k) = 0$. Hence there exists $j \in J$ such that $\mathrm{tr}(e_1)e_2je_3 \neq 0$. In this case we consider the subalgebra with trace U generated by the elements

$$e_1, \quad e_2, \quad e_3, \quad e_2je_3.$$

The linear map $\varphi\colon U \to L_3^{t_{0,\alpha,0}}$, defined by

$$\varphi(e_1) = e_{22}, \qquad \varphi(e_2) = e_{11}, \qquad \varphi(e_3) = e_{33}, \qquad \varphi(e_2je_3) = e_{13},$$

is an isomorphism of algebras with trace. Hence $L_3^{t_{0,\alpha,0}} \in \mathrm{var}^{tr}(A), \alpha \neq 0$, and we are done in this case.

In the last case $A_iJA_jJA_k \neq 0$ and $\mathrm{tr}(A_i) = \mathrm{tr}(A_i) = \mathrm{tr}(A_k) = 0$, by [8] we get that $UT_3^{t_{0,0,0}} \in \mathrm{var}^{tr}(A)$ and this direction of the theorem is proved.

The opposite implication clearly follows by Remark 23. □

Our next goal is to prove that the above list of algebras cannot be reduced.

Proposition 25. *If* $B, C \in \left\{UT_3^{t_{0,0,0}}, L_3^{t_{0,\xi,0}}, D_3^{t_{\alpha,\beta,\gamma}}, D_3^{t_{\delta,\epsilon,0}}, M_2^{t_\kappa}\right\}$ *are distinct, then* $\mathrm{Id}^{tr}(B) \not\subseteq \mathrm{Id}^{tr}(C)$.

Proof. In order to prove the result we need to observe the following facts.

- If $B \in \left\{UT_3^{t_{0,0,0}}, L_3^{t_{0,\xi,0}}, D_3^{t_{\alpha,\beta,\gamma}}, D_3^{t_{\delta,\epsilon,0}}\right\}$, then $\mathrm{Id}^{tr}(B) \not\subseteq \mathrm{Id}^{tr}(M_2^{t_\kappa})$ since $\exp^{tr}(B) = 3 < 4 = \exp^{tr}(M_2^{t_\kappa})$.
- Since $UT_3^{t_{0,0,0}}$ has zero trace, we get that $\mathrm{Id}^{tr}(UT_3^{t_{0,0,0}}) \not\subseteq \mathrm{Id}^{tr}(C)$, $C \in \left\{L_3^{t_{0,\xi,0}}, D_3^{t_{\alpha,\beta,\gamma}}, D_3^{t_{\delta,\epsilon,0}}\right\}$.
- Since D_3 is commutative, we get that $\mathrm{Id}^{tr}(D_3^{t_{\alpha,\beta,\gamma}}) \not\subseteq \mathrm{Id}^{tr}(C)$, $C \in \left\{UT_3^{t_{0,0,0}}, L_3^{t_{0,\xi,0}}, M_2^{t_\kappa}\right\}$. Here γ can be zero.
- $\mathrm{Id}^{tr}(D_3^{t_{\delta,\epsilon,0}}) \not\subseteq \mathrm{Id}^{tr}(D_3^{t_{\alpha,\beta,\gamma}})$. In fact, the polynomial g_4 of [13, Theorem 19] is an identity of $D_3^{t_{\delta,\epsilon,0}}$ but $g_4(e_{11}, e_{22}, e_{33}, e_{33}) \neq 0$ on $D_3^{t_{\alpha,\beta,\gamma}}$.
- $\mathrm{Id}^{tr}(D_3^{t_{\delta,\epsilon,0}}) \not\supseteq \mathrm{Id}^{tr}(D_3^{t_{\alpha,\beta,\gamma}})$. In fact, the identity f of $D_3^{t_{\alpha,\beta,\gamma}}$ (cited in [12, Lemma 7]) does not vanish on $D_3^{t_{\delta,\epsilon,0}}$.
- $\mathrm{Id}^{tr}(M_2^{t_\kappa}) \not\subseteq \mathrm{Id}^{tr}(C)$, $C \in \left\{UT_3^{t_{0,0,0}}, L_3^{t_{0,\xi,0}}, D_3^{t_{\alpha,\beta,\gamma}}, D_3^{t_{\delta,\epsilon,0}}\right\}$, $\kappa \neq 0$. In fact, if $\kappa \neq \gamma$, we have that the polynomial $CH_2^{\kappa^{-1}}$ is an identity of $M_2^{t_\kappa}$ but $CH_2^{\kappa^{-1}}(e_{33}, e_{33}) \neq 0$ on C. In case $\kappa = \gamma$ we have that $CH_2^{\kappa^{-1}}(e_{11}, e_{22}) \neq 0$ on $D_3^{t_{\alpha,\beta,\kappa}}$ (notice that $CH_2^{\kappa^{-1}}(e_{33}, e_{33}) = 0$ on $D_3^{t_{\alpha,\beta,\kappa}}$).
- $\mathrm{Id}^{tr}(M_2^{t_0}) \not\subseteq \mathrm{Id}^{tr}(C)$, $C \in \left\{L_3^{t_{0,\xi,0}}, D_3^{t_{\alpha,\beta,\gamma}}, D_3^{t_{\delta,\epsilon,0}}\right\}$ since $\mathrm{Tr}(x)$ is an identity for $M_2^{t_0}$ but not for C.
- $\mathrm{Id}^{tr}(M_2^{t_0}) \not\subseteq \mathrm{Id}^{tr}(UT_3^{t_{0,0,0}})$ since $\left[[x_1, x_2]^2, x_3\right]$ is an identity for $M_2^{t_0}$ but $\left[[e_{12}, e_{21}]^2, e_{23}\right] \neq 0$ on $UT_3^{t_{0,0,0}}$.
- $\mathrm{Id}^{tr}(L_3^{t_{0,\xi,0}}) \not\subseteq \mathrm{Id}^{tr}(UT_3^{t_{0,0,0}})$. In fact, the polynomial $[x_1, x_2][x_3, x_4] \equiv 0$ on $L_3^{t_{0,\xi,0}}$ but $[e_{11}, e_{12}][e_{23}, e_{33}] \neq 0$ on $UT_3^{t_{0,0,0}}$.
- $\mathrm{Id}^{tr}(L_3^{t_{0,\xi,0}}) \not\subseteq \mathrm{Id}^{tr}(D_3^{t_{\alpha,\beta,\gamma}})$. In fact, the polynomial $h = \xi\mathrm{Tr}(x_1x_2) - \mathrm{Tr}(x_1)\mathrm{Tr}(x_2) \equiv 0$ on $L_3^{t_{0,\xi,0}}$ but $h(e_{11}, e_{22}) \neq 0$ on $D_3^{t_{\alpha,\beta,\gamma}}$. Here γ can be zero.

□

As a consequence of Theorem 24, we get the following corollary.

Corollary 26. *Let $A = \bar{A} + J$ be a unitary finite dimensional algebra with trace tr over a field F of characteristic zero with tr$(J) = 0$. Then* $\exp^{tr}(A) = 2$ *if and only if*

1. $B \notin var^{tr}(A)$, *for all* $B \in \left\{UT_3^{t_{0,0,0}}, L_3^{t_{0,\xi,0}}, D_3^{t_{\alpha,\beta,\gamma}}, D_3^{t_{\delta,\epsilon,0}}, M_2^{t_\kappa}\right\}$, *where* $\alpha, \beta, \gamma, \delta, \epsilon, \xi \in F \setminus \{0\}$, $\kappa \in F$,
2. $C \in var^{tr}(A)$, *for at least one algebra* $C \in \left\{D_2^{t_{\alpha',0}}, D_2^{t_{\beta',\beta'}}, D_2^{t_{\gamma',\delta'}}, UT_2\right\}$, α', β', γ', $\delta' \in F \setminus \{0\}$, $\gamma' \neq \delta'$.

Now we present the analogous characterization concerning the pure trace exponent.

Theorem 27. *Let $A = \bar{A} + J$ be a unitary finite dimensional algebra with trace tr over a field F of characteristic zero with tr$(J) = 0$. Then* $\exp^{ptr}(A) > 2$ *if and only if* $D_3^{t_{\alpha,\beta,\gamma}}$ *or* $M_2^{t_\delta}$ *belong to* $var^{ptr}(A)$, $\alpha, \beta, \gamma, \delta \in F \setminus \{0\}$.

Proof. As in the proof of Theorem 24 we may assume that the field F is algebraically closed and $\bar{A} = A_1 \oplus \cdots \oplus A_m = M_{n_1}(F) \oplus \cdots \oplus M_{n_1}(F)$ and, for any $i = 1, \ldots, m$, the trace on A_i is $t_{\alpha_i} = \alpha_i t_1^i$. If for some $j \in \{1, \ldots, m\}$ $\dim_F A_j \geq 4$ then, by [11, Lemma 28], $M_2^{t_{\alpha_j}} \in \mathrm{var}^{ptr}(A)$ and we are done. Hence we may assume that $\dim_F A_j < 4$, for every $j = 1, \ldots, m$, and, so, $A_j \cong F$, for any $j \in \{1, \ldots, m\}$. Since $\exp^{tr}(A) > 2$, there exist distinct A_i, A_j, A_k such that $\mathrm{tr}(A_i)\mathrm{tr}(A_j)\mathrm{tr}(A_k) \neq 0$. As in Theorem 24, we get that $D_3^{t_{\alpha,\beta,\gamma}} \in \mathrm{var}^{ptr}(A), \alpha, \beta, \gamma \neq 0$ and we are done.

The opposite implication clearly follows by Remark 23. □

Notice that $\mathrm{Id}^{ptr}(D_3^{t_{\alpha,\beta,\gamma}}) \not\subseteq \mathrm{Id}^{ptr}(M_2^{t_\delta})$ since $\exp^{ptr}(D_3^{t_{\alpha,\beta,\gamma}}) = 3 < 4 = \exp^{ptr}(M_2^{t_\delta})$. On the other hand, the pure trace polynomial $\mathrm{Tr}(x_1[x_2, x_3])$ is an identity of $D_3^{t_{\alpha,\beta,\gamma}}$ but $\mathrm{Tr}(e_{11}[e_{12}, e_{21}]) = \delta \neq 0$ on $M_2^{t_\delta}$. This says that the above list of algebras cannot be reduced.

As a consequence of Theorem 27 we get the following.

Corollary 28. *Let $A = \bar{A} + J$ be a unitary finite dimensional algebra with trace tr over a field F of characteristic zero with $tr(J) = 0$. Then $\exp^{ptr}(A) = 2$ if and only if $D_3^{t_{\alpha,\beta,\gamma}}, M_2^{t_\delta} \notin var^{ptr}(A)$, for any choice of $\alpha, \beta, \gamma, \delta \in F \setminus \{0\}$ and either $D_2^{t_{\beta',\beta'}}$ or $D_2^{t_{\gamma',\delta'}} \in var^{ptr}(A)$, for some $\beta', \gamma', \delta' \in F \setminus \{0\}$, $\gamma' \neq \delta'$.*

Now we recall that a variety $\mathcal{V}^{tr}$ of algebras with trace is minimal with respect to the trace exponent if for any proper subvariety $\mathcal{U}^{tr}$, generated by a unitary finite dimensional trace algebra with zero trace on its Jacobson radical, we have that $\exp^{tr}(\mathcal{V}^{tr}) > \exp^{tr}(\mathcal{U}^{tr})$. Here the trace exponent of a variety is the trace exponent of a generating algebra. In a similar way one defines a minimal variety with respect to the pure trace exponent.

By using this definition we get the following.

Corollary 29.

1. *The algebras $D_2^{t_{\alpha,0}}$, $D_2^{t_{\beta,\beta}}$, $D_2^{t_{\gamma,\delta}}$, UT_2, for every choice of α, β, γ, $\delta \in F \setminus \{0\}$, $\gamma \neq \delta$, are the only algebras, up to T^{tr}-equivalence, generating minimal varieties of trace exponent 2.*
2. *The algebras $D_2^{t_{\beta,\beta}}$, $D_2^{t_{\gamma,\delta}}$, for every choice of β, γ, $\delta \in F \setminus \{0\}$, $\gamma \neq \delta$, are the only algebras, up to T^{tr}-equivalence, generating minimal varieties of pure trace exponent 2.*
3. *The algebras $UT_3^{t_{0,0,0}}, L_3^{t_{0,\xi,0}}, D_3^{t_{\alpha,\beta,\gamma}}, D_3^{t_{\delta,\epsilon,0}}$, where $\alpha, \beta, \gamma, \delta, \epsilon, \xi \in F \setminus \{0\}$, are the only algebras, up to T^{tr}-equivalence, generating minimal varieties of trace exponent 3.*
4. *The algebras $D_3^{t_{\alpha,\beta,\gamma}}$ $\alpha, \beta, \gamma, \delta \in F \setminus \{0\}$, are the only algebras, up to T^{tr}-equivalence, generating minimal varieties of pure trace exponent 3.*
5. *The algebras $M_2^{t_\kappa}$, $\kappa \in F$, generate minimal varieties of trace exponent 4.*
6. *The algebras $M_2^{t_\delta}$, $\delta \in F \setminus \{0\}$ generate minimal varieties of pure trace exponent 4.*

Proof. We prove only the statement 3., since any other statement can be proved in the same way. Let

$$B \in \{UT_3^{t_{0,0,0}}, L_3^{t_{0,\xi,0}}, D_3^{t_{\alpha,\beta,\gamma}}, D_3^{t_{\delta,\epsilon,0}}\}$$

and let $\mathcal{V}^{tr}$ be a proper subvariety of $\mathrm{var}^{tr}(B)$. Clearly $B \notin \mathcal{V}^{tr}$. By Proposition 25, $UT_3^{t_{0,0,0}}$, $L_3^{t_{0,\xi,0}}$, $D_3^{t_{\alpha,\beta,\gamma}}$, $D_3^{t_{\delta,\epsilon,0}}$, $M_2^{t_\kappa} \notin \mathcal{V}^{tr}$. Then, from Theorem 24, $\exp^{tr}(\mathcal{V}^{tr}) \leq 2$ and we are done. Now suppose that there exists a minimal variety $\mathcal{U}^{tr}$ of trace exponent 3 which is not generated by any of the algebras in 3. Then, since its trace exponent is 3, $M_2^{t_\kappa} \notin \mathcal{U}^{tr}$, $\kappa \in F$. Moreover, since it is minimal of exponent 3, $UT_3^{t_{0,0,0}}, L_3^{t_{0,\xi,0}}, D_3^{t_{\alpha,\beta,\gamma}}, D_3^{t_{\delta,\epsilon,0}} \notin \mathcal{U}^{tr}$. Then by Theorem 24 we should have $\exp^{tr}(\mathcal{U}) \leq 2$, a contradiction. □

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