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An innovative step-by-step stochastic linearization procedure for solving deterministic nonlinear differential equations

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ABSTRACT

Deterministic nonlinear differential equations are widely used in engineering to model the response of different mechanical systems. In most cases, exact analytical solutions are not available, and approximate techniques are therefore required. Deterministic linearization methods constitute a common class of approximate techniques. However, the development of methods that combine high accuracy with computational efficiency remains a challenging task.

In this paper, an innovative procedure based on a stochastic framework is proposed for solving deterministic nonlinear differential equations. The key idea is to replace the initial condition by a Gaussian random variable with small variance and mean equal to the deterministic initial value. This transformation turns the deterministic response into a stochastic process whose probability density function evolves according to the diffusion-free Fokker-Planck-Kolmogorov equation. Starting from this formulation, the evolution equation of the first-order statistics is derived, and an innovative numerical method, based on a step-by-step stochastic linearization, is introduced. The method is based on a subdivision of the mean value of the nonlinear term in different steps, so that within each step a closed-form linearized solution can be obtained together with the corresponding time instant. It is shown that, for sufficiently small variance of the initial condition, the mean value of the stochastic process coalesces with the solution of the original deterministic nonlinear equation.

The reliability of the proposed procedure is assessed through numerical applications to Kelvin-Voigt-type models with one and two nonlinear springs. The results demonstrate excellent agreement with both direct Runge-Kutta integration and quasi-linearization method, thereby confirming their accuracy and effectiveness. Additionally, it is worth highlighting the enhanced performance of the proposed procedure, as it strongly reduces computational time.

1. Introduction

In mechanical engineering applications, the equation of motion is often ruled by nonlinear differential equations [1–3]. Therefore, in general, no exact analytical solutions are available. Such an example, for a suspended cable loaded by a transversal load the equation of motion presents a nonlinear term $k_\nu x^\nu(t)$ (with $\nu = 3$). The latter represents the hardening behavior, that is modeled as a cubic spring. Another example is the experimental test of the steel, in which the behavior of the material is described through three different power laws of the kind $k_\nu x^\nu$. Specifically, for moderate level of stress, the power law becomes linear ($\nu = 1$). At the yield limit, the plastic behavior is described by a constant law ($\nu = 0$), while the work-hardening behavior is described by a power law $k_\nu x^\nu$ in which $0 < \nu < 1$. In biomechanics, the aorta has two different phases described by a power law: the hardening phase ($\nu > 1$) and the softening phase ($\nu < 1$). In any case, there is a maximum level of stress (or strain) in which the structure collapses. Further engineering applications in which nonlinear differential equations are extensively employed include robotics [4,5], fluid-conveying pipes [6,7], vibration isolation [8,9], bridge dynamics [10], beam structures [11], and energy harvesting systems [12,13].

In order to obtain the solutions of the nonlinear differential equations, approximations, numerical solutions, or both are usually needed. Among the approximation methods, linearization techniques are widely used. Initially, they were developed in a deterministic framework, and then they have been extended to the stochastic field.

Deterministic linearization methods can be broadly classified into three families: local tangent linearization, which is based on first-order

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expansion around a reference state; equivalent or harmonic linearization, which replaces the nonlinear system by an amplitude-dependent linear oscillator; and iterative quasi-linearization, which solves a sequence of linear problems converging toward the nonlinear solution. These methods are conceptually different, but they share the same general objective: to exploit the analytical and computational simplicity of linear systems while retaining, at least approximately, the dominant effects of nonlinear dynamics.

The most direct deterministic approach is local or tangent linearization. In this case, the nonlinear equations are expanded around an equilibrium configuration, a nominal trajectory, or a prescribed operating point, and only the first-order terms of the Taylor expansion are retained. This approach is particularly useful for local stability analysis, control design, and small-amplitude perturbations around known configurations. However, its validity is intrinsically local. In fact, when the response amplitude increases, or when the system explores regions far from the linearization point, the accuracy of the tangent approximation may rapidly deteriorate.

A different and more global idea is provided by deterministic equivalent linearization. Rather than approximating the nonlinear function only in an infinitesimal neighborhood of a prescribed point, the method replaces the nonlinear system by an equivalent linear system whose parameters are selected according to an integral, energetic, harmonic, or residual-minimization criterion. The origin of this idea is commonly associated with the works of Krylov and Bogoliubov and with the asymptotic methods later systematized by Bogoliubov and Mitropolsky [14,15]. For weakly nonlinear oscillations, these methods exploit the separation between fast oscillatory motion and slowly varying amplitude and phase variables, allowing the nonlinear dynamics to be approximated by an equivalent system.

For periodic and quasi-periodic motions, equivalent linearization is closely related to harmonic balance and describing-function methods. In harmonic balance, the nonlinear term is projected onto a finite set of harmonic components, and the governing nonlinear differential equation is transformed into a set of algebraic equations for the unknown harmonic amplitudes. In the describing-function method, only the fundamental harmonic of the nonlinear response to a sinusoidal input is retained. Therefore, the nonlinear element is replaced by an amplitude-dependent complex gain, which provides an equivalent linear representation of the dominant harmonic behavior [16,17].

Another type of linearization is the quasi-linearization method. In this approach, the nonlinear problem is not replaced by a single equivalent linear system, but by a sequence of linear problems obtained through iterative linearization of the nonlinear terms. Bellman and Kalaba formalized quasi-linearization as a Newton-type procedure for nonlinear boundary-value problems, leading to successive linear approximations that converge to the nonlinear solution under suitable conditions [18]. Compared with harmonic or equivalent-linear methods, quasi-linearization is more directly numerical and iterative. It is especially useful when the objective is to solve a deterministic nonlinear problem accurately, rather than to construct a reduced equivalent model with explicit physical interpretation.

Although several deterministic linearization techniques have been proposed in the literature, developing methods that simultaneously ensure high accuracy and computational efficiency remains a nontrivial problem.

Over the past decades, deterministic linearization techniques have been extended to the stochastic field, and statistical linearization methods (SLM) [19] has emerged as one of the most widely adopted techniques. The central idea of SLM is to replace the nonlinear terms with statistically equivalent linear representations by minimizing the mean-square error between the original nonlinear function and its linear approximation. This transformation enables the application of well-established linear stochastic analysis techniques while preserving, to a reasonable extent, the essential statistical characteristics of the nonlinear response.

In this paper, an innovative procedure based upon a stochastic framework is proposed to find the response of deterministic nonlinear differential equations. It can be described as a stochastic reformulation of deterministic nonlinear ordinary differential equations through randomized initial conditions, followed by a moment-based statistical linearization strategy for time integration. Indeed, the initial condition is assumed as a random variable (RV) having Gaussian distribution with small variance and mean value corresponding to the deterministic initial condition. Consequently, the initial deterministic response becomes a stochastic process, fully characterized in a probabilistic framework by its probability density function. The time evolution of this PDF is governed by the drift term of the Fokker-Planck-Kolmogorov equation [20,21]. The solution of the FPK equation, including the contribution of the diffusion term, can be obtained using various methods available in the literature [22–36]. This formulation allows to reformulate the problem in probability space, where the FPK equation is linear both in time and space, while the nonlinear terms of the original dynamics appear as variable coefficients in the new space. Starting from the FPK equation, it is also possible to derive the differential equation governing the evolution of the first-order statistics of the process. By exploiting the smallness of the variance of the initial condition, an innovative numerical method, based on stochastic linearization techniques, can then be employed to compute the solution.

To assess the reliability of the proposed procedure, two numerical applications were performed, showing that the results are in excellent agreement with those obtained by using a Runge-Kutta integration and the quasi-linearization method. Additionally, it is worth highlighting the enhanced performance of the proposed procedure, as it strongly reduces computational time.

2. Nonlinear deterministic equations handled by stochastic approach: Mathematical formulation

Consider a deterministic nonlinear differential equation

$$\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = \overline{x} \end{cases}, \tag{1}$$

where $x(t)$ represents the structural response in terms of displacement, $\dot{x}(t)$ is the structural response in terms of velocity, $x(0) = \overline{x}$ is the initial condition in terms of displacement and $f(x(t))$ is any nonlinear function such that the system is stable.

In the proposed approach, the initial condition is replaced by a random variable (RV) X_0, whose average is

$$E[X_0] = \int_{-\infty}^{\infty} x\, p_{X_0}(x) dx = \overline{x}, \tag{2}$$

being $E[\cdot]$ the stochastic average operator and $p_{X_0}(x)$ the assigned probability density function (PDF) of the RV X_0. It is to be remarked that $p_{X_0}(x)$ must be symmetric with respect to the average $E[X_0]$, and that the variance of X_0, denoted as $\sigma^2_{X_0}$, must be sufficiently small. In this regard, Gaussian distributions or uniform distributions with low values of variance represent suitable choices.

By assuming that the initial condition is a RV, it follows that the structural response, in terms of displacement, is a stochastic process $X(t)$, as well. Therefore, the governing equation of motion is expressed in the form

$$\begin{cases} \dot{X}(t) = f(X(t)) \\ X(0) = X_0 \end{cases} \tag{3}$$

constrained by Eq. (2).

Further, the response process $X(t)$ is fully described, in probabilistic setting, by the PDF $p_X(x,t)$, whose evolution in time is governed by the diffusion-free Fokker-Planck-Kolmogorov (FPK) equation as:

$$\begin{cases} \dfrac{\partial p_X(x,t)}{\partial t} = -\dfrac{\partial}{\partial x}(f(x)p_X(x,t)) \\ p_X(x,0) = p_{X_0}(x) \end{cases}. \tag{4}$$

In this form, Eq. (4) is also related to the main system at hand, given in Eq. (1). At first glance, it seems that Eq. (1) is simpler than Eq. (4). However, Eq. (1) is nonlinear for the presence of the nonlinear function $f(x(t))$, while Eq. (4) is a linear differential equation both in time and in terms of PDF response. Moreover, it is to be stressed that, in Eq. (4), $f(x)$ plays the role of variable coefficient in the probabilistic space.

Once that the solution of Eq. (4) has been evaluated, the solution of Eq. (1) is recovered as

$$x(t) = E[X(t)] = \int_{-\infty}^{\infty} x\, p_X(x,t)dx. \tag{5}$$

3. Test on the validity of the proposed approach

In order to check the validity of the proposed approach, a classical Kelvin-Voigt model is considered. The latter is obtained by connecting, in parallel, a purely elastic element (spring) and a purely viscous element (dashpot). The governing equation of motion is ruled by the linear differential equation

$$\begin{cases} c\dot{x}(t) = -k\,x(t) \\ x(0) = \overline{x} \end{cases}, \tag{6}$$

in which k is the stiffness of the spring, while c represents the damping coefficient of the dashpot. Dividing both members of Eq. (6) by c, Eq. (6) is rewritten as

$$\begin{cases} \dot{x}(t) = -\widetilde{k}\,x(t) \\ x(0) = \overline{x} \end{cases}, \tag{7}$$

being $\widetilde{k} = k/c$. First of all, the solution of Eq. (7) is given in the form

$$x(t) = \overline{x}\, e^{-\widetilde{k}t}. \tag{8}$$

Then, by considering $f(X(t)) = -\widetilde{k}\,X(t)$, Eq. (3) can be particularized as

$$\begin{cases} \dot{X}(t) = -\widetilde{k}\,X(t) \\ X(0) = X_0 \end{cases}, \tag{9}$$

while Eq. (4) becomes

$$\begin{cases} \dfrac{\partial p_X(x,t)}{\partial t} = -\dfrac{\partial}{\partial x}(-\widetilde{k}\,x\,p_X(x,t)) \\ p_X(x,0) = p_{X_0}(x) \end{cases}. \tag{10}$$

Consider a Gaussian distribution of X_0 with a standard deviation σ_{X_0}, that is

$$p_{X_0}(x) = \frac{1}{\sqrt{2\pi\sigma_{X_0}^2}} \exp\left(-\frac{(x-\overline{x})^2}{2\sigma_{X_0}^2}\right). \tag{11}$$

Since Eq. (9) is linear, the system response is a Gaussian process. Consequently, its PDF is Gaussian and is also the solution of Eq. (10). Therefore, it can be expressed as

$$p_X(x,t) = \frac{1}{\sqrt{2\pi\sigma_{X_0}^2}} \exp\left(\widetilde{k}t - \frac{\left(\overline{x} - x\, e^{\widetilde{k}t}\right)^2}{2\sigma_{X_0}^2}\right). \tag{12}$$

By multiplying both sides of Eq. (12) by $x\,dx$ and integrating between $-\infty$ and ∞, yields

$$E[X(t)] = \frac{1}{\sqrt{2\pi\sigma_{X_0}^2}} \int_{-\infty}^{\infty} x \exp\left(\widetilde{k}t - \frac{\left(\overline{x} - x\, e^{\widetilde{k}t}\right)^2}{2\sigma_{X_0}^2}\right)dx = \overline{x}\, e^{-\widetilde{k}t}, \tag{13}$$

that perfectly coincides with the solution of Eq. (7) reported in Eq. (8).

Once this result has been established, the solution of Eq. (1) can be obtained by determining the evolution equation for the mean, given below

$$\begin{cases} \dot{E}[X(t)] = E[f(X(t))] \\ E[X(0)] = E[X_0] \end{cases}. \tag{14}$$

The latter equation has been obtained simply by taking the stochastic average $E[\cdot]$ of both sides of Eq. (3) and by commuting the expectation operator with the derivative, but it could also have been derived through the following steps.

Starting from Eq. (4) and multiplying both sides of Eq. (4) by $x\,dx$ and integrating between $-\infty$ and ∞,

$$\begin{cases} \displaystyle\int_{-\infty}^{\infty} x\frac{\partial p_X(x,t)}{\partial t}dx = -\int_{-\infty}^{\infty} x\frac{\partial}{\partial x}(f(x)p_X(x,t))dx \\ \displaystyle\int_{-\infty}^{\infty} x p_X(x,0)dx = \int_{-\infty}^{\infty} x p_{X_0}(x)dx \end{cases}, \tag{15}$$

i.e.

$$E[\dot{X}(t)] = -\int_{-\infty}^{\infty} x\frac{\partial}{\partial x}(f(x)p_X(x,t))dx \tag{16}$$

with initial condition

$$E[X(0)] = E[X_0]. \tag{17}$$

Integrating by parts the second member of Eq. (16), yields

$$-\int_{-\infty}^{\infty} x\frac{\partial}{\partial x}(f(x)p_X(x,t))dx = -\left[x f(x)p_X(x,t)\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} f(x)p_X(x,t)dx. \tag{18}$$

If

$$\lim_{x\to\pm\infty}(x f(x)p_X(x,t)) = 0, \tag{19}$$

then Eq. (18) becomes

$$-\int_{-\infty}^{\infty} x\frac{\partial}{\partial x}(f(x)p_X(x,t))dx = \int_{-\infty}^{\infty} f(x)p_X(x,t)dx = E[f(X(t))]. \tag{20}$$

Through a direct comparison between Eq. (16) and Eq. (20), and commuting the expectation operator with the derivative, it leads to

$$\begin{cases} \dot{E}[X(t)] = E[f(X(t))] \\ E[X(0)] = E[X_0] \end{cases}, \tag{21}$$

that coincides with Eq. (14).

Up to this point, a key result has been established: the solution of Eq. (1) is recovered from the solution of Eq. (14). This statement constitutes not only a significant theoretical result, but also provides the basis for an innovative numerical method, based on stochastic linearization techniques, as detailed in the next section.

4. Innovative numerical method

For the system under consideration, being stable and unexcited, the maximum value of the response coincides with the initial condition, namely $E[X(0)] = E[X_0]$. The corresponding value $E[f(X_0)]$ is first approximated as $f(E[X_0])$ and subsequently partitioned into n steps, not

necessarily constant.

At the i-th step, with $i = 1,2,\ldots,n$, the values of $E[f(X_i)] \approx f(E[X_i])$ and $E[X_i]$ are known a priori from the adopted subdivision. Thus, only the time instant t_i at which the event $E[X_i]$ occurs is unknown, to be determined.

To this end, Eq. (14) is replaced, over the time interval $t \in [t_{i-1}, t_i]$, by the locally linearized problem

$$\begin{cases} \dot{E}[X(t)] = -k_{eq}^{(i)} E[X(t)] \\ E[X(t_{i-1})] = E[X_{i-1}] \end{cases}; \quad t \in [t_{i-1}, t_i]. \tag{22}$$

The equivalent coefficient $k_{eq}^{(i)}$ is identified by direct comparison between Eq. (14) and Eq. (22), which yields

$$E[f(X(t))] = -k_{eq}^{(i)} E[X(t)]. \tag{23}$$

By imposing $t = t_{i-1}$, the equivalent coefficient $k_{eq}^{(i)}$ is evaluated as

$$k_{eq}^{(i)} = -\frac{E[f(X_{i-1})]}{E[X_{i-1}]} \approx -\frac{f(E[X_{i-1}])}{E[X_{i-1}]}. \tag{24}$$

Further, the solution of Eq. (22), characterized at the time instant $t = t_i$, is expressed in the form

$$E[X_i] = E[X_{i-1}]\, e^{-k_{eq}^{(i)}\,\Delta t_i}, \tag{25}$$

from which the i-th time step Δt_i is computed as

$$\Delta t_i = -\frac{1}{k_{eq}^{(i)}} \ln\left[\frac{E[X_i]}{E[X_{i-1}]}\right] = \frac{E[X_{i-1}]}{E[f(X_{i-1})]} \ln\left[\frac{E[X_i]}{E[X_{i-1}]}\right]. \tag{26}$$

Finally, the time instant t_i corresponding to the value $E[X_i]$ is obtained by

$$t_i = t_{i-1} + \Delta t_i. \tag{27}$$

It should be emphasized that, in order to obtain a discrepancy between $E[f(X_0)]$ and $f(E[X_0])$ lower than a prescribed value, denoted by $d_{\%}$, the value of $\sigma_{X_0}^2$ must be chosen such that $\sigma_{X_0}^2 < (d_{\%}/50)(f(\bar{x})/|f''(\bar{x})|)$. Moreover, since $\sigma_{X_i}^2$, at each step, decays faster than $E[X_i]$, this choice also ensures that the discrepancy between $E[f(X_i)]$ and $f(E[X_i])$ remains below the aforementioned value.

5. Numerical applications

In order to both assess the reliability of the proposed approach and its computational efficiency, in this section, numerical simulations on two different mechanical systems are reported.

5.1. Numerical application: Kelvin-Voigt model with nonlinear spring

Consider the Kelvin-Voigt model in which the linear spring is replaced by a nonlinear spring, depicted in Fig. 1, and whose motion is governed by

$$\begin{cases} c\dot{x}(t) = -k_\nu\, x^\nu(t) \\ x(0) = \bar{x} \end{cases}, \tag{28}$$

where k_ν is the coefficient of the nonlinear spring, while the exponent ν defines the softening behavior ($0 < \nu < 1$), the linear behavior ($\nu = 1$) or the hardening ($\nu > 1$) behavior. It is to be stressed that the physical dimension of k_ν depends on the exponent ν.

Dividing both members of Eq.(28) by c,

$$\begin{cases} \dot{x}(t) = -\tilde{k}_\nu\, x^\nu(t) \\ x(0) = \bar{x} \end{cases}, \tag{29}$$

$\tilde{k}_\nu = k_\nu/c$ is introduced.

To obtain the solution of Eq. (29), the proposed procedure was used. Specifically, the initial condition has been replaced by a Gaussian RV X_0. Therefore, the differential equation that rules the mean of the response process $X(t)$ can be written in the form

$$\begin{cases} \dot{E}[X(t)] = -\tilde{k}_\nu E[X^\nu(t)] \\ E[X(0)] = E[X_0] \end{cases}. \tag{30}$$

For the first numerical application, the three different cases reported in Table 1 have been considered.

For the RV X_0, the selected mean value is $E[X_0] = \bar{x} = 3$ m. The variance of X_0 has been chosen by imposing $d_{\%}$ equal to 0.5 %. Consequently, for the first case $\sigma_{X_0}^2 \le 0.015$ m^2, for the second case any positive finite value can be selected, since the problem is linear. Finally, for the third case $\sigma_{X_0}^2 \le 0.36$ m^2. Therefore, $\sigma_{X_0}^2 = 0.01$ m^2 has been selected for all three cases. It is worth noting that the selection of $\sigma_{X_0}^2$ does not affect the computational time, but only the accuracy of the approximations in Eq. (24).

For the discretization of the nonlinear function, first it has been subdivided into 100 equally spaced intervals between its maximum value ($f(E[X_0])$) and a minimum value set equal to $f\left(E\left[5 \times 10^{-3} X_0\right]\right)$. Each interval was then refined using a reference grid of 10^5 points. Starting from two points for each interval, i.e. the initial and final point, the number of points within each interval is progressively doubled until the mean percentage discrepancy between the piecewise-linear approximation and the refined nonlinear curve becomes smaller than 0.1 %.

In Figs. 2–4, the solution of Eq. (30) for the cases reported in Table 1 is compared with the solutions obtained from both a Runge-Kutta integration of Eq. (28) and the quasi-linearization method. In order to ensure a consistent comparison, the time vector obtained from the proposed procedure has been used for both the Runge-Kutta integration and the quasi-linearization method.

From Figs. 2–4, it can be observed that the results obtained by using the proposed procedure perfectly coincide with those obtained by a Runge-Kutta integration of Eq. (28) and those obtained from the quasi-linearization method.

In Table 2, the computational time (CT) required by the proposed procedure is compared with the corresponding times required by the Runge–Kutta integration and the quasi-linearization method. The table also reports the corresponding reduction in computational time, evaluated with respect to the Runge–Kutta integration ($RED_{\%}^{(PP-RK)}$) and to the

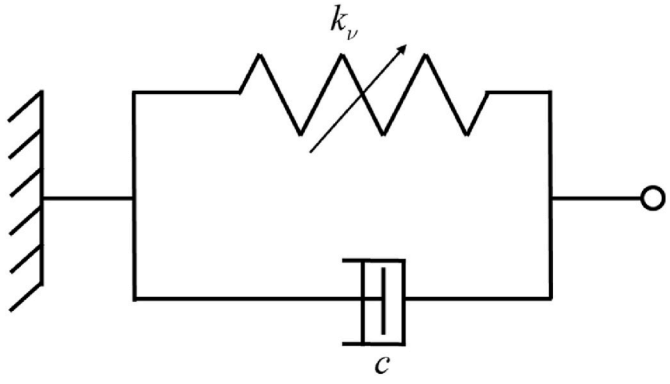

Fig. 1. – Kelvin-Voigt model with nonlinear spring.

Table 1
– Cases considered for the first numerical application.

Case	Parameters of the model		
	ν	k_ν	c
1	0.5	100 kg m$^{0.5}$s^{-2}	200 kg s^{-1}
2	1.0	50 kg s^{-2}	100 kg s^{-1}
3	3.0	10 kg m^{-2}s^{-2}	20 kg s^{-1}

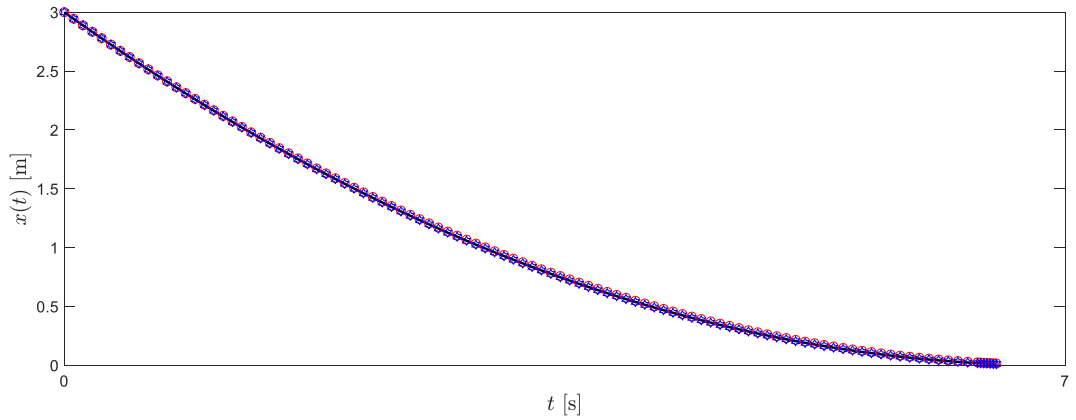

Fig. 2. – Comparison between the results obtained by using the proposed procedure (circular marker), the results obtained by performing a Runge-Kutta integration of the deterministic nonlinear differential equation (continuous line), and the results obtained from the quasi-linearization method (hexagonal marker): Case 1 ($\nu = 0.5$).

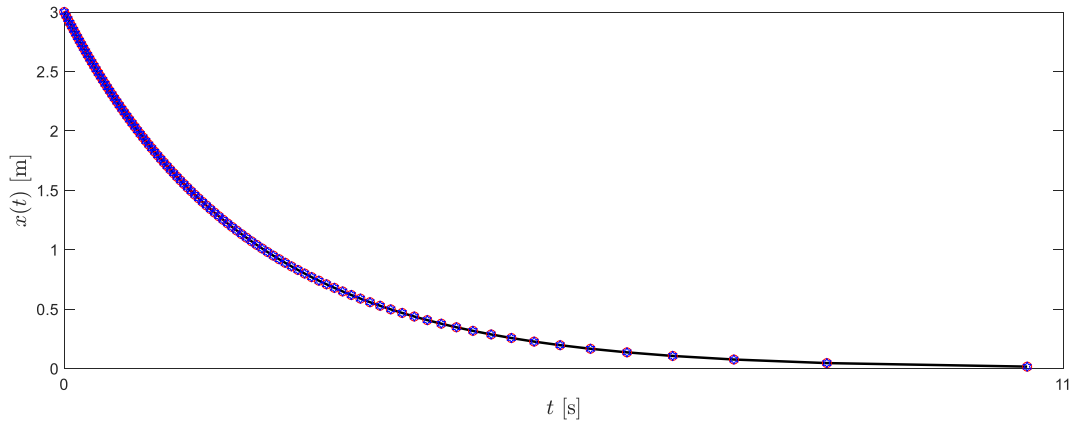

Fig. 3. – Comparison between the results obtained by using the proposed procedure (circular marker), the results obtained by performing a Runge-Kutta integration of the deterministic nonlinear differential equation (continuous line), and the results obtained from the quasi-linearization method (hexagonal marker): Case 2 ($\nu = 1.0$).

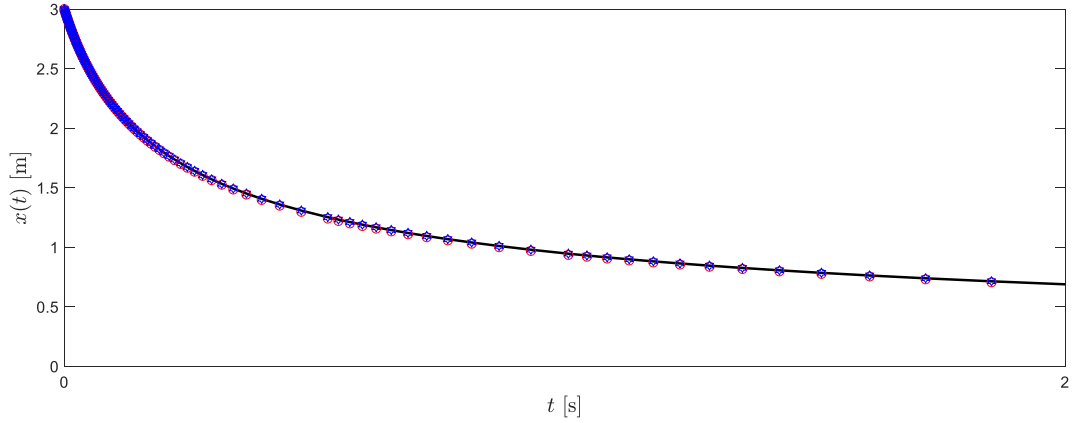

Fig. 4. – Comparison between the results obtained by using the proposed procedure (circular marker), the results obtained by performing a Runge-Kutta integration of the deterministic nonlinear differential equation (continuous line), and the results obtained from the quasi-linearization method (hexagonal marker): Case 3 ($\nu = 3.0$).

quasi-linearization method ($RED_{\%}^{(PP-QL)}$) as

$$RED_{\%}^{(PP-RK)} = 100\left(1 - \frac{CT^{(PP)}}{CT^{(RK)}}\right) \tag{31a}$$

and

$$RED_{\%}^{(PP-QL)} = 100\left(1 - \frac{CT^{(PP)}}{CT^{(QL)}}\right), \tag{31b}$$

respectively. In Eqs. (31a and 31b), $CT^{(PP)}$ denotes the computational time required by the proposed procedure, $CT^{(RK)}$ represents the

Table 2
– Comparison between the computational time required by the proposed procedure, the computational time required by a Runge-Kutta integration and that required by the quasi-linearization method.

Case	$CT^{(PP)}$	$CT^{(RK)}$	$CT^{(QL)}$	$RED_{\%}^{(PP-RK)}$	$RED_{\%}^{(PP-QL)}$
1	5.27×10^{-4} s	7.68×10^{-3} s	3.06×10^{-3} s	93.13 %	82.78 %
2	5.55×10^{-4} s	2.15×10^{-3} s	8.16×10^{-3} s	74.22 %	93.20 %
3	1.24×10^{-3} s	6.61×10^{-2} s	1.28×10^{-1} s	98.13 %	99.04 %

computational time required by the Runge-Kutta integration, and $CT^{(QL)}$ is the computational time required by the quasi-linearization method.

From Tables 2 and it can be noted that the proposed procedure requires substantially less computational time than the other methods.

5.2. Numerical application: Kelvin-Voigt model with two nonlinear springs

Consider the Kelvin-Voigt model in which the linear spring is replaced by two nonlinear springs, depicted in Fig. 5, and whose motion is governed by

$$\begin{cases} c\dot{x}(t) = -k_{\nu_1} x^{\nu_1}(t) - k_{\nu_2} x^{\nu_2}(t) \\ x(0) = \overline{x} \end{cases}, \tag{32}$$

where k_{ν_1} and k_{ν_2} are the coefficients of the nonlinear springs.

Dividing both members of Eq.(32) by c,

$$\begin{cases} \dot{x}(t) = -\widetilde{k}_{\nu_1} x^{\nu_1}(t) - \widetilde{k}_{\nu_2} x^{\nu_2}(t) \\ x(0) = \overline{x} \end{cases}, \tag{33}$$

$\widetilde{k}_{\nu_1} = k_{\nu_1}/c$ and $\widetilde{k}_{\nu_2} = k_{\nu_2}/c$ are introduced.

To obtain the solution of Eq. (33), the proposed procedure was used. Specifically, the initial condition has been replaced by a Gaussian RV X_0. Therefore, the differential equation that rules the mean of the response process $X(t)$ can be written in the form

$$\begin{cases} \dot{E}[X(t)] = -\widetilde{k}_{\nu_1} E[X^{\nu_1}(t)] - \widetilde{k}_{\nu_2} E[X^{\nu_2}(t)] \\ E[X(0)] = E[X_0] \end{cases}. \tag{34}$$

For the second numerical application, the three different cases reported in Table 3 have been considered.

For the RV X_0, the selected mean value is $E[X_0] = \overline{x} = 3$ m. The variance of X_0 has been chosen by imposing $d_{\%}$ equal to 0.5 %.

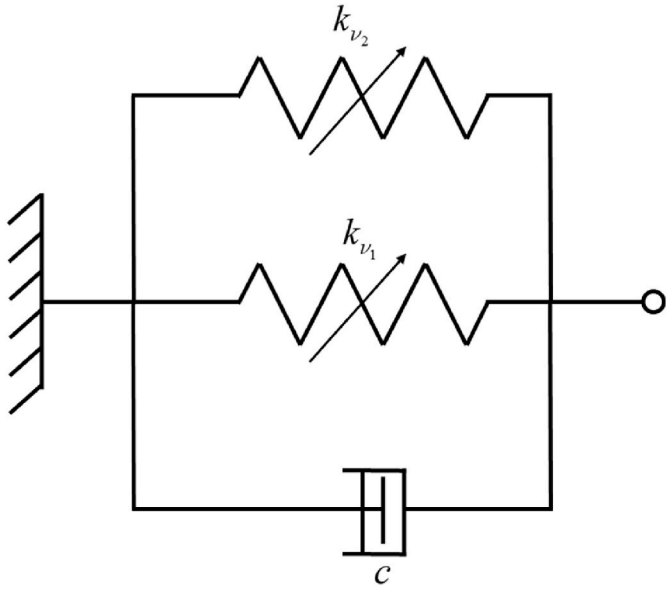

Fig. 5. – Kelvin-Voigt model with two nonlinear springs.

Table 3
– Cases considered for the second numerical application.

Case	Parameters of the model				
	ν_1	ν_2	k_{ν_1}	k_{ν_2}	c
1	3.0	1.0	25 kg m^{-2}s^{-2}	10 kg s^{-2}	50 kg s^{-1}
2	2.0	0.5	50 kg m^{-1}s^{-2}	20 kg m$^{0.5}$s^{-2}	100 kg s^{-1}
3	1.0	0.5	100 kg s^{-2}	40 kg m$^{0.5}$s^{-2}	200 kg s^{-1}

Consequently, for the first case $\sigma_{X_0}^2 \leq 0.016$ m^2, for the second case $\sigma_{X_0}^2 \leq 0.049$ while, for the third case $\sigma_{X_0}^2 \leq 1.92$ m^2. Therefore, $\sigma_{X_0}^2 = 0.01$ m^2 has been selected for all three cases.

For the discretization of the nonlinear function, the same procedure adopted in the numerical application reported in Section 4.1 has been used.

In Figs. 6–8, the solution of Eq. (34) for the cases reported in Table 3 is compared with the solutions obtained from both a Runge-Kutta integration of Eq. (32) and the quasi-linearization method. In order to ensure a consistent comparison, the time vector obtained from the proposed procedure has been used for both the Runge-Kutta integration and the quasi-linearization method.

From Figs. 6–8, it can be observed that the results obtained by using the proposed procedure perfectly coincide with those obtained by a Runge-Kutta integration of Eq. (32) and those obtained from the quasi-linearization method.

In Table 4, the computational time (CT) required by the proposed procedure is compared with those required by the Runge–Kutta integration and the quasi-linearization method, together with the corresponding reductions in computational time computed according to Eqs. (31a and 31b).

From Tables 4 and it can be noted that, once again, the proposed procedure requires substantially lower computational times than the other methods.

6. Concluding remarks

In this paper, an innovative procedure for solving deterministic nonlinear differential equations via stochastic approach has been proposed. The initial step is to substitute the deterministic initial condition by a Gaussian random variable with small variance and mean equal to the deterministic initial value. This transformation allows the response to be described in probabilistic terms through the diffusion-free Fokker-Planck-Kolmogorov equation, while the deterministic solution is recovered from the first-order statistics of the associated stochastic process.

The main theoretical result is that, for a small variance of the initial condition, the mean response of the stochastic process coalesces with the solution of the original deterministic nonlinear equation. On this basis, an innovative numerical method has been developed. The method does not rely on a conventional discretization of the time axis. Rather, it constructs the solution by subdividing the range of the mean nonlinear term and solving in closed form, within each interval, a linearized problem associated with an equivalent coefficient and a corresponding time instant.

The numerical applications to Kelvin-Voigt-type models with one and two nonlinear springs, including softening, linear, and hardening

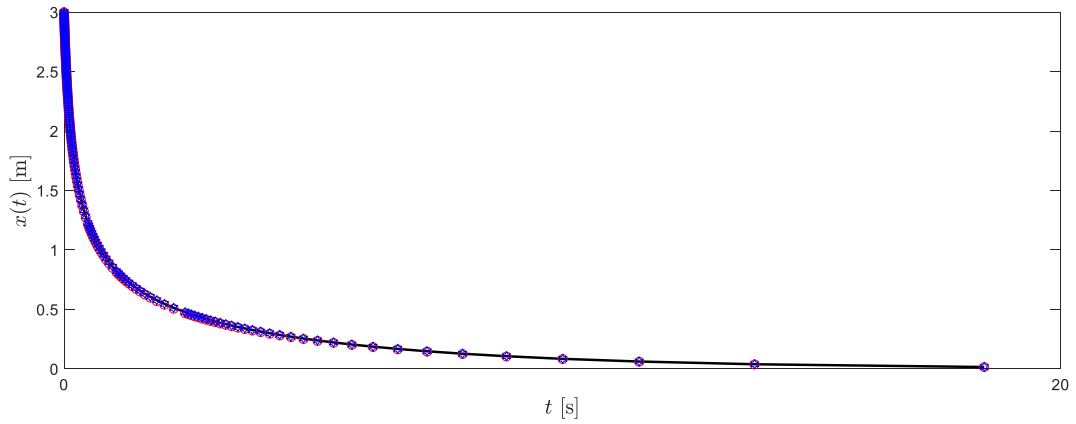

Fig. 6. – Comparison between the results obtained by using the proposed procedure (circular marker), the results obtained by performing a Runge-Kutta integration of the deterministic nonlinear differential equation (continuous line), and the results obtained from the quasi-linearization method (hexagonal marker): Case 1 ($\nu_1 = 3.0$; $\nu_2 = 1.0$).

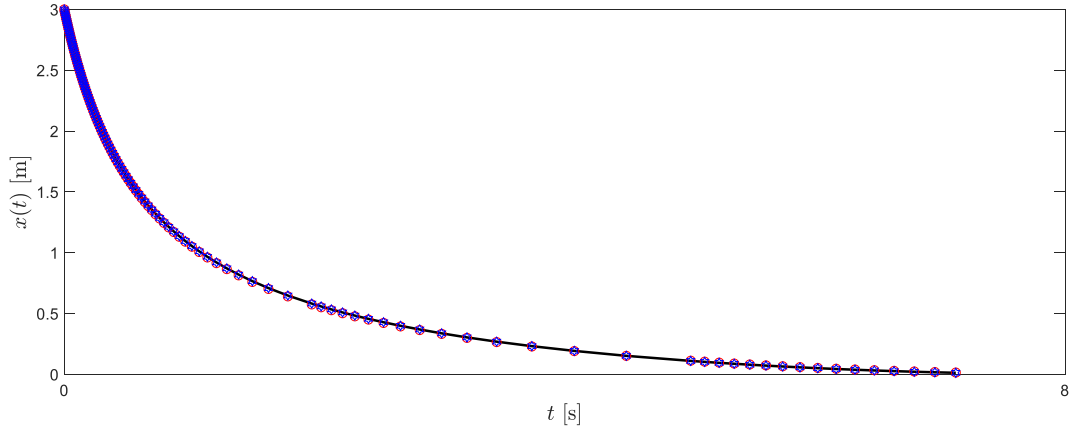

Fig. 7. – Comparison between the results obtained by using the proposed procedure (circular marker), the results obtained by performing a Runge-Kutta integration of the deterministic nonlinear differential equation (continuous line), and the results obtained from the quasi-linearization method (hexagonal marker): Case 2 ($\nu_1 = 2.0$; $\nu_2 = 0.5$).

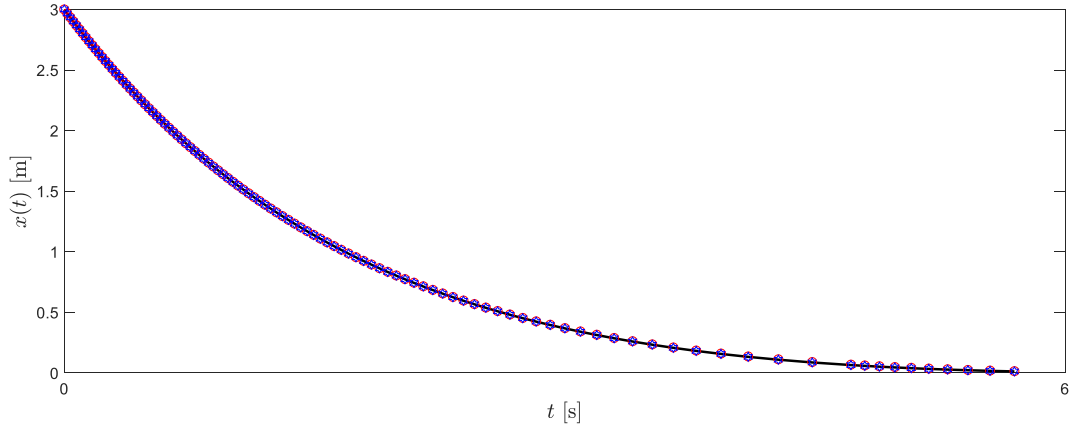

Fig. 8. – Comparison between the results obtained by using the proposed procedure (circular marker), the results obtained by performing a Runge-Kutta integration of the deterministic nonlinear differential equation (continuous line), and the results obtained from the quasi-linearization method (hexagonal marker): Case 3 ($\nu_1 = 1.0$; $\nu_2 = 0.5$).

behaviors, have shown that the proposed procedure not only reproduces with excellent agreement the solutions obtained by both a Runge-Kutta integration and a quasi-linearization method, but also strongly reduces the computational time. Therefore, the proposed procedure can be considered a reliable and computationally efficient approach that opens a different perspective for the treatment of deterministic nonlinear problems, showing that stochastic tools can be employed even in the absence of external randomness.

Future developments may include extending the proposed approach to multi-degree-of-freedom systems and to systems characterized by

Table 4
– Comparison between the computational time required by the proposed procedure, the computational time required by a Runge-Kutta integration and that required by the quasi-linearization method.

Case	$CT^{(PP)}$	$CT^{(RK)}$	$CT^{(QL)}$	$RED_{\%}^{(PP-RK)}$	$RED_{\%}^{(PP-QL)}$
1	6.70×10^{-4} s	2.41×10^{-3} s	1.51×10^{-1} s	72.16 %	99.56 %
2	5.98×10^{-4} s	2.13×10^{-3} s	4.10×10^{0} s	71.93 %	99.99 %
3	5.64×10^{-4} s	2.02×10^{-3} s	9.88×10^{-2} s	72.10 %	99.43 %

more complex nonlinearities.

CRediT authorship contribution statement

Salvatore Russotto: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Validation, Writing – original draft, Writing – review & editing. **Mario Di Paola:** Conceptualization, Investigation, Methodology, Supervision, Writing – original draft, Writing – review & editing. **Antonina Pirrotta:** Conceptualization, Investigation, Methodology, Supervision, Validation, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

References

[1] A. Pirrotta, Multiplicative cases from additive cases: extension of kolmogorov-feller equation to parametric poisson white noise processes, Probab. Eng. Mech. 22 (2) (2007) 127–135.
[2] Paola M. Di, G. Falsone, A. Pirrotta, Stochastic response analysis of nonlinear systems under gaussian inputs, Probab. Eng. Mech. 7 (1) (1992) 15–21.
[3] A. Pirrotta, Non-linear systems under parametric white noise input: digital simulation and response, Int. J. Non Lin. Mech. 40 (8) (2005) 1088–1101.
[4] J. Wang, J. Liu, B. Ji, Y. He, S. Xia, Y. Zhou, Vibration suppression and boundary control for nonlinear flexible rotating manipulator in three-dimensional space subject to output restrictions, Communications in Nonlinear Science and Numerical Simulation 120 (2023) 107151.
[5] N. Ji, J. Liu, Consensus control and vibration suppression for multiple flexible nonlinear timoshenko manipulators under undirected communication topology, Communications in Nonlinear Science and Numerical Simulation 138 (2024) 108200.
[6] M.J. Pourmohammadi, M. Eftekhari, M. Gholami, A novel approach to vibration control in fluid-conveying pipes using piezoelectric actuators, Communications in Nonlinear Science and Numerical Simulation 149 (2025) 108914.
[7] S. Mirhashemi, M. Saeidiha, H. Ahmadi, Dynamics of a harmonically excited nonlinear pipe conveying fluid equipped with a local nonlinear energy sink, Communications in Nonlinear Science and Numerical Simulation 118 (2023) 107035.
[8] C. Liu, K. Yu, B. Liao, R. Hu, Enhanced vibration isolation performance of quasi-zero-stiffness isolator by introducing tunable nonlinear inerter, Communications in Nonlinear Science and Numerical Simulation 95 (2021) 105654.
[9] C. Masnata, S.D. Di Trapani, S. Russotto, A. Pirrotta, Optimal placement and design of M-STLCDs in n-story buildings: a comparative study with M-TMDs and M-TLCDs, Frontiers in Built Environment 11 (2025) 1652972.
[10] L.H. Dongmo Nguebem, S.C. Mba Feulefack, A.M. Ngounou, B.R. Nana Nbendjo, Dynamic analysis and performance evaluation of inerter fractional nonlinear quasi-zero stiffness isolator on a multi-span bridge under moving load, Communications in Nonlinear Science and Numerical Simulation 135 (2024) 108074.
[11] H. Kang, Y. Wang, Y. Zhao, Nonlinear vibration analysis of a double-cable beam structure with nonlinear energy sinks, Communications in Nonlinear Science and Numerical Simulation 142 (2025) 108529.
[12] Y. Zhang, G. Zhang, W. Wang, A piezoelectric cantilever-beam-spring-pendulum oscillator for multi-directional vibration energy harvesting, Communications in Nonlinear Science and Numerical Simulation 138 (2024) 108199.
[13] D. Wang, F. Chen, Z. Hao, M. Wiercigroch, Nonlinear energy harvesting from rotating system, Communications in Nonlinear Science and Numerical Simulation 127 (2023) 107521.
[14] N.M. Krylov, N.N. Bogoliubov, Introduction to Non-linear Mechanics, Princeton University Press, Princeton, 1950.
[15] N.N. Bogoliubov, Y.A. Mitropolsky, Asymptotic Methods in the Theory of Non-linear Oscillations, Gordon and Breach, New York, 1961.
[16] A. Gelb, W.E. Vander Velde, Multiple-Input Describing Functions and Nonlinear System Design, McGraw-Hill, New York, 1968.
[17] A.H. Nayfeh, D.T. Mook, Nonlinear Oscillations, John Wiley & Sons, New York, 1979.
[18] R.E. Bellman, R.E. Kalaba, Quasilinearization and Nonlinear boundary-value Problems, American Elsevier, New York, 1965.
[19] J.B. Roberts, P.D. Spanos, Random Vibration and Statistical Linearization, Dover Publications Inc., New York, 2003.
[20] Y.K. Lin, Probabilistic Theory of Structural Dynamics, McGraw-Hill, New York, 1967.
[21] C.W. Gardiner, Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences, Springer-Verlag, Berlin, 1992.
[22] M. Di Paola, Fokker planck equation solved in terms of complex fractional moments, Probab. Eng. Mech. 38 (2014) 70–76.
[23] Matteo A. Di, M. Di Paola, A. Pirrotta, Probabilistic characterization of nonlinear systems under poisson white noise via complex fractional moments, Nonlinear Dyn. 77 (3) (2014) 729–738.
[24] M. Di Paola, F.P. Pinnola, Riesz fractional integrals and complex fractional moments for the probabilistic characterization of random variables, Probab. Eng. Mech. 29 (2012) 149–156.
[25] S. Russotto, A. Pirrotta, Mellin transform for the probabilistic characterization of random variables and stochastic processes, Probab. Eng. Mech. 80 (2025) 103766.
[26] L. Niu, W. Xu, M. Di Paola, A. Pirrotta, Generalized complex fractional moment for the probabilistic characteristic of random vectors, Eng. Struct. 318 (2024) 118685.
[27] G. Alotta, M. Di Paola, Probabilistic characterization of nonlinear systems under α-stable white noise via complex fractional moments, Physica A 420 (2015) 265–276.
[28] A. Di Matteo, M. Di Paola, A. Pirrotta, Poisson white noise parametric input and response by using complex fractional moments, Probab. Eng. Mech. 38 (2014) 119–126.
[29] A. Pirrotta, R. Santoro, Probabilistic response of nonlinear systems under combined normal and poisson white noise via path integral method, Probab. Eng. Mech. 26 (1) (2011) 26–32.
[30] I.A. Kougioumtzoglou, P.D. Spanos, Nonstationary stochastic response determination of nonlinear systems: a wiener path integral formalism, J. Eng. Mech. 140 (2014) 04014064.
[31] I.A. Kougioumtzoglou, A. Di Matteo, P.D. Spanos, A. Pirrotta, M. Di Paola, An efficient wiener path integral technique formulation for stochastic response determination of nonlinear MDOF systems, J. Appl. Mech. 82 (2015) 101005.
[32] I.A. Kougioumtzoglou, A wiener path integral solution treatment and effective material properties of a class of one-dimensional stochastic mechanics problems, J. Eng. Mech. 143 (2017) 04017014.
[33] A. Di Matteo, A. Pirrotta, Efficient path integral approach via analytical asymptotic expansion for nonlinear systems under gaussian white noise, Nonlinear Dyn. 112 (16) (2024) 13995–14018.
[34] A. Di Matteo, Path integral approach via Laplace's method of integration for nonstationary response of nonlinear systems, Meccanica 54 (9) (2019) 1351–1363.
[35] I.A. Kougioumtzoglou, A.F. Apostolos, P.D. Spanos, Path Integrals in Stochastic Engineering Dynamics, Springer Nature Switzerland AG, Cham, 2024.
[36] L. Niu, J. Chen, M. Di Paola, A. Pirrotta, Y. Shi, W. Xu, Probabilistic methods for nonlinear differential equations based on shift characteristic function, Communications in Nonlinear Science and Numerical Simulation 161 (2026) 110148.