

VARIABLE EXPONENT $p(x)$-KIRCHHOFF TYPE PROBLEM WITH CONVECTION

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ABSTRACT. We study a nonlinear $p(x)$-Kirchhoff type problem with Dirichlet boundary condition, in the case of a reaction term depending also on the gradient (convection). Using a topological approach based on the Galerkin method, we discuss the existence of two notions of solutions: strong generalized solution and weak solution. Strengthening the bound on the Kirchhoff type term (positivity condition), we establish existence of weak solution, this time using the theory of operators of monotone type.

1. INTRODUCTION

It is well-known that the mechanisms of fluid movement are quite relevant in dealing with diffusive problems. A largely investigated phenomenon in the study of porous media, as liquids and gases, is referred as "convection". Briefly, the phenomenon of convection consists in the fact that an energy transfer is accomplished by moving particles. Mainly, it occurs when the temperature gradient exceeds some critical values. In order to give the possibility to include this phenomenon, we introduce in problem (1) below a reaction term $f(x, z, y)$ depending on the gradient of the solution. On the other hand, we generalize our study by considering in the leading operator of problem (1), a Kirchhoff type term. We recall that the interest for the Kirchhoff type problems also originates from physical applications (again related to diffusive processes). We mention that Kirchhoff [13] studied an extension of the D'Alembert wave equation for free vibrations of elastic strings, of the form

$$\rho\frac{\partial^2}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2}{\partial x^2} = 0,$$

under a suitable set of physical parameters ρ, P_0, h, E, L (namely, mass density, initial tension, area of the cross-section, Young modulus of the material, length of the string).

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following nonlinear $p(x)$-Kirchhoff type problem with Dirichlet boundary condition and with gradient dependence (convection) in the reaction term

$$-\Delta^K_{p(x)} u(x) = f(x, u(x), \nabla u(x)) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{1}$$

In this problem, $\Delta_{p(x)}$ denotes the $p(x)$-Laplace differential operator defined by

$$\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) \quad \text{for all } u \in W_0^{1,p(x)}(\Omega),$$

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and we consider a Kirchhoff type term of the form

$$K(p,u)=a_p-b_p\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx, \quad \text{for some } a_p,b_p>0. \tag{2}$$

Therefore by $\Delta^K_{p(x)}$, we denote the $p(x)$-Kirchhoff type operator defined by

$$\Delta^K_{p(x)}=K(p,u)\Delta_{p(x)}u=\left(a_p-b_p\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$$

for all $u\in W_0^{1,p(x)}(\Omega)$. Here, we assume that the variable exponent $p\in C(\overline{\Omega})$ is finite with:

$$1<p^-:=\inf_{x\in\overline{\Omega}}p(x)\le p(x)\le p^+:=\sup_{x\in\overline{\Omega}}p(x)<+\infty.$$

Since we have a reaction depending on the gradient, then we cannot apply the usual variational methods (for example, critical point theory and mountain pass theorem) in the analysis of problem (1), but we follow a topological method. This approach is based on fixed-point arguments and the theory of operators with monotone type features.

Our first result establishes the existence of a strong generalized solution of problem (1) (see Definition 1 on Section 3). We work with a Galerkin basis of $W_0^{1,p(x)}(\Omega)$ and use some consequences of the classical Brouwer fixed point theorem. The similar approach is presented by Motreanu [17] for a quasilinear Dirichlet problem with convection, whose leading operator is the operator $-\Delta_p+\Delta_q$. This operator is the sum of a negative p-Laplacian and of a q-Laplacian. Motreanu establishes the existence of both a generalized solution and of a strong generalized solution, in the case of constant exponents $1<q<p<+\infty$. Some complementary results in the case of the most classical operator $-\Delta_p-\Delta_q$ were obtained by Faria-Miyagaki-Motreanu [5] (positive solutions), Liu-Papageorgiou [15] (resonant reaction) and Gasiński-Winkert [10] (double phase operator). In details, [5] develops an approximating process using a Schauder basis of $W_0^{1,p(x)}(\Omega)$ and uses a generalized version of the strong maximum principle. The Leray-Schauder alternative principle solves the main problem in [15], in combination with the method of frozen variable. The work in [10] focuses on pseudomonotone operators and their regularity properties. One more reference on the solution of problems with convection in the reaction, using Leray-Schauder alternative principle, is Fragnelli-Mugnai-Papageorgiou [7] (Robin problems). Passing to the framework setting of variable exponent, Wang-Hou-Ge [19] studied problem (1) without the Kirchhoff type term. On the other hand, Hamdani-Harrabi-Mtiri-Repovš [11] studied the same operator on the left hand side of (1), but without gradient dependence in the reaction. Consequently, the problem in [11] is variational, and hence the authors establish the existence of a weak solution and of infinitely many solutions using Palais-Smale compactness condition, mountain pass theorem and Fountain theorem (that is, they work mainly with critical point theory).

We point out that there is a large literature considering the presence of a Kirchhoff term weighting the leading operator of elliptic and parabolic equations, usually all these works consider only a positive defined Kirchhoff term (see also (24) below). Indeed, there has been a revival interest for the Kirchhoff work after the publication of Lions' book [14]. Here we recall the recent works of Molica Bisci-Pizzimenti [16] (infinitely many solutions) and Figueiredo-Nascimento [6] (sign-changing solution). At the best of our knowledge, [11] is the first attempt to leave the above mentioned sign restriction on the Kirchhoff term, and we follow this new feature. So, our work here is the first attempt to consider the new nonlocal term (2) in the case of a reaction with convection (that is, without applying the variational methods).

In the second part of this paper, we will also prove some results in the case of a positive defined Kirchhoff term. Again, we will use topological tools but this time mainly related to the theory of operators of monotone type.

2. Mathematical Background - Hypotheses

Here, we collect some notions and notation relevant for the study of elliptic equations in the context of variable exponent spaces. Further details and motivations, can be found in the works of Diening-Harjulehto-Hästö-Růžička [1] and Rădulescu-Repovš [18].

Problem (1) arises naturally in the framework spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. Precisely, we refer to:

• The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ defined by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \ : \ u \text{ is measurable and } \rho_p(u) := \int_\Omega |u(x)|^{p(x)} dx < +\infty \right\}.$$

On this space, we consider the norm

$$\|u\|_{L^{p(x)}(\Omega)} := \inf \left\{ \lambda > 0 : \rho_p \left(\frac{u}{\lambda} \right) \leq 1 \right\}.$$

• The Sobolev space $W^{1,p(x)}(\Omega)$ given as

$$W^{1,p(x)}(\Omega) := \{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \}.$$

On this space, we consider the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)} \quad (\text{where } \|\nabla u\|_{L^{p(x)}(\Omega)} = \| |\nabla u| \|_{L^{p(x)}(\Omega)}).$$

Suitable definitions of norms and norm inequalities are the key tools to make possible calculations on bounds and a priori estimates. From Diening-Harjulehto-Hästö-Růžička [1] (see Theorem 8.2.18, p. 263), we know that

$$\|u\|_{L^{p(x)}(\Omega)} \leq c_1 \|\nabla u\|_{L^{p(x)}(\Omega)} \quad \text{for all } u \in W_0^{1,p(x)}(\Omega), \text{ some } c_1 > 0, \tag{3}$$

where $W_0^{1,p(x)}(\Omega)$ is the $W^{1,p(x)}$-norm closure of $C_0^\infty(\Omega)$. Essentially the norms $\|u\|_{W^{1,p(x)}(\Omega)}$ and $\|\nabla u\|_{L^{p(x)}(\Omega)}$ are equivalent on the space $W_0^{1,p(x)}(\Omega)$, and hence we can replace $\|u\|_{W^{1,p(x)}(\Omega)}$ by $\|\nabla u\|_{L^{p(x)}(\Omega)}$. We set

$$\|u\| = \|\nabla u\|_{L^{p(x)}(\Omega)} \quad \text{in } W_0^{1,p(x)}(\Omega).$$

If X is a reflexive Banach space and X^* is the topological dual, then $\langle \cdot, \cdot \rangle$ means the duality brackets of the pair (X^*, X). According to this notation, for the operator $-\Delta_{p(x)}^K : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega) = W^{1,p(x)}(\Omega)^*$, where $p'(\cdot)$ is such that $\frac{1}{p'(\cdot)} + \frac{1}{p(\cdot)} = 1$, we have

$$\langle -\Delta_{p(x)}^K u, h \rangle = K(p,u) \int_\Omega |\nabla u|^{p(x)-2} (\nabla u, \nabla h)_{\mathbb{R}^N} dx \quad \text{for all } u, h \in W_0^{1,p(x)}(\Omega).$$

Continuity of some embeddings and density of Banach spaces are the objects of the following result (see Gasiński-Papageorgiou [9], p. 141, Lemma 2.2.27).

Theorem 1. *Let X, Y be two Banach spaces with $X \subseteq Y$. If X is dense in Y and the embedding is continuous, then the embedding $Y^* \subseteq X^*$ is continuous too. In addition, X reflexive implies that Y^* is dense in X^*.*

Our technique of proof is also based on the following consequence of Brouwer fixed point theorem.

Proposition 1. *Let $(X, \|\cdot\|_X)$ be a normed finite-dimensional space and let $V : X \to X^*$ be a continuous map. If there exists some $R > 0$ such that*

$$\langle V(h), h\rangle \geq 0 \quad \textit{for all } h \in X \textit{ with } \|h\|_X = R,$$

then the equation $V(u) = 0$ admits a solution $u \in X$ such that $R \geq \|u\|_X$.

Turning to variable exponent Lebesgue and Sobolev spaces, Fan-Zhao [2] proved that $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, endowed with the above norms, are separable, reflexive and uniformly convex Banach spaces. In the same paper, one can find the following interesting results.

Proposition 2. *Assume that $p \in C(\overline{\Omega})$ with $p(x) > 1$ for each $x \in \overline{\Omega}$. If $\alpha \in C(\overline{\Omega})$ and $1 < \alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then there exists a continuous and compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$.*

If $p^- > 1$, we recall the Hölder inequality

$$\int_\Omega uh dx \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|u\|_{L^{p(x)}(\Omega)}\|h\|_{L^{p'(x)}(\Omega)} \leq 2\|u\|_{L^{p(x)}(\Omega)}\|h\|_{L^{p'(x)}(\Omega)}, \tag{4}$$

for $u \in L^{p(x)}(\Omega)$, $h \in L^{p'(x)}(\Omega)$, which is useful in the proof of some embedding results. We mention that [2, Theorem 1.11] leads to the fact that the embedding $L^{p_1(x)}(\Omega) \hookrightarrow L^{p_2(x)}(\Omega)$ is continuous, whenever $p_1, p_2 \in C(\overline{\Omega})$ with $p_1(x) \geq p_2(x) > 1$ for all $x \in \overline{\Omega}$. In addition, [2, Theorem 1.3] relates $\|\cdot\|_{L^{p(x)}(\Omega)}$ to $\rho_p(\cdot)$ as follows.

Theorem 2. *Let $u \in L^{p(x)}(\Omega)$. Then, the following relations hold:*

(i) *$\|u\|_{L^{p(x)}(\Omega)} < 1$ $(= 1,\ > 1) \Leftrightarrow \rho_p(u) < 1$ $(= 1,\ > 1)$;*

(ii) *if $\|u\|_{L^{p(x)}(\Omega)} > 1$, then $\|u\|^{p^-}_{L^{p(x)}(\Omega)} \leq \rho_p(u) \leq \|u\|^{p^+}_{L^{p(x)}(\Omega)}$;*

(iii) *if $\|u\|_{L^{p(x)}(\Omega)} < 1$, then $\|u\|^{p^+}_{L^{p(x)}(\Omega)} \leq \rho_p(u) \leq \|u\|^{p^-}_{L^{p(x)}(\Omega)}$.*

Remark 1. By manipulating the information in Theorem 2 we get easily the bound inequalities:

$$\|u\|^{p^+}_{L^{p(x)}(\Omega)} + 1 \geq \rho_{p(x)}(u) \geq \|u\|^{p^-}_{L^{p(x)}(\Omega)} - 1. \tag{5}$$

Moreover, we can obtain other results by routine calculations. Indeed (5) gives us

$$\left[\|u\|^{p^+}_{L^{p(x)}(\Omega)} + 1\right] + 1 \geq \int_\Omega (|u|^{p(x)-1})^{\frac{p(x)}{p(x)-1}} dx + 1 \geq \||u|^{p(x)-1}\|^{(p')^-}_{L^{p'(x)}(\Omega)},$$

where we used the fact that $u \in L^{p(x)}(\Omega)$ implies $|u|^{p(x)-1} \in L^{p'(x)}(\Omega)$. It follows that

$$\||u|^{p(x)-1}\|_{L^{p'(x)}(\Omega)} \leq 2 + \|u\|^{p^+}_{L^{p(x)}(\Omega)}. \tag{6}$$

Following a similar argument, one can derive the following inequality

$$\||\nabla u|^{\frac{p(x)}{\alpha'(x)}}\|_{L^{\alpha'(x)}(\Omega)} \leq 2 + \||\nabla u|\|^{p^+}_{L^{p(x)}(\Omega)}, \quad \alpha \in C(\overline{\Omega}) \text{ with } \alpha(x) > 1 \text{ for all } x \in \overline{\Omega}. \tag{7}$$

To conclude this section, we introduce the precise hypotheses on the reaction term. We start with the usual Carathéodory assumptions. For a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ we mean a function such that

- for all $(z, y) \in \mathbb{R} \times \mathbb{R}^N$, $x \to f(x, z, y)$ is measurable;

- for almost all $x \in \Omega$, $(z, y) \to f(x, z, y)$ is continuous,

and hence it is jointly measurable (see Hu-Papageorgiou [12], p. 142).

Now, we have to introduce the growth bounds on this reaction. But before we make the assumption about the exponent p.

$H(p)$: There exists $\xi_0 \in \mathbb{R}^N \setminus \{0\}$ such that for all $x \in \Omega$ the function $p_x : \Omega_x \to \mathbb{R}$ defined by $p_x(z) = p(x + z\xi_0)$ is monotone, where $\Omega_x := \{z \in \mathbb{R} : x + z\xi_0 \in \Omega\}$.

$H(p)$ is significant because, according to [3, Theorem 3.3], leads to the Rayleigh quotient

$$\widehat{\lambda} := \inf_{u \in W_0^{1,p(x)}(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^{p(x)} dx}{\int_\Omega |u|^{p(x)} dx} > 0. \tag{8}$$

For a different condition on the exponent function p leading again to (8) see also [3, Theorem 3.4]. Now, we have:

$H(f)$: $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that

(i) there exist $\sigma \in L^{\alpha'(x)}(\Omega)$, $1 < \alpha(x) < p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{otherwise,} \end{cases}$ and $c > 0$ such that

$$|f(x, z, y)| \leq c(\sigma(x) + |z|^{\alpha(x)-1} + |y|^{\frac{p(x)}{\alpha'(x)}}) \quad \text{for a.a. } x \in \Omega, \text{ all } z \in \mathbb{R}, \text{ all } y \in \mathbb{R}^N;$$

(ii) there exist $a_0 \in L^1(\Omega)$ and $b_1, b_2 \geq 0$ such that

$$|f(x, z, y)z| \leq a_0(x) + b_1|z|^{p(x)} + b_2|y|^{p(x)} \quad \text{for a.a. } x \in \Omega, \text{ all } z \in \mathbb{R}, \text{ all } y \in \mathbb{R}^N.$$

3. Main results

We recall that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution to (1) if

$$\langle -\Delta^K_{p(x)} u, h \rangle = \int_\Omega f(x, u(x), \nabla u(x)) h(x) dx \quad \text{for all } h \in W_0^{1,p(x)}(\Omega). \tag{9}$$

We note that if $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution to (1), then there exists $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(x)}(\Omega)$ such that

(i) $u_n \xrightarrow{w} u$ in $W_0^{1,p(x)}(\Omega)$, as $n \to +\infty$;
(ii) $-\Delta^K_p u_n - f(\cdot, u_n(\cdot), \nabla u_n(\cdot)) \xrightarrow{w} 0$ in $W^{-1,p'(x)}(\Omega)$, as $n \to +\infty$;
(iii) $\lim_{n\to+\infty} \langle -\Delta^K_p u_n, u_n - u \rangle = 0$.

However, such a kind of solution (that is, $u \in W_0^{1,p(x)}(\Omega)$ satisfying (i), (ii), (iii) above), is known as "strong generalized solution" to problem (1), according to the terminology of Motreanu [17] (notion given for the operator $-\Delta_p + \Delta_q$). Consequently, the set of weaker solutions to (1) is a subset of the strong generalized solutions to (1) (it follows choosing $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(x)}(\Omega)$ with $u_n := u$ for all $n \in \mathbb{N}$).

Changing the point of view in this reasoning (that is, starting from the notion of strong generalized solution), it is a natural question to ask when a strong generalized solution to (1) leads to the classical notion of weak solution.

We answer to this question in the positive, presenting the following condition:

$$\int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx \not\to \frac{a_p}{b_p} \quad \text{as } n \to +\infty, \ \{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(x)}(\Omega). \tag{10}$$

We model this condition, having in mind the classical $(S)_+$-property of operators as the $p(x)$-Laplacian one. Indeed $\Delta_{p(x)}$ has the $(S)_+$-property, which says us that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p(x)}(\Omega) \text{ and } \limsup_{n\to+\infty}\langle -\Delta_p u_n, u_n - u\rangle \leq 0 \quad \Rightarrow \quad u_n \to u \text{ in } W_0^{1,p(x)}(\Omega).$$

It is clear that, at least for a subsequence,

$$\lim_{n\to+\infty}\langle -\Delta_p^K u_n, u_n - u\rangle = 0 \text{ and (10)} \quad \Rightarrow \quad \lim_{n\to+\infty}\langle -\Delta_p u_n, u_n - u\rangle = 0,$$

and hence we retrieve the $(S)_+$-property of the $p(x)$-Laplacian operator, provided that $u_n \xrightarrow{w} u$ in $W_0^{1,p(x)}(\Omega)$, as $n \to +\infty$. This argument will be used later in the proof of Theorem 4.

Now, we are ready to establish the existence of strong generalized solution to problem (1) under hypotheses $H(p)$ and $H(f)$. We develop a topological approach based on a Galerkin basis of $W_0^{1,p(x)}(\Omega)$ and an appropriate definition of Nemitsky map corresponding to the Carathéodory function f. For the reader's convenience, we prefer to repeat in the form of a definition, the above suggested notion of strong generalized solution.

Definition 1. We say that $u \in W_0^{1,p(x)}(\Omega)$ is a strong generalized solution to (1), whenever we can find a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(x)}(\Omega)$ satisfying the following convergence statements:

(i) $u_n \xrightarrow{w} u$ in $W_0^{1,p(x)}(\Omega)$, as $n \to +\infty$;
(ii) $-\Delta_p^K u_n - f(\cdot, u_n(\cdot), \nabla u_n(\cdot)) \xrightarrow{w} 0$ in $W^{-1,p'(x)}(\Omega)$, as $n \to +\infty$;
(iii) $\lim_{n\to+\infty}\langle -\Delta_p^K u_n, u_n - u\rangle = 0$.

The following Proposition 3 provides an useful growth estimate, related to the reaction term $f(\cdot, u(\cdot), \nabla u(\cdot))$. The calculations follow by an application of Hölder inequality.

Proposition 3. *If hypothesis $H(f)(i)$ holds, then for all $u, h \in W_0^{1,p(x)}(\Omega)$ we have the following estimate:*

$$\left|\int_\Omega f(x,u,\nabla u)h dx\right| \leq 2c\|h\|_{L^{\alpha(x)}(\Omega)}\left[\|\sigma\|_{L^{\alpha'(x)}(\Omega)} + \|u\|^{\alpha^+}_{L^{\alpha(x)}(\Omega)} + \|\nabla u\|^{p^+}_{L^{p(x)}(\Omega)} + 4\right],$$

for some $c > 0$.

Proof. Using $H(f)(i)$ we get

$$\begin{aligned}\left|\int_\Omega f(x,u,\nabla u)h dx\right| &\leq c\int_\Omega [|\sigma(x)| + |u|^{\alpha(x)-1} + |\nabla u|^{\frac{p(x)}{\alpha'(x)}}]\,|h|dx \\ &\leq 2c\|h\|_{L^{\alpha(x)}(\Omega)}\left[\|\sigma\|_{L^{\alpha'(x)}(\Omega)} + \left\||u|^{\alpha(x)-1}\right\|_{L^{\alpha'(x)}} + \left\||\nabla u|^{\frac{p(x)}{\alpha'(x)}}\right\|_{L^{\alpha'(x)}(\Omega)}\right] \\ &\qquad\qquad \text{(by Hölder inequality)} \\ &\leq 2c\|h\|_{L^{\alpha(x)}(\Omega)}\left[\|\sigma\|_{L^{\alpha'(x)}(\Omega)} + \|u\|^{\alpha^+}_{L^{\alpha(x)}(\Omega)} + \|\nabla u\|^{p^+}_{L^{p(x)}(\Omega)} + 4\right] \qquad (11) \\ &\qquad\qquad \text{(by (6) and (7))},\end{aligned}$$

for all $u, h \in W_0^{1,p(x)}(\Omega)$ and some $c > 0$. □

Now, by $N_f^* : W_0^{1,p(x)}(\Omega) \subset L^{\alpha(x)}(\Omega) \to L^{\alpha'(x)}(\Omega)$ we mean the Nemitsky map corresponding to the Carathéodory function f, that is,

$$N_f^*(u)(\cdot) = f(\cdot, u(\cdot), \nabla u(\cdot)) \quad \text{for all } u \in W_0^{1,p(x)}(\Omega).$$

By $H(f)\,(i)$ (see also Galewski [8]), we get that $N_f^*(\cdot)$ is bounded and continuous. Then, we consider the operator $N_f : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by

$$N_f = i^* \circ N_f^*,$$

where $i^* : L^{\alpha'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is a continuous embedding (see Lemma 1). It follows that $N_f : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is bounded and continuous.

On the other side (recall Proposition 3), we have the bound

$$\|N_f(h)\|_{W^{-1,p'(x)}(\Omega)} \le 2c\left[\|\sigma\|_{L^{\alpha'(x)}(\Omega)} + \|h\|^{\alpha^+}_{L^{\alpha(x)}(\Omega)} + \|\nabla h\|^{p^+}_{L^{p(x)}(\Omega)} + 4\right],$$

for all $h \in W_0^{1,p(x)}(\Omega)$, some $c > 0$.

Recalling that $W_0^{1,p(x)}(\Omega)$ is a separable Banach space, then we can find a Galerkin basis of $W_0^{1,p(x)}(\Omega)$, that is, a sequence $\{X_n\}_{n\in\mathbb{N}}$ of vector subspaces of $W_0^{1,p(x)}(\Omega)$ with

- $\dim(X_n) < +\infty$ for all $n \in \mathbb{N}$;
- $X_n \subset X_{n+1}$ for all $n \in \mathbb{N}$;
- $\overline{\cup_{n=1}^{\infty} X_n} = W_0^{1,p(x)}(\Omega)$.

Proposition 4. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a Galerkin basis of $W_0^{1,p(x)}(\Omega)$. If hypotheses $H(p)$ and $H(f)$ hold, then for all $n \in \mathbb{N}$ we can find $u_n \in X_n$ with*

$$\langle -\Delta_p^K u_n, h\rangle = \int_\Omega f(x, u_n(x), \nabla u_n(x))h(x)dx \quad \textit{for all } h \in X_n. \tag{12}$$

Proof. Fixed $n \in \mathbb{N}$, let $V_n : X_n \to X_n^*$ be the operator defined by

$$\langle V_n(u), h\rangle = \langle -\Delta_p^K u, h\rangle - \int_\Omega f(x, u(x), \nabla u(x))h(x)dx \quad \text{for all } u, h \in X_n.$$

Now, $H(f)(ii)$ implies that

$$\begin{aligned}
\langle -V_n(h), h\rangle &= \left(b_p \int_\Omega \frac{1}{p(x)}|\nabla h|^{p(x)}dx - a_p\right)\int_\Omega |\nabla h|^{p(x)}dx + \int_\Omega f(x,h,\nabla h)hdx \\
&\ge \left(b_p \int_\Omega \frac{1}{p(x)}|\nabla h|^{p(x)}dx - a_p\right)\int_\Omega |\nabla h|^{p(x)}dx - \int_\Omega |f(x,h,\nabla h)h|dx \\
&\ge \frac{b_p}{p^+}\left(\int_\Omega |\nabla h|^{p(x)}dx\right)^2 - a_p\int_\Omega |\nabla h|^{p(x)}dx - \int_\Omega |a_0(x)|dx \\
&\quad - b_1\int_\Omega |h|^{p(x)}dx - b_2\int_\Omega |\nabla h|^{p(x)}dx \quad (\text{by } H(f)(ii)) \\
&\ge \frac{b_p}{p^+}\rho_p^2(\nabla h) - a_p\rho_p(\nabla h) - b_1\widehat{\lambda}^{-1}\rho_p(\nabla h) - b_2\rho_p(\nabla h) - \|a_0\|_{L^1(\Omega)} \\
&= \frac{b_p}{p^+}\rho_p^2(\nabla h) - (a_p + b_1\widehat{\lambda}^{-1} + b_2)\rho_p(\nabla h) - \|a_0\|_{L^1(\Omega)},
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad \langle -V_n(h), h\rangle &\ge \frac{b_p}{p^+}\rho_p^2(\nabla h) - (a_p + b_1\widehat{\lambda}^{-1} + b_2)\rho_p(\nabla h) - \|a_0\|_{L^1(\Omega)} \\
&\quad \text{for all } h \in X_n.
\end{aligned}$$

If $\rho_p(\nabla h) > 1$, then

$$\langle -V_n(h), h\rangle \ge \frac{b_p}{p^+}\rho_p^2(\nabla h) - (a_p + b_1\widehat{\lambda}^{-1} + b_2 + \|a_0\|_{L^1(\Omega)})\rho_p(\nabla h)$$

$$= \left[\frac{b_p}{p^+}\rho_p(\nabla h) - (a_p + b_1\widehat{\lambda}^{-1} + b_2 + \|a_0\|_{L^1(\Omega)})\right]\rho_p(\nabla h),$$

$$\Rightarrow \quad \langle -V_n(h), h\rangle \geq 0 \quad \text{if } \rho_p(\nabla h) \geq \frac{p^+}{b_p}(a_p + b_1\widehat{\lambda}^{-1} + b_2 + \|a_0\|_{L^1(\Omega)}).$$

Fixed $R = \max\left\{\left[\frac{p^+}{b_p}(a_p + b_1\widehat{\lambda}^{-1} + b_2 + \|a_0\|_{L^1(\Omega)})\right]^{1/p^-}, 1\right\}$, then for all $h \in X_n$ such that $\|h\| = R$ we get

$$\langle -V_n(h), h\rangle \geq 0 \quad (\text{recall that } \|h\| = \|\nabla h\|_{L^{p(x)}(\Omega)} \leq \rho_p^{\frac{1}{p^-}}(\nabla h)).$$

Next, by Lemma 1 we deduce that the equation $-V_n(u) = 0$ (or equivalently, $V_n(u) = 0$) admits a solution $u_n \in X_n$, and hence (12) holds true. □

Remark 2. From (5) we see that $S \subseteq W_0^{1,p(x)}(\Omega)$ is bounded if there exists a constant $C > 0$ such that $\rho_p(\nabla u) \leq C$ for all $u \in S$.

Let $\{u_n\}_{n\in\mathbb{N}} \subset \cup_{n=1}^{\infty} X_n$ be the sequence originated in the proof of Proposition 4. Then, the following proposition establishes the boundedness of such a sequence in $W_0^{1,p(x)}(\Omega)$.

Proposition 5. *If hypotheses $H(p)$ and $H(f)$ hold, then $\{u_n\}_{n\in\mathbb{N}} \subset \cup_{n=1}^{\infty} X_n$ is bounded in $W_0^{1,p(x)}(\Omega)$.*

Proof. We prove that

$$\rho_p(\nabla u_n) \leq \max\left\{\frac{p^+}{b_p}(a_p + b_1\widehat{\lambda}^{-1} + b_2 + \|a_0\|_{L^1(\Omega)}), 1\right\} \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

If $\rho_p(\nabla u_n) \leq 1$ for all $n \in \mathbb{N}$, by Remark 2 the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$. So, we assume that there is some $n \in \mathbb{N}$ such that $\rho_p(\nabla u_n) > 1$. From (12) for $h = u_n$ we have

$$\begin{aligned}
&\frac{b_p}{p^+}\left(\int_\Omega |\nabla u_n|^{p(x)}dx\right)^2 \leq a_p \int_\Omega |\nabla u_n|^{p(x)}dx - \int_\Omega f(x, u_n, \nabla u_n)u_n dx \\
&\leq a_p\rho_p(\nabla u_n) + \int_\Omega |f(x, u_n, \nabla u_n)u_n| dx \\
&\leq a_p\rho_p(\nabla u_n) + \int_\Omega [|a_0(x)| + b_1|u_n|^{p(x)} + b_2|\nabla u_n|^{p(x)}]dx \quad (\text{by } H(f)(ii)) \\
&\leq (a_p + b_1\widehat{\lambda}^{-1} + b_2)\rho_p(\nabla u_n) + \|a_0\|_{L^1(\Omega)} \quad (\text{by } (8)), \\
\Rightarrow \quad &\frac{b_p}{p^+}\rho_p^2(\nabla u_n) \leq (a_p + b_1\widehat{\lambda}^{-1} + b_2 + \|a_0\|_{L^1(\Omega)})\rho_p(\nabla u_n) \quad (\text{recall } \rho_p(\nabla u_n) > 1), \\
\Rightarrow \quad &\frac{b_p}{p^+}\rho_p(\nabla u_n) \leq a_p + b_1\widehat{\lambda}^{-1} + b_2 + \|a_0\|_{L^1(\Omega)}, \\
\Rightarrow \quad &\rho_p(\nabla u_n) \leq \frac{p^+}{b_p}(a_p + b_1\widehat{\lambda}^{-1} + b_2 + \|a_0\|_{L^1(\Omega)}) \quad (\text{and hence (13) holds}), \\
\Rightarrow \quad &\{u_n\}_{n\in\mathbb{N}} \subset \cup_{n=1}^{\infty} X_n \text{ is bounded in } W_0^{1,p(x)}(\Omega).
\end{aligned}$$

□

Consequently, we establish the following existence result.

Theorem 3. *If hypotheses $H(p)$ and $H(f)$ hold, then problem (1) admits a strong generalized solution $u \in W_0^{1,p(x)}(\Omega)$.*

Proof. From Proposition 5, we know that $\{u_n\}_{n\in\mathbb{N}} \subset \cup_{n=1}^{\infty} X_n$ is bounded in $W_0^{1,p(x)}(\Omega)$. Since $W_0^{1,p(x)}(\Omega)$ is reflexive, we can assume that

$$u_n \to u \text{ in } L^{\alpha(x)}(\Omega) \text{ and } u_n \xrightarrow{w} u \text{ in } W_0^{1,p(x)}(\Omega), \text{ for some } u \in W_0^{1,p(x)}(\Omega). \tag{14}$$

On the other hand, we already mentioned that the Nemitsky map is bounded and so

$$\{N_f(u_n)\}_{n\in\mathbb{N}} \quad \text{is bounded in } W^{-1,p'(x)}(\Omega).$$

The boundedness of the operator $-\Delta_{p(x)}^K : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ implies that

$$\{-\Delta_{p(x)}^K u_n - N_f(u_n)\}_{n\in\mathbb{N}} \quad \text{is bounded in } W^{-1,p'(x)}(\Omega).$$

Again, the reflexivity of the space $W^{-1,p'(x)}(\Omega)$ leads to the convergence

$$-\Delta_{p(x)}^K u_n - N_f(u_n) \xrightarrow{w} \kappa \text{ in } W^{-1,p'(x)}(\Omega), \text{ for some } \kappa \in W^{-1,p'(x)}(\Omega), \tag{15}$$

true at least for a relabeled subsequence of $\{-\Delta_{p(x)}^K u_n - N_f(u_n)\}_{n\in\mathbb{N}}$.

If we choose $h \in \cup_{n=1}^{\infty} X_n$, then we can find $n(h) \in \mathbb{N}$ such that $h \in X_{n(h)}$. So, Proposition 4 implies that (12) is satisfied for all $n \geq n(h)$. We can pass to the limit as $n \to +\infty$ in (12), then we get

$$\langle \kappa, h\rangle = 0 \quad \text{for all } h \in \cup_{n=1}^{\infty} X_n.$$

Since $\cup_{n=1}^{\infty} X_n$ is dense in $W_0^{1,p(x)}(\Omega)$ (recall that $\{X_n\}_{n\in\mathbb{N}}$ is a Galerkin basis), then we conclude that $\kappa = 0$. Therefore, by (15) we deduce that

$$-\Delta_{p(x)}^K u_n - N_f(u_n) \xrightarrow{w} 0 \text{ in } W^{-1,p'(x)}(\Omega). \tag{16}$$

Next, we choose $h = u_n$ in (12) and have

$$a_p \int_\Omega |\nabla u_n|^{p(x)} dx = b_p \left(\int_\Omega |\nabla u_n|^{p(x)} dx \right)^2 + \int_\Omega f(x, u_n, \nabla u_n) u_n dx \quad \text{for all } n \in \mathbb{N}. \tag{17}$$

The convergence in (16) gives us

$$\begin{aligned} &\langle -\Delta_{p(x)}^K u_n - N_f(u_n), u\rangle \to 0 \quad \text{as } n \to +\infty, \\ \Rightarrow \quad &\lim_{n\to+\infty} \langle -\Delta_{p(x)}^K u_n - N_f(u_n), u_n - u\rangle = 0 \quad \text{(by (17))}. \end{aligned} \tag{18}$$

From Proposition 3 (putting u_n in place of u), choosing the test function $h = (u_n - u) \in W_0^{1,p(x)}(\Omega)$ we get

$$\begin{aligned} &\left| \int_\Omega f(x, u_n, \nabla u_n)(u_n - u) dx \right| \\ &\leq 2c \|u_n - u\|_{L^{\alpha(x)}(\Omega)} \left[\|\sigma\|_{L^{\alpha'(x)}(\Omega)} + \|u_n\|_{L^{\alpha(x)}(\Omega)}^{\alpha^+} + \|\nabla u_n\|_{L^{p(x)}(\Omega)}^{p^+} + 4 \right], \end{aligned} \tag{19}$$

for some $c > 0$, all $n \in \mathbb{N}$.

Now, we know that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$ and hence in $L^{\alpha(x)}(\Omega)$. Also, $\{|\nabla u_n|\}_{n\in\mathbb{N}}$ is bounded in $L^{p(x)}(\Omega)$. Consequently, from (19) we deduce that

$$\left| \int_\Omega f(x, u_n, \nabla u_n)(u_n - u) dx \right| \leq M \|u_n - u\|_{L^{\alpha(x)}(\Omega)} \quad \text{for some } M > 0. \tag{20}$$

It follows that

$$\lim_{n\to+\infty}\int_\Omega f(x,u_n,\nabla u_n)(u_n-u)dx=0 \quad \text{(by (20))},$$

$$\Rightarrow \quad \lim_{n\to+\infty}\langle -\Delta^K_{p(x)}u_n, u_n-u\rangle = 0 \quad \text{(recall (18))}. \tag{21}$$

Clearly (14), (16) and (21) mean that $u\in W_0^{1,p(x)}(\Omega)$ is a strong generalized solution to problem (1). □

We conclude this section, turning to the initial question asking when a strong generalized solution to (1) leads to a weak solution to (1). We establish the following theorem.

Theorem 4. *Let $u\in W_0^{1,p(x)}(\Omega)$ be a strong generalized solution to problem* (1)*, associated to the sequence $\{u_n\}_{n\in\mathbb{N}}\subseteq W_0^{1,p(x)}(\Omega)$ satisfying* (10)*. If hypotheses $H(f)$ holds, then $u\in W_0^{1,p(x)}(\Omega)$ is a weak solution to problem* (1)*.*

Proof. Since the sequence $\{u_n\}_{n\in\mathbb{N}}\subseteq W_0^{1,p(x)}(\Omega)$ is bounded, at least for a relabeled subsequence we may assume that

$$\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \neq \frac{a_p}{b_p} \quad \text{for all } n\in\mathbb{N}$$

and

$$\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \to t_0 \neq \frac{a_p}{b_p} \quad \text{as } n\to+\infty, \text{ for some } t_0>0 \text{ (by (10))},$$

$$\Rightarrow \quad a_p - b_p\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \to a_p - b_p t_0 \neq 0.$$

Therefore, we can find $\delta>0$ such that

$$\left| a_p - b_p\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right| \geq \delta > 0 \quad \text{for all } n\in\mathbb{N}. \tag{22}$$

Considering that the sequence $\left\{a_p - b_p\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right\}$ is bounded. Since $u\in W_0^{1,p(x)}(\Omega)$ is a strong generalized solution to problem (1) we get

$$\lim_{n\to+\infty}\langle -\Delta^K_{p(x)}u_n, u_n-u\rangle = 0,$$

which means

$$\lim_{n\to+\infty}\left[\left(a_p - b_p\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\langle -\Delta_{p(x)}u_n, u_n-u\rangle\right]=0,$$

$$\Rightarrow \quad \lim_{n\to+\infty}\langle -\Delta_{p(x)}u_n, u_n-u\rangle = 0 \quad \text{(by (22))},$$

$$\Rightarrow \quad u_n\to u \text{ in } W_0^{1,p(x)}(\Omega) \quad \text{(since } -\Delta_{p(x)} \text{ has the } (S)_+\text{-property)}.$$

Using again the definition of strong generalized solution, we deduce that

$$-\Delta^K_{p(x)}u_n - f(\cdot,u_n(\cdot),\nabla u_n(\cdot)) \xrightarrow{w} 0 \text{ in } W^{-1,p'(x)}(\Omega),$$

$$\Rightarrow \quad -\Delta^K_{p(x)}u - f(\cdot,u(\cdot),\nabla u(\cdot)) = 0,$$

$$\Rightarrow \quad u\in W_0^{1,p(x)}(\Omega) \text{ is a weak solution to problem (1) (recall (9))}.$$

□

4. Complementary results

We show that our results remain true under a different hypothesis on the variable exponent p. Precisely, we substitute hypothesis $H(p)$ with the following one:

$H'(p)$: $p \in C(\overline{\Omega})$ is finite with $p^+ < 2p^-$.

We point out that a similar hypothesis can be found in Hamdani-Harrabi-Mtiri-Repovš [11], see (1.4) of Theorem 1.1 (in absence of convection).

The change of $H(p)$ by $H'(p)$ requires only slight adaptations in the proofs of Propositions 4 and 5. For reader convenience, we give the precise calculations in the following two propositions.

Proposition 6. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a Galerkin basis of $W_0^{1,p(x)}(\Omega)$. If hypotheses $H'(p)$ and $H(f)$ hold, then for all $n \in \mathbb{N}$ we can find $u_n \in X_n$ with*

$$\langle -\Delta_p^K u_n, h\rangle = \int_\Omega f(x, u_n(x), \nabla u_n(x))h(x)dx \quad \textit{for all } h \in X_n. \tag{23}$$

Proof. As in the proof of Proposition 4, fixed $n \in \mathbb{N}$, we consider the operator $V_n : X_n \to X_n^*$ defined by

$$\langle V_n(u), h\rangle = \langle -\Delta_p^K u, h\rangle - \int_\Omega f(x, u(x), \nabla u(x))h(x)dx \quad \text{for all } u, h \in X_n.$$

Using again $H(f)(ii)$, we deduce that

$$\langle -V_n(h), h\rangle \geq \frac{b_p}{p^+}\left(\int_\Omega |\nabla h|^{p(x)}dx\right)^2 - (a_p+b_2)\int_\Omega |\nabla h|^{p(x)}dx - b_1\int_\Omega |h|^{p(x)}dx - \|a_0\|_{L^1(\Omega)}$$

for all $h \in X_n$.

Now, if $\|h\| = \|\nabla h\|_{L^{p(x)}(\Omega)} > 1$ we get

$$\begin{aligned}
\langle -V_n(h), h\rangle &\geq \frac{b_p}{p^+}\|h\|^{2p^-} - (a_p + b_2)\|h\|^{p^+} - b_1 \max\{\|h\|^{p^+}_{L^{p(x)}(\Omega)}, \|h\|^{p^-}_{L^{p(x)}(\Omega)}\} - \|a_0\|_{L^1(\Omega)} \\
&\geq \frac{b_p}{p^+}\|h\|^{2p^-} - (a_p + b_2 + b_1 C_h + \|a_0\|_{L^1(\Omega)})\|h\|^{p^+} \\
&\quad (\text{some } C_h = C_h(p^-, p^+, c_1) > 0, \text{ where } c_1 \text{ is the constant in (3)}) \\
&= \|h\|^{p^+}\left[\frac{b_p}{p^+}\|h\|^{2p^- - p^+} - (a_p + b_2 + b_1 C_h + \|a_0\|_{L^1(\Omega)})\right], \\
\Rightarrow \quad & \langle -V_n(h), h\rangle \geq 0 \quad \text{if } \|h\| \geq \left[\frac{p^+}{b_p}(a_p + b_2 + b_1 C_h + \|a_0\|_{L^1(\Omega)})\right]^{1/(2p^- - p^+)}.
\end{aligned}$$

Fixed $R = \max\left\{\left[\frac{p^+}{b_p}(a_p + b_2 + b_1 C_h + \|a_0\|_{L^1(\Omega)})\right]^{1/(2p^- - p^+)}, 1\right\}$, then for all $h \in X_n$ such that $\|h\| = R$ we get

$$\langle -V_n(h), h\rangle \geq 0.$$

Next, by Lemma 1 we deduce that the equation $-V_n(u) = 0$ (or equivalently, $V_n(u) = 0$) admits a solution $u_n \in X_n$, and hence (23) holds true. $\square$

Proposition 7. *If hypotheses $H'(p)$ and $H(f)$ hold, then $\{u_n\}_{n\in\mathbb{N}} \subset \cup_{n=1}^\infty X_n$ is bounded in $W_0^{1,p(x)}(\Omega)$.*

Proof. If $\|\nabla u_n\|_{L^{p(x)}(\Omega)} \leq 1$ for all $n \in \mathbb{N}$, then the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$. So, we assume that there is some $n \in \mathbb{N}$ such that $\|\nabla u_n\|_{L^{p(x)}(\Omega)} > 1$. From (23) for $h = u_n$ we have

$$\begin{aligned}
&\frac{b_p}{p^+}\|\nabla u_n\|^{2p^-}_{L^{p(x)}(\Omega)} \leq \frac{b_p}{p^+}\left(\int_\Omega |\nabla u_n|^{p(x)}dx\right)^2 \\
&\qquad\qquad \leq a_p \int_\Omega |\nabla u_n|^{p(x)}dx - \int_\Omega f(x,u_n,\nabla u_n)u_n dx, \\
\Rightarrow\quad &\frac{b_p}{p^+}\|\nabla u_n\|^{2p^-}_{L^{p(x)}(\Omega)} \leq (a_p + b_1C_h + b_2)\|\nabla u_n\|^{p^+}_{L^{p(x)}(\Omega)} + \|a_0\|_{L^1(\Omega)}, \\
\Rightarrow\quad &\|\nabla u_n\|^{2p^- - p^+}_{L^{p(x)}(\Omega)} \leq \frac{p^+}{b_p}(a_p + b_1C_h + b_2 + \|a_0\|_{L^1(\Omega)}), \\
\Rightarrow\quad &\|\nabla u_n\|_{L^{p(x)}(\Omega)} \leq \max\left\{\left[\frac{p^+}{b_p}(a_p + b_1C_h + b_2 + \|a_0\|_{L^1(\Omega)})\right]^{1/(2p^- - p^+)}, 1\right\}, \\
\Rightarrow\quad &\{u_n\}_{n\in\mathbb{N}} \subset \cup_{n=1}^\infty X_n \text{ is bounded in } W_0^{1,p(x)}(\Omega).
\end{aligned}$$

□

Consequently, we have the following counterpart of Theorem 3.

Theorem 5. *If hypotheses $H'(p)$ and $H(f)$ hold, then problem* (1) *admits a strong generalized solution $u \in W_0^{1,p(x)}(\Omega)$.*

5. Positive Kirchhoff term

As already mentioned in the Introduction, usually in the literature the Kirchhoff term in $p(x)$-Laplacian type equations (and others) satisfies a bound condition (positivity condition) of the form

$$\widetilde{K}(p,u) \geq k_0 > 0 \quad \text{for all } u \in W_0^{1,p(x)}(\Omega),$$

equivalently this changes (2) as follows

$$\widetilde{K}(p,u) = a_p + b_p \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx, \quad \text{for some } a_p, b_p > 0. \tag{24}$$

In respect of (24), we can complement our previous results, discussing the existence and uniqueness of weak solutions to the problem

$$-\Delta^{\widetilde{K}}_{p(x)}u(x) = f(x,u(x),\nabla u(x)) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{25}$$

Recall that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution to (25) if

$$\langle -\Delta^{\widetilde{K}}_{p(x)}u, h\rangle = \int_\Omega f(x,u(x),\nabla u(x))h(x)dx \quad \text{for all } h \in W_0^{1,p(x)}(\Omega).$$

To start the preliminary work, we collect some notions about the theory of operators of monotone type.

For convenience, we consider the operator $-\Delta^{\widetilde{K}}_{p(x)} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega) = W^{1,p(x)}(\Omega)^*$ defined by

$$\langle -\Delta^K_{p(x)}u, h\rangle = \widetilde{K}(p,u)\langle -\Delta_{p(x)}u, h\rangle$$

$$= \widetilde{K}(p,u)\int_\Omega |\nabla u|^{p(x)-2}(\nabla u, \nabla h)_{\mathbb{R}^N}dx \quad \text{for all } u,h \in W_0^{1,p(x)}(\Omega).$$

This operator possesses some features of regularity as consequence of the properties of the operator $-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$. Recall the following result of Fan-Zhang.

Proposition 8. [4, Theorem 3.1 (i)-(ii)] *The operator $-\Delta_{p(x)}(\cdot)$ is continuous, bounded, strictly monotone convex and of type $(S)_+$.*

From the previous proposition we deduce that the operator $-\Delta_{p(x)}^{\widetilde{K}} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is continuous, bounded and of type $(S)_+$.

Using again the Nemitsky map $N_f : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$, we consider the operator $V : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by

$$V(u) = -\Delta_{p(x)}^{\widetilde{K}} u - N_f(u) \quad \text{for all } u \in W_0^{1,p(x)}(\Omega).$$

By definition, $V(\cdot)$ is a bounded and continuous operator. Now, we prove that V is surjective. To this aim we proceed as follows: first we show that V is pseudomonotone, then we show that V is strongly coercive. Indeed, referring to the book of Gasiński-Papageorgiou [9], (p. 336), we recall the following proposition.

Proposition 9. *If $V : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is pseudomonotone and strongly coercive (that is, $\lim_{\|u\|\to+\infty} \frac{\langle V(u),u\rangle}{\|u\|} = +\infty$), then V is surjective (that is, $R(V) = W^{-1,p'(x)}(\Omega)$).*

About the pseudomonotonicity, we get it as a byproduct of generalized pseudomonotonicity (that is, $V : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ has the following property: "if $u_n \xrightarrow{w} u$ in $W_0^{1,p(x)}$ for some $u \in W_0^{1,p(x)}$, and $V(u_n) \xrightarrow{w} u^*$ in $W^{-1,p'(x)}(\Omega)$ for some $u^* \in W^{-1,p'(x)}(\Omega)$, and $\limsup_{n\to+\infty}\langle V(u_n), u_n - u\rangle \le 0$, then $u^* = V(u)$ and $\langle V(u_n), u_n\rangle \to \langle V(u), u\rangle$"), see also [9, Definition 3.2.45]. Indeed, from [9, Proposition 3.2.49] we deduce the following result.

Proposition 10. *If $V : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is a bounded generalized pseudomonotone operator, then V is also a pseudomonotone operator.*

We establish the following existence theorem.

Theorem 6. *If hypotheses $H(p)$ and $H(f)$ hold, then problem (25) admits at least a weak solution.*

Proof. We show that the operator $V(\cdot) = -\Delta_{p(x)}^{\widetilde{K}}(\cdot) - N_f(\cdot)$ is pseudomonotone. Since $V(\cdot)$ is well-defined and bounded, it only takes to prove that $V(\cdot)$ is generalized pseudomonotone (recall the statement of [9, Proposition 3.2.49]). We assume that $u_n \xrightarrow{w} u$ in $W_0^{1,p(x)}(\Omega)$, $V(u_n) \xrightarrow{w} u^*$ in $W^{-1,p'(x)}(\Omega)$ and

$$\limsup_{n\to+\infty}\langle V(u_n), u_n - u\rangle \le 0. \tag{26}$$

From (26) we get

$$\limsup_{n\to+\infty}[\langle -\Delta_{p(x)}^{\widetilde{K}} u_n, u_n - u\rangle - \int_\Omega f(x,u_n,\nabla u_n)(u_n-u)dx] \le 0. \tag{27}$$

From (11) putting u_n in place of u and choosing $h = u_n - u \in W_0^{1,p(x)}(\Omega)$, the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $W_0^{1,p(x)}(\Omega)$ and since $u_n \to u$ in $L^{\alpha(x)}(\Omega)$, we have

$$\int_\Omega f(x, u_n, \nabla u_n)(u_n - u)dx \to 0 \quad \text{as } n \to +\infty.$$

Consequently, (27) gives us

$$\limsup_{n\to+\infty} \langle -\Delta_{p(x)}^{\widetilde{K}} u_n, u_n - u\rangle \leq 0,$$

$$\Rightarrow \quad u_n \to u \text{ in } W_0^{1,p(x)}(\Omega) \quad (\text{since } -\Delta_{p(x)}^{\widetilde{K}} \text{ has the } (S)_+\text{-property}). \tag{28}$$

By (28) and the continuity of V, we deduce that $u^* = V(u)$ and $\langle V(u_n), u_n\rangle \to \langle V(u), u\rangle$, and hence $V(\cdot)$ is generalized pseudomonotone, thus pseudomonotone.

Next step is to show that $V(\cdot)$ is strongly coercive too. To this aim, we involve hypothesis $H(f)(ii)$ to derive the following inequalities

$$\begin{aligned}
\langle V(u), u\rangle &= \left(a_p + b_p \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right) \int_\Omega |\nabla u|^{p(x)} dx - \int_\Omega f(x, u, \nabla u) u dx \\
&\geq \frac{b_p}{p^+} \left(\int_\Omega |\nabla u|^{p(x)} dx\right)^2 + a_p \int_\Omega |\nabla u|^{p(x)} dx - \int_\Omega f(x, u, \nabla u) u dx \\
&\geq \frac{b_p}{p^+} \rho_p^2(\nabla u) + (a_p - b_1 \widehat{\lambda}^{-1} - b_2)\rho_p(\nabla u) - \int_\Omega |a_0(x)| dx \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{by } H(f)(ii) \text{ and } (8)) \\
&= \left[\frac{b_p}{p^+} \rho_p(\nabla u) + a_p - b_1 \widehat{\lambda}^{-1} - b_2\right] \rho_p(\nabla u) - \|a_0\|_{L^1(\Omega)}, \\
\Rightarrow \quad \langle V(u), u\rangle &\geq \left[\frac{b_p}{p^+} \|u\|^{p^-} - 1 + a_p - b_1 \widehat{\lambda}^{-1} - b_2\right] (\|u\|^{p^-} - 1) - \|a_0\|_{L^1(\Omega)} \ (\text{by } (5)).
\end{aligned}$$

Since $p^- > 1$, then $V(\cdot)$ is strongly coercive.

Proposition 9 says that a pseudomonotone strongly coercive operator is a surjection. Thus, the equation $V(u) = 0$ admits a solution $\widehat{u} \in W_0^{1,p(x)}(\Omega)$, which is a weak solution to Problem 1. □

Changing $H(p)$ by $H'(p)$ we have the following counterpart of Theorem 5.

Theorem 7. *If hypotheses $H'(p)$ and $H(f)$ hold, then problem* (25) *admits at least a weak solution.*

Proof. Proceeding as in the proof of Theorem 6, we prove that the operator $V(\cdot) = -\Delta_{p(x)}^{\widetilde{K}}(\cdot) - N_f(\cdot)$ is pseudomonotone.

Next step is to show that $V(\cdot)$ is strongly coercive too. To this aim, we involve hypothesis $H(f)(ii)$ and $H'(p)$ to derive the following inequalities

$$\begin{aligned}
\langle V(u), u\rangle &= \left(a_p + b_p \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right) \int_\Omega |\nabla u|^{p(x)} dx - \int_\Omega f(x, u, \nabla u) u dx \\
&\geq \frac{b_p}{p^+} \left(\int_\Omega |\nabla u|^{p(x)} dx\right)^2 + (a_p - b_2) \int_\Omega |\nabla u|^{p(x)} dx - b_1 \int_\Omega |u|^{p(x)} dx - \int_\Omega |a_0(x)| dx \\
&\geq \frac{b_p}{p^+} \|u\|^{2p^-} - C\|u\|^{p^+} \quad \text{for some } C > 0 \text{ if } \|u\| > 1,
\end{aligned}$$

$\Rightarrow \quad V(\cdot)$ is strongly coercive.

$\square$

Remark 3. Note that for the problem (25), each weak solution is a strong generalized solution. Clearly, the same holds in the opposite sense.

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