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A Mathematical Analysis of the Intermediate Behaviour of the Energy Cascades of Quantum Turbulence

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Abstract
We propose a mathematical interpolation between several regimes of energy cascade in quantum turbulence in He II. On the basis of a physical interpretation of such mathematical expression we discuss in which conditions it is expected to appear an intermediate k^2 regime (equipartition regime) in the transition region between the hydrodynamic regime and the Kelvin wave regime (namely, between the $k^{-5/3}$ and k^{-1} regions in coflow situations and between the k^{-3} and k^{-1} regions in counterflow situations). It is seen that if the energy rate transfer from the hydrodynamic region to the Kelvin wave region is sufficiently slow, such equipartition region will be present, but for higher values of such energy rate transfer it will disappear. For high rates of the energy rate transfer, the transition regime between the hydrodynamic and the Kelvin wave regimes will be monotonous, characterized by a negative exponent of k between $-5/3$ and -1 (or between -3 and -1), instead of the positive 2 exponent of the equipartition regime.

Keywords Superfluid helium · Heat transfer · Quantum turbulence · Energy spectrum · Kolmogorov cascade · Quantized vortices

1 Introduction

The aim of this paper is to contribute to the current lively discussions on the energy cascades in turbulent superfluid helium, where different regimes and situations have been found, by combining experimental data, dimensional analysis, and interpolation proposals between the several regimes. Interpolation proposals have much interest for the analysis of systems whose behaviour is well known at two different scales of energy, length or time, but whose intermediate behaviour is still uncertain and the subject of research. In this case, mathematical proposals combined with physical interpretations are a valuable way of progress. In

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contrast with the classical dimensional description of the Kolmogorov energy spectrum in the inertial range of classical turbulence — giving the energy distribution $E(k)$ in terms of the dissipation rate ϵ and wavevector k —, in quantum turbulence there appear additional relevant physical quantities (namely, the quantum of circulation κ and the average separation ℓ between quantized vortices), in such a way that dimensional arguments are inconclusive to derive the form of the energy spectrum, which shows three different regions depending on the value of k (namely, $k\ell \gg 1$, $k\ell \approx 1$ and $k\ell \ll 1$) and of the kind of flow. However, a posteriori, such arguments may be useful to interpret some observations and to grasp some of the outstanding qualitative features, so that it is convenient not dismissing them a priori.

In more concrete terms, we focus our attention on the results of theoretical, numerical and experimental observations for the energy cascade in coflow, counterflow and superflow situations. In coflow, there is a net mass flux, whereas in counterflow there is heat flux but not mass flux [1–7]. The direct experimental evidence of all the energy spectra of superfluid helium is a hard task. Most of the experiments have performed to check the existence of a Kolmogorov spectrum in superfluid helium [1–7], thus focusing on a limited range of wavevectors. The arguments of this paper are timely as confirmed by the recent publication in Ref. [8, 9]. In particular, in Ref. [8] the authors try to derive a phenomenological theory of quantum turbulence in superfluid helium.

Recently, turbulence in Bose-Einstein condensates has also attracted much attention [10–15]. In some experiments it shows some features typical of two-dimensional turbulence.

Quantum turbulence has several relevant differences with respect to the classical one. One of them refers to the nature of vortices: the core of a vortex line is of the order of the atomic size of helium and the circulation of the superfluid velocity around any vortex line is given by the quantum of circulation $\kappa = \dfrac{h}{m}$ (h being the Planck constant and m the mass of helium atom) [16–20]. In macroscopic terms, the turbulent state is described by the vortex length density L, namely the total length of vortex lines per unit volume, or alternatively, by $\ell = L^{-1/2}$, with ℓ the average separation between neighbouring vortices. Thus, instead of having $E(\epsilon, k)$ the spectra are expected to be of the kind $E(\epsilon, k, \ell, \kappa)$. Note that ℓ is related to κ, but independent of it, since it also depends on the external conditions. For instance, in a counterflow situation, the average separation of the quantized vortices, which is of the order of $L^{-1/2}$, with L the vortex length density per unit volume, is given by $\ell = L^{-1/2} = constant\ (\kappa/v_{ns})$, with v_{ns} the counterflow velocity and the constant is related to the coefficients of the production and destruction terms in the Vinen evolution equation for L [21]. Thus, if quantum effects were null (namely if one sets $\hbar = 0$), κ would be zero and ℓ would be zero as well. But ℓ depends not only on κ, but also on v_{ns}, which is proportional to the heat flux q (speaking in terms of one-fluid extended model), and it decreases for increasing v_{ns}.

A second main difference between superfluid helium at very low temperature and any classical fluid is related to the smallest scales where energy is dissipated: in a classical fluid the ultimate scale is that where energy is dissipated because of viscosity; in contrast, in superfluid helium at very low temperature the energy is dissipated by sound (phonon emission) at very low scales [22, 23]. Indeed, when the transfer of energy reaches scales of the order of intervortex space ℓ energy is transferred to the Kelvin waves running along the vortex lines [18, 23, 24]. A nonlinear interaction between Kelvin waves leads to the so-called Kelvin wave cascade [23, 25] and hence the loss of energy to the ultimate scales by means of sound emission. For higher temperature, namely $1K \lesssim T \leq T_\lambda$, viscosity and mutual friction cannot be neglected and Kelvin waves cascade does not take place because waves are damped. In the context of the two-fluid model there would be another kind of turbulence, that of the normal component, which is temperature-dependent because its density depends on the temperature and it is maximum near T_λ and negligible below 1 K. This means that two kinds of turbulence may appear for $T > 1$ K, which interact through the mutual friction force. These arguments do not occur in the one-fluid extended model (proposed on the basis of Extended Thermodynamics), because there the two independent vectorial fields are the mean velocity $\mathbf{v}$ and heat flux $\mathbf{q}$, and thus two kinds of turbulence may be interpreted in terms of turbulence in $\mathbf{v}$ and in $\mathbf{q}$.

A third feature of turbulence in superfluid helium is that besides the several classical ways to trigger turbulence in a classical fluid by stirring the fluid, quantum turbulence may appear through an external applied heat flux beyond some critical value, which usually refers to counterflow experiments [1] [16–19]. It has been seen that in mechanical driving energy is concentrated at large scales, whereas in thermal driving it is concentrated at mesoscales [26]. This possibility of driving at different scales is another difference of quantum turbulence with respect to classical turbulence.

The presence of ℓ (with dimensions of length) and of κ (with dimensions of length^2 $\times$ time^{-1}) makes that instead of having a single energy cascade from the stirring excitation scale L_0 to the viscous dissipation scale, the cascade is split in three main regions: the "hydrodynamic region" from the stirring scale L_0 to ℓ ($L_0 \ll k^{-1} \ll \ell$), the "Kelvin wave region" from ℓ to the acoustic dissipation scale l_p ($\ell \ll k^{-1} \ll l_P$) and an "intermediate region" ($k^{-1} \simeq \ell$) strongly affected by interaction and reconnections between quantized vortices. The cascade in the hydrodynamic region may be seen as a cascade between eddies of different sizes (thought to be made by bundles of vortex lines [27]), whereas in the quantum scale it may be interpreted as a cascade of nonlinear Kelvin waves of different wavelengths along quantized vortex lines.

Apart from the three regions introduced above, the behaviour of superfluid helium depends also on temperature and on how energy is supplied to the largest scales. Moreover, there is not a unique model for studying the energy spectrum at different scales, and the limit of calculation in the numerical experiments makes these results still not fully confirmed and in some situations open to debate.

We can state that the Kolmogorov spectrum ($k^{-5/3}$) is practically confirmed for superfluid helium both numerically, experimentally and theoretically in the hydrodynamic regime when energy is supplied to He II in the same way as in a classical fluid [1–7]. This is mainly related to the so-called coflow experiments, when normal and superfluid components are locked by means of the mutual friction, arising from the interaction between vortex lines and the quasiparticles (rotons and photons) constituting the normal component. However, this spectrum is found in a small range of wavenumbers, and the fundamental reasons for its appearance are not yet completely clear (see for instance the discussion in [28]). Also, a temperature-dependent enhancement of intermittency has been found in some range of scales.

The behaviour of energy spectra is expected to be different in counterflow and superflow experiments, where the two mentioned components are not mutually locked [1, 29]. There, a k^{-3}-slope for the energy spectrum is proposed theoretically [29]. In fact, the spectra for turbulence in counterflow have recently become a hot topic [30–32]. It has been recently pointed out that the large-scale energy spectra in the counterflow are strongly anisotropic and in certain conditions may become quasi two-dimensional, with different mechanisms defining the spectra in the several dimensions. Regarding the superflow regime, it has been shown to probably include a direct (not cascade) energy transfer between large and small scale. Note that regimes without an energy cascade were already considered in [33].

At quantum length-scale ($k^{-1} \simeq \ell$) the behaviour of He II depends strongly on the temperature, and at very low temperature where the friction is practically null, the Kelvin wave cascade predominates and two theoretical proposals have been open to debate for the energy spectrum ($k^{-7/5}$ and $k^{-5/3}$) [34, 35]. For higher temperature, Kelvin waves are damped by the mutual friction and the Kelvin wave cascade is affected by these interactions. In this region, it is expected that spectral energy decays as k^{-1} [36]. This is the case for one-dimensional Kelvin waves cascade along the vortex line; however, for three-dimensional Kelvin wave cascades it has been shown that the spectrum will be analogous to the classical Kolmogorov one, namely, $k^{-5/3}$ [37].

The intermediate region is unclear because the theoretical k^2-slope for the energy spectrum (corresponding to energy equipartition) is not experimentally observed, probably because of the limited resolution of the devices used, as argued in [38]. This region corresponds to a kind of bottleneck between the 3D-energy in the hydrodynamical region and the 1D energy cascade of the Kelvin waves along the vortex line [24]. The exchange of energy between these scales happens through reconnections and interactions between vortices. A detailed analysis of these features would require a clear microscopic understanding of the several mechanisms; this is far deeper than the relatively modest abilities of dimensional analysis. Despite this limitation, we illustrate how a discussion of a phenomenological interpolation between spectra at large scales and long scales may suggest in which conditions a k^2 intermediate spectrum may arise, or what other kinds of behaviours could be expected.

The paper is organized as follows. In Sect. 2 we deal with the energy spectrum of the superfluid component of Helium II in coflow, counterflow and superflow, and in the absence of friction of the quasiparticles with quantum turbulence which occurs at very low temperature and we propose mathematical interpolation of the spectra between $k\ell \ll 1$ and $k\ell \gg 1$ regimes. In Sect. 3 we discuss a physical interpretation of the mathematical interpolation and on its basis we explore how the energy rate transfer may influence on the possibility of a k^2 intermediate region. This is helpful to understand whether such a region is not practically observed because it is very narrow, or whether the conditions in which it may exist are limited. Section 4 is devoted to conclusions.

2 Energy Cascades in Turbulent Superfluid Helium: Dependence on the Kind of Flow and on the Average Vortex Separation

In this section we discuss some of the main features of the energy spectra of the superfluid component in Helium II in coflow, superflow and counterflow by dimensional analysis, namely, the $k^{-5/3}$, k^{-3}, k^{-1} and k^2 regimes, and interpolations between several regions of values of k. The results summarized in Fig. 4 of [1] are qualitatively mimicked in our Figs. 1 and 2, corresponding respectively to our phenomenological analytical proposals (2.5) and (2.11).

2.1 Coflow

In coflow it is usually assumed that the two components of superfluid helium flow in the same direction and, that because of the presence of quantized vortices and hence of the mutual friction between vortices and normal component, the two components match their velocities in such a way that their relative velocity (counterflow velocity) $\mathbf{v}_{ns} = \mathbf{v}_n - \mathbf{v}_s$ is negligible. Since the heat flux in the two-fluid model is proportional to $\mathbf{v}_{ns}$, because $\mathbf{q} = \rho_s ST(\mathbf{v}_n - \mathbf{v}_s)$, with S the entropy per unit volume, then $\mathbf{q} = 0$; thus, energy is mainly associated to the mean velocity field $\rho\mathbf{v} = \rho_n\mathbf{v}_n + \rho_s\mathbf{v}_s$.

The evolution equation of the velocity $\mathbf{v}$ in the one-fluid theory is

$$\frac{\partial}{\partial t}\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v}+\frac{1}{\rho}\nabla p-\nu\nabla^2\mathbf{v}-\frac{\nu}{ST}\nabla^2\mathbf{q}=0, \tag{2.1}$$

which, neglecting the contribution of the heat flux in the last term, becomes the usual Navier-Stokes equation, ν being the kinematic viscosity.

Let the energy $E(t)=\frac{1}{V}\int\frac{1}{2}\mathbf{v}^2dV=\int_0^\infty E(k,t)dk$ so that $E(k,t)$ gives the fraction of kinetic energy in modes with wavevectors between k and $k+dk$. In the hydrodynamic regime it is assumed that viscous dissipation does not take place in a wide range of values of k (the so-called inertial range), thus $E(k,t)$ satisfies the continuity equation

$$\frac{\partial}{\partial t}E(k,t)+\frac{\partial}{\partial k}\epsilon(k,t)=0 \tag{2.2}$$

where $\epsilon(k,t)$ is the energy flux between different modes in spectral space.

In the stationary case, equation (2.2) leads to $\frac{\partial}{\partial k}\epsilon(k)=0$ and therefore $\epsilon(k)\equiv\epsilon$ will be constant, independent of k and t. Using arguments from classical turbulence, $E(k)$, which has the dimension (length)^3(time)$^{-2}$ (because it is given by kinetic energy per unit mass per unit wavevector), is a function of the dissipation rate ϵ, which has dimension of (length)^2(time)$^{-3}$, and of the wavevector k (which has dimension of (length)$^{-1}$). Then, it follows that in this case

$$E(k)=C_{K41}\epsilon^a k^b=C_{K41}\epsilon^{2/3}k^{-5/3} \tag{2.3}$$

which is the Kolmogorov cascade and $C_{K41}\simeq 1$. Dimensional arguments are sufficient in this case to lead to the values $a=2/3$, $b=-5/3$ for the exponents a and b in (2.3).

In the so-called intermittent turbulence this cascade changes to [39]

$$E(k)\sim\epsilon^a k^b\sim\epsilon^{2/3}k^{-5/3}(kL_0)^\beta \tag{2.4}$$

where L_0 is the characteristic size of the stirring process supplying energy to the system, and the exponent β is related to the fractal dimension of intermittent turbulence.

The transfer of energy supplied to the largest scales down to smallest scales is still open to debate. It has been observed that if energy is supplied in the same way as a classical fluid (using a fork or a grid for instance) superfluid helium behaves as a classical fluid, namely the transfer of energy from bigger to smaller vortices follows the classical Kolmogorov spectrum.

The energy spectrum (2.3) is observed in the hydrodynamic region $k\ell\ll 1$, whereas k^{-1} behaviour is observed for $k\ell\gg 1$ (in Sect. 2.2 we will comment about the k^{-1} behaviour). In the intermediate region of $k\ell\simeq 1$, it is observed a relatively narrow transition region with equipartition behaviour k^2 [6]. As an interpolation function between the $k^{-5/3}$ and k^{-1} regions, with an intermediate k^2 shoulder we propose

$$E(k)=\frac{A_1k^{-1}+A_{K41}k^{-5/3}}{2}\left[1+\frac{A_1k^{-1}-A_{K41}k^{-5/3}}{A_1k^{-1}+A_{K41}k^{-5/3}}\tanh\left[A(k\ell-1)\right]\right] \tag{2.5}$$

where $A_{K41}=C_{K41}\epsilon^{2/3}$, as given in (2.3), $A_1=c_1(\kappa/\ell)^2$ with c_1 and A dimensionless constant. In Fig. 1 there is the plot of (2.5). For $A=4$, this figure reproduces in a qualitative

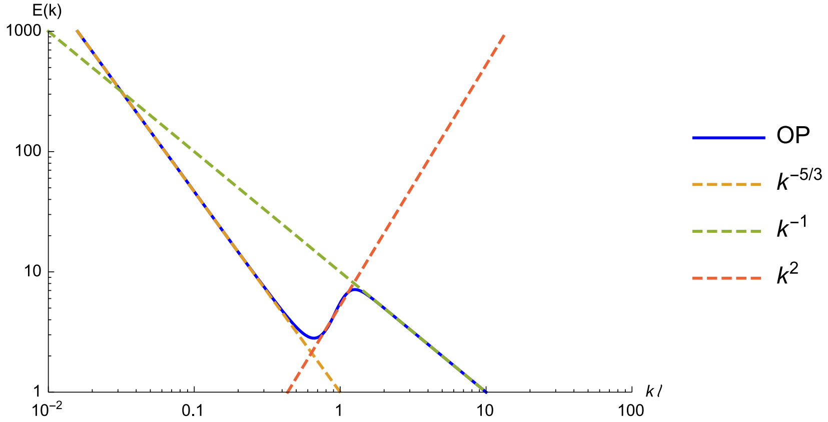

Fig. 1 Plot of (2.5) for $A_1 = 10$, $A_{K41} = 1$ and $A = 4$. The legend on the right-hand side refers to the power-law behaviour of the straight-lines in the Log-Log plot, and OP stands for "Our Proposal" (2.5), which reproduces the observed behaviour represented in Fig. 4 of the paper [1]. (Color figure online)

way the observed behaviour reported in Ref. [1], namely a transition from $k^{-5/3}$ to k^{-1} behaviour with a k^2 shoulder. In Sect. 3 we further discuss the physics behind the parameter A. In Fig. 1 of [8] there is an analysis of the transition from the behaviour k^2 to the behaviour $k^{-5/3}$, but there is no information about a region k^{-1} at shorter values of k.

2.2 Counterflow and Superflow

In terms of the two-fluid model, counterflow and superflow experiments are similar because both are characterized by a relative velocity between the normal and superfluid components. In counterflow, both components move in the opposite directions in such a way that the average velocity $\mathbf{v}$ is zero on a transversal section of the channel; in superflow the normal component is at rest on the average and only the superfluid component flows. In terms of the one-fluid extended model [19, 40], instead, in counterflow heat $\mathbf{q}$ flows whereas the velocity $\mathbf{v}$ fluctuates around the null value, whereas in superflow both heat $\mathbf{q}$ and velocity $\mathbf{v}$ flow in such a way that $\mathbf{v} + \dfrac{1}{\rho ST}\mathbf{q}$ fluctuates around the null value [19].

In [1, 29], it is seen that counterflow and superflow are in some sense the same situation, because the main important ingredient is the relative velocity between the two components, the counterflow velocity $\mathbf{v}_{ns}$, whereas the former situation is characterized by $\overline{\mathbf{v}} = 0$ and the latter situation by $\overline{\mathbf{v}_n} = 0$. In terms of the one-fluid theory, $\overline{\mathbf{v}} = 0$ and $\mathbf{q} \neq 0$ in counterflow and $\overline{\mathbf{v}} = constant$ and $\mathbf{q} \neq 0$ in superflow. Thus, instead of being the main field (as in coflow) here $\mathbf{v}$ contributes to the heat $\mathbf{q}$ through its fluctuations around the constant values of $\mathbf{v}$.

For the superfluid component, or in heat-fluctuations, the relevant physical quantities are κ (the quantum of circulation, having dimensions (length)^2(time)$^{-1}$), the vortex average separation ℓ and the wavevector k.

From κ, a reciprocal of time can be easily obtained as (time)$^{-1} \sim \kappa/$(length)^2. Then, one may have several different combinations with the dimension of $E(k, \ell)$, namely (length)^3(time)$^{-2}$. For instance

$$E(k) \sim \left(\frac{\kappa}{\ell^2}\right)^2 k^{-3}(\ell k)^{\beta} \tag{2.6}$$

or

$$E(k) \sim \frac{\kappa^2}{\ell^2} k^{-1} (\ell k)^{\beta} \tag{2.7}$$

or, in general terms, one could write

$$E(k) \sim \kappa^a \ell^b k^{-c}. \tag{2.8}$$

From dimensional analysis it follows $a = 2$, $b + c = -1$.

The exponent k^{-3} may be easily understood for $k \gg \ell$. The energy per unit mass of a vortex with quantized circulation κ is $v^2 \sim (\kappa/\ell)^2$, with κ/ℓ being of the order of the speed of the superfluid around the vortex core at a distance ℓ from the core. For $k \gg \ell$, in a surface of area k^{-2} there will be k^{-2}/ℓ^2 quantized vortex lines. Thus, the energy per unit mass and unit k will be

$$E(k) \sim \left(\frac{\kappa}{\ell}\right)^2 \frac{k^{-2}}{\ell^2} k^{-1} \sim \kappa^2 \ell^{-4} k^{-3}. \tag{2.9}$$

The behaviour in k^{-1} (for $k\ell < 1$) may also be interpreted on the same grounds as k^{-3}, with the difference that the factor in $(k^{-1}/\ell)^2$ giving the number of vortices in the area k^{-2} should be changed to 1, because in the area k^{-2} only a vortex will be found. Then we simply have $(\kappa/\ell)^2 k^{-1}$. However, in the case of k^{-1} considerably smaller than ℓ one expects to have intermittent behaviour, because in an area k^{-2} one will find sometimes one vortex, and some times zero vortices.

In (2.6) and (2.7) we have singled out scalings in k^{-3} and k^{-1} with the corresponding ℓ^{-4} and ℓ^{-2} scalings, and k^2 in the transition regime, corresponding to ℓ scaling. Of course, since ℓk is a dimensionless quantity, other exponents for k may be obtained, so that in the quantum case the cascade is not directly given by dimensional arguments. The exponent β could be related to some kind of intermittence [41–43]. Alternatively, following the analogy with (2.4) intermittence could be associated to a dependence of the kind $(kd)^{\beta}$, with d being the diameter of the container rather than to $(k\ell)^{\beta}$. Thus, for narrow channels ($\ell \gg d$) one could have $E(k, \ell, d)$.

A simple interpolation between k^{-3} and k^{-1} behaviour in (2.6) and (2.7) could be

$$E(k) = \left(\frac{\kappa}{\ell^2}\right)^2 k^{-3} \left(C_3 + C_1 \ell^2 k^2\right), \tag{2.10}$$

with C_1 and C_3 numerical constants, but it does not describe the shoulder in the intermediate region around $k\ell \approx 1$, where there is a k^2-behaviour.

Here, we propose an expression more general than (2.10), similar to (2.5), leading from the k^{-3} behaviour to the k^{-1} behaviour (2.7) through an intermediate regime in k^2. Our proposal reads as

$$E(k) \propto \frac{A_1 k^{-1} + A_3 k^{-3}}{2} \left[1 + \frac{A_1 k^{-1} - A_3 k^{-3}}{A_1 k^{-1} + A_3 k^{-3}} \tanh\left[A(k\ell - 1)\right]\right] \tag{2.11}$$

where $A_1 = C_1(\kappa/\ell)^2$ (see (2.7) with $\beta = 0$) and $A_3 = C_3 \kappa^2/\ell^4$ (see (2.7) with $\beta = -2$). Expression (2.11) is plotted in Fig. 2 for $A_1 = 10$, $A_3 = 10^{-3.5}$, $A = 2.8$. This figure reproduces in a qualitative way the behaviour reported in Fig. 4 in Ref. [1]. For other values of A, instead, the intermediate shoulder regions has not the equipartition form k^2 (for instance,

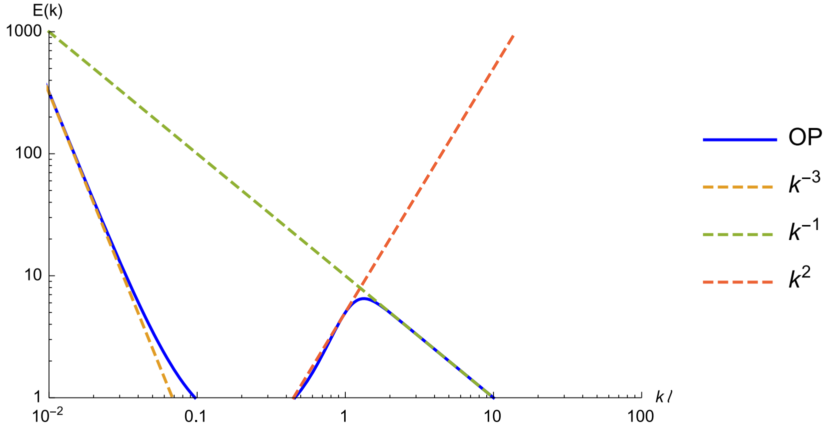

Fig. 2 Plot of (2.11) for $A_2 = 10$, $A_3 = 10^{-3.5}$ and $A = 2.8$. The legend on the right-hand side explains the power-decay of the straight-lines in the Log-Log plot, and OP stands for "Our Proposal" (2.11), which reproduces the observed behaviour represented in Fig. 4 of the paper [1]. (Color figure online)

for $A = 5$, the intermediate region has the form k^3). The expression (2.11) for $k\ell \gg 1$ and $k\ell \ll 1$ are the same as the limiting expression from (2.10). In Sect. 3 we discuss a possible physical interpretation of parameter A.

3 Discussion of Our Interpolation Proposals in Transition Regimes

In this section we discuss the mathematical relation between the value of the parameter A in the interpolation expressions (2.5) and (2.11) and the exponent m of k^m in the transition zone (the elbow in the plots in Figs. 1 and 2) as well as some physical comments about the possible physical meaning of A.

The relation between A and m in interpolation expression (2.5) for coflow between $k^{-5/3}$ and k^{-1} can be found calculating the slope of the straight line in the log-log plot of the figure. The cumbersome expression is:

$$m = -\frac{1}{6\coth^{-1}\left(\frac{2A}{\cosh^{-1}(2)}\right)}\left[3\log\left(-\frac{10\left(1-\frac{\cosh^{-1}(2)}{2A}\right)^{2/3}}{\sqrt{3}}+10\left(1-\frac{\cosh^{-1}(2)}{2A}\right)^{2/3}\right.\right.$$
$$\left.+\frac{1}{\sqrt{3}}+1\right)-3\log\left(\frac{10\left(\frac{\cosh^{-1}(2)}{2A}+1\right)^{2/3}}{\sqrt{3}}+10\left(\frac{\cosh^{-1}(2)}{2A}+1\right)^{2/3}-\frac{1}{\sqrt{3}}+1\right)$$
$$\left.+10\coth^{-1}\left(\frac{2A}{\cosh^{-1}(2)}\right)\right] \tag{3.12}$$

which for $A = 4$ becomes $m \simeq 2$. For $A \simeq 0.66$, m reaches its minimum value, $m \simeq -1.4$. The first case corresponds to the transition regime with a k^2 regime; in the second case, the behaviour from $-5/3$ and -1 is always with negative slope, and the slope at $k\ell \simeq 1$ is intermediate between $-5/3$ and -1.

The relation between A and m in interpolation expression (2.11) for counterflow between k^{-3} and k^{-1} can be found following the same procedure adopted in (3.12). The result is:

$$m = \frac{\log\left(\frac{2.114A\left(1.A^2-1.317A+0.434\right)}{(1.A-0.658)^3}\right) - \log\left(\frac{7.886A\left(1.A^2+1.317A+0.434\right)}{(1.A+0.658)^3}\right)}{\log\left(1-\frac{\log\left(2+\sqrt{3}\right)}{2A}\right) - \log\left(\frac{\log\left(2+\sqrt{3}\right)}{2A}+1\right)}, \tag{3.13}$$

which for $A \simeq 2.8$ becomes $m \simeq 2$. For $A \simeq 0.66$, m reaches its minimum value $m \simeq -1.9$. As well as in the previous case, the transition may have a k^2 regime ($m = 2$) or it may be always with a negative slope such that the slope at $k\ell \simeq 1$ is intermediate between -3 and -1.

On physical grounds, we conjecture that the value of A could depend on the ratio between some residence time, t_{res}, of the energy in the transition region, and the characteristic time of exchange of energy, t_{ex}, between the several physical modes of the system. If $t_{res} \gg t_{ex}$ there is time enough to distribute the energy equally in all the physical modes (equipartition). If, instead, $t_{res} \ll t_{ex}$, the energy goes directly from the initial mode to the final mode without time to be redistributed in the several other modes, and a region k^2 will not be achieved.

Thus, we heuristically suggest that A could be related to t_{res}/t_{ex} as

$$A = \frac{4 + 4(t_{res}/t_{ex})^{\gamma}}{6.06 + (t_{res}/t_{ex})^{\gamma}} \tag{3.14}$$

for (2.5) and for (2.11)

$$A = \frac{2.8 + 2.8(t_{res}/t_{ex})^{\gamma}}{4.24 + (t_{res}/t_{ex})^{\gamma}} \tag{3.15}$$

with γ an exponent which is not relevant in the asymptotic regimes.

We have considered our proposal (2.5) and (2.11) for different values of the coefficient A, according to our proposal (3.14) and (3.15), which are related to the ratio t_{res}/t_{ex} and the exponent γ. More precisely, assuming $\gamma = 1$, for the sake of simplicity, we consider three different value of t_{res}/t_{ex}, namely $t_{res}/t_{ex} = 10$, 1 and 0.1, for understanding how changes the behaviour of our proposal. The results for (2.5) are plotted in Fig. 3 for three values of A, namely $A = 2.73$ (blue line for $t_{res}/t_{ex} = 10$), $A = 1.13$ (red line for $t_{res}/t_{ex} = 1$) and $A = 0.71$ (purple line for $t_{res}/t_{ex} = 0.1$). In Fig. 4 instead (2.11) is plotted for three values of A, namely $A = 2.16$ (blue line for $t_{res}/t_{ex} = 10$), $A = 1.07$ (red line for $t_{res}/t_{ex} = 1$) and $A = 0.71$ (purple line for $t_{res}/t_{ex} = 0.1$). Note the different behaviour of our proposals in these figures with respect to the ones plotted in Fig. 1 and Fig. 2.

Indeed, for $t_{res} \gg t_{ex}$ one finds the values corresponding to equipartition (k^2 behaviour) whereas for $t_{res} \ll t_{ex}$ one finds a negative slope intermediate between the negative slopes of the $k\ell \gg 1$ region and the $k\ell \ll 1$ region. Experimentally exploring the role of t_{res}/t_{ex} on the intermediate behaviour of the energy cascade would require being able to modify t_{res}/t_{ex}. A possibility of doing so would be by changing the temperature, but we do not know at present the influence of temperature on t_{res} nor t_{ex}. A possible guess is that at lower temperature the exchange time of energy t_{ex} will become longer, because the friction is smaller. A deeper analysis of this topic should be carried out.

The idea that in the intermediate region there could be a bottleneck in the energy transfer between the $k\ell \gg 1$ and $k\ell \ll 1$ regimes has been proposed by L'Vov et al. [24]. In their suggestion, the transfer of energy from three-dimensional or two-dimensional excitations

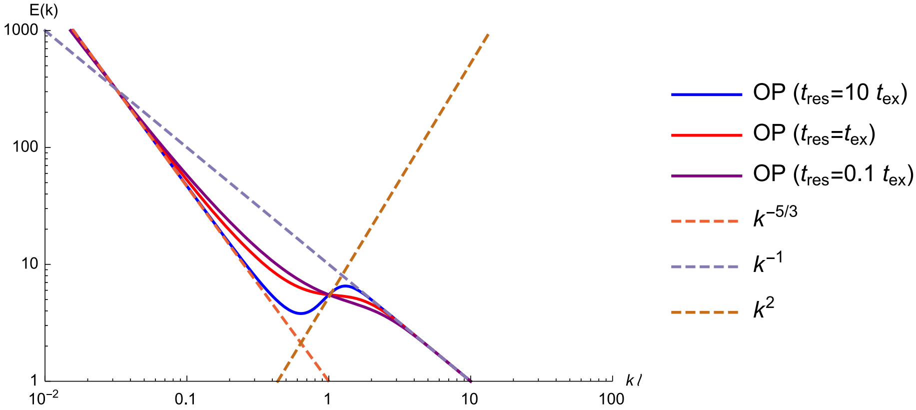

Fig. 3 Plot of (2.5) for $A_1 = 10$, $A_{K41} = 1$ and $A = 2.73$ (blue line), $A = 1.13$ (red line) and $A = 0.71$ (purple line), found for $\gamma = 1$ and $t_{res}/t_{ex} = 10$, 1 and 0.1, respectively in (3.14) for "Our Proposal" (2.5). (Color figure online)

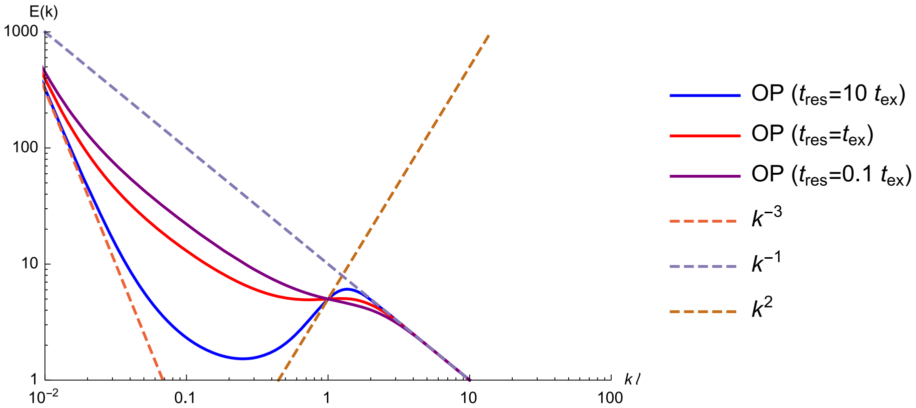

Fig. 4 Plot of (2.11) for $A_1 = 10$, $A_3 = 10^{-3.5}$ and $A = 2.16$ (blue line), $A = 1.07$ (red line) and $A = 0.71$ (purple line), found for $\gamma = 1$ and $t_{res}/t_{ex} = 10$, 1 and 0.1, respectively in (3.15) for "Our Proposal" (2.11). (Color figure online)

($k^{-5/3}$ and k^{-3} scaling, respectively) at $k\ell \gg 1$ to one-dimensional behaviour (Kelvin-wave excitation along one-dimensional vortex lines) for $k\ell \ll 1$ regime would be slow enough in order to have time enough to exchange energy between the several modes. However, in the case that the energy transfer rate in the bottleneck is not sufficiently low, our interpretation of A suggests how the spectrum could be modified.

4 Conclusions

In this paper we have proposed simple dimensional interpretations of some of the salient features of the energy cascade in the hydrodynamic regime and the quantum regime in coflow and counterflow and superflow experiments. We have used the quantities ϵ, κ, ℓ and k, but the diameter d of the container could also appear for turbulence in narrow channels. We have also proposed interpolation expressions (2.5) and (2.11) leading from the hydrodynamic regime to the Kelvin wave regime (namely, from $k^{-5/3}$ to k^{-1} in coflow, and from k^{-3} to k^{-1} in counterflow), with an intermediate k^2 shoulder.

On the other side, we have proposed (3.14) and (3.15) for the physical interpretation of the parameter A playing a central role in the interpolation expressions (2.5) and (2.11) in terms of $t_{res/tex}$, with t_{res} and t_{ex} being the característic residence time and exchange time. Such interpretation may be helpful for the analysis of the k^2 behaviour of the shoulder connecting the hydrodynamical region from L_0 to ℓ and the quantum region from ℓ to l_P as energy equipartition between the several k-modes. This region will appear if the energy does not flow in a fast way from high vortices to small vortices, i.e. if $t_{res} \gg t_{ex}$, but that it has some time to redistribute in the several vortices, either smaller or higher. In this region, the vortex lines begin to be sufficiently separated from each other, and there is a combination of energy exchange between different vortices, and energy exchange between Kelvin-wave excitations inside the same vortex. In contrast, if the energy transfer rate is not low enough, i.e. if $t_{res} \ll t_{ex}$, a k^2 region is not expected to appear. This is helpful to understand whether such a k^2 region becomes narrower and narrower when the energy transfer rate increases, in such a way that it is not practically observed because it is too narrow, or whether the k^2 region simply disappears because the exponent in the intermediate region of the cascade changes with the energy rate transfer. Our analysis suggests that the latter is indeed the case. These conclusions do not depend in a crucial way on the use of a tanh function for interpretation in (2.5) and (2.11), but they would be analogous if alternative interpolations were used.

Dimensional analysis is not able to yield univocal predictions for the cascade, but anyway it has some limited but potentially useful predictive power. Indeed, in (2.6), (2.7), the behaviour in k is linked in a definite way to the behaviour in ℓ. Thus, the scaling laws in k and in ℓ are linked. Future researches on the dependence of the cascade on ℓ, for a given k, could test whether these scaling laws are indeed related to the essential physics of the problem in the several range of wavevectors. For instance, it has been suggested that a reason for a bottleneck in the energy transfer from the hydrodynamic region to the Kelvin region (and therefore a low value of the rate of the energy transfer) is the contrast between the three dimensional character of the former one and the one-dimensional character of the latter one. Thus, our suggestion is that turbulence in wide channels would exhibit a k^2 intermediate region, whereas for narrower channels the energy rate transfer between both regions would become higher and the k^2 region would disappear, by changing the exponent 2 to lower values of the exponent.

Instead, for coflow one observes the classical $k^{-5/3}$ behaviour for $k\ell$ small and k^{-1} for $k\ell$ high. Note that $k\ell$ small means that the average vortex separation ℓ is much smaller than the wavelength, i.e. there are many vortices per unit length. When k increases (and wavelength diminishes), there are fewer vortices per unit length, and k^{-1} behaviour dominates. In this case, the energy cascade is in the form of Kelvin waves in quantized vortex lines, rather than in hydrodynamic vortices. Indeed, if the vortex lines are too much separated their mutual interaction is small, and the cascade is due to processes taking place separately in any single vortex line.

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Declarations

Competing Interests Authors declare not to have financial interests that are directly or indirectly related to the work submitted for publication.

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