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A bifurcation phenomenon for the critical Laplace and p-Laplace equation in the ball: part 2

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Abstract. In this paper we continue the discussion started in [12] concerning a bifurcation phenomenon for radial solutions of the following nonlinear eigenvalue problem:

$$\begin{cases} \Delta_p u(x) + \lambda \mathcal{K}(|x|)\, u(x)\, |u(x)|^{q-2} = 0\,, \\ u(x) > 0\,, & |x| < 1\,, \\ u(x) = 0\,, & |x| = 1\,, \end{cases}$$

where $q = \frac{np}{n-p}$ is the critical exponent and $x \in \mathbb{R}^n$. Our main purpose is to remove the restriction in the range of parameters, i.e. $\frac{2n}{n+2} \leq p \leq 2$, and to consider the whole range $p > 1$. The proofs rely on a dynamical systems approach and the main technical contribution is the construction of an unstable manifold in a non-smooth context.

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1. Introduction

In this paper we continue the discussion started in [12] concerning positive radial solutions for the following problem

$$\begin{cases} \Delta_p u(x) + \mathcal{K}(|x|)|x|^{\delta} u(x)^{q-1} = 0\,, \\ u(x) > 0\,, & |x| < \mathcal{R}\,, \\ u(x) = 0\,, & |x| = \mathcal{R}\,, \end{cases} \tag{1}$$

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where $\Delta_p u = \operatorname{div}(\nabla u |\nabla u|^{p-2})$ denotes the p-Laplace operator, $x \in \mathbb{R}^n$, $1 < p < n$, $\delta > -p$ and q is the Hardy-Sobolev critical exponent

$$q = p^*(\delta) = p\,\frac{n+\delta}{n-p}. \tag{2}$$

Further, $\mathcal{K} : [0, +\infty[\to]0, +\infty[$ is C^1 in $]0, +\infty[$, and satisfies the following assumption that will play a key role in our analysis:

$(\mathbf{H}_\ell)$ there are some constants $A_0, B_0, \ell > 0$ such that

$$\mathcal{K}(r) = A_0 + B_0 r^\ell + h(r) \qquad \text{and} \qquad \lim_{r\to 0} \frac{|h(r)| + r|h'(r)|}{r^\ell} = 0\,.$$

In some of the main results we assume one of the following additional hypotheses:

$(\mathbf{H}_\uparrow)$ the function $\mathcal{K}$ is increasing for any $r \geq 0$, strictly in some interval;
$(\mathbf{W}_s)$ $\overline{K} = \sup\{\mathcal{K}(r) \mid r \geq 0\} < \infty$;
$(\mathbf{B}_l)$ there is $\rho_1 > 0$ such that $\mathcal{K}(r) \geq \mathcal{K}(\rho_1)$ for any $r \geq \rho_1$.

In [12] the authors showed how the existence and the multiplicity of radial solutions of (1) vary with the length of the radius $\mathcal{R}$, and depicted some bifurcation diagrams depending on the flatness order ℓ in $(\mathbf{H}_\ell)$.

In fact, with a standard rescaling argument, the results concerning problem (1) can be restated to address the following equivalent nonlinear eigenvalue problem

$$\begin{cases} \Delta_p u(x) + \lambda\mathcal{K}(|x|)|x|^\delta u(x)^{q-1} = 0\,, & \\ u(x) > 0\,, & |x| < 1, \\ u(x) = 0\,, & |x| = 1\,. \end{cases} \tag{3}$$

Consequently, the existence and the multiplicity of radial solutions of (3) vary with $\lambda > 0$, and undergo a bifurcation scenario depending on ℓ.

We emphasize that, due to technical reasons, in [12] condition $(\mathbf{W}_s)$ was strengthened into the following stronger condition:

$(\mathbf{W}'_s)$ The function $\mathcal{K}(\mathrm{e}^t)$ is uniformly continuous in $[0, +\infty[$ and there are $\overline{K} > \underline{K} > 0$ such that $\underline{K} < \mathcal{K}(\mathrm{e}^t) < \overline{K}$ for any $t \in \mathbb{R}$.

In [12] we restricted to the motivating case $\delta = 0$, so that $q = p^*$ is the Sobolev critical exponent; further we also required the simplifying but technical assumption $\frac{2n}{n+2} \leq p \leq 2$. The purpose of this paper is to extend the results of [12] to the case $\delta > -p$ and, mainly, to remove the restriction on p, so that $p > 1$ is allowed.

Similar non-linear eigenvalue problems are nowadays a classical topic, see e.g. the papers [1,5,22], involving critical Sobolev exponents, and [6,13,14,23,29], where the reaction term $\mathcal{K}(r)u^{q-1}$ is replaced by a sum of a linear term, and a term supercritical with respect to the Sobolev exponent. We address the interested reader to the introduction of [13,14] for a discussion of several possible diagrams, appearing as different reaction terms are considered.

The critical problem (1) subject to condition $(\mathbf{H}_\ell)$ has been already investigated in the early 1990s by Bianchi and Egnell in [4], and Lin and Lin in

[28], who determined the existence of the flatness order's threshold $\ell_2^* = n-2$ in the Laplacian setting $p=2$, when $\delta = 0$.

In fact, Bianchi and Egnell, and Lin and Lin have been able to prove that, when $\ell < n-2$, problem (1) admits a radial solution for every $\mathcal{R} > 0$, but, when $\ell \geq n-2$, there exists a sufficiently small constant $r_0 > 0$ such that problem (1) does not admit any radial solution for $\mathcal{R} < r_0$. Further, if $n-2 \leq \ell < n$, there is at least one solution of (1) provided that $\mathcal{R} > 0$ is sufficiently large.

Recently, in [12], we have completed their study in the following directions. Firstly, we generalized their results to the p-Laplace setting, assuming $\frac{2n}{n+2} \leq p \leq 2$: in this case the critical flatness order becomes

$$\ell_p^* = \frac{n-p}{p-1}\,. \tag{4}$$

Secondly, we observed the presence of a second solution in the supercritical case for $\mathcal{R}$ large. Finally, we have shown that the restriction $\ell < n$ can be weakened in $\ell \leq \frac{n}{p-1}$, and might be neglected if $\mathcal{K}$ is monotonically increasing for any $r > 0$.

We now recall the main results found in [12].

Theorem A. ([12]) *Assume that $\mathcal{K}$ satisfies* $(\mathbf{H}_\ell)$, *$\delta = 0$, and $\frac{2n}{n+2} \leq p \leq 2$.*

If $\ell < \ell_p^*$, then problem (1) admits a radial solution for every $\mathcal{R} > 0$.
If $\ell \geq \ell_p^*$, then there exists $\mathcal{R}_0 > 0$ such that problem (1) does not admit any radial solution when $\mathcal{R} < \mathcal{R}_0$.

Theorem B. ([12]) *Assume that $\mathcal{K}$ satisfies* $(\mathbf{H}_\ell)$ *and* $(\mathbf{H}_\uparrow)$*. Further, assume $\delta = 0$, and $\frac{2n}{n+2} \leq p \leq 2$.*

If $\ell = \ell_p^*$, then there exists $\mathcal{R}_0 > 0$ such that problem (1)

- *does not admit any radial solution when $\mathcal{R} < \mathcal{R}_0$;*
- *admits at least a radial solution when $\mathcal{R} > \mathcal{R}_0$.*

If $\ell > \ell_p^*$, then there exists $\mathcal{R}_0 > 0$ such that problem (1)

- *does not admit any radial solution when $\mathcal{R} < \mathcal{R}_0$;*
- *admits at least a radial solution when $\mathcal{R} = \mathcal{R}_0$;*
- *admits at least 2 radial solutions when $\mathcal{R} > \mathcal{R}_0$.*

If the global condition $(\mathbf{H}_\uparrow)$ is dropped, the result of Theorem B holds true by restricting the interval in which ℓ varies, and by imposing some further weak asymptotic conditions on $\mathcal{K}(r)$ for r large.

Theorem C. ([12]) *Assume that $\mathcal{K}$ satisfies* $(\mathbf{H}_\ell)$ *with $\ell_p^* \leq \ell \leq \frac{n+\delta}{p-1}$, and that either* $(\mathbf{W}'_s)$ *or* $(\mathbf{B}_l)$ *holds. Further, suppose $\delta = 0$ and $\frac{2n}{n+2} \leq p \leq 2$, then we get the same conclusion as in Theorem B.*

Since we just consider radial solutions, we can rewrite the differential equation in (1) as a singular ordinary differential equation of the form

$$\left(r^{n-1}u'(r)|u'(r)|^{p-2}\right)' + r^{n-1}\mathcal{K}(r)r^{\delta}\,u\,|u|^{q-2} = 0\,, \tag{5}$$

where " $'$ " denotes the differentiation with respect to $r = |x|$, and, with a slight abuse of notation, $u(r) = u(x)$.

According to [12], we adopt the standard change of variables

$$w(s) = u(r), \qquad s = \frac{r}{\lambda^{\frac{1}{\delta+p}}}, \tag{6}$$

and $\mathscr{K}(s) = \mathcal{K}(s\,\lambda^{\frac{1}{\delta+p}})$, to convert (5) to the eigenvalue equation

$$\left(s^{n-1}w'(s)|w'(s)|^{p-2}\right)' + \lambda s^{n-1}\,\mathscr{K}(s)\,s^{\delta}\; w|w|^{q-2} = 0. \tag{7}$$

Note that u solves equation (5) if and only if w solves (7), and $w(1) = 0$ if and only if $u(\mathcal{R}) = 0$, where $\mathcal{R} := \lambda^{\frac{1}{\delta+p}}$. In particular, Theorems A, B, and C can be rewritten as follows.

Corollary D. ([12]) *Assume that $\mathcal{K}$ satisfies* $(\mathbf{H}_\ell)$, *$\delta = 0$ and $\frac{2n}{n+2} \le p \le 2$.*

If $\ell < \ell_p^*$, then problem (3) admits a radial solution for every $\lambda > 0$.
If $\ell \ge \ell_p^*$, then there exists $\lambda_0 > 0$ such that problem (3) does not admit any radial solution when $\lambda < \lambda_0$.

Corollary E. ([12]) *Assume that $\mathcal{K}$ satisfies* $(\mathbf{H}_\ell)$ *and* $(\mathbf{H}_\uparrow)$. *Further, assume $\delta = 0$ and $\frac{2n}{n+2} \le p \le 2$.*
If $\ell = \ell_p^*$, then there exists $\lambda_0 > 0$ such that problem (3)

- *does not admit any radial solution when $\lambda < \lambda_0$;*
- *admits at least a radial solution when $\lambda > \lambda_0$.*

If $\ell > \ell_p^*$, then there exists $\lambda_0 > 0$ such that problem (3)

- *does not admit any radial solution when $\lambda < \lambda_0$;*
- *admits at least a radial solution when $\lambda = \lambda_0$;*
- *admits at least two radial solutions when $\lambda > \lambda_0$.*

Corollary F. ([12]) *Assume that $\mathcal{K}$ satisfies* $(\mathbf{H}_\ell)$ *with $\ell_p^* \le \ell \le \frac{n+\delta}{p-1}$, and that either* $(\mathbf{W}'_s)$ *or* $(\mathbf{B}_l)$ *holds. Further, suppose $\delta = 0$ and $\frac{2n}{n+2} \le p \le 2$, then we get the same conclusions as in Corollary E.*

The main result of the present paper is the following.

Theorem 1. *Theorems A, B, C, and Corollaries D, E, F hold in the general case $\delta > -p$ and $p > 1$. Further, assumption* $(\mathbf{W}'_s)$ *can be replaced by* $(\mathbf{W}_s)$ *in Theorem C and in Corollary F.*

Notice that the value ℓ_p^* in (4) does not depend on the choice of δ.

Existence results, under the flatness condition $(\mathbf{H}_\ell)$ in absence of upper-bounds on ℓ for a scalar curvature problem (with $p = 2$) on four dimensional manifolds, can be found in [8]. We address the reader to [12] for a more detailed discussion of the literature on the topic, and the applications of the results.

For a comparison with [12], we refer to the end of this section, where we highlight the main differences with our previous paper.

A solution $u(r)$ of (5) is called *regular* if $\lim_{r\to 0} u(r) = d > 0$ for a suitable $d > 0$: in this case it will be denoted by $u(r; d)$.

Remark 2. If $u(r)$ is regular, then $u'(r)r^{-\frac{\delta+1}{p-1}}$ converges to a negative constant as $r \to 0$. In particular, $u(r)$ is decreasing for r small, and $u'(r)r^{\frac{n}{p}} \to 0$ as $r \to 0$.

Let us now recall that local existence and uniqueness of regular solutions $u(r;d)$, and their continuous dependence on d are ensured for any $p > 1$ (see also [20, Proposition A4]), while the smoothness follows from the proof of Theorem 1.3 in [27].

Lemma 3. ([21, Proposition A.2], [27, Section 5]) *For any $d > 0$ there exists a unique regular solution $u(r;d)$ of (5). Further, $u(r;d)$ and $u'(r;d)$ are C^1 in r and d as long as $u(r;d)$ is positive.*

Again, from [21, Section 2], we also find the following standard fact.

Remark 4. Any regular solution $u(r;d)$ is decreasing as long as it is positive.

We call *crossing solution* a regular solution of (5) which has a positive zero. We set

$$J = \{d > 0 \mid u(r;d) \ \text{ is a crossing solution}\}, \tag{8}$$

and denote by $R(d)$ the *first zero* of $u(r;d)$. The proofs of our results rely on a study of the properties of the function $R : J \to]0, +\infty[$, and, in particular, on the asymptotic estimates of $R(d)$ for d large. In fact, under suitable assumptions, e.g. $(\mathbf{H}_{\uparrow})$, we are able to show the following bifurcation phenomenon:

$$\begin{cases} \lim\limits_{d\to+\infty} R(d) = 0\,, & \text{if } 0 < \ell < \ell_p^*\,, \\ \liminf\limits_{d\to+\infty} R(d) > 0\,, & \text{if } \ell = \ell_p^*\,, \\ \lim\limits_{d\to+\infty} R(d) = +\infty\,, & \text{if } \ell > \ell_p^*\,. \end{cases} \tag{9}$$

See §2 for an overview of the properties of the function R.

Remark 5. When $\ell = \ell_p^*$ the information available on (9) is less sharp, consequently in Theorem B we are not able to say if there is a solution for $\mathcal{R} = \mathcal{R}_0$.

Main techniques and differences with [12]. To obtain the estimates on $R(d)$ we follow the way paved by [25] and developed later by [2,3,18]: we rely on a change of variables, known as Fowler transformation, which allows us to pass from the singular differential equation (5) to a planar Hamiltonian system from which the singularity is removed, see (14) below. In this way the term r^δ disappears, so that our results are immediately extended to the case where $\delta > -p$.

We see that regular solutions $u(r;d)$ of (5) correspond to trajectories $\phi(t)$ converging to the origin as $t \to -\infty$: the asymptotic estimates (9) are then proved through phase plane analysis and dynamical system techniques.

Since the nonlinearity in (14) below is given by the couple $(y\,|y|^{\frac{2-p}{p-1}}, -\mathcal{K}(\mathrm{e}^t)\,x|x|^{q-2})$, the main technical issue we face in this paper is that, when $p > 2$, the term $y|y|^{\frac{2-p}{p-1}}$ is not locally Lipschitz and the same happens to $x|x|^{q-2}$ when $1 < p < \frac{2n}{n+2+\delta}$: hence, local uniqueness of the solutions leaving from the coordinate axes is not ensured. In particular, we cannot rely anymore on the results of invariant manifolds theory and integral manifolds, available from the classical theory of non-autonomous dynamical systems, see [9], which have been crucial in [12].

Therefore, we need to borrow some ideas from [15,16], and [10]: we fix $\tau \leq 0$ and we construct via Ważewski principle the compact and connected set $\hat{W}^u(\tau)$, then we show a bijective correspondence between the points $\boldsymbol{Q} \in \hat{W}^u(\tau)$ and the values $u(\mathrm{e}^\tau; d)$ of the regular solutions, thus leading us to parametrize $\hat{W}^u(\tau)$ by $d > 0$. Then, we need to prove a sort of continuous dependence of $\hat{W}^u(\tau)$ on τ using some box arguments, while in [12] it was guaranteed by classical tools of invariant manifold theory.

We wish to mention that an alternative approach to overcome regularity issues for radial solutions ruled by p-Laplace equations was developed in a different context by Itakura et al. in [24].

Plan of the paper. The paper is divided as follows; in §2 we enumerate the properties of $R(d)$ used to draw Figures 1, 2, and we prove Theorem 1; however the proofs concerning $R(d)$ are postponed to §5. In §3 we introduce the Fowler transformation, whereby we turn to consider a dynamical system, and the Hamiltonian $\mathcal{H}$, which is the dynamical transposition of the Pohozaev function. In §4 we face the main technical issues: we first construct the unstable sets $\hat{M}^u(\tau)$ and $\hat{W}^u(\tau)$ in a non-smooth context using some adaptation of the Ważewski's principle, then we show that they correspond to regular solutions $u(r; d)$ of (5) and that they can be parametrized by d. More precisely, in §4.1 we construct a small branch, i.e. $\tilde{M}^u(\tau)$, for any $\tau \leq 0$, then in §4.2 we build a larger one, $\tilde{W}^u(\tau)$, for $\tau \ll 0$. Finally, in §5 we adapt the argument of [12] to prove the asymptotic properties of $R(d)$: in §5.1 we consider the case $\ell < \ell_p^*$, in §5.2 the simpler case where $\ell \geq \ell_p^*$ and $(\mathbf{H}_\uparrow)$ holds, in §5.3 the case where $\ell \geq \ell_p^*$ and either $(\mathbf{W}_s)$ or $(\mathbf{B}_l)$ holds, which involves the construction of the stable set $\tilde{M}^s(\tau)$. At the beginning of §2 the reader can find a table (Table 1) where the most used notation concerning the construction of the unstable and stable sets is collected.

2. Proof of the main results

In this section, following the scheme introduced in [12], we first illustrate the main properties of the function $R(d)$, with a special focus on the relative asymptotic estimates, and, finally, we show that our main result follows straightforwardly from them. We postpone all the proofs concerning the behaviour of $R(d)$ to §5, since they involve the use of the Fowler transformation, and the dynamical system techniques introduced and developed in the next two sections.

From a standard transversality argument, we easily get the following.

Lemma 6. *The set J is open and $R : J \to [0, +\infty[$ is continuous.*

Table 1. Most frequently used notations in the paper. The *Section* column indicates where each item is defined; in the *Figure* column, we refer to the figure that illustrates it

Set	Description	Section	Figure
$\mathcal{F}(\tau)$	Compact set used to construct $\tilde{M}(\tau)$, defined for $\tau \le 0$	§4, Def. 21	3
$\mathfrak{f}(\tau)$	$\mathcal{F}(\tau) \cap \{x = \overline{E}_x(0)\}$	§4, Def. 21	3
$\tilde{M}^u(\tau)$	Unstable set contained in $\mathcal{F}(\tau)$	§4, (31)	3
$\hat{M}^u(\tau)$	Connected component of $\tilde{M}^u(\tau)$	§4, Prop. 25	3
$\tilde{\Xi}_M(\tau)$	$\tilde{M}(\tau) \cap \mathfrak{f}(\tau)$	§4, (33)	3
$\hat{\boldsymbol{\xi}}(\tau)$	$\hat{M}(\tau) \cap \mathfrak{f}(\tau)$	§4, Prop. 25	3
$\mathcal{G}(\tau)$	Compact set used to construct $\tilde{W}^u(\tau)$, defined for $\tau \ll 0$	§4.2, Def. 44	5
$\mathcal{L}(a)$, $a > 0$	Segment of $y = -a$ between $x = 0$ and the isocline $\dot{x} = 0$	§4.2, (43)	5
$\mathfrak{g}(\tau, a)$	$\mathcal{L}(a) \cap \mathcal{G}(\tau)$	§4.2, Def. 44	5
$\tilde{W}^u(\tau)$	Unstable set contained in $\mathcal{G}(\tau)$	§4.2, Lemma 48	
$\hat{W}^u(\tau)$	Connected component of $\tilde{W}^u(\tau)$	§4.2, Prop. 49	
$\tilde{\Xi}_W(\tau, a)$	$\tilde{W}^u(\tau) \cap \mathfrak{g}(\tau, a)$	§4.2, (45)	
$\mathcal{F}_s$	Compact set used to construct $\tilde{M}^s(\tau)$	§5.3, Def. 60	6
$\mathfrak{f}_s$	$\mathcal{F}_s \cap \{x = \overline{E}_{x,s}\}$	§5.3, Def.60	6
$\tilde{M}^s(\tau)$	Stable set contained in $\mathcal{F}_s$	§5.3, (71)	6

The framework is clearer when $(\mathbf{H}_\uparrow)$ holds: in this case it is well known that $J =]0, +\infty[$, cf. [21] and [12, Proposition 43]. Then, it is not difficult to show that

$$\lim_{d \to 0} R(d) = +\infty\,, \tag{10}$$

see, e.g., [26, Proposition 2.4]. In absence of condition $(\mathbf{H}_\uparrow)$, it is still possible to deduce that, cf. [12, Proposition 24], for every accumulation point D^* for J with $D^* \notin J$, then

$$\lim_{\substack{d \to D^* \\ d \in J}} R(d) = +\infty. \tag{11}$$

The bifurcation phenomena depend on the asymptotic behavior of $R(d)$ for d large, which, in turn, depends crucially on the parameter ℓ, see e.g. (9): these relations are our main contribution and the core of the paper.

More precisely, in the ℓ-subcritical case we generalize [12, Proposition 37] and [7, Remark 4.1], showing that $R(d)$ is infinitesimal for large d.

Proposition 7. *Assume* $(\mathbf{H}_\ell)$ *and* $\ell < \ell_p^*$, *then there exists* $D^* \ge 0$ *such that* $]D^*, +\infty[\subset J$ *and* $\lim_{d \to +\infty} R(d) = 0$.

We stress that in Proposition 7, whose proof is postponed to §5.1, hypothesis $(\mathbf{H}_\uparrow)$ is dropped through a truncation argument.

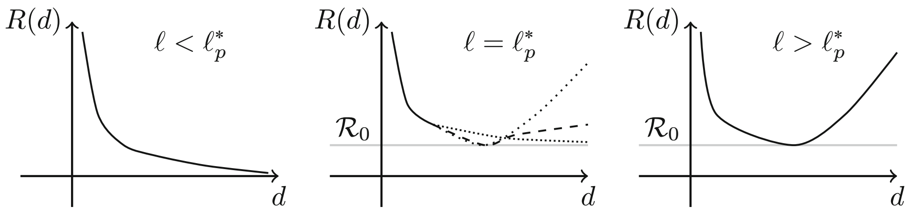

FIGURE 1. A sketch of the graph of the function $R(d)$, when $\mathbf{(H_\uparrow)}$ and $\mathbf{(H_\ell)}$ are assumed

Actually, assumption $\mathbf{(H_\uparrow)}$ is needed just to ensure that $J =]0, +\infty[$. In absence of condition $\mathbf{(H_\uparrow)}$, we can define

$$\tilde{R}(d) = \begin{cases} R(d) & \text{if } d \in J\,, \\ +\infty & \text{if } d \notin J \end{cases} \tag{12}$$

to obtain the following ℓ-critical and supercritical result, proved in §5.2, extending [12, Proposition 44] to the non-smooth case.

Proposition 8. *Assume $\mathbf{(H_\ell)}$ with $\ell \geq \ell_p^*$, then there is $\mathcal{R}_0 > 0$ such that $\tilde{R}(d) \geq \mathcal{R}_0 > 0$ for any $d \geq 0$.*

Further, if $\ell > \ell_p^*$, we get a more precise estimate by imposing an extra assumption, as, e.g., $\mathbf{(H_\uparrow)}$.

Proposition 9. *Assume $\mathbf{(H_\uparrow)}$ and $\mathbf{(H_\ell)}$ with $\ell > \ell_p^*$. Then, $J =]0, +\infty[$ and $\lim_{d \to +\infty} R(d) = +\infty$.*

The proof is postponed to §5.2 and contributes to extend [12, Proposition 43] to the non-smooth case.

Summing up, when $\mathbf{(H_\uparrow)}$ holds, we are able to prove the asymptotic estimate (9). So, we can draw the graph of $R(d)$, see Figure 1, and prove Theorems A and B when $\delta > -p$ and $p > 1$.

Remark 10. The interested reader may get some information on the $\|\cdot\|_\infty$ norm of the solutions of (1) and (3), since according to Remark 4 one has $d = \|u(r; d)\|_\infty$. In particular, in the supercritical case $\ell > \ell_p^*$, when λ is large enough we have two solutions of (3), one whose L^∞ norm is going to zero and the other whose L^∞ norm is going to infinity as $\lambda \to +\infty$.

Proof of Theorem A in the general setting of Theorem 1. If $\ell < \ell_p^*$, Proposition 7, (10)-(11) lead to $\lim_{d \to D^*} R(d) = +\infty$. From the continuity of $R(d)$, we conclude that $R(J) =]0, +\infty[$. Thus, for every $\mathcal{R} > 0$ there is $d \in J$ such that $R(d) = \mathcal{R}$, i.e. $u(\cdot; d)$ solves problem (1).

On the other hand, if $\ell \geq \ell_p^*$, the second assertion follows immediately from Proposition 8. □

Proof of Theorem B in the general setting of Theorem 1. The proof takes advantage of (10) combined with Propositions 8 and 9.

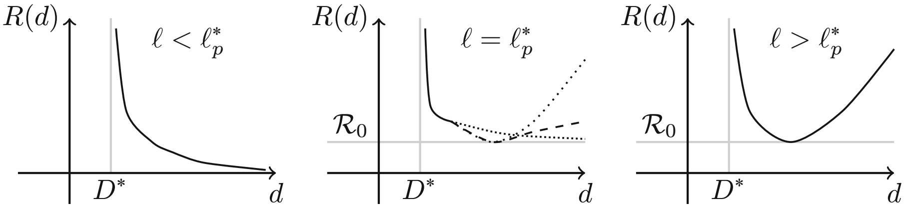

FIGURE 2. A sketch of the graph of the function $R(d)$, in the setting of Theorem C

If $\ell = \ell_p^*$, we distinguish two alternatives: either $R(\,]0,+\infty[\,) = [\mathcal{R}_0,+\infty[$ where $\mathcal{R}_0$ is an internal minimum of R, or $R(\,]0,+\infty[\,) =]\mathcal{R}_0,+\infty[$ since $\mathcal{R}_0 = \liminf_{d\to+\infty} R(d) > 0$: the theorem is proved in both the cases.

On the other hand, if $\ell > \ell_p^*$, from $\lim_{d\to 0} R(d) = \lim_{d\to+\infty} R(d) = +\infty$, we deduce that the function R has an internal minimum $\mathcal{R}_0$, and the pre-image $R^{-1}(\mathcal{R})$ has at least two elements for every $\mathcal{R} > \mathcal{R}_0$, thus giving the multiplicity result. $\square$

Following [12], we are able to drop the global assumption $(\mathbf{H}_\uparrow)$ by restricting the range of ℓ, i.e. strengthening the local assumption. More precisely, we find the following two results, which generalize [12, Lemma 45 and Proposition 46], whose proofs rely again on dynamical systems tools adapted for a non-smooth context.

Lemma 11. *Assume that $\mathcal{K}$ satisfies $(\mathbf{H}_\ell)$ with $\ell \le \frac{n+\delta}{p-1}$, and that either $(\mathbf{W}_s)$ or $(\mathbf{B}_l)$ holds. Then, there is $D^* \ge 0$ such that $]D^*,+\infty[\subset J$.*

Proposition 12. *Assume that $\mathcal{K}$ satisfies $(\mathbf{H}_\ell)$, and that either $(\mathbf{W}_s)$ or $(\mathbf{B}_l)$ holds.*

- *If $\ell = \ell_p^*$, then $\liminf\limits_{d\to+\infty} R(d) > 0$.*
- *If $\ell_p^* < \ell \le \frac{n+\delta}{p-1}$, then $\lim\limits_{d\to+\infty} R(d) = +\infty$.*

For the proofs of Lemma 11 and Proposition 12 we refer to §5.3.

Now, we are ready to draw Figure 2 and prove Theorem C.

Proof of Theorem C in the general setting of Theorem 1. Lemma 11, together with (10)-(11), leads to $\lim_{d\to D^*} R(d) = +\infty$. Then, we use Proposition 12 to complete the proof. $\square$

Remark 13. We think it is worthwhile to emphasize that in the setting of Theorem C we may have $D^* > 0$ such that $]D^*,+\infty[\subset J$ and $D^* \notin J$: it is easy to show that in this case $u(r;D^*)$ is a ground state, i.e. $u(r;D^*) > 0$ for any $r > 0$ and $\lim_{r\to+\infty} u(r;D^*) = 0$.

Corollaries D, E, F in the general setting of Theorem 1 follow from Theorems A, B, C proved in the general context $p > 1$, $\delta > -p$, combined with the change of variable (6). The proof of Theorem 1 is so completed.

3. Fowler transformation

In this section, we repeat some constructions borrowed from [12], for convenience of the reader. Firstly, we introduce the change of variables known as Fowler transformation:

$$\alpha = \frac{n-p}{p}, \qquad \beta = \frac{n(p-1)}{p},$$

$$x = u(r)r^{\alpha}, \qquad y = u'(r)|u'(r)|^{p-2}r^{\beta}, \qquad r = \mathrm{e}^t. \tag{13}$$

According to (13), we can rewrite (5) as the following dynamical system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y\,|y|^{\frac{2-p}{p-1}} \\ -K(t)\,x|x|^{q-2} \end{pmatrix}, \tag{14}$$

where $K(t) = \mathcal{K}(\mathrm{e}^t)$, $q = p^*(\delta) = p\,\frac{n+\delta}{n-p}$ as in (2), and "$\cdot$" denotes the differentiation with respect to t.

Remark 14. Note that system (14) is C^1 if and only if $\frac{2n}{2+n+\delta} \le p \le 2$, i.e. if $p \le 2 \le q$.

Following [12], for any $\tau \in \mathbb{R}$ and any $\boldsymbol{Q} \in \mathbb{R}^2$ we denote by

$$\boldsymbol{\phi}(t;\tau,\boldsymbol{Q}) = \big(x(t;\tau,\boldsymbol{Q}), y(t;\tau,\boldsymbol{Q})\big) \tag{15}$$

the trajectory of (14) such that $\boldsymbol{\phi}(\tau;\tau,\boldsymbol{Q}) = \boldsymbol{Q}$, when it is uniquely defined. We focus our attention on the dynamics in the set $\{x > 0, y < 0\}$, so the notation will be consistent as long as the trajectory remains in this set. Similarly, also continuous dependence on initial data is implicitly guaranteed there.

Let $\boldsymbol{\phi}(t;d)$ denote the trajectory of (14) corresponding to the regular solution $u(r;d)$ of (5). Notice that $\lim_{r\to 0} u(r) = d$ and $\lim_{r\to 0}(u'(r))^{p-1}r^{\beta} = 0$, see Remark 2, so $\lim_{t\to-\infty} \boldsymbol{\phi}(t;d) = (0,0)$.

The Hamiltonian function

$$\mathcal{H}(x,y;t) := \alpha xy \,+\, \frac{p-1}{p}\,|y|^{\frac{p}{p-1}} \,+\, K(t)\,\frac{|x|^q}{q} \tag{16}$$

satisfies

$$\frac{d}{dt}\mathcal{H}\big(x(t),y(t);t\big) = \dot{K}(t)\,\frac{|x(t)|^q}{q} \tag{17}$$

along the solutions $(x(t), y(t))$ of (14).

We also need to consider the autonomous system obtained by *freezing* the t-dependence of $K(\cdot)$. In particular, fixed $\varpi > 0$, we consider the *frozen* autonomous system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y\,|y|^{\frac{2-p}{p-1}} \\ -\varpi\,x|x|^{q-2} \end{pmatrix}. \tag{18}$$

Let us fix $\varpi > 0$ in (18), let $\tau \in \mathbb{R}$, and $\boldsymbol{Q} \in \mathbb{R}^2$; we denote by

$$\boldsymbol{\phi}_{\boldsymbol{\varpi}}(t;\tau,\boldsymbol{Q}) = \big(x_{\varpi}(t;\tau,\boldsymbol{Q}), y_{\varpi}(t;\tau,\boldsymbol{Q})\big) \tag{19}$$

the trajectory of (18) such that $\boldsymbol{\phi}_{\boldsymbol{\varpi}}(\tau;\tau,\boldsymbol{Q}) = \boldsymbol{Q}$.

System (18) exhibits an equilibrium point in $\{x>0, y<0\}$ of coordinates

$$\boldsymbol{E}(\varpi) = (E_x(\varpi), E_y(\varpi)) = \left(\left(\frac{\alpha^p}{\varpi}\right)^{\frac{1}{q-p}}, -\left(\frac{\alpha^q}{\varpi}\right)^{\frac{p-1}{q-p}} \right). \tag{20}$$

When $\frac{2n}{2+n+\delta} \le p \le 2$, the origin is a saddle. We will see that we have a geometrically similar situation in the general case $p>1$. Notice that, for any $\varpi > 0$, $\boldsymbol{E}(\varpi)$ is a centre and it lies in the interior of an homoclinic orbit lying in the fourth quadrant

$$\boldsymbol{\Gamma}(\varpi) = \{(x,y) \mid \mathcal{H}_\varpi(x,y) = 0, x>0\}, \qquad \text{where} \tag{21}$$

$$\mathcal{H}_\varpi(x,y) := \alpha x y + \frac{p-1}{p} |y|^{\frac{p}{p-1}} + \varpi \frac{|x|^q}{q}. \tag{22}$$

From a direct computation, we get the following useful scaling result.

Remark 15. For every trajectory $\boldsymbol{\phi_1}(t) = (x_1(t), y_1(t))$ of (18) with $\varpi = 1$, the function

$$\boldsymbol{\phi_c}(t) = (c^{\frac{1}{p-q}} x_1(t), c^{\frac{p-1}{p-q}} y_1(t))$$

is a trajectory of (18) with $\varpi = c > 0$.

Further, restricting $\boldsymbol{\phi_1}$ and $\boldsymbol{\phi_c}$ to the half-plane $\{x>0\}$, both the points $\boldsymbol{\phi_1}(t_0)$ and $\boldsymbol{\phi_c}(t_0)$ lie in the same parabola $y = -m(t_0) x^{p-1}$, where $m(t_0) = \frac{|y_1(t_0)|}{|x_1(t_0)|^{p-1}}$.

It is also well known (see, among others, [17]), that all the regular solutions $u_\varpi(r;d)$ of (5) with $\mathcal{K}(r) \equiv \varpi$ correspond to trajectories $\boldsymbol{\phi}_\varpi(t;d) = (x_\varpi(t;d), y_\varpi(t;d))$ of (18) which are homoclinic to the origin. In fact (see, among others, [31, §1], [16, Eq. (2.3)]) we also know their exact expression, in particular

$$x_\varpi(t;d) = d \left\{ \left[\mathrm{e}^{-t\frac{p+\delta}{p}} + C\, \mathrm{e}^{\frac{p+\delta}{p(p-1)}t} \right] \right\}^{-\frac{n-p}{p+\delta}}, \tag{23}$$

where

$$C = C(\varpi, d) = c\, \varpi^{\frac{1}{p-1}} d^{\frac{p(p+\delta)}{(n-p)(p-1)}} \qquad \text{with } c = \frac{p-1}{n-p}(n+\delta)^{-\frac{1}{p-1}}.$$

Notice that $\boldsymbol{\phi}_\varpi(t;d)$ has always the same image for any $d>0$, indeed we find $\boldsymbol{\Gamma}(\varpi) = \{\boldsymbol{\phi}_\varpi(t;d) \mid t \in \mathbb{R}\}$.

Remark 16. The following estimates hold:

$$\left.\begin{aligned} x_\varpi(t;d) &= d\mathrm{e}^{\alpha t} + o(\mathrm{e}^{\alpha t}), \\ y_\varpi(t;d) &= -\zeta_1(\varpi,d)\, \mathrm{e}^{\alpha(q-1)t} + o(\mathrm{e}^{\alpha(q-1)t}), \end{aligned}\right. \qquad \text{as } t \to -\infty,$$

$$\left.\begin{aligned} x_\varpi(t;d) &= \zeta_2(\varpi,d)\, \mathrm{e}^{-\frac{\alpha}{p-1}t} + o(\mathrm{e}^{-\frac{\alpha}{p-1}t}), \\ y_\varpi(t;d) &= -\zeta_3(\varpi,d)\, \mathrm{e}^{-\alpha t} + o(\mathrm{e}^{-\alpha t}), \end{aligned}\right. \qquad \text{as } t \to +\infty,$$

where we have introduced the constants (which, however, are not our primary focus here)

$$\zeta_1(\varpi,d) = (n+\delta)^{-1} \varpi\, d^{q-1},$$

$$\zeta_2(\varpi,d) = c^{-\frac{n-p}{p+\delta}} \varpi^{-\frac{n-p}{(p+\delta)(p-1)}} d^{-\frac{1}{p-1}},$$

$$\zeta_3(\varpi, d) = (n+\delta)^{\frac{n-p}{p+\delta}} \left(\frac{p-1}{n-p}\right)^{-\frac{n+\delta}{p+\delta}(p-1)} \varpi^{-\frac{n-p}{p+\delta}} d^{-1}.$$

Let $\Omega \subset]0,+\infty[$ be a compact interval, and denote with $\mathbf{\Gamma}^+(\varpi)$ the subset of $\mathbf{\Gamma}(\varpi)$ lying in $\dot{x} \geq 0$. Then, there exists a constant $c^+ = c^+(\Omega) > 0$ such that, for $h = \alpha \min\{1, q-1\}$,

$$\sup\{\|\boldsymbol{\phi}_{\boldsymbol{\varpi}}(\theta; 0, \boldsymbol{Q})\| e^{-h\theta} \mid \varpi \in \Omega,\ \theta \leq 0,\ \boldsymbol{Q} \in \mathbf{\Gamma}^+(\varpi)\} \leq c^+, \tag{24}$$

$$\sup\{\|\dot{\boldsymbol{\phi}}_{\boldsymbol{\varpi}}(\theta; 0, \boldsymbol{Q})\| e^{-h\theta} \mid \varpi \in \Omega,\ \theta \leq 0,\ \boldsymbol{Q} \in \mathbf{\Gamma}^+(\varpi)\} \leq c^+. \tag{25}$$

The results can be easily deduced from the explicit formula (23) and from (18).

In particular, rephrasing Lemma 2.9 in [10], we obtain the following remark.

Remark 17. The set $\mathbf{\Gamma}(\varpi)$ can be parametrized by $d > 0$. Namely, let $\boldsymbol{Q}(d; \varpi) := \boldsymbol{\phi}_{\boldsymbol{\varpi}}(0; d)$, then the function $\boldsymbol{Q}(\cdot; \varpi) : [0, +\infty) \to \mathbf{\Gamma}(\varpi)$ is bijective and C^1, providing a smooth parametrization of $\mathbf{\Gamma}(\varpi)$. Indeed, $\boldsymbol{Q}(0; \varpi) = (0, 0)$, and $\boldsymbol{Q}(d; \varpi)$ enters $\dot{x} > 0$ as d increases, then it rotates clockwise around the critical point $\boldsymbol{E}(\varpi)$ and converges back to the origin as $d \to +\infty$.

We note that from [10] we just find the continuity in d, while the smoothness is ensured by Lemma 3, which relies on [27].

Let us now list some immediate consequences of assumption $(\mathbf{H}_\uparrow)$.

Remark 18. Assume $(\mathbf{H}_\uparrow)$. Then, the homoclinic orbit $\mathbf{\Gamma}(K(\tau_2))$ belongs to the region enclosed by $\mathbf{\Gamma}(K(\tau_1))$, whenever $\tau_2 > \tau_1$.

Lemma 19. *Assume $(\mathbf{H}_\uparrow)$ and $\mathcal{K}(r) > \mathcal{K}(0)$ for every $r > 0$. If $u(r; d)$ is a regular solution, then the corresponding trajectory $\boldsymbol{\phi}(\cdot; d) = (x(\cdot; d), y(\cdot; d))$ of system (14) satisfies $\lim_{t\to-\infty} \mathcal{H}(\boldsymbol{\phi}(t; d); t) = 0$, and*

$$\mathcal{H}(\boldsymbol{\phi}(t; d); t) > 0 \qquad \text{for every } t \leq T(d),$$

where $T(d) := \sup\{\tau \in \mathbb{R} : x(t; d) > 0, \forall t \leq \tau\}$ is the first zero of $x(\cdot; d)$. Moreover,

$$\mathcal{H}_{K(\tau)}(\boldsymbol{\phi}(t; d)) = \mathcal{H}(\boldsymbol{\phi}(t; d); \tau) \geq \mathcal{H}(\boldsymbol{\phi}(t; d); t) > 0 \qquad \text{for every } t \leq \min\{\tau, T(d)\},$$

where the equality occurs only when $K(t) = K(\tau)$ holds.

Lemma 20. *Assume $(\mathbf{H}_\uparrow)$ and $\mathcal{K}(r) > \mathcal{K}(0)$ for every $r > 0$. Then, the trajectory $\boldsymbol{\phi}(t; d)$ of system (14) lies in the exterior of the region enclosed by $\mathbf{\Gamma}(K(\tau))$ for every $t \leq \min\{\tau, T(d)\}$.*

4. Construction of the unstable set

In the whole section we assume the existence of the limit $\lim_{\tau\to-\infty} K(\tau) = A_0 > 0$, which is satisfied if $(\mathbf{H}_\ell)$ holds. We fix an arbitrary continuous increasing function $\sigma : \mathbb{R} \to]0, \frac{1}{2}]$ satisfying $\lim_{\mathcal{T}\to-\infty} \sigma(\mathcal{T}) = 0$. Fix $\mathcal{T} \in \mathbb{R}$ and set

$$\begin{aligned} \underline{K}(\mathcal{T}) &:= (1 - \sigma(\mathcal{T})) \inf\{K(t) \mid t \leq \mathcal{T}\}, \\ \overline{K}(\mathcal{T}) &:= (1 + \sigma(\mathcal{T})) \sup\{K(t) \mid t \leq \mathcal{T}\}; \end{aligned} \tag{26}$$

hence $\underline{K}(\mathcal{T}) < K(t) < \overline{K}(\mathcal{T})$ for any $t \le \mathcal{T}$ and

$$\lim_{\tau\to-\infty} \underline{K}(\tau) = \lim_{\tau\to-\infty} \overline{K}(\tau) = \lim_{\tau\to-\infty} K(\tau) = A_0\,.$$

Our construction of the unstable set is based on a comparison between the non-autonomous system (14), and the frozen autonomous systems (18) with $\varpi \equiv \underline{K}(\mathcal{T})$ and $\varpi \equiv \overline{K}(\mathcal{T})$. Hence, in this section, we underline all the sets and quantities of (18) with $\varpi \equiv \underline{K}(\mathcal{T})$, and we overline all the sets and quantities of (18) with $\varpi \equiv \overline{K}(\mathcal{T})$, bracketing the time dependence $\mathcal{T}$. E.g., we denote by $\underline{\boldsymbol{E}}(\mathcal{T})$ and $\overline{\boldsymbol{E}}(\mathcal{T})$ the critical points of (18) where $\varpi \equiv \underline{K}(\mathcal{T})$ and $\varpi \equiv \overline{K}(\mathcal{T})$, respectively.

By definition, $\underline{K}(\cdot)$ is decreasing and $\overline{K}(\cdot)$ is increasing, and, from Remark 18, the homoclinic orbit $\overline{\boldsymbol{\Gamma}}(\mathcal{T}_2)$ belongs to the region enclosed by $\overline{\boldsymbol{\Gamma}}(\mathcal{T}_1)$, whenever $\mathcal{T}_2 \ge \mathcal{T}_1$. Moreover, $\overline{\boldsymbol{\Gamma}}(\mathcal{T})$ belongs to the region enclosed by $\underline{\boldsymbol{\Gamma}}(\mathcal{T})$ for every $\mathcal{T} \in \mathbb{R}$.

Set $x_o = \overline{E}_x(0)$, fix $\mathcal{T} \le 0$, and focus on the half-line $x = x_o$ contained in $\dot{x} > 0$, see Figure 3. The half-line intersects transversely the path $\overline{\boldsymbol{\Gamma}}(\mathcal{T})$ in a point, say $\overline{\boldsymbol{P}}(\mathcal{T})$, and then the path $\underline{\boldsymbol{\Gamma}}(\mathcal{T})$ in another point, say $\underline{\boldsymbol{P}}(\mathcal{T})$. Notice that $\underline{\boldsymbol{P}}(\mathcal{T})$ lies above $\overline{\boldsymbol{P}}(\mathcal{T})$.

Definition 21. We denote by $\partial\mathcal{F}^{\downarrow}(\mathcal{T})$ the branch of $\overline{\boldsymbol{\Gamma}}(\mathcal{T})$ between the origin and $\overline{\boldsymbol{P}}(\mathcal{T})$, by $\partial\mathcal{F}^{\uparrow}(\mathcal{T})$ the branch of $\underline{\boldsymbol{\Gamma}}(\mathcal{T})$ between the origin and $\underline{\boldsymbol{P}}(\mathcal{T})$, by $\mathfrak{f}(\mathcal{T})$ the vertical segment between $\overline{\boldsymbol{P}}(\mathcal{T})$ and $\underline{\boldsymbol{P}}(\mathcal{T})$. Then, we denote by $\mathcal{F}(\mathcal{T})$ the compact set enclosed by $\partial\mathcal{F}^{\downarrow}(\mathcal{T})$, $\partial\mathcal{F}^{\uparrow}(\mathcal{T})$ and by $\mathfrak{f}(\mathcal{T})$, see Figure 3.

Let us now set

$$\begin{aligned} \overline{\mathcal{H}}_{\mathcal{T}}(x,y) &:= \alpha x y + \tfrac{p-1}{p}|y|^{\frac{p}{p-1}} + \overline{K}(\mathcal{T})\tfrac{|x|^q}{q}\,, \\ \underline{\mathcal{H}}_{\mathcal{T}}(x,y) &:= \alpha x y + \tfrac{p-1}{p}|y|^{\frac{p}{p-1}} + \underline{K}(\mathcal{T})\tfrac{|x|^q}{q}\,. \end{aligned} \tag{27}$$

We emphasize that if $\boldsymbol{\phi}(t) = (x(t), y(t))$ is a trajectory of (14), we find

$$\begin{aligned} \tfrac{d}{dt}\overline{\mathcal{H}}_{\mathcal{T}}(\boldsymbol{\phi}(t)) &= \big(\overline{K}(\mathcal{T}) - K(t)\big)\,|x(t)|^{q-2}\,x(t)\;\dot{x}(t)\,, \\ \tfrac{d}{dt}\underline{\mathcal{H}}_{\mathcal{T}}(\boldsymbol{\phi}(t)) &= \big(\underline{K}(\mathcal{T}) - K(t)\big)\,|x(t)|^{q-2}\,x(t)\;\dot{x}(t)\,. \end{aligned} \tag{28}$$

So, we easily get the following.

Lemma 22. *Fix $\mathcal{T} \le 0$. If $\boldsymbol{Q} \in [(\partial\mathcal{F}^{\uparrow}(\mathcal{T}) \cup \partial\mathcal{F}^{\downarrow}(\mathcal{T})) \setminus \{(0,0)\}]$ and $T \le \mathcal{T}$, then there is $\mu > 0$ such that the trajectory $\boldsymbol{\phi}(t; T, \boldsymbol{Q}) \notin \mathcal{F}(\mathcal{T})$ for any $t \in]T-\mu, T[$. Moreover, the flow on $\mathfrak{f}(\tau)$ is transversal and it aims towards the exterior of $\mathcal{F}(\tau)$.*

Proof. By construction, if $\boldsymbol{P}$ is in the interior of $\mathcal{F}(\mathcal{T})$, then $\overline{\mathcal{H}}_{\mathcal{T}}(\boldsymbol{P}) > 0 > \underline{\mathcal{H}}_{\mathcal{T}}(\boldsymbol{P})$; further $\dot{x}(t; T, \boldsymbol{Q}) > 0$ as long as $\boldsymbol{\phi}(t; T, \boldsymbol{Q})$ is in $\mathcal{F}(\mathcal{T}) \setminus \{(0,0)\}$. Hence, the lemma easily follows from (28) and an analysis of the phase portrait, cf. Figure 3. $\square$

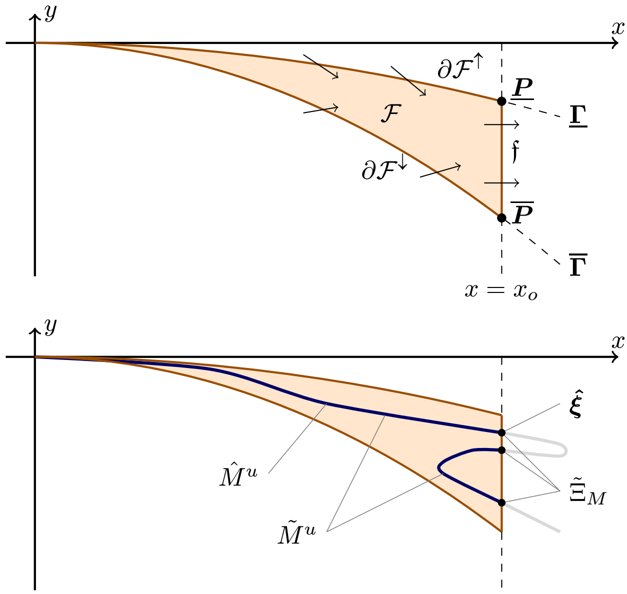

FIGURE 3. The construction of the sets $\tilde{M}^u(\mathcal{T}) \subset \mathcal{F}(\mathcal{T})$. The arrows suggest the direction of the vector fields of the dynamical system (14) along the boundary of $\mathcal{F}(\mathcal{T})$. We can draw $\tilde{M}^u(\mathcal{T})$ and its connected component $\hat{M}^u(\mathcal{T})$

Lemma 23. *Fix $\mathcal{T} \le 0$. Let $\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q})$ be such that $\lim_{t\to-\infty}\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q}) = (0,0)$ and $x(t;\mathcal{T},\boldsymbol{Q}) > 0$ for $t \ll 0$, then there is $S_0 = S_0(\mathcal{T},\boldsymbol{Q})$, $S_0 \le \mathcal{T}$ such that $\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q}) \in \mathcal{F}(S_0) \subseteq \mathcal{F}(\mathcal{T})$ for any $t \le S_0$. Further we can choose S_0 as*

$$S_0 := \max\{s \in]-\infty,\mathcal{T}] :\ 0 < x(t;\mathcal{T},\boldsymbol{Q}) \le x_o \quad \forall\, t \in]-\infty, s]\}. \tag{29}$$

Proof. Let $\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q})$ be as in the statement, then there is $S \le \mathcal{T}$ such that $0 < x(t;\mathcal{T},\boldsymbol{Q}) \le x_o$ for any $t \in]-\infty, S]$: we claim that

$$\dot{x}(t;\mathcal{T},\boldsymbol{Q}) > 0 \quad \text{when } t \le S, \tag{30}$$

possibly choosing a smaller S. Assume by contradiction that there is a sequence $t_k \to -\infty$ such that $\dot{x}(t_k;\mathcal{T},\boldsymbol{Q}) \le 0$: possibly redefining the sequence, we can assume that $\dot{x}(t_k;\mathcal{T},\boldsymbol{Q}) = 0$ for any k and that $x(t_k;\mathcal{T},\boldsymbol{Q})$ is a local maximum for k even, and a minimum for k odd. Then,

$$\ddot{x}(t_{2j};\mathcal{T},\boldsymbol{Q}) = \frac{|y(t_{2j};\mathcal{T},\boldsymbol{Q})|^{(2-p)/(p-1)}}{p-1}\dot{y}(t_{2j};\mathcal{T},\boldsymbol{Q}) \le 0 \implies \dot{y}(t_{2j};\mathcal{T},\boldsymbol{Q}) \le 0.$$

However, since at $t = t_{2j}$ we have $\dot{x} = 0$, $x > 0$, and $\dot{y} \le 0$, recalling (20) we deduce that $x(t_{2j};\mathcal{T},\boldsymbol{Q}) \ge E_x(K(t_{2j}))$. Hence, for j large, we have $x(t_{2j};\mathcal{T},\boldsymbol{Q}) > E_x(A_0)/2$, which contradicts $x(t_{2j};\mathcal{T},\boldsymbol{Q}) \to 0$ and proves the claim. Then, using (30) and (28), and recalling that both $\overline{\mathcal{H}}_S(\boldsymbol{\phi}(t))$ and $\underline{\mathcal{H}}_S(\boldsymbol{\phi}(t))$ converge to 0 as $t \to -\infty$, we conclude that $\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q}) \in \mathcal{F}(S)$ for any $t \le S$.

In particular, $\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q}) \in \mathcal{F}(\mathcal{T})$ for any $t \leq S$. Combining Lemma 22 with the fact that $\dot{x}(t;\mathcal{T},\boldsymbol{Q}) > 0$ as long as $\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q}) \in \mathcal{F}(\mathcal{T})$, from the argument of the first part of the proof we infer that $\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q}) \in \mathcal{F}(\mathcal{T})$ for any $t \leq S_0$, where S_0 is defined in (29). $\square$

Let us define the unstable set

$$\tilde{M}^u(\mathcal{T}) := \{\boldsymbol{Q} \mid \boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q}) \in \mathcal{F}(\mathcal{T}), \text{ for any } t \leq \mathcal{T}\} \subset \mathcal{F}(\mathcal{T}). \tag{31}$$

Note that $\lim_{t\to-\infty} \boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q}) = (0,0)$ for every $\boldsymbol{Q} \in \tilde{M}^u(\mathcal{T})$.

Now, we state some results (Lemma 24, Propositions 25, 27, and Lemma 32 below) whose proofs are rather technical in this non-smooth context, so they are developed in Section 4.1.

The following lemma states that the trajectories $\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q})$ with $\boldsymbol{Q} \in \tilde{M}^u(\mathcal{T})$ correspond to regular solutions of (5). The result is in fact known, see [11,15,16]; however, we will sketch its proof in Section 4.1 to be more self-contained.

Lemma 24. *Fix $\mathcal{T} \leq 0$, then $\tilde{M}^u(\mathcal{T})$ is compact. Let $u(r)$ be a solution of (5) and let $\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q})$ be the corresponding trajectory of (14).*

If $\boldsymbol{Q} \in \tilde{M}^u(\mathcal{T})$, then $u(r)$ is a regular solution, i.e. there is $d = d_M(\boldsymbol{Q},\mathcal{T}) \geq 0$ such that $\lim_{r\to 0} u(r) = d$. Further, $d = 0$ if and only if $\boldsymbol{Q} = (0,0)$. So, we can define the homeomorphism

$$d_M(\cdot,\mathcal{T}) : \tilde{M}^u(\mathcal{T}) \to I_M(\mathcal{T}),$$

where $I_M(\mathcal{T}) \subset \mathbb{R}$ is a suitable, possibly disconnected, compact set, whose minimum is 0 and whose positive maximum is denoted by $\tilde{D}_M(\mathcal{T})$. Conversely, if $u(r)$ is a regular solution, there is $\mathcal{T}$ ($\mathcal{T} \ll 0$) such that $\boldsymbol{Q} \in \tilde{M}^u(\mathcal{T})$.

The next step is to show that $\tilde{M}^u(\mathcal{T})$ contains a connected set, denoted by $\hat{M}^u(\mathcal{T})$, intersecting $\mathfrak{f}(\mathcal{T})$ and containing the origin. We also get a C^1 parametrization of $\hat{M}^u(\mathcal{T})$. The proofs of the following two propositions are postponed to Section 4.1.

Proposition 25. *Fix $\mathcal{T} \leq 0$. Then, there is a compact connected set $\hat{M}^u(\mathcal{T}) \subset \tilde{M}^u(\mathcal{T})$ such that $(0,0) \in \hat{M}^u(\mathcal{T})$ and $\hat{M}^u(\mathcal{T}) \cap \mathfrak{f}(\mathcal{T}) = \{\hat{\boldsymbol{\xi}}(\mathcal{T})\}$ is a singleton.*

Further, there is $\hat{D}_M(\mathcal{T}) \leq \tilde{D}_M(\mathcal{T})$ such that the restriction

$$d_M(\cdot,\mathcal{T}) : \hat{M}^u(\mathcal{T}) \to [0,\hat{D}_M(\mathcal{T})]$$

is a homeomorphism.

Remark 26. The functions $\hat{D}_M(\cdot)$ and $\tilde{D}_M(\cdot)$ may be discontinuous, so their ranges, say $\hat{J}$ and $\tilde{J}$, may be disconnected.

Proposition 27. *The functions $\hat{D}_M(\mathcal{T})$ and $\tilde{D}_M(\mathcal{T})$ are both strictly decreasing and $\tilde{D}_M(\mathcal{T}) \geq \hat{D}_M(\mathcal{T}) \to +\infty$ as $\mathcal{T} \to -\infty$. Hence, their respective inverses, $\hat{\tau}_M : \hat{J} \to]-\infty,0]$ and $\tilde{\tau}_M : \tilde{J} \to]-\infty,0]$, satisfy*

$$\lim_{d\to+\infty} \hat{\tau}_M(d) = -\infty, \qquad \lim_{d\to+\infty} \tilde{\tau}_M(d) = -\infty. \tag{32}$$

Remark 28. Let $\boldsymbol{\phi}(t;d)$ be the trajectory of (14) corresponding to the solution $u(r;d)$ of (5) and let $T(d)$ be as in Lemma 19. Then, using Lemma 3 we see that the function

$$\boldsymbol{\phi}(t;d) : \{(t,d) \mid d>0,\ \ t<T(d)\} \to \mathbb{R}^2$$

is C^1 and $\boldsymbol{\phi}(t;\cdot)$ is one to one in its domain.

We define

$$\tilde{\Xi}_M(\mathcal{T}) := \tilde{M}^u(\mathcal{T}) \cap \mathfrak{f}(\mathcal{T}), \tag{33}$$

and we observe that $\hat{\boldsymbol{\xi}}(\mathcal{T}) = \boldsymbol{\phi}(\mathcal{T};\hat{D}_M(\mathcal{T}))$ and $\tilde{\boldsymbol{\xi}}(\mathcal{T}) := \boldsymbol{\phi}(\mathcal{T};\tilde{D}_M(\mathcal{T}))$, cf. Figure 3.

Remark 29. In the smooth case analyzed in [12] we have $\hat{D}_M(\mathcal{T}) = \tilde{D}_M(\mathcal{T})$, so that $\hat{\boldsymbol{\xi}}(\mathcal{T}) = \tilde{\boldsymbol{\xi}}(\mathcal{T})$ when $\mathcal{T} \ll 0$; further, in the smooth case $\hat{\boldsymbol{\xi}}(\mathcal{T})$ is C^1. In the setting of this paper this is no longer ensured, so $\hat{\boldsymbol{\xi}}(\mathcal{T})$ and $\tilde{\boldsymbol{\xi}}(\mathcal{T})$ may be different and discontinuous.

Lemma 30. *Let $d \geq \tilde{D}_M(0)$, then there is a unique $\tau = \tau_M(d) \leq 0$ such that $\boldsymbol{\phi}(\tau;d) \in \tilde{\Xi}_M(\tau)$.*

Proof. Let $d > \tilde{D}_M(0)$, and consider the trajectory $\boldsymbol{\phi}(t;d)$ of (14) corresponding to the solution $u(r;d)$ of (5). From Lemma 23, we see that there is $S \leq 0$ such that $\boldsymbol{\phi}(t;d) \in \mathcal{F}(S) \subseteq \mathcal{F}(0)$ for any $t \leq S$.

To prove the existence of $\tau_M(d)$ we argue by contradiction, and we assume that there is no $\tau \leq 0$ such that $\boldsymbol{\phi}(\tau;d) \in \tilde{\Xi}_M(\tau)$. Then, $\boldsymbol{\phi}(t;d)$ does not intersect the half-line $x = x_o$ for any $t \leq 0$. Hence, $\boldsymbol{\phi}(t;d) \in \mathcal{F}(0)$ for any $t \leq 0$ or, equivalently, $\boldsymbol{\phi}(0;d) \in \tilde{M}^u(0)$. So, $d \in I_M(0)$ and $d \leq \tilde{D}_M(0)$, a contradiction, and the existence of $\tau_M(d)$ is proved.

Now, we address the uniqueness. Firstly, we observe that, once fixed $d \geq \tilde{D}_M(0)$, the function $\boldsymbol{\phi}(t;d)$ is uniquely defined; then, from Lemma 22, we see that the time $t = \tau_M(d)$ at which $\boldsymbol{\phi}(t;d)$ crosses $\mathfrak{f}(\tau)$ in $\tilde{\Xi}_M(\tau)$ is unique. □

Lemma 31. *The function $\tau_M(d)$ of Lemma 30 satisfies $\lim_{d\to+\infty} \tau_M(d) = -\infty$.*

Proof. Let us consider a sequence $d_n \to +\infty$ with $d_n \geq \tilde{D}_M(0)$. By definition of $\tau_M(\cdot)$, we find that $\boldsymbol{\phi}(\tau_M(d_n);d_n) \in \tilde{\Xi}_M(\tau_M(d_n)) \subset \tilde{M}^u(\tau_M(d_n))$. Setting $\tilde{d}_n := \tilde{D}_M(\tau_M(d_n))$, we note that $d_n \leq \tilde{d}_n$ and $\tau_M(d_n) = \tilde{\tau}_M(\tilde{d}_n)$. Then, from Proposition 27, we find

$$\lim_{n\to+\infty} \tau_M(d_n) = \lim_{n\to+\infty} \tilde{\tau}_M(\tilde{d}_n) = -\infty,$$

and the lemma is proved. □

Let us recall that $\mathcal{K}(0) = \lim_{t\to-\infty} K(t) = A_0 > 0$. Denote by $\hat{\boldsymbol{\xi}}(-\infty)$ the transversal intersection between $\boldsymbol{\Gamma}(A_0)$ and the half-line $x = x_o$ contained in $\dot{x} > 0$.

Lemma 32. *According to the notation in* (15) *and* (19)*, we have*

$$\lim_{\tau\to-\infty}\sup_{\theta\le 0}\left(\left\|\boldsymbol{\phi}\big(\theta+\tau;\tau,\boldsymbol{\xi}\big)-\boldsymbol{\phi}_{\boldsymbol{A_0}}\big(\theta;0,\hat{\boldsymbol{\xi}}(-\infty)\big)\right\|\mathrm{e}^{-h\theta}\right)=0\,, \tag{34}$$

with $h=\alpha\min\{1,q-1\}$*, whenever* $\boldsymbol{\xi}\in\tilde{\Xi}_M(\tau)$*.*

The proof is postponed to the end of Section 4.1.

4.1. Proof of Lemmas 24, 32 and of Propositions 25, 27

The argument of this section is modeled on [15] and [16], where the unstable set $\hat{M}^u(\mathcal{T})$ is proved to be just a compact connected set joining the origin and $\mathfrak{f}(\mathcal{T})$. The novelty here consists in showing that $\hat{M}^u(\mathcal{T})$ is the image of a continuous curve parametrized by $d\in[0,\hat{D}_M(\mathcal{T})]$, where $d>0$ is the initial condition of the regular solution $u(r;d)$ of (5); further $\tilde{D}_M(\mathcal{T})\ge\hat{D}_M(\mathcal{T})>0$ are decreasing functions which diverge to infinity as $\mathcal{T}\to-\infty$.

Recalling the definition of $\tilde{M}^u(\mathcal{T})$ in (31), for every point $\boldsymbol{Q}\in\mathcal{F}(\mathcal{T})\setminus\tilde{M}^u(\mathcal{T})$ the trajectory $\boldsymbol{\phi}(\cdot;\mathcal{T},\boldsymbol{Q})$ must exit $\mathcal{F}(\mathcal{T})$ in the past. In particular, we can define a *backward exit point function* as follows.

Lemma 33. *For any* $\boldsymbol{Q}\in\mathcal{F}(\mathcal{T})\setminus\tilde{M}^u(\mathcal{T})$ *there is* $T=T(\boldsymbol{Q})\le\mathcal{T}$ *such that* $\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q})\in\mathcal{F}(\mathcal{T})$ *for every* $t\in[T(\boldsymbol{Q}),\mathcal{T}]$*, and* $\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q})\notin\mathcal{F}(\mathcal{T})$ *when* $t\in]T(\boldsymbol{Q})-\mu,T(\boldsymbol{Q})[$ *for a suitable* $\mu>0$*. Moreover, the function* $\psi:[\mathcal{F}(\mathcal{T})\setminus\tilde{M}^u(\mathcal{T})]\to\partial\mathcal{F}^{\uparrow}(\mathcal{T})\cup\partial\mathcal{F}^{\downarrow}(\mathcal{T})$ *defined by* $\psi(\boldsymbol{Q})=\boldsymbol{\phi}(T(\boldsymbol{Q});\mathcal{T},\boldsymbol{Q})$ *is continuous.*

Proof. The first assertion is an immediate consequence of Lemma 22.

We now show the continuity of the function $T(\cdot):[\mathcal{F}(\mathcal{T})\setminus\tilde{M}^u(\mathcal{T})]\to\,]-\infty,\mathcal{T}]$. Let $\boldsymbol{Q}\in\mathcal{F}(\mathcal{T})\setminus\tilde{M}^u(\mathcal{T})$, using the continuous dependence of the flow of (14) on initial data when it is restricted to $\mathcal{F}(\mathcal{T})\setminus\{(0,0)\}$, we deduce the existence of $\sigma=\sigma(\boldsymbol{Q},\mu)>0$ such that $\boldsymbol{\phi}(T(\boldsymbol{Q})-\mu/2;\mathcal{T},\boldsymbol{P})\notin\mathcal{F}(\mathcal{T})$ for any $\boldsymbol{P}\in\mathcal{F}(\mathcal{T})$ with $|\boldsymbol{P}-\boldsymbol{Q}|<\sigma$; so $\boldsymbol{P}\notin\tilde{M}^u(\mathcal{T})$. Hence, there is $T(\boldsymbol{P})\in\,]T(\boldsymbol{Q})-\mu/2,\mathcal{T}]$ such that $\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{P})\in\mathcal{F}(\mathcal{T})$ when $t\in[T(\boldsymbol{P}),\mathcal{T}]$ and it crosses transversely its border at $t=T(\boldsymbol{P})$. Then, the continuity of $T(\boldsymbol{P})$ follows from the transversality of the crossing and continuous dependence on initial data. The continuity of ψ is an immediate consequence. $\square$

Remark 34. Observe that if $\boldsymbol{Q}\in\tilde{M}^u(\mathcal{T})$, then there is no $t_0<\mathcal{T}$ such that $\boldsymbol{\phi}(t_0;\mathcal{T},\boldsymbol{Q})=(0,0)$. This is obvious if (14) is C^1, but it holds in a non-smooth context too: in fact assume, by contradiction, that such a t_0 exists and let $u(r)$ be the solution of (5) corresponding to $\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q})$, then $u(\mathrm{e}^{t_0})=0$ and $u'(r)<0<u(r)$ in a right neighborhood of $r=\mathrm{e}^{t_0}$, a contradiction.
Further, $\psi(\boldsymbol{Q})\neq(0,0)$, whenever $\boldsymbol{Q}\in\mathcal{F}(\mathcal{T})\setminus\tilde{M}^u(\mathcal{T})$.

Now, we can continuously extend ψ to the whole set $\mathcal{F}(\mathcal{T})$.

Lemma 35. *The function* $\tilde{\psi}:\mathcal{F}(\mathcal{T})\to\partial\mathcal{F}^{\uparrow}(\mathcal{T})\cup\partial\mathcal{F}^{\downarrow}(\mathcal{T})$ *defined as*

$$\tilde{\psi}(\boldsymbol{Q})=\begin{cases}\psi(\boldsymbol{Q})\ \text{if}\ \boldsymbol{Q}\notin\tilde{M}^u(\mathcal{T})\,,\\(0,0)\ \ \text{if}\ \boldsymbol{Q}\in\tilde{M}^u(\mathcal{T})\end{cases}$$

is continuous.

Proof. Consider $\boldsymbol{Q_n} \notin \tilde{M}^u(\mathcal{T})$ and $\boldsymbol{Q} \in \tilde{M}^u(\mathcal{T})$ with $\boldsymbol{Q_n} \to \boldsymbol{Q}$. We claim that $T(\boldsymbol{Q_n}) \to -\infty$. In fact, according to Lemma 22 and Remark 34, $\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q}) \in \mathcal{F}(\mathcal{T}) \setminus [\partial\mathcal{F}^{\uparrow}(\mathcal{T}) \cup \partial\mathcal{F}^{\downarrow}(\mathcal{T})]$ for any $t \le \mathcal{T}$, hence the claim easily follows by a contradiction argument combined with the continuous dependence of (14) on initial data.

For any $\varepsilon > 0$, we can find $\mu \in]0,\varepsilon[$ such that

$$\boldsymbol{P} \in \{(x,y) \in \mathcal{F}(\mathcal{T}) \mid 0 \le x \le \mu\} \quad \Rightarrow \quad |\boldsymbol{P}| \le \varepsilon\,. \tag{35}$$

Once fixed such a μ, recalling that $\lim_{t\to-\infty}\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q}) = (0,0)$, we can choose $M > 0$ so that $0 < x(t;\mathcal{T},\boldsymbol{Q}) < \mu/2$, whenever $t \le -M$. Moreover, we can choose $N \in \mathbb{N}$ with the following property: if $n \ge N$, then $T(\boldsymbol{Q_n}) < -M$ and, again by continuous dependence on initial data, $|x(-M;\mathcal{T},\boldsymbol{Q_n}) - x(-M;\mathcal{T},\boldsymbol{Q})| < \mu/2$. Hence, we get $|x(-M;\mathcal{T},\boldsymbol{Q_n})| < \mu$. Since $\dot{x} > 0$ in $\mathcal{F}(\mathcal{T}) \setminus \{(0,0)\}$, we see that $\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q_n}) \in \{(x,y) \in \mathcal{F}(\mathcal{T}) \mid 0 \le x \le \mu\}$ for any $t \in [T(\boldsymbol{Q_n}),-M]$. Thus, from (35), $|\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q_n})| \le \varepsilon$ for any $t \in [T(\boldsymbol{Q_n}),-M]$. In particular, $|\psi(\boldsymbol{Q_n})| = |\boldsymbol{\phi}(T(\boldsymbol{Q_n});\mathcal{T},\boldsymbol{Q_n})| \le \varepsilon$, so for the arbitrariness of ε, the lemma is proved. □

Let us define the counterimages

$$\Omega^{\uparrow}(\mathcal{T}) = \tilde{\psi}^{-1}[\partial\mathcal{F}^{\uparrow}(\mathcal{T}) \setminus \{(0,0)\}] \quad \text{and} \quad \Omega^{\downarrow}(\mathcal{T}) = \tilde{\psi}^{-1}[\partial\mathcal{F}^{\downarrow}(\mathcal{T}) \setminus \{(0,0)\}]\,.$$

Notice that $\tilde{M}^u(\mathcal{T}) = \tilde{\psi}^{-1}[\{(0,0)\}]$, by Remark 34. From the continuity of $\tilde{\psi}$, we have the following.

Remark 36. The sets $\Omega^{\uparrow}(\mathcal{T})$ and $\Omega^{\downarrow}(\mathcal{T})$ are relatively open in $\mathcal{F}(\mathcal{T})$, while $\tilde{M}^u(\mathcal{T})$ is a non-empty, compact set, for any $\mathcal{T} \in \mathbb{R}$.

Now, we need the following topological Lemma, borrowed from [30, Lemma 4], see also [15, Lemma 3.2] or [16, Lemma 3.3].

Lemma 37. *Let $\boldsymbol{\mathcal{R}}$ be a closed set homeomorphic to a full triangle. We call $\boldsymbol{O}$, $\boldsymbol{A}$ and $\boldsymbol{B}$ the vertices and $\boldsymbol{o}$, $\boldsymbol{a}$, $\boldsymbol{b}$ the edges which are opposite to the respective vertex. Let $\boldsymbol{\mathcal{S}} \subset \boldsymbol{\mathcal{R}}$ be a closed set such that $\boldsymbol{\gamma} \cap \boldsymbol{\mathcal{S}} \neq \emptyset$, for any path $\boldsymbol{\gamma} \subset \boldsymbol{\mathcal{R}}$ joining $\boldsymbol{a} \setminus \{\boldsymbol{O}\}$ with $\boldsymbol{b} \setminus \{\boldsymbol{O}\}$. Then, $\boldsymbol{\mathcal{S}}$ contains a closed connected set which contains $\boldsymbol{O}$ and at least one point of $\boldsymbol{o}$.*

Now, we are ready to show the existence of an unstable connected set $\hat{M}^u(\mathcal{T})$.

Proposition 38. *The compact set $\tilde{M}^u(\mathcal{T})$ contains a compact connected set, denoted by $\hat{M}^u(\mathcal{T})$, containing the origin and intersecting $\mathfrak{f}(\mathcal{T})$.*

Proof. We want to apply Lemma 37 where $\boldsymbol{\mathcal{R}} = \mathcal{F}(\mathcal{T})$, $\boldsymbol{o} = \mathfrak{f}(\mathcal{T})$, $\boldsymbol{a} = \partial\mathcal{F}^{\downarrow}(\mathcal{T})$, $\boldsymbol{b} = \partial\mathcal{F}^{\uparrow}(\mathcal{T})$ and $\boldsymbol{\mathcal{S}} = \tilde{M}^u(\mathcal{T})$. We already know that $\tilde{M}^u(\mathcal{T})$ is closed. So, let γ be a generic continuous path such that $\gamma : [0,1] \to \mathcal{F}(\mathcal{T})$ satisfies $\gamma(0) \in \partial\mathcal{F}^{\downarrow}(\mathcal{T}) \setminus \{(0,0)\}$ and $\gamma(1) \in \partial\mathcal{F}^{\uparrow}(\mathcal{T}) \setminus \{(0,0)\}$. In order to apply Lemma 37, we claim that there is $\bar{s} \in]0,1[$ such that $\gamma(\bar{s}) \in \tilde{M}^u(\mathcal{T})$.

Now, we prove the claim; so, let us define

$$A_\gamma := \{s \in [0,1] \mid \gamma(s) \in \Omega^{\downarrow}(\mathcal{T})\}\,,$$
$$B_\gamma := \{s \in [0,1] \mid \gamma(s) \in \Omega^{\uparrow}(\mathcal{T})\}.$$

The sets A_γ and B_γ are relatively open in $[0,1]$, since γ is continuous and $\Omega^{\downarrow}(\mathcal{T})$ and $\Omega^{\uparrow}(\mathcal{T})$ are relatively open in $\mathcal{F}(\mathcal{T})$. Further, according to Lemma 22, $0 \in A_\gamma$ and $1 \in B_\gamma$. Since $[0,1]$ is connected, we deduce the existence of $\bar{s} \in]0,1[$ with $\bar{s} \notin (A_\gamma \cup B_\gamma)$. Therefore, $\gamma(\bar{s}) \in \tilde{M}^u(\mathcal{T})$, and the claim is proved. Then, Proposition 38 follows from Lemma 37. □

Proof of Lemma 24. In this proof we will use some notation introduced at the beginning of Section 4. Fix $\mathcal{T} \le 0$ and $\boldsymbol{Q} = (Q_x, Q_y) \in \tilde{M}^u(\mathcal{T}) \setminus \{(0,0)\}$. Let us denote by $\boldsymbol{Q}^{\uparrow}$ and $\boldsymbol{Q}^{\downarrow}$ the intersections of the line $x = Q_x$ with $\partial\mathcal{F}^{\uparrow}(\mathcal{T})$ and $\partial\mathcal{F}^{\downarrow}(\mathcal{T})$, respectively, see Figure 4.

Then, consider the trajectories $\underline{\boldsymbol{\phi}}_{\boldsymbol{\mathcal{T}}}(t;\mathcal{T},\boldsymbol{Q}^{\uparrow})$ and $\overline{\boldsymbol{\phi}}_{\boldsymbol{\mathcal{T}}}(t;\mathcal{T},\boldsymbol{Q}^{\downarrow})$ of the autonomous system (18) frozen at $\varpi \equiv \underline{K}(\mathcal{T})$ and at $\varpi \equiv \overline{K}(\mathcal{T})$, respectively. We set for short

$$\begin{aligned} \boldsymbol{\phi}^{\uparrow}(t) &= (x^{\uparrow}(t), y^{\uparrow}(t)) = \underline{\boldsymbol{\phi}}_{\boldsymbol{\mathcal{T}}}(t;\mathcal{T},\boldsymbol{Q}^{\uparrow}) \quad \text{and} \\ \boldsymbol{\phi}^{\downarrow}(t) &= (x^{\downarrow}(t), y^{\downarrow}(t)) = \overline{\boldsymbol{\phi}}_{\boldsymbol{\mathcal{T}}}(t;\mathcal{T},\boldsymbol{Q}^{\downarrow})\,. \end{aligned} \tag{36}$$

Moreover, let $u^{\uparrow}(r)$ and $u^{\downarrow}(r)$ be the corresponding solutions of the frozen equation (5) with $\mathcal{K}(\cdot) \equiv \underline{K}(\mathcal{T})$ and $\mathcal{K}(\cdot) \equiv \overline{K}(\mathcal{T})$, respectively. Let $u(r)$ be the solution of the original equation (5) corresponding to the trajectory $\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q})$ of the non-autonomous system.

We first claim that

$$x^{\uparrow}(t) < x(t;\mathcal{T},\boldsymbol{Q}) < x^{\downarrow}(t)\,, \quad \text{for any } t < \mathcal{T}\,. \tag{37}$$

In fact, by construction, see (14), we have $\dot{x}^{\downarrow}(t) < \dot{x}(t;\mathcal{T},\boldsymbol{Q}) < \dot{x}^{\uparrow}(t)$ at $t = \mathcal{T}$. So, let us define

$$T = \inf\{\tau \in \mathbb{R} \mid x^{\uparrow}(t) < x(t;\mathcal{T},\boldsymbol{Q}) < x^{\downarrow}(t)\,, \text{ for any } t \in]\tau,\mathcal{T}[\,\}.$$

Thus, T is well defined and $T < \mathcal{T}$. Assume, by contradiction, that $T \in \mathbb{R}$, then either $x^{\uparrow}(T) = x(T;\mathcal{T},\boldsymbol{Q})$ or $x(T;\mathcal{T},\boldsymbol{Q}) = x^{\downarrow}(T)$; assume the former, the latter being analogous. By construction, cf. Figure 4, we find $y^{\uparrow}(T) > y(T;\mathcal{T},\boldsymbol{Q})$; hence $\dot{x}^{\uparrow}(T) > \dot{x}(T;\mathcal{T},\boldsymbol{Q})$ which gives $x^{\uparrow}(t) > x(t;\mathcal{T},\boldsymbol{Q})$ in a right neighborhood of T, but this is a contradiction with the definition of T, so the claim in (37) is proved.

From Remark 17, we know that there are $d^{\uparrow}_{Q_x}(\mathcal{T})$, $d^{\downarrow}_{Q_x}(\mathcal{T}) > 0$ such that $u^{\uparrow}(0) = d^{\uparrow}_{Q_x}(\mathcal{T})$ and $u^{\downarrow}(0) = d^{\downarrow}_{Q_x}(\mathcal{T})$. So, from (37), we see that $0 < d^{\uparrow}_{Q_x}(\mathcal{T}) \le d^{\downarrow}_{Q_x}(\mathcal{T})$. Further, again from (37), we see that $u^{\uparrow}(r) < u(r) < u^{\downarrow}(r)$ for any $0 < r < \mathrm{e}^{\mathcal{T}}$. According to (13), note that $u'(r) < 0$ when $0 < r < \mathrm{e}^{\mathcal{T}}$, so the limit $\lim_{r\to 0} u(r) = d$ exists and

$$d^{\uparrow}_{Q_x}(\mathcal{T}) \le d \le d^{\downarrow}_{Q_x}(\mathcal{T})\,. \tag{38}$$

So, we have defined a one to one function $d_M(\cdot,\mathcal{T}) : \tilde{M}^u(\mathcal{T}) \to [0,+\infty[$.

Notice that $d_M(\cdot,\mathcal{T})$ is the inverse of the restriction to $I_M(\mathcal{T}) := d_M(\tilde{M}^u(\mathcal{T}),\mathcal{T})$ of the continuous function $\boldsymbol{\phi}(\mathcal{T};\cdot)$ defined in Remark 28. Observe that $\tilde{M}^u(\mathcal{T})$ is closed, so $I_M(\mathcal{T}) = [\boldsymbol{\phi}(\mathcal{T};\cdot)]^{-1}[\tilde{M}^u(\mathcal{T})]$ is closed too. Further, if $d \in I_M(\mathcal{T})$, then $d \le d^{\downarrow}_{x_o}(\mathcal{T})$, so $I_M(\mathcal{T})$ is compact. Hence,

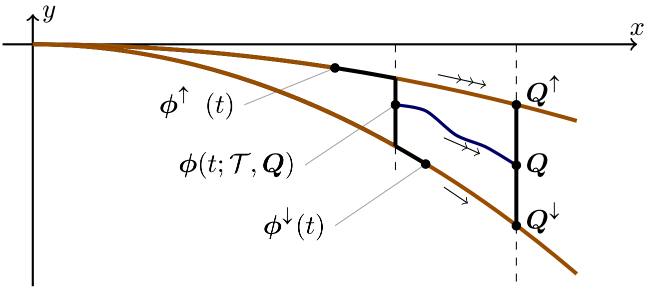

Figure 4. A sketch of the first part of the proof of Lemma 24. The trajectories $\boldsymbol{\phi}^{\uparrow}, \boldsymbol{\phi}(\cdot;\mathcal{T},\boldsymbol{Q}), \boldsymbol{\phi}^{\downarrow}$, cf. (36), have, in the order, increasing horizontal speed, as suggested by the arrows. Hence, going backward in time, we expect to have the validity of (37)

$d_M(\cdot,\mathcal{T}) : \tilde{M}^u(\mathcal{T}) \to I_M(\mathcal{T})$ is continuous, since it is the inverse of a continuous function defined on a compact set.

Conversely, assume that $u(r)$ is a regular solution, then from Remark 4 we see that $u(r)$ is decreasing for $r > 0$ small. Thus, the corresponding trajectory $\boldsymbol{\phi}(t) = (x(t), y(t))$ lies in the 4^{th} quadrant when $t \ll 0$, satisfies $\lim_{t\to-\infty}\boldsymbol{\phi}(t) = (0,0)$ and $\dot{x}(t) > 0$ when $t \ll 0$; so from Lemma 23 we deduce the existence of $\mathcal{T} \le 0$ such that $\boldsymbol{\phi}(t) \in \mathcal{F}(\mathcal{T})$ for any $t \le \mathcal{T}$; hence $\boldsymbol{\phi}(\mathcal{T}) = \boldsymbol{Q} \in \tilde{M}^u(\mathcal{T})$. $\square$

Using the rescaling in Remark 15, it is possible to find a relation between the trajectories introduced in (36).

Lemma 39. *Let $\mathcal{T} \le 0$, $\boldsymbol{Q} = (Q_x, Q_y) \in \tilde{M}^u(\mathcal{T})$ and let $\phi^{\uparrow}(t)$, $\phi^{\downarrow}(t)$ be as in (36), then there is $T_0^{Q_x}(\mathcal{T}) > 0$ such that*

$$\begin{aligned} x^{\uparrow}(t) < x^{\downarrow}(t) &= [\underline{K}(\mathcal{T})/\overline{K}(\mathcal{T})]^{\frac{1}{q-p}}\, x^{\uparrow}(t + T_0^{Q_x}(\mathcal{T}))\,, \\ |y^{\uparrow}(t)| < |y^{\downarrow}(t)| &= [\underline{K}(\mathcal{T})/\overline{K}(\mathcal{T})]^{\frac{p-1}{q-p}}\, |y^{\uparrow}(t + T_0^{Q_x}(\mathcal{T}))| \end{aligned} \tag{39}$$

for any $t < \mathcal{T}$. Further, $T_0^{Q_x}(\mathcal{T}) \to 0$ as $\mathcal{T} \to -\infty$ uniformly when $Q_x \in [0, x_o]$.

Proof. The inequalities in (39) follow from (37). From Remark 15, we see that

$$\boldsymbol{\phi}^{\star}(t) = (x^{\star}(t), y^{\star}(t)) := \left([\overline{K}(\mathcal{T})/\underline{K}(\mathcal{T})]^{\frac{1}{q-p}}\, x^{\downarrow}(t), [\overline{K}(\mathcal{T})/\underline{K}(\mathcal{T})]^{\frac{p-1}{q-p}}\, y^{\downarrow}(t)\right)$$

is a solution of (18) where $\varpi = \underline{K}(\mathcal{T})$. In particular, $x^{\star}(\mathcal{T}) = [\overline{K}(\mathcal{T})/\underline{K}(\mathcal{T})]^{\frac{1}{q-p}} Q_x > Q_x = x^{\uparrow}(\mathcal{T})$. Then, from the invariance of (18) for t-translations, we easily get the existence of $T_0^{Q_x}(\mathcal{T}) > 0$ such that $\boldsymbol{\phi}^{\star}(t) \equiv \boldsymbol{\phi}^{\uparrow}(t + T_0^{Q_x}(\mathcal{T}))$ for any $t \in \mathbb{R}$. Then, we immediately find (39).

Finally, the fact that $T_0^{Q_x}(\mathcal{T}) \to 0$ as $\mathcal{T} \to -\infty$ easily follows from (23). $\square$

From the previous argument, we also find the following result.

Remark 40. Fix $\mathcal{T} \leq 0$ and $\boldsymbol{Q} = (Q_x, Q_y) \in \tilde{M}^u(\mathcal{T}) \setminus \{(0,0)\}$. Using the notation of the proof of Lemma 24, cf. (38), we see that

$$d^{\uparrow}_{Q_x}(\mathcal{T}) \leq d_M(\boldsymbol{Q}, \mathcal{T}) \leq d^{\downarrow}_{Q_x}(\mathcal{T}) \tag{40}$$

and, as a consequence of Lemma 39, $d^{\uparrow}_{Q_x}(\mathcal{T}) - d^{\downarrow}_{Q_x}(\mathcal{T}) \to 0$ as $\mathcal{T} \to -\infty$, uniformly with respect to Q_x. In particular, setting $d^{\uparrow}_M(\mathcal{T}) = d^{\uparrow}_{x_o}(\mathcal{T})$ and $d^{\downarrow}_M(\mathcal{T}) = d^{\downarrow}_{x_o}(\mathcal{T})$, for any $\boldsymbol{Q} \in \tilde{\Xi}_M(\mathcal{T})$, we find

$$\begin{aligned} &d^{\uparrow}_M(\mathcal{T}) \leq \hat{D}_M(\mathcal{T}) \leq d_M(\boldsymbol{Q}, \mathcal{T}) \leq \tilde{D}_M(\mathcal{T}) \leq d^{\downarrow}_M(\mathcal{T}), \quad \text{for any } \mathcal{T} \leq 0, \\ &d^{\downarrow}_M(\mathcal{T}) \geq d_M(\boldsymbol{Q}, \mathcal{T}) \geq d^{\uparrow}_M(\mathcal{T}) \to +\infty \qquad \text{as } \mathcal{T} \to -\infty. \end{aligned} \tag{41}$$

Proof of Proposition 25. The existence of the set $\hat{M}^u(\mathcal{T})$ is proved in Proposition 38. Since $\hat{M}^u(\mathcal{T})$ is compact and connected, its image via $d_M(\cdot, \mathcal{T})$ is a compact interval, say $\hat{I}_M(\mathcal{T})$. Let

$$\hat{D}_M(\mathcal{T}) := \min\{d \in \hat{I}_M(\mathcal{T}) \mid d = d_M(\boldsymbol{Q}, \mathcal{T}) \in \mathfrak{f}(\mathcal{T})\},$$

so that $0 < \hat{D}_M(\mathcal{T}) \leq \tilde{D}_M(\mathcal{T})$. According to Lemma 24, it is not restrictive to redefine $\hat{M}^u(\mathcal{T}) := d_M^{-1}([0, \hat{D}_M(\mathcal{T})], \mathcal{T}) \subseteq d_M^{-1}(\hat{I}_M(\mathcal{T}), \mathcal{T})$. Notice that the revised definition may yield a smaller set than the original one. In this setting, $d_M(\cdot, \mathcal{T}) : \hat{M}^u(\mathcal{T}) \to [0, \hat{D}_M(\mathcal{T})]$ is a homeomorphism. Then, we denote by $\hat{\boldsymbol{\xi}}(\mathcal{T})$ the unique point in $\hat{M}^u(\mathcal{T}) \cap \mathfrak{f}(\mathcal{T})$ so that $d_M(\hat{\boldsymbol{\xi}}(\mathcal{T}), \mathcal{T}) = \hat{D}_M(\mathcal{T})$. $\square$

Proof of Proposition 27. First, we show that $\tilde{D}_M(\mathcal{T})$ and $\hat{D}_M(\mathcal{T})$ are both strictly decreasing. Observe first that we have the following characterization

$$\begin{aligned} \hat{D}_M(\mathcal{T}) &= \min\{d > 0 \mid \boldsymbol{\phi}(\mathcal{T}; d) \in \tilde{\Xi}_M(\mathcal{T})\}, \\ \tilde{D}_M(\mathcal{T}) &= \max\{d > 0 \mid \boldsymbol{\phi}(\mathcal{T}; d) \in \tilde{M}^u(\mathcal{T})\}. \end{aligned} \tag{42}$$

Since $\dot{x} > 0$ on $\mathfrak{f}(\mathcal{T})$, there is $\mu > 0$ (independent of $\mathcal{T}$) such that $x(t; \hat{D}_M(\mathcal{T})) = x(t; \mathcal{T}, \hat{\boldsymbol{\xi}}(\mathcal{T})) > x_o$ whenever $t \in]\mathcal{T}, \mathcal{T} + \mu[$, hence $\boldsymbol{R}(t) = \boldsymbol{\phi}(t; \hat{D}_M(\mathcal{T})) \notin \mathcal{F}(t)$ for any $t \in]\mathcal{T}, \mathcal{T}+\mu[$. So, $\boldsymbol{R}(t) \notin \tilde{M}^u(t)$ and, consequently, $\boldsymbol{R}(t) \notin \hat{M}^u(t)$ for any $t \in]\mathcal{T}, \mathcal{T} + \mu[$. It follows that $\hat{D}_M(\mathcal{T}) \notin [0, \hat{D}_M(t)]$ when $t \in]\mathcal{T}, \mathcal{T} + \mu[$, that is $\hat{D}_M(\mathcal{T}) > \hat{D}_M(t)$ whenever $t \in]\mathcal{T}, \mathcal{T} + \mu[$: so $\hat{D}_M(t)$ is strictly decreasing in t.

Now let $\tau_1 < \tau_2 \leq 0$ and fix any $d > \tilde{D}_M(\tau_1)$. From (42), we deduce $\boldsymbol{\phi}(\tau_1; d) \notin \tilde{M}^u(\tau_1)$ and so, recalling (31), there is $t_0 \leq \tau_1$ such that $\boldsymbol{\phi}(t_0; d) \notin \mathcal{F}(\tau_1)$.

By Lemmas 22, 23, we see that there is $S_0 = S_0(\tau_1, \boldsymbol{\phi}(\tau_1; d))$, $S_0 < t_0 \leq \tau_1$, such that $\boldsymbol{\phi}(t; d) \in \mathcal{F}(S_0)$ for any $t \leq S_0$ and it leaves $\mathcal{F}(S_0)$ at $t = S_0$ crossing transversely $x = x_o$. So, by (31), we get $\boldsymbol{\phi}(\tau_2; d) \notin \tilde{M}^u(\tau_2)$.

We have thus proved that $\boldsymbol{\phi}(\tau_2; d) \notin \tilde{M}^u(\tau_2)$ for every $d > \tilde{D}_M(\tau_1)$. Hence, we deduce from (42) that $\tilde{D}_M(\tau_2) \leq \tilde{D}_M(\tau_1)$ for every $\tau_1 < \tau_2$. Since the flow on $\mathfrak{f}(\tau_1)$ is transversal, we deduce that $\tilde{D}_M$ cannot be constant on any interval, so it is strictly monotone decreasing.

Now, since $\hat{D}_M(t)$ and $\tilde{D}_M(t)$ are strictly monotone, we can define their respective inverse functions $\hat{\tau}_M(d)$ and $\tilde{\tau}_M(d)$. Further, from (41) we know that $\tilde{D}_M(\mathcal{T}) \geq \hat{D}_M(\mathcal{T}) \to +\infty$ as $\mathcal{T} \to -\infty$, so (32) follows. $\square$

Remark 41. If $\frac{2n}{n+2+\delta} \leq p \leq 2$, i.e. if (14) is C^1, then $\hat{M}^u(\mathcal{T})$ is a branch of the unstable leaf, so it is a C^1 embedded manifold. Notice, further, that in the smooth case we can find a sufficiently large $N > 0$ such that $\tilde{M}^u(\mathcal{T})$ is connected and intersects $\mathfrak{f}(\mathcal{T})$ in a singleton, for every $\mathcal{T} < -N$, using [12, Remark 14]; hence $\tilde{M}^u(\mathcal{T}) = \hat{M}^u(\mathcal{T})$ when $\mathcal{T} < -N$.

However, for a generic $\mathcal{T} \leq 0$, even if the flow of (14) on $\mathfrak{f}(\mathcal{T})$ is always transversal, $\tilde{M}^u(\mathcal{T})$ may have tangencies on $\mathfrak{f}(\mathcal{T})$: so $\tilde{M}^u(\mathcal{T}) \cap \mathfrak{f}(\mathcal{T})$ may not be a singleton and $\tilde{M}^u(\mathcal{T})$ may be disconnected also in the smooth case.

We believe that using the techniques of [10, Appendix], or of [24] it might be possible to prove that $\hat{M}^u(\tau)$ is a C^1 manifold in the general non-smooth case $p > 1$, but this result is not needed for the present paper.

We are now ready to prove Lemma 32. We recall that $\lim_{t\to-\infty} K(t) = A_0 > 0$ and the notation $\hat{\boldsymbol{\xi}}(-\infty)$ for the transversal intersection between $\boldsymbol{\Gamma}(A_0)$ and the half-line $x = x_o$ contained in $\dot{x} > 0$.

Proof of Lemma 32. Let us keep the notation introduced in the proof of Lemma 24, cf. (36). Let $\boldsymbol{\xi} \in \tilde{\Xi}_M(\tau) = \tilde{M}^u(\tau) \cap \mathfrak{f}(\tau)$, then from (37) and Lemma 39 we see that there is $T_0^{x_o}(\tau) > 0$ such that

$$x^{\uparrow}(t) < x(t;\tau,\boldsymbol{\xi}) < x^{\downarrow}(t) < x^{\uparrow}(t + T_0^{x_o}(\tau)),$$

$$|y^{\uparrow}(t)| < |y(t;\tau,\boldsymbol{\xi})| < |y^{\downarrow}(t)| < |y^{\uparrow}(t + T_0^{x_o}(\tau))|,$$

for every $t < \tau$. Analogously, we also find that, for every $t < \tau$,

$$x^{\uparrow}(t) < x_{A_0}(t;\tau,\hat{\boldsymbol{\xi}}(-\infty)) < x^{\downarrow}(t) < x^{\uparrow}(t + T_0^{x_o}(\tau)),$$
$$|y^{\uparrow}(t)| < |y_{A_0}(t;\tau,\hat{\boldsymbol{\xi}}(-\infty))| < |y^{\downarrow}(t)| < |y^{\uparrow}(t + T_0^{x_o}(\tau))|.$$

Hence, for every $\theta \leq 0$, we get

$$\begin{aligned}
&\|\boldsymbol{\phi}(\theta+\tau;\tau,\boldsymbol{\xi}) - \boldsymbol{\phi}_{\boldsymbol{A_0}}(\theta;0,\hat{\boldsymbol{\xi}}(-\infty))\| = \\
&= \|\boldsymbol{\phi}(\theta+\tau;\tau,\boldsymbol{\xi}) - \boldsymbol{\phi}_{\boldsymbol{A_0}}(\theta+\tau;\tau,\hat{\boldsymbol{\xi}}(-\infty))\| \leq \\
&\leq \|\boldsymbol{\phi}^{\uparrow}(\theta+\tau+T_0^{x_o}(\tau)) - \boldsymbol{\phi}^{\uparrow}(\theta+\tau)\| \leq \\
&\leq T_0^{x_o}(\tau) \max_{\sigma\in[0,T_0^{x_o}(\tau)]} \|\dot{\boldsymbol{\phi}}^{\uparrow}(\theta+\tau+\sigma)\|.
\end{aligned}$$

Now, let $\varpi = \underline{K}(\tau)$, $\widehat{\tau} = \tau + T_0^{x_o}(\tau)$, $\boldsymbol{R}_\tau = \boldsymbol{\phi}^{\uparrow}(\widehat{\tau})$, and recall the definition of $\underline{\boldsymbol{P}}(\tau)$ given at the beginning of Section 4. Then, from (36), we obtain

$$\begin{aligned}
\boldsymbol{\phi}^{\uparrow}(\theta+\tau+\sigma) &= \boldsymbol{\phi}_{\boldsymbol{\varpi}}(\theta+\tau+\sigma;\tau,\underline{\boldsymbol{P}}(\tau)) = \boldsymbol{\phi}_{\boldsymbol{\varpi}}(\theta+\tau+\sigma;\widehat{\tau},\boldsymbol{R}_\tau) = \\
&= \boldsymbol{\phi}_{\boldsymbol{\varpi}}(\theta+\widehat{\tau}-T_0^{x_o}(\tau)+\sigma;\widehat{\tau},\boldsymbol{R}_\tau) = \\
&= \boldsymbol{\phi}_{\boldsymbol{\varpi}}(\theta+\sigma-T_0^{x_o}(\tau);0,\boldsymbol{R}_\tau).
\end{aligned}$$

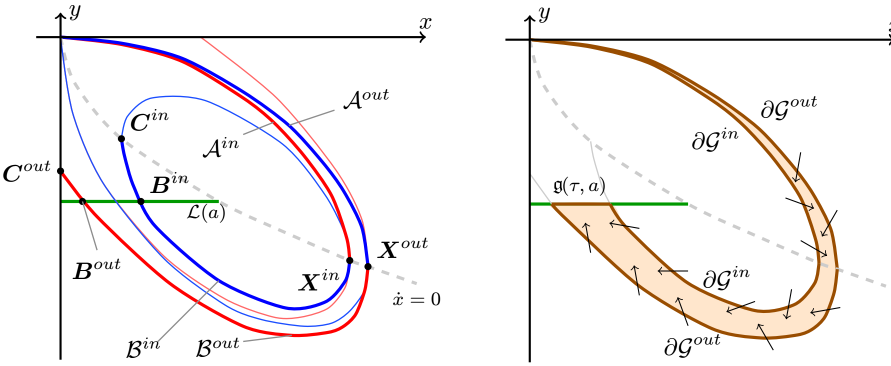

FIGURE 5. The construction of the set $\mathcal{G}(\mathcal{T})$ for suitable $\mathcal{T} \ll 0$. The arrows suggest the direction of the vector fields of the dynamical system (14) along the boundary of $\mathcal{G}(\mathcal{T})$

Further, $T_0^{x_o}(\tau) \to 0$ and $\varpi \to A_0$ as $\tau \to -\infty$, see Lemma 39; hence, from (25), for every $\theta \le 0$ we deduce the existence of $c > 0$ such that

$$\begin{aligned}&\|\boldsymbol{\phi}(\theta+\tau;\tau,\boldsymbol{\xi}) - \boldsymbol{\phi}_{\boldsymbol{A_0}}(\theta;0,\hat{\boldsymbol{\xi}}(-\infty))\|\mathrm{e}^{-h\theta} \le \\ &\le T_0^{x_o}(\tau) \max_{s\in[-T_0^{x_o}(\tau),0]} \|\dot{\boldsymbol{\phi}}_{\boldsymbol{\varpi}}(\theta+s;0,\boldsymbol{R}_\tau)\|\mathrm{e}^{-h\theta} \le c\, T_0^{x_o}(\tau) \to 0,\end{aligned}$$

as $\tau \to -\infty$. So, (34) is proved and the proof is concluded. $\square$

4.2. From the unstable set close to the origin to a larger one.

We denote by $\hat{M}^u(-\infty)$ the branch of $\boldsymbol{\Gamma}(A_0)$ between the origin and $\hat{\boldsymbol{\xi}}(-\infty)$. Further, if we let $\tau \to -\infty$, then the whole set $\mathcal{F}(\tau)$ shrinks to $\hat{M}^u(-\infty)$.

Now, following [19], we can use the continuous dependence on initial data of the flow of (14) to extend the unstable set $\hat{M}^u(\mathcal{T})$ outside $\mathcal{F}(\mathcal{T})$. However, the argument works as long as the trajectories do not cross the coordinate axes (since afterwards uniqueness of the solutions is not ensured anymore).

Following [12], for any fixed $a > 0$, we denote by $\mathcal{L}(a)$ the segment of $y = -a$ lying between the negative y semi-axis and the isocline $\dot{x} = 0$ of system (14), i.e.

$$\mathcal{L}(a) = \{(s,-a) \mid 0 \le s \le \tilde{X}(a)\}, \qquad \text{where } \tilde{X}(a) := \tfrac{1}{\alpha}\, a^{\frac{1}{p-1}}. \tag{43}$$

Recalling the definition of equilibrium point $\boldsymbol{E}(A_0)$ given in (20), we observe that for any $0 < a < |E_y(A_0)|$ the homoclinic orbit $\boldsymbol{\Gamma}(A_0)$ intersects $\mathcal{L}(a)$ transversely in a point, say $\boldsymbol{Q}(-\infty,a)$.

Now, we aim to construct an unstable curve $\hat{W}^u(\tau)$ joining the origin and $\mathcal{L}(a)$.

Firstly, we introduce some sets sketched in Figure 5. Let us fix $\tau \le 0$ and denote by $\boldsymbol{X}^{\mathrm{out}}(\tau)$ and $\boldsymbol{X}^{\mathrm{in}}(\tau)$ the intersections of the isocline $\dot{x} = 0$ with $\underline{\boldsymbol{\Gamma}}(\tau)$ and $\overline{\boldsymbol{\Gamma}}(\tau)$, respectively. Since $\|\boldsymbol{X}^{\mathrm{out}}(\tau) - \boldsymbol{X}^{\mathrm{in}}(\tau)\| \to 0$ as $\tau \to -\infty$, we can find $\mathcal{T}_0$ such that $\boldsymbol{X}^{\mathrm{in}}(\tau)$ lies on the right of $\underline{\boldsymbol{E}}(\tau)$ for any $\tau \le \mathcal{T}_0$.

Then, we denote by $\mathcal{A}^{\rm out}(\tau)$ and by $\mathcal{A}^{\rm in}(\tau)$, respectively, the paths of $\underline{\mathbf{\Gamma}}(\tau)$ and of $\overline{\mathbf{\Gamma}}(\tau)$ lying in $\dot{x} \geq 0$, so that $\mathcal{A}^{\rm out}(\tau)$ connects the origin with $\boldsymbol{X}^{\rm out}(\tau)$, while $\mathcal{A}^{\rm in}(\tau)$ connects the origin with $\boldsymbol{X}^{\rm in}(\tau)$, for any $\tau \leq \mathcal{T}_0$.

Remark 42. Recalling (26) and (27), we have $H^+(\tau) := \overline{\mathcal{H}}_\tau(\boldsymbol{X}^{\rm out}(\tau)) > 0$, while $H^-(\tau) := \underline{\mathcal{H}}_\tau(\boldsymbol{X}^{\rm in}(\tau)) < 0$. Further, $\lim_{\tau\to-\infty} H^+(\tau) = \lim_{\tau\to-\infty} H^-(\tau) = 0$.

Let us focus on the semiplane $x > 0$; the level set $\underline{\mathcal{H}}_\tau(\cdot) = H^-(\tau)$ intersects the isocline $\dot{x} = 0$ in $\boldsymbol{X}^{\rm in}(\tau)$ and in a second point, say $\boldsymbol{C}^{\rm in}(\tau)$, while the level set $\overline{\mathcal{H}}_\tau(\cdot) = H^+(\tau)$ intersects the isocline $\dot{x} = 0$ just in $\boldsymbol{X}^{\rm out}(\tau)$, and then it intersects the y negative semi-axis in a point, say $\boldsymbol{C}^{\rm out}(\tau) = (0, -a_0(\tau))$.

Notice further that both $\boldsymbol{C}^{\rm in}(\tau)$ and $\boldsymbol{C}^{\rm out}(\tau)$ converge to the origin as $\tau \to -\infty$, so, possibly choosing a smaller $\mathcal{T}_0$, we can assume that both $\boldsymbol{C}^{\rm out}(\tau)$ and $\boldsymbol{C}^{\rm in}(\tau)$ lie above $\boldsymbol{E}(A_0)$ for any $\tau \leq \mathcal{T}_0$.

Remark 43. Let $\bar{a}_0 := \min\{a_0(\mathcal{T}_0), \frac{|\overline{E}_y(\mathcal{T}_0)|}{2}\}$; then for any $0 < a \leq \bar{a}_0$ we can find $N_1(a) \geq |\mathcal{T}_0| > 0$ large enough so that, for any $\tau \leq -N_1(a)$, the level sets $\underline{\mathcal{H}}_\tau(\cdot) = H^-(\tau)$ and $\overline{\mathcal{H}}_\tau(\cdot) = H^+(\tau)$ intersect the segment $\mathcal{L}(a)$ transversely in points denoted by $\boldsymbol{B}^{\rm in}(\tau)$ and $\boldsymbol{B}^{\rm out}(\tau)$, respectively.

We denote by $\mathcal{B}^{\rm in}(\tau)$ the branch of the level set $\underline{\mathcal{H}}_\tau(\cdot) = H^-(\tau)$ between $\boldsymbol{X}^{\rm in}(\tau)$ and $\boldsymbol{B}^{\rm in}(\tau)$ and by $\mathcal{B}^{\rm out}(\tau)$ the branch of the level set $\overline{\mathcal{H}}_\tau(\cdot) = H^+(\tau)$ between $\boldsymbol{X}^{\rm out}(\tau)$ and $\boldsymbol{B}^{\rm out}(\tau)$ (both lying in $\dot{x} \leq 0$).

Definition 44. We denote by $\partial\mathcal{G}^{\rm out}(\tau) = \mathcal{A}^{\rm out}(\tau) \cup \mathcal{B}^{\rm out}(\tau)$, by $\partial\mathcal{G}^{\rm in}(\tau) = \mathcal{A}^{\rm in}(\tau) \cup \mathcal{B}^{\rm in}(\tau)$ and by $\partial\mathcal{G}(\tau) = \partial\mathcal{G}^{\rm out}(\tau) \cup \partial\mathcal{G}^{\rm in}(\tau)$. We denote by $\mathfrak{g}(\tau, a)$ the horizontal segment between $\boldsymbol{B}^{\rm out}(\tau)$ and $\boldsymbol{B}^{\rm in}(\tau)$. Let $\mathcal{G}(\tau)$ be the compact set enclosed by $\partial\mathcal{G}(\tau)$ and $\mathfrak{g}(\tau, a)$, see Figure 5.

Recalling that $\boldsymbol{Q}(-\infty, a)$ was defined as the unique intersection between $\mathbf{\Gamma}(A_0)$ and the segment $\mathcal{L}(a)$, we have the following.

Remark 45. The segment $\mathfrak{g}(\tau, a)$ shrinks to $\boldsymbol{Q}(-\infty, a)$ as $\tau \to -\infty$ and to the origin as $a \to 0$ (and, consequently, $\tau \to -\infty$).

Now, we need some observations concerning the flow of (14) on the border of $\mathcal{G}(\tau)$.

Remark 46. Fix $a \in]0, \bar{a}_0]$ and, then, $\tau \leq -N_1(a)$.

The flow of (14) on $\partial\mathcal{G}(\tau) \setminus \{(0,0), \boldsymbol{X}^{\rm in}(\tau), \boldsymbol{X}^{\rm out}(\tau), \boldsymbol{B}^{\rm in}(\tau), \boldsymbol{B}^{\rm out}(\tau)\}$ points towards the interior of $\mathcal{G}(\tau)$. The flow on $\mathfrak{g}(\tau, a)$ points towards the exterior of $\mathcal{G}(\tau)$, i.e. upwards: this follows from the assumption $a < |\overline{E}_y(\mathcal{T}_0)|$ and an analysis of the isocline $\dot{y} = 0$.

Lemma 47. *Fix $0 < a \leq \bar{a}_0$ and $\tau \leq -N_1(a)$; let $\boldsymbol{Q} \in \partial\mathcal{G}(\tau) \setminus \{(0,0), \boldsymbol{B}^{in}(\tau), \boldsymbol{B}^{out}(\tau)\}$ and $T < \tau$; then there is $\mu > 0$ such that $\boldsymbol{\phi}(t; T, \boldsymbol{Q})$ is in the exterior of $\mathcal{G}(\tau)$ for any $t \in [T - \mu, T[$ and it is the interior of $\mathcal{G}(\tau)$ for any $t \in]T, T + \mu]$.*

Proof. The part of the Lemma concerning $\partial\mathcal{G}(\tau)\backslash\{(0,0),\boldsymbol{X}^{\text{in, out}}(\tau),\boldsymbol{B}^{\text{in, out}}(\tau)\}$ immediately follows from Remark 46. So, we turn to consider $\boldsymbol{\phi}(t;T,\boldsymbol{X}^{\text{out}}(\tau))$. By construction and using (28), we get

$$\underline{\mathcal{H}}_\tau(\boldsymbol{X}^{\text{out}})=0\,,\qquad \frac{d\underline{\mathcal{H}}_\tau}{dt}(\boldsymbol{\phi}(t;T,\boldsymbol{X}^{\text{out}}))\lfloor_{t=T}=0\,,$$
$$\overline{\mathcal{H}}_\tau(\boldsymbol{X}^{\text{out}})=H^+(\tau)\,,\qquad \frac{d\overline{\mathcal{H}}_\tau}{dt}(\boldsymbol{\phi}(t;T,\boldsymbol{X}^{\text{out}}))\lfloor_{t=T}=0\,.$$

Now, let us set for brevity $\boldsymbol{\phi}(t)=(x(t),y(t))$ for $\boldsymbol{\phi}(t;T,\boldsymbol{X}^{\text{out}})$. Using the fact that $\dot{y}(T)<0=\dot{x}(T)$, we find

$$\ddot{x}(T)=\alpha\dot{x}(T)+\frac{|y(T)|^{\frac{2-p}{p-1}}}{p-1}\dot{y}(T)=\frac{|y(T)|^{\frac{2-p}{p-1}}}{p-1}\dot{y}(T)<0.$$

Hence, if $t_1<T<t_2$ and t_2-t_1 is small enough, we have $\dot{x}(t_1)>0>\dot{x}(t_2)$, and so

$$\begin{aligned}\boldsymbol{\phi}(t_1)&\in\{(x,y)\mid \alpha x+y|y|^{\frac{2-p}{p-1}}>0\},\\ \boldsymbol{\phi}(t_2)&\in\{(x,y)\mid \alpha x+y|y|^{\frac{2-p}{p-1}}<0\}.\end{aligned}\tag{44}$$

Further,

$$\frac{d^2}{dt^2}\underline{\mathcal{H}}_\tau(\boldsymbol{\phi}(t))\lfloor_{t=T}=[\underline{K}(\tau)-K(T)]x^{q-1}(T)\ddot{x}(T)>0\,,$$
$$\frac{d^2}{dt^2}\overline{\mathcal{H}}_\tau(\boldsymbol{\phi}(t))\lfloor_{t=T}=[\overline{K}(\tau)-K(T)]x^{q-1}(T)\ddot{x}(T)<0.$$

So, using a second order Taylor expansion, we see that $\underline{\mathcal{H}}_\tau(\boldsymbol{\phi}(t_1))>0$ and $\overline{\mathcal{H}}_\tau(\boldsymbol{\phi}(t_2))<H^+(\tau)$ for t_2-t_1 small enough. Since in a neighborhood $\mathcal{U}$ of $\boldsymbol{X}^{\text{out}}(\tau)$ the set $\mathcal{G}(\tau)$ can be described as

$$\begin{aligned}\mathcal{G}(\tau)\cap\mathcal{U}=\{(x,y)\in\mathcal{U}\mid \alpha x+y|y|^{\frac{2-p}{p-1}}\geq 0 \text{ and } \underline{\mathcal{H}}_\tau(x,y)\leq 0\\ \text{or } \alpha x+y|y|^{\frac{2-p}{p-1}}\leq 0 \text{ and } \overline{\mathcal{H}}_\tau(x,y)\leq H^+(\tau)\},\end{aligned}$$

recalling (44), the claim concerning $\boldsymbol{X}^{\text{out}}(\tau)$ follows.

The proof for $\boldsymbol{X}^{\text{in}}(\tau)$ is analogous and it is omitted. □

Now, arguing as in the proof of Lemma 38 we obtain the following.

Lemma 48. *Let $a\in\,]0,\bar{a}_0]$ and $N_1(a)>0$ be as in Remark 43. Then, for any $\tau\leq -N_1(a)$ the set*

$$\tilde{W}^u(\tau):=\{\boldsymbol{Q}\mid\boldsymbol{\phi}(t;\tau,\boldsymbol{Q})\in\mathcal{G}(\tau)\ \textit{for any}\ t\leq\tau\}$$

is non-empty and compact, and it intersects $\mathfrak{g}(\tau,a)$.

The following proposition rephrases the content of Lemma 24 and Proposition 25 for the set $\tilde{W}^u(\tau)$.

Proposition 49. *Let $a\in\,]0,\bar{a}_0]$ and $N_1(a)>0$ be as in Remark 43. Then, for any $\tau\leq -N_1(a)$, there is a compact connected set $\hat{W}^u(\tau)\subset\tilde{W}^u(\tau)$ which contains the origin and intersects $\mathfrak{g}(\tau,a)$ in a singleton, say $\{\boldsymbol{Q}^{\boldsymbol{u}}(\tau,a)\}$.*

For any fixed $\tau \le -N_1(a)$, we can define the homeomorphism

$$d_W(\cdot,\tau) : \tilde{W}^u(\tau) \to I_W(\tau)\,,$$

where $I_W(\tau)$ is a suitable compact set whose positive maximum is denoted by $\tilde{D}_W(\tau)$.

Further, there is $\hat{D}_W(\tau) \le \tilde{D}_W(\tau)$ such that the restriction

$$d_W(\cdot,\tau) : \hat{W}^u(\tau) \to [0, \hat{D}_W(\tau)]$$

is a homeomorphism.

Proof. Fix $\tau \le -N_1(a)$. The construction of $\hat{W}^u(\tau) \subset \tilde{W}^u(\tau)$ is obtained profiting from Lemma 37, combined with Remark 46 and Lemma 47. Observe that $\tilde{M}^u(\tau) \subset \tilde{W}^u(\tau)$. Using continuous dependence on initial data and Lemma 23, we can find $\mathcal{T}(a) \ge 0$ such that $\boldsymbol{\phi}(\tau - \mathcal{T}(a); \tau, \boldsymbol{Q}) \in \tilde{M}^u(\tau - \mathcal{T}(a))$ for any $\boldsymbol{Q} \in \tilde{W}^u(\tau)$; further, we can choose $\mathcal{T}(a)$ independent of $\boldsymbol{Q}$.

Hence, we can set $d_W(\boldsymbol{Q},\tau) := d_M(\boldsymbol{\phi}(\tau - \mathcal{T}(a); \tau, \boldsymbol{Q}), \tau - \mathcal{T}(a))$ so that

$$\boldsymbol{\phi}(t; d_W(\boldsymbol{Q},\tau)) \equiv \boldsymbol{\phi}(t; d_M(\boldsymbol{\phi}(\tau - \mathcal{T}(a); \tau, \boldsymbol{Q}), \tau - \mathcal{T}(a))) \equiv \boldsymbol{\phi}(t; \tau, \boldsymbol{Q})$$

for any $t \le \tau$.

Notice that, by construction, $d_W(\cdot,\tau)$ is continuous, cf. Lemma 24. So, as in the proof of Proposition 25, we set $\hat{D}_W(\tau) := \min\{d_W(\boldsymbol{Q},\tau) \mid \boldsymbol{Q} \in \mathfrak{g}(\tau,a) \cap \tilde{W}^u(\tau)\}$ and we redefine $\hat{W}^u(\tau) := \{\boldsymbol{\phi}(\tau; d) \mid d \in [0, \hat{D}_W(\tau)]\}$. Once again, the revised definition may yield a smaller set than the original one. Define $\boldsymbol{Q^u}(\tau,a) = \boldsymbol{\phi}(\tau; \hat{D}_W(\tau))$ so that $\mathfrak{g}(\tau,a) \cap \tilde{W}^u(\tau) = \{\boldsymbol{Q^u}(\tau,a)\}$. Notice that, by construction, $\hat{D}_W(\tau) \le \tilde{D}_W(\tau) \le \tilde{D}_M(\tau - \mathcal{T}(a))$, so we conclude the proof. $\square$

Remark 50. By construction, $\hat{M}^u(\tau) \subset \hat{W}^u(\tau) \subset \tilde{W}^u(\tau)$ for any $\tau \le -N_1(a)$.

Let us define

$$\tilde{\Xi}_W(\tau,a) := \tilde{W}^u(\tau) \cap \mathcal{L}(a). \tag{45}$$

Observe that $\boldsymbol{Q^u}(\tau,a) = \boldsymbol{\phi}(\tau; \hat{D}_W(\tau)) \in \tilde{\Xi}_W(\tau,a)$, but $\tilde{\Xi}_W(\tau,a)$ might not be a singleton.

Proposition 51. *The functions $\hat{D}_W(\tau)$ and $\tilde{D}_W(\tau)$ defined in Proposition 49 are both strictly decreasing and $\lim_{\tau\to-\infty} \hat{D}_W(\tau) = +\infty$. Further, their respective inverses, $\hat{\tau}_W(d)$ and $\tilde{\tau}_W(d)$, satisfy*

$$\lim_{d\to+\infty} \hat{\tau}_W(d) = -\infty, \qquad \lim_{d\to+\infty} \tilde{\tau}_W(d) = -\infty\,. \tag{46}$$

Proof. First, observe that from Remark 50 we find

$$\tilde{D}_W(\tau) \ge \hat{D}_W(\tau) > \hat{D}_M(\tau) \to +\infty \quad \text{ as } \tau \to -\infty.$$

Then, observe that we have a characterization analogous to (42), i.e.

$$\begin{aligned} \hat{D}_W(\tau) &= \min\{d > 0 \mid \boldsymbol{\phi}(\tau; d) \in \tilde{\Xi}_W(\tau)\}, \\ \tilde{D}_W(\tau) &= \max\{d > 0 \mid \boldsymbol{\phi}(\tau; d) \in \tilde{W}^u(\tau)\}, \end{aligned} \tag{47}$$

so we conclude the proof of this Proposition repeating the argument of the proof of Proposition 27. $\square$

Lemma 52. *Let* $a \in]0, \bar{a}_0]$, $d \geq \tilde{D}_W(-N_1(a))$, *then there is a unique* $\tau = \tau_W(d) \leq -N_1(a)$ *such that* $\boldsymbol{\phi}(\tau; d) \in \tilde{\Xi}_W(\tau, a)$. *Further,* $\lim_{d \to +\infty} \tau_W(d) = -\infty$.

Proof. The Lemma is proved arguing as in Lemmas 30 and 31. □

Fix $a \in]0, \bar{a}_0]$, recall that $\boldsymbol{Q}(-\infty, a)$ denotes the unique intersection between $\boldsymbol{\Gamma}(A_0)$ and $\mathcal{L}(a)$, while $\hat{\boldsymbol{\xi}}(-\infty)$ is the unique intersection between $\boldsymbol{\Gamma}(A_0)$ and the half-line $x = x_o = \overline{E}_x(0)$ contained in $\dot{x} > 0$.

Lemma 53. *Let* $a \in]0, \bar{a}_0]$. *Set* $h = \alpha \min\{1, q-1\}$, *then for every* $\tilde{\boldsymbol{Q}} \in \tilde{\Xi}_W(\tau, a)$

$$\lim_{\tau \to -\infty} \sup_{\theta \leq 0} \left(\left\| \boldsymbol{\phi}(\theta + \tau; \tau, \tilde{\boldsymbol{Q}}) - \boldsymbol{\phi}_{\boldsymbol{A_0}}(\theta; 0, \boldsymbol{Q}(-\infty, a)) \right\| \mathrm{e}^{-h\theta} \right) = 0 . \tag{48}$$

Proof. For any $\tau \leq -N_1(a)$, let us denote by $\Theta_0(\tau) < 0$ the value such that the trajectory $\boldsymbol{\phi}(\cdot; \tau, \tilde{\boldsymbol{Q}})$ intersects transversely $\mathfrak{f}(\Theta_0(\tau) + \tau)$ in a point $\bar{\boldsymbol{\xi}}(\Theta_0(\tau) + \tau) \in \tilde{\Xi}_M(\Theta_0(\tau) + \tau)$ at $t = \Theta_0(\tau) + \tau$, and it is in $\mathcal{F}(\Theta_0(\tau) + \tau)$ for any $t \leq \Theta_0(\tau) + \tau$. Let $\Theta_0(-\infty) < 0$ be the time at which the trajectory $\boldsymbol{\phi}_{\boldsymbol{A_0}}(\cdot; 0, \boldsymbol{Q}(-\infty, a))$ intersects transversely the line $\{x = x_o\}$ in $\hat{\boldsymbol{\xi}}(-\infty)$ (independent of τ), and it is in $\mathcal{F}(\Theta_0(\tau) + \tau)$ for any $t \leq \Theta_0(-\infty)$.

From Remark 45, we know that $\tilde{\boldsymbol{Q}} \to \boldsymbol{Q}(-\infty, a)$ as $\tau \to -\infty$ (uniformly for $\tilde{\boldsymbol{Q}} \in \tilde{\Xi}_W(\tau, a)$); then, using continuous dependence on data and parameters, we see that $\Theta_0(\tau) \to \Theta_0(-\infty)$ as $\tau \to -\infty$, and $\bar{\boldsymbol{\xi}}(\Theta_0(\tau) + \tau) \to \hat{\boldsymbol{\xi}}(-\infty)$ as $\tau \to -\infty$. Moreover, using also the compactness of the interval $[\Theta_0(-\infty) - 1, 0]$, since $[\Theta_0(\tau), 0] \subset [\Theta_0(-\infty) - 1, 0]$ as $\tau \to -\infty$, we deduce that

$$\lim_{\tau \to -\infty} \sup_{\theta \in [\Theta_0(\tau), 0]} \left\| \boldsymbol{\phi}(\theta + \tau; \tau, \tilde{\boldsymbol{Q}}) - \boldsymbol{\phi}_{\boldsymbol{A_0}}(\theta; 0, \boldsymbol{Q}(-\infty, a)) \right\| = 0 ;$$

then, since the interval is bounded, we get

$$\lim_{\tau \to -\infty} \sup_{\theta \in [\Theta_0(\tau), 0]} \left[\left\| \boldsymbol{\phi}(\theta + \tau; \tau, \tilde{\boldsymbol{Q}}) - \boldsymbol{\phi}_{\boldsymbol{A_0}}(\theta; 0, \boldsymbol{Q}(-\infty, a)) \right\| \mathrm{e}^{-h\theta} \right] = 0 . \tag{49}$$

It remains to prove that

$$\lim_{\tau \to -\infty} \sup_{\theta \leq 0} \left[\left\| \boldsymbol{\phi}(\theta + \Theta_0(\tau) + \tau; \tau, \tilde{\boldsymbol{Q}}) - \boldsymbol{\phi}_{\boldsymbol{A_0}}(\theta + \Theta_0(\tau); 0, \boldsymbol{Q}(-\infty, a)) \right\| \mathrm{e}^{-h\theta} \right] = 0 . \tag{50}$$

From Lemma 32, replacing τ with $\Theta_0(\tau) + \tau$ in (34), we have

$$\lim_{\tau \to -\infty} \sup_{\theta \leq 0} \left[\left\| \boldsymbol{\phi}(\theta + \Theta_0(\tau) + \tau; \Theta_0(\tau) + \tau, \bar{\boldsymbol{\xi}}(\Theta_0(\tau) + \tau)) - \boldsymbol{\phi}_{\boldsymbol{A_0}}(\theta; 0, \hat{\boldsymbol{\xi}}(-\infty)) \right\| \mathrm{e}^{-h\theta} \right] = 0 . \tag{51}$$

So, recalling that

$$\boldsymbol{\phi}\big(t; \Theta_0(\tau) + \tau, \bar{\boldsymbol{\xi}}(\Theta_0(\tau) + \tau)\big) \equiv \boldsymbol{\phi}\big(t; \tau, \tilde{\boldsymbol{Q}}\big) ,$$
$$\boldsymbol{\phi}_{\boldsymbol{A_0}}\big(t; 0, \hat{\boldsymbol{\xi}}(-\infty)\big) \equiv \boldsymbol{\phi}_{\boldsymbol{A_0}}\big(t + \Theta_0(-\infty); 0, \boldsymbol{Q}(-\infty, a)\big) ,$$

we can rephrase (51) as

$$\lim_{\tau \to -\infty} \sup_{\theta \leq 0} \left[\left\| \boldsymbol{\phi}(\theta + \Theta_0(\tau) + \tau; \tau, \tilde{\boldsymbol{Q}}) - \boldsymbol{\phi}_{\boldsymbol{A_0}}(\theta + \Theta_0(-\infty); 0, \boldsymbol{Q}(-\infty, a)) \right\| \mathrm{e}^{-h\theta} \right] = 0 . \tag{52}$$

Then, since $\Theta_0(\tau) \to \Theta_0(-\infty)$, we deduce, from (25),

$$\lim_{\tau\to-\infty} \sup_{\theta\le 0} \left[\left\| \boldsymbol{\phi}_{\boldsymbol{A}_{\mathbf{0}}}(\theta+\Theta_0(-\infty);0,\boldsymbol{Q}(-\infty,a)) - \boldsymbol{\phi}_{\boldsymbol{A}_{\mathbf{0}}}(\theta+\Theta_0(\tau);0,\boldsymbol{Q}(-\infty,a)) \right\| \mathrm{e}^{-h\theta} \right] = 0\,. \tag{53}$$

Hence, (52) and (53) give (50), concluding the proof of (48). □

5. Asymptotic estimates

In the whole Section we assume $(\mathbf{H}_\ell)$, so that for any $a \le \bar{a}_0$ there is $N_1(a)$ such that we can construct $\hat{W}^u(\tau)$ when $\tau \le -N_1(a)$. In [12], $\hat{W}^u(\tau)$ was constructed through the classical integral manifold theory and a standard transversality argument, while here we had to use an ad hoc argument. However, following the argument performed in the smooth case, see [12, Proposition 26], we obtain some estimates of the energy in $\tilde{\Xi}_W(\tau,a)$.

Lemma 54. *Assume* $(\mathbf{H}_\ell)$. *Then, for any* $0 < a \le \bar{a}_0$ *there is* $c(a) > 0$ *such that for any* $\tilde{\boldsymbol{Q}} \in \tilde{\Xi}_W(\tau,a)$ *we get*

$$\mathcal{H}(\tilde{\boldsymbol{Q}};\tau) = c(a)\,\mathrm{e}^{\ell\tau} + o(\mathrm{e}^{\ell\tau}) \qquad \text{as } \tau\to-\infty\,. \tag{54}$$

Proof. Let $\tilde{\boldsymbol{Q}} \in \tilde{\Xi}_W(\tau,a)$ with $\tau \le -N_1(a)$. From (17) and Lemmas 19 and 53, we find

$$\begin{aligned}\mathcal{H}(\tilde{\boldsymbol{Q}};\tau) &= \int_{-\infty}^0 \frac{d\mathcal{H}}{dt}(\boldsymbol{\phi}(\theta+\tau;\tau,\tilde{\boldsymbol{Q}});\theta+\tau)\,d\theta \\ &= \frac{1}{q}\int_{-\infty}^0 [x(\theta+\tau;\tau,\tilde{\boldsymbol{Q}})]^q \dot{K}(\theta+\tau)\,d\theta \\ &= \frac{1}{q}\int_{-\infty}^0 [x_{A_0}(\theta;0,\boldsymbol{Q}(-\infty,a)) + \omega_1(\theta,\tau)]^q \dot{K}(\theta+\tau)\,d\theta,\end{aligned}$$

where $\omega_1(\theta,\tau) \to 0$ uniformly as $\tau\to-\infty$. Then, using the first estimate in Remark 16 and $(\mathbf{H}_\ell)$, we find $d(a)$, $c(a) > 0$ and negligible functions ω_2, ω_3 such that

$$\mathcal{H}(\tilde{\boldsymbol{Q}};\tau) = \frac{B_0\ell}{q}\int_{-\infty}^0 [d(a)\mathrm{e}^{\alpha\theta} + \omega_2(\theta,\tau)]^q (1+\omega_3(\theta,\tau))\mathrm{e}^{\ell(\theta+\tau)}\,d\theta = c(a)\mathrm{e}^{\ell\tau} + o(\mathrm{e}^{\ell\tau}),$$

see the proof of [12, Proposition 26] for more details. □

5.1. Asymptotic behavior of $R(d)$ for d large when $\ell < \ell_p^*$.

In this and in the next subsection we consider the asymptotic behavior of the first zero $R(d)$ of $u(r;d)$, for d large: we just need to repeat the argument developed in [12], which we sketch here for convenience of the reader.

Following [12, §4.1], we assume firstly $(\mathbf{H}_\ell)$, $(\mathbf{W}_s)$ and

$$\dot{K}(t) > 0 \qquad \text{for any } t\in\mathbb{R}; \tag{55}$$

later on we drop (55) and $(\mathbf{W}_s)$ and we prove Theorem A in the general case $\delta > -p$ and $p > 1$ with a truncation argument.

From now on, we denote by $\tilde{\boldsymbol{Q}}(\tau,a)$ a point such that $\tilde{\boldsymbol{Q}}(\tau,a) \in \tilde{\Xi}_W(\tau,a)$.

We recall that, when $(\mathbf{H}_{\uparrow})$ holds (and a fortiori when (55) is satisfied), all the regular solutions of (5) are crossing solutions. Let us take $a > 0$ such that

$$0 < a \leq a_1(\overline{K}) := \min\left\{\frac{|E_y(\overline{K})|}{2}, \bar{a}_0\right\}, \tag{56}$$

where $\overline{K} = \sup_{t\in\mathbb{R}} K(t)$, and consider the closed region $\mathcal{V}$ delimited by the isocline $\dot{x} = 0$, the line $\mathcal{L}(a)$, and the axis $x = 0$. From the choice in (56) and by $(\mathbf{W}_{\boldsymbol{s}})$, we see that the trajectories of system (14) satisfy $\dot{x} < 0$ and $\dot{y} > 0$ there, except for the points $\boldsymbol{Q}$ belonging to the isocline where $\dot{x} = 0$, $\dot{y} > 0$, and notably $\mathcal{H}(\boldsymbol{Q}; t) \leq \mathcal{H}_{\overline{K}}(\boldsymbol{Q}) < 0$ for every t, by construction.

Let $\tau \leq -N_1(a)$, then, from (17) and (55), we deduce that the trajectory $\boldsymbol{\phi}(t; \tau, \tilde{\boldsymbol{Q}}(\tau, a))$ satisfies $\mathcal{H}(\boldsymbol{\phi}(t; \tau, \tilde{\boldsymbol{Q}}(\tau, a)); t) > 0$ for every t, so it remains inside $\mathcal{V}$ in a right neighborhood of τ until it crosses the y negative semi-axis transversely at a certain $t = T(\tilde{\boldsymbol{Q}}(\tau, a), a) > \tau$ in a point, say $\boldsymbol{R}(\tau, a) = (0, R_y(\tau, a))$. Moreover,

$$\dot{x}(t; \tau, \tilde{\boldsymbol{Q}}(\tau, a)) < 0 < \dot{y}(t; \tau, \tilde{\boldsymbol{Q}}(\tau, a)), \qquad \text{for any } \tau < t < T(\tilde{\boldsymbol{Q}}(\tau, a), a). \tag{57}$$

Let us now consider the trajectory $\boldsymbol{\phi}_{\boldsymbol{K}(\boldsymbol{\tau})}(t; \tau, \tilde{\boldsymbol{Q}}(\tau, a))$ of the system (18) frozen at $\varpi = K(\tau)$: its graph is contained in the level set

$$\{(x, y) \mid \mathcal{H}(x, y; \tau) = \mathcal{H}(\tilde{\boldsymbol{Q}}(\tau, a); \tau) > 0\}, \tag{58}$$

so it lies in the exterior of the homoclinic orbit $\boldsymbol{\Gamma}(K(\tau))$ defined in (21), since $\mathcal{H}(\tilde{\boldsymbol{Q}}(\tau, a); \tau) > 0$ from Lemma 19.

Hence, there exists $T^1 = T^1(\tilde{\boldsymbol{Q}}(\tau, a), a) > \tau$ such that $\boldsymbol{\phi}_{\boldsymbol{K}(\boldsymbol{\tau})}(\cdot; \tau, \tilde{\boldsymbol{Q}}(\tau, a))$ lies in the 4^{th} quadrant when $t \in [\tau, T^1[$ and it crosses transversely the y negative semi-axis at $t = T^1$ in a point, say

$$\boldsymbol{R}^1(\tau, a) = (0, R^1_y(\tau, a)) = \boldsymbol{\phi}_{\boldsymbol{K}(\boldsymbol{\tau})}(T^1; \tau, \tilde{\boldsymbol{Q}}(\tau, a))\,. \tag{59}$$

Then, similarly to (28), from (55), we deduce that $\frac{\partial}{\partial t}\mathcal{H}_{K(\tau)}\big(\boldsymbol{\phi}(t; \tau, \tilde{\boldsymbol{Q}}(\tau, a))\big) > 0$ for any $\tau < t < T(\tilde{\boldsymbol{Q}}(\tau, a), a)$; hence, according to (58),

$$\begin{aligned}\mathcal{H}_{K(\tau)}(\boldsymbol{R}(\tau, a)) &= \mathcal{H}_{K(\tau)}(\boldsymbol{\phi}(T(\tau, a)); \tau, \tilde{\boldsymbol{Q}}(\tau, a)) > \\ &> \mathcal{H}_{K(\tau)}(\boldsymbol{\phi}(\tau; \tau, \tilde{\boldsymbol{Q}}(\tau, a))) = \mathcal{H}_{K(\tau)}(\tilde{\boldsymbol{Q}}(\tau, a)) = \mathcal{H}_{K(\tau)}(\boldsymbol{R}^1(\tau, a))\,.\end{aligned}$$

Thus, $\boldsymbol{R}^1(\tau, a)$ lies above $\boldsymbol{R}(\tau, a)$, i.e. $R_y(\tau, a) < R^1_y(\tau, a) < 0$. Notice that both $\boldsymbol{R}(\tau, a)$ and $\boldsymbol{R}^1(\tau, a)$ in fact depend on the choice of $\tilde{\boldsymbol{Q}}(\tau, a)$ in $\tilde{\Xi}_W(\tau, a)$.

Following [12, Lemma 30], we deduce that

$$T(\tilde{\boldsymbol{Q}}(\tau, a), a) < T^1(\tilde{\boldsymbol{Q}}(\tau, a), a)\,, \qquad \forall \tau \leq -N_1(a)\,. \tag{60}$$

In fact, in [12] the set $\tilde{\Xi}_W(\tau, a)$ reduces to a singleton, since (14) is smooth. However, the proofs go through also in this non-smooth setting without further changes and the estimates are uniform for any $\tilde{\boldsymbol{Q}}(\tau, a) \in \tilde{\Xi}_W(\tau, a)$, since $\tilde{\Xi}_W(\tau, a)$ is compact.

Thus, following the proof of [12, Lemma 31], we get this result.

Lemma 55. *([12, Lemma 31]) Assume* $(\mathbf{H}_\ell)$, (55) *and* $(\mathbf{W}_s)$. *For any* $\varepsilon > 0$ *sufficiently small, define*

$$a = a(\varepsilon) := \left(\frac{\alpha^q \varepsilon}{(1+\varepsilon)\overline{K}}\right)^{\frac{p-1}{q-p}} \leq a_1(\overline{K}), \tag{61}$$

where $\overline{K} = \sup_{t\in\mathbb{R}} K(t)$ *and* $a_1(\overline{K})$ *was introduced in* (56). *Then, there is* $\mathcal{T}(\varepsilon) := -N_1(a(\varepsilon))$ *such that for any* $\tau \leq \mathcal{T}(\varepsilon)$

$$T(\tilde{\boldsymbol{Q}}(\tau,a),a) > \tau + \frac{1}{\alpha}\ln\left(\frac{a}{|R_y(\tau,a)|}\right), \tag{62}$$

$$T^1(\tilde{\boldsymbol{Q}}(\tau,a),a) < \tau + \frac{1+\varepsilon}{\alpha}\ln\left(\frac{a}{|R^1_y(\tau,a)|}\right). \tag{63}$$

The previous result can be proved using (57) and showing with a Gronwall argument that

$$a\mathrm{e}^{-\alpha(t-\tau)} < |y(t)| < a\,\mathrm{e}^{-\frac{\alpha}{1+\varepsilon}(t-\tau)}, \qquad \text{whenever } \tau < t < T(\tilde{\boldsymbol{Q}}(\tau,a),a). \tag{64}$$

Lemma 56. ([12, Lemma 33]) *Assume* $(\mathbf{H}_\ell)$, $(\mathbf{H}_\uparrow)$, *and* $(\mathbf{W}_s)$. *Let* $\varepsilon > 0$ *and* $a = a(\varepsilon) > 0$ *be as in Lemma 55, then there are positive constants* $C = C(a,\varepsilon)$ *and* $N_C = N_C(a,\varepsilon)$ *such that*

$$T^1(\tilde{\boldsymbol{Q}}(\tau,a),a) < C(a,\varepsilon) + |\tau|\left(\ell\,\frac{1+\varepsilon}{\alpha}\,\frac{p-1}{p} - 1\right) \quad \textit{for any } \tau \leq -N_C. \tag{65}$$

Proof. We sketch the proof for convenience of the reader. From Lemma 54, taking into account (16) and (58), we find

$$\frac{p-1}{p}|R^1_y(\tau,a)|^{\frac{p}{p-1}} = \mathcal{H}(\boldsymbol{R}^1(\tau,a);\tau) = \mathcal{H}(\tilde{\boldsymbol{Q}}(\tau,a);\tau) = c(a)\mathrm{e}^{\ell\tau} + o\left(\mathrm{e}^{\ell\tau}\right),$$

as $\tau \to -\infty$. So, there is a constant $c^1(a) > 0$, independent of τ, such that

$$|R^1_y(\tau,a)| = c^1(a)\mathrm{e}^{\frac{p-1}{p}\ell\tau} + o\left(\mathrm{e}^{\frac{p-1}{p}\ell\tau}\right), \qquad \text{as } \tau \to -\infty. \tag{66}$$

Then, from (63), we find $C(a,\varepsilon) > 0$ and $N_C = N_C(a,\varepsilon) > 0$ such that

$$T^1(\tilde{\boldsymbol{Q}}(\tau,a),a) < \tau + \frac{1+\varepsilon}{\alpha}\ln\left(\frac{2a}{c^1(a)\mathrm{e}^{\frac{p-1}{p}\ell\tau}}\right) = C(a,\varepsilon) + |\tau|\left(\ell\,\frac{1+\varepsilon}{\alpha}\,\frac{p-1}{p} - 1\right) \tag{67}$$

for any $\tau \leq -N_C$. □

Now, when $\ell < \ell^*_p = \frac{n-p}{p-1}$, there is a sufficiently small $\varepsilon > 0$ such that $\ell < \frac{p}{(p-1)}\frac{\alpha}{(1+\varepsilon)} = \frac{\ell^*_p}{1+\varepsilon} < \ell^*_p$.

Then, recalling (60) and using a truncation argument, we get the following.

Proposition 57. *Assume* $(\mathbf{H}_\ell)$ *with* $\ell < \ell^*_p$, *and let* $0 < a \leq \bar{a}_1 := a_1(\overline{K}(\mathcal{T}_0))$, *where* $\mathcal{T}_0$ *was introduced at page 23. Then, for any* $M > 0$ *there is* $\tilde{N}(a) > 0$ *such that* $T(\tilde{\boldsymbol{Q}}(\tau,a),a) < -M$ *if* $\tilde{\boldsymbol{Q}}(\tau,a) \in \tilde{\Xi}_W(\tau,a)$ *and* $\tau < -\tilde{N}(a)$, *i.e.*

$$\lim_{\tau\to-\infty} T(\tilde{\boldsymbol{Q}}(\tau,a),a) = -\infty, \qquad \textit{uniformly for any } \tilde{Q}(\tau,a) \in \tilde{\Xi}_W(\tau,a).$$

Proof of Proposition 7. Let $0 < a \le \bar{a}_1$; from Lemma 52 for any $d \ge \tilde{D}_W(-N_1(a))$ there is $\tau_W(d) \le -N_1(a)$ such that $\tilde{Q}(d) = \tilde{Q}(\tau_W(d), a) := \boldsymbol{\phi}(\tau_W(d); d) \in \tilde{\Xi}_W(\tau_W(d), a)$; so $d \in [\hat{D}_W(\tau_W(d)), \tilde{D}_W(\tau_W(d))]$. Thus, combining Lemma 52 and Proposition 57, we find

$$\lim_{d\to+\infty} T(\tilde{Q}(d), a) = \lim_{\tau_W(d)\to-\infty} T(\tilde{Q}(\tau_W(d), a), a) = -\infty.$$

Hence, $\lim_{d\to+\infty} R(d) = \exp[\lim_{d\to+\infty} T(\tilde{Q}(d), a)] = 0$, and the proposition is proved. □

5.2. Asymptotic behavior of $R(d)$ for d large when $\ell \ge \ell_p^*$.

Let $\ell \ge \ell_p^*$, choose $\varepsilon \in]0, \frac{p+\delta}{p\alpha}[$ small enough so that

$$a = a(\varepsilon) := \left(\frac{\alpha^q \varepsilon}{[A_0 + 1](1+\varepsilon)} \right)^{\frac{p-1}{q-p}} < \bar{a}_1 \,. \tag{68}$$

Let $\tilde{\boldsymbol{Q}} \in \tilde{\Xi}_W(\tau, a)$, and consider the trajectory $\boldsymbol{\phi}(\cdot; \tau, \tilde{\boldsymbol{Q}})$. We denote by $T(\tilde{\boldsymbol{Q}})$ the time, if it exists, at which such a trajectory crosses the negative y semi-axis. Otherwise, we set $T(\tilde{\boldsymbol{Q}}) = +\infty$.

Then, we set

$$\underline{T}(\tau, a) = \min\{T(\tilde{\boldsymbol{Q}}) \mid \tilde{\boldsymbol{Q}} \in \tilde{\Xi}_W(\tau, a)\}.$$

Proposition 58. *Assume* $(\mathbf{H}_\ell)$*, fix* $\varepsilon \in\,]0, \frac{p+\delta}{p\alpha}[$*, and* $a = a(\varepsilon)$ *as in* (68)*. If* $\ell \ge \ell_p^*$*, we have* $\liminf_{\tau\to-\infty} \underline{T}(\tau, a) > -\infty$.

Proof. Arguing by contradiction, assume that there is a sequence $\tau_n \to -\infty$ and a sequence of points $\tilde{\boldsymbol{Q}}_n \in \tilde{\Xi}_W(\tau_n, a)$ such that $T(\tilde{\boldsymbol{Q}}_n) = \underline{T}(\tau_n, a)$ and $\lim_n \underline{T}(\tau_n, a) = -\infty$. Following [12, Lemmas 39-40], we can provide the following estimate of the energy at the crossing point $\tilde{\boldsymbol{R}}_n = \boldsymbol{\phi}(T(\tilde{\boldsymbol{Q}}_n); \tau_n, \tilde{\boldsymbol{Q}}_n)$

$$\mathcal{H}\big(\tilde{\boldsymbol{R}}_n; T(\tilde{\boldsymbol{Q}}_n)\big) \le 2\mathcal{H}(\tilde{\boldsymbol{Q}}_n; \tau_n)\,.$$

Then, from (54) and (64), arguing as in the proof of Lemma 56 (see also [12, proof of Proposition 41]), we see that there is a positive constant $\check{C}_1(\varepsilon)$ such that

$$\underline{T}(\tau_n, a) = T(\tilde{\boldsymbol{Q}}_n) \ge \check{C}_1(\varepsilon) + \tau_n \left[1 - \frac{1}{\alpha} \frac{p-1}{p} \ell \right] \ge \check{C}_1(\varepsilon)\,, \tag{69}$$

where the last inequality holds since we have assumed $\ell \ge \ell_p^* = \frac{n-p}{p-1} = \frac{\alpha p}{p-1}$. We get a contradiction, thus concluding the proof. □

Now, we are ready to prove Propositions 8 and 9.

Proof of Proposition 8. Let us choose $a > 0$ as in (68). Given a sequence $(d_n)_n$ with $\lim_n d_n = +\infty$, we can argue as in the proof of Proposition 7: from Lemma 52 and Proposition 49 we can find $\tilde{\boldsymbol{Q}}(\tau_n, a) \in \tilde{\Xi}_W(\tau_n, a)$ and $(\tau_n)_n$ with $\lim_n \tau_n = -\infty$ such that $d_n = d_W(\tilde{\boldsymbol{Q}}(\tau_n, a), \tau_n)$ for every n large. Then, from (12) and Proposition 58, we see that

$$\liminf_{n\to+\infty} \tilde{R}(d_n) = \liminf_{n\to+\infty} \mathrm{e}^{T(\tilde{\boldsymbol{Q}}(\tau_n, a))} \ge \liminf_{\mathcal{T}\to-\infty} \mathrm{e}^{\underline{T}(\mathcal{T}, a)} > 0,$$

so we conclude. □

Proof of Proposition 9. The Proposition is proved using Proposition 49 and Lemma 52, and following Propositions 42-43 in [12]. □

5.3. Construction of the stable set and proofs of Lemma 11 and Proposition 12.

To prove Theorem C, as in the smooth case considered in [12], we need to construct the stable set: this will enable us to prove Lemma 11 and Proposition 12. So, in this section we use the ideas introduced in [10,15,16] to build up a stable set $\hat{M}^s(\mathcal{T})$ analogous to the unstable set $\hat{M}^u(\mathcal{T})$ defined in Proposition 25. Then, we follow [12, §4.2.1] to prove Lemma 11.

Assume first $(\mathbf{W}_s)$; the construction of the stable set relies again on a comparison between the non-autonomous system (14) and the frozen autonomous systems (18) with $\varpi \equiv 0$ and $\varpi \equiv 2\overline{K}$.

Remark 59. System (18) can be solved explicitly when $\varpi = 0$. In particular, the solutions $\underline{\boldsymbol{\phi}}(t) = (\underline{x}(t), \underline{y}(t))$ of this system which converge to the origin as $t \to +\infty$, and such that $\underline{x}(t) > 0$ when $t \gg 0$, can be characterized as follows

$$(\underline{x}(t), \underline{y}(t)) = (x_0 \mathrm{e}^{-\frac{\alpha}{p-1}t}, y_0 \mathrm{e}^{-\alpha t}), \quad \text{where } \ y_0 = -\left(\frac{\alpha p}{p-1} x_0\right)^{p-1}$$

for any $x_0 > 0$. Further, the graph of all these trajectories $\underline{\boldsymbol{\phi}}(t)$ is given by $\underline{\boldsymbol{\Psi}} := \{(x,y) \mid y = -\left(\frac{\alpha p}{p-1} x\right)^{p-1}, \ x > 0\}$.

By $\overline{\boldsymbol{E}}_{\boldsymbol{s}} = (\overline{E}_{x,s}, \overline{E}_{y,s})$ and $\overline{\boldsymbol{\Gamma}}_{\boldsymbol{s}}$ we respectively denote the critical point and the graph of the homoclinic trajectory of the autonomous dynamical system (18) where $\varpi \equiv 2\overline{K}$. The solutions of this system are denoted by $\overline{\boldsymbol{\phi}}(t) = (\overline{x}(t), \overline{y}(t))$.

The reader can follow the next construction by looking at Figure 6. Focus on the semi-line $x = \overline{E}_{x,s}$ contained in $\dot{x} < 0$: it intersects transversely $\underline{\boldsymbol{\Psi}}$ in a point, say $\underline{\boldsymbol{P}}_{\boldsymbol{s}}$, and it intersects the path $\overline{\boldsymbol{\Gamma}}_{\boldsymbol{s}}$ in another point, say $\overline{\boldsymbol{P}}_{\boldsymbol{s}}$. Notice that $\overline{\boldsymbol{P}}_{\boldsymbol{s}}$ lies above $\underline{\boldsymbol{P}}_{\boldsymbol{s}}$.

Definition 60. We denote by $\partial\mathcal{F}_s^{\uparrow}$ the branch of $\overline{\boldsymbol{\Gamma}}_{\boldsymbol{s}}$ between the origin and $\overline{\boldsymbol{P}}_{\boldsymbol{s}}$, by $\partial\mathcal{F}_s^{\downarrow}$ the branch of $\underline{\boldsymbol{\Psi}}$ between the origin and $\underline{\boldsymbol{P}}_{\boldsymbol{s}}$, by $\mathfrak{f}_s$ the vertical segment between $\overline{\boldsymbol{P}}_{\boldsymbol{s}}$ and $\underline{\boldsymbol{P}}_{\boldsymbol{s}}$. Then, we denote by $\mathcal{F}_s$ the compact set enclosed by $\partial\mathcal{F}_s^{\downarrow}$, $\partial\mathcal{F}_s^{\uparrow}$ and $\mathfrak{f}_s$, see Figure 6.

Lemma 61. *Assume* $(\mathbf{W}_s)$ *and fix* $T \in \mathbb{R}$. *If* $\boldsymbol{Q} \in [(\partial\mathcal{F}_s^{\uparrow} \cup \partial\mathcal{F}_s^{\downarrow}) \setminus \{(0,0)\}]$, *then there is* $\delta = \delta(\boldsymbol{Q}) > 0$ *such that the trajectory* $\boldsymbol{\phi}(t; T, \boldsymbol{Q}) \notin \mathcal{F}_s$ *for any* $t \in]T, T+\delta[$.

Proof. Let us set

$$\overline{\mathcal{H}}^s(x,y) = \alpha x y + \tfrac{p-1}{p}|y|^{\frac{p}{p-1}} + 2\overline{K}\tfrac{|x|^q}{q},$$
$$\underline{\mathcal{H}}^s(x,y) = \alpha x y + \tfrac{p-1}{p}|y|^{\frac{p}{p-1}}.$$

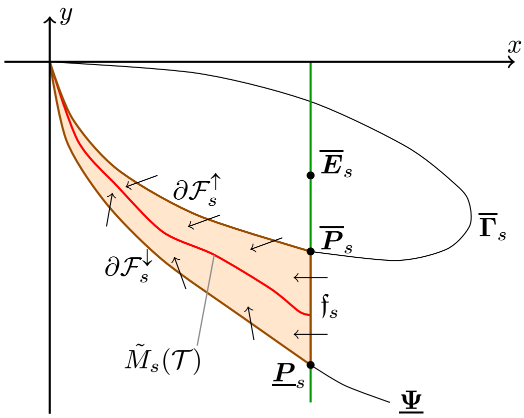

Figure 6. The construction of the sets $\mathcal{F}_s$ and $\tilde{M}^s(\mathcal{T})$. The arrows suggest the direction of the vector fields of (14) along the boundary of $\mathcal{F}_s$

Notice that $\overline{\boldsymbol{\Gamma}}_s$ and $\underline{\boldsymbol{\Psi}}$ lie respectively in the 0-level set of $\overline{\mathcal{H}}^s$ and $\underline{\mathcal{H}}^s$. Further, if $\boldsymbol{\phi}(t) = (x(t), y(t))$ is a trajectory of (14), we find the following estimates analogous to (28)

$$\begin{aligned} &\tfrac{d}{dt}\overline{\mathcal{H}}^s(\boldsymbol{\phi}(t)) = \big(2\overline{K} - K(t)\big)\, x(t)^{q-1}\ \dot{x}(t)\,, \\ &\tfrac{d}{dt}\underline{\mathcal{H}}^s(\boldsymbol{\phi}(t)) = -K(t)\, x(t)^{q-1}\ \dot{x}(t)\,, \end{aligned} \tag{70}$$

so the lemma follows. □

Remark 62. Fix $0 < a < a_s(\overline{K}) := \min\{\bar{a}_0, |\overline{E}_{y,s}|\}$, then $\overline{\boldsymbol{\Gamma}}_s \cap \mathcal{L}(a) \neq \emptyset$. Further, there exists $\bar{N}_1(a) > N_1(a) > 0$ such that $\tilde{\Xi}_W(\tau, a) \subset \mathfrak{g}(\tau, a) \subset \mathcal{F}_s$ for every $\tau < -\bar{N}_1(a)$, cf. Remark 45.

Let us define the stable set

$$\tilde{M}^s(\mathcal{T}) := \{\boldsymbol{Q} \mid \boldsymbol{\phi}(t; \mathcal{T}, \boldsymbol{Q}) \in \mathcal{F}_s \quad \text{for any } t \geq \mathcal{T}\}. \tag{71}$$

From an analysis of the phase portrait we get the following.

Remark 63. If $\boldsymbol{Q} \in \tilde{M}^s(\mathcal{T})$, then $\dot{x}(t; \mathcal{T}, \boldsymbol{Q}) < 0 < \dot{y}(t; \mathcal{T}, \boldsymbol{Q})$ for any $t \geq \mathcal{T}$, and $\boldsymbol{\phi}(t; \mathcal{T}, \boldsymbol{Q})$ converges to the origin as $t \to +\infty$.

Now, using Lemma 61, we note that for any $\boldsymbol{Q} \in \mathcal{F}_s \setminus \tilde{M}_s(\mathcal{T})$ there is $T_s(\boldsymbol{Q}, \mathcal{T})$ such that $\boldsymbol{\phi}(t; \tau, \boldsymbol{Q}) \in \mathcal{F}_s$ for any $t \in [\tau, T_s(\boldsymbol{Q}, \mathcal{T})]$, and leaves $\mathcal{F}_s$ crossing transversely $\partial\mathcal{F}_s^{\uparrow} \cup \partial\mathcal{F}_s^{\downarrow}$ at $t = T_s(\boldsymbol{Q}, \mathcal{T})$. We first define the map:

$$\psi_s : \mathcal{F}_s \setminus \tilde{M}_s(\mathcal{T}) \to \partial\mathcal{F}_s^{\uparrow} \cup \partial\mathcal{F}_s^{\downarrow}, \qquad \psi_s(\boldsymbol{Q}) = \boldsymbol{\phi}(T_s(\boldsymbol{Q}, \mathcal{T}); \mathcal{T}, \boldsymbol{Q}),$$

and from Lemma 61 we deduce that ψ_s is continuous.

Then we define $\tilde{\psi}_s : \mathcal{F}_s \to \partial\mathcal{F}_s^{\uparrow} \cup \partial\mathcal{F}_s^{\downarrow}$ as

$$\tilde{\psi}_s(\boldsymbol{Q}) := \begin{cases} \psi_s(\boldsymbol{Q}) \text{ if } \boldsymbol{Q} \in \mathcal{F}_s \setminus \tilde{M}_s(\mathcal{T}) \\ (0,0) \qquad \text{if } \boldsymbol{Q} \in \tilde{M}_s(\mathcal{T}) \end{cases},$$

and, reasoning as in Lemma 35, we show that $\tilde{\psi}_s$ is continuous, too.

We can repeat the reasoning in Remark 36, Lemma 37, and Proposition 38 to deduce the following property of $\tilde{M}^s(\mathcal{T})$.

Proposition 64. *Assume* $(\mathbf{W}_s)$*; for any* $\mathcal{T} \in \mathbb{R}$ *the set* $\tilde{M}^s(\mathcal{T})$ *is a non-empty compact set. Further, there is a compact connected set* $\hat{M}^s(\mathcal{T}) \subset \tilde{M}^s(\mathcal{T})$ *which contains the origin and intersects* $\mathfrak{f}_s$*.*

Since we know the exact expressions of the trajectories, cf. respectively Remark 59 and (23) together with Remark 16, we can find $c = \overline{E}_{x,s} > 0$ and $C > 0$, both independent of τ, such that

$$c\,\mathrm{e}^{-\frac{\alpha}{p-1}(t-\tau)} = \underline{x}(t;\tau,\underline{\boldsymbol{P}}_{\boldsymbol{s}}) \le \overline{x}(t;\tau,\overline{\boldsymbol{P}}_{\boldsymbol{s}}) \le C\,\mathrm{e}^{-\frac{\alpha}{p-1}(t-\tau)} \quad \text{for any } t \ge \tau\,. \tag{72}$$

Via (72) we obtain similar estimates for the solutions of the non-autonomous system (14).

Lemma 65. *Assume* $(\mathbf{W}_s)$*. Then, there are positive constants* $C > c > 0$ *such that for every* $\tau \in \mathbb{R}$ *and* $\tilde{\boldsymbol{Q}} \in \mathfrak{f}_s \cap \tilde{M}^s(\tau)$ *we have*

$$c\,\mathrm{e}^{-\frac{\alpha}{p-1}(t-\tau)} \le x(t;\tau,\tilde{\boldsymbol{Q}}) \le C\,\mathrm{e}^{-\frac{\alpha}{p-1}(t-\tau)} \quad \textit{for any } t \ge \tau\,. \tag{73}$$

Proof. We claim that

$$\underline{x}(t;\tau,\underline{\boldsymbol{P}}_{\boldsymbol{s}}) \le x(t;\tau,\tilde{\boldsymbol{Q}}) \le \overline{x}(t;\tau,\overline{\boldsymbol{P}}_{\boldsymbol{s}}) \quad \text{for any } t \ge \tau\,. \tag{74}$$

Then, the conclusion will follow from (72).

We just give the proof of the first inequality, the other being similar. Observe that we have the equalities in (74) at $t = \tau$; let $T = \sup\{t \ge \tau : \underline{x}(t;\tau,\underline{\boldsymbol{P}}_{\boldsymbol{s}}) \le x(t;\tau,\tilde{\boldsymbol{Q}})\}$. Assume by contradiction that $T \in \mathbb{R}$. Then, since $\tilde{\boldsymbol{Q}} \in \tilde{M}^s(\tau)$, by construction the trajectory remains in $\mathcal{F}_s$ so that $\underline{x}(T;\tau,\underline{\boldsymbol{P}}_{\boldsymbol{s}}) = x(T;\tau,\tilde{\boldsymbol{Q}})$ implies $\dot{\underline{x}}(T;\tau,\underline{\boldsymbol{P}}_{\boldsymbol{s}}) < \dot{x}(T;\tau,\tilde{\boldsymbol{Q}})$. Hence, we have $\underline{x}(t;\tau,\underline{\boldsymbol{P}}_{\boldsymbol{s}}) < x(t;\tau,\tilde{\boldsymbol{Q}})$ in a right neighborhood of T, leading to a contradiction. $\square$

Let $0 < a < a_s(\overline{K})$ and set

$$\Lambda(a) := \{(x,y) \mid 0 \le x \le \tfrac{1}{\alpha}\,|y|^{\frac{1}{p-1}},\ \ -a \le y \le 0\}$$

(i.e. $\dot{x} \le 0 \le x$ and $-a \le y \le 0$).

Definition 66. Let $\tau \in \mathbb{R}$; denote by $\tilde{\Xi}^s_W(\tau,a) = \tilde{M}^s(\tau) \cap \mathcal{L}(a)$ and by $\hat{W}^s(\tau,a)$ the connected branch of $\hat{M}^s(\tau)$ joining the origin to $\mathcal{L}(a)$ so that $\hat{W}^s(\tau,a) \subset [\Lambda(a) \cap \tilde{M}^s(\tau)]$.

Remark 67. Let $0 < a < a_s(\overline{K})$, $\tau \in \mathbb{R}$, and $\boldsymbol{Q} \in \tilde{\Xi}^s_W(\tau,a)$, then $\dot{y}(t;\tau,\boldsymbol{Q}) > 0$ for any $t \ge \tau$, therefore $\boldsymbol{\phi}(t;\tau,\boldsymbol{Q}) \in \Lambda(a)$ for any $t \ge \tau$. Further, adapting slightly the argument of Lemma 65, we can find $C(a) > c(a) > 0$ such that

$$c(a)\,\mathrm{e}^{-\frac{\alpha}{p-1}(t-\tau)} \le x(t;\tau,\boldsymbol{Q}) \le C(a)\,\mathrm{e}^{-\frac{\alpha}{p-1}(t-\tau)} \quad \text{for any } t \ge \tau\,. \tag{75}$$

Now, using (75), (17), and arguing as in the proof of Lemma 49 in [12], we find the following.

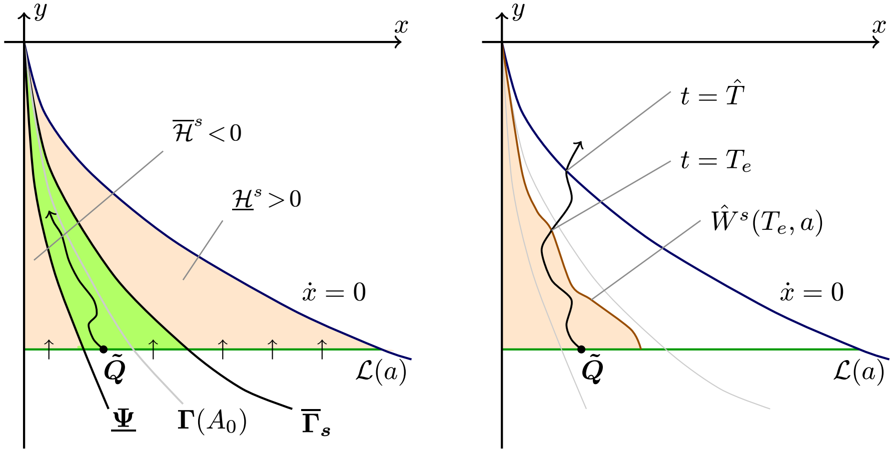

FIGURE 7. A sketch of the arguments in the proof of Lemma 70. *On the left:* if $\hat{T} = +\infty$, the trajectory $\boldsymbol{\phi}(t; \tau, \tilde{\boldsymbol{Q}})$ converges to the origin as $t \to +\infty$. From (70), we have $\frac{d}{dt}\overline{\mathcal{H}}^s(\boldsymbol{\phi}(t; \tau, \tilde{\boldsymbol{Q}})) < 0$ and $\frac{d}{dt}\underline{\mathcal{H}}^s(\boldsymbol{\phi}(t; \tau, \tilde{\boldsymbol{Q}})) > 0$ in the colored region. Since $\overline{\mathcal{H}}^s(\boldsymbol{\phi}(t; \tau, \tilde{\boldsymbol{Q}})) < 0$ in the pink region on the right, while $\underline{\mathcal{H}}^s(\boldsymbol{\phi}(t; \tau, \tilde{\boldsymbol{Q}})) > 0$ in the pink region on the left, then the trajectory necessarily remains in the green region in the middle which is contained in $\mathcal{F}_s$, so $\tilde{Q} \in \tilde{M}^s(\tau)$. *On the right:* if $\hat{T} < +\infty$, we argue by contradiction and introduce the set $\mathcal{C}(\tau, a)$. In the picture, the trajectory $\boldsymbol{\phi}(t; \tau, \tilde{\boldsymbol{Q}})$ leaves $\mathcal{C}(T_e, a)$ at time $T_e < \hat{T}$ through $\hat{W}^s(T_e, a)$

Lemma 68. *Assume* $(\mathbf{H}_\ell)$ *with* $\ell \le \frac{\alpha}{p-1} q = \frac{n+\delta}{p-1}$ *and* $(\mathbf{W}_s)$. *Then, for every* $0 < a < a_s(\overline{K})$ *there is* $N_2(a) > 0$ *such that* $\mathcal{H}(\boldsymbol{Q}; \tau) < 0$ *for any* $\tau < -N_2(a)$ *and* $\boldsymbol{Q} \in \tilde{\Xi}^s_W(\tau, a)$.

Remark 69. Assume $(\mathbf{H}_\ell)$. Then, according to Lemma 54, for every $0 < a < \bar{a}_0$ there exists $\hat{N}_1(a) > N_1(a) > 0$ such that

$$\mathcal{H}(\tilde{\boldsymbol{Q}}; \tau) > 0 \quad \text{for any } \tau < -\hat{N}_1(a) \text{ and } \tilde{\boldsymbol{Q}} \in \tilde{\Xi}_W(\tau, a). \tag{76}$$

Now, we adapt the ideas of Lemma 50 in [12] to prove the following.

Lemma 70. *Assume* $(\mathbf{H}_\ell)$ *with* $\ell \le \frac{n+\delta}{p-1}$ *and* $(\mathbf{W}_s)$. *Then, for every* $0 < a < a_s(\overline{K})$, *there is* $N_3(a) > 0$ *such that, for any* $\tau < -N_3(a)$ *and* $\tilde{\boldsymbol{Q}} \in \tilde{\Xi}_W(\tau, a)$, *the trajectory* $\boldsymbol{\phi}(t; \tau, \tilde{\boldsymbol{Q}})$ *corresponds to a crossing solution, i.e. there is* $T = T(\tau, a) > \tau$ *such that* $x(t; \tau, \tilde{\boldsymbol{Q}}) > 0$ *for any* $t < T$ *and it becomes null at* $t = T$.

Further, $\dot{x}(t; \tau, \tilde{\boldsymbol{Q}}) < 0 < \dot{y}(t; \tau, \tilde{\boldsymbol{Q}})$ *for any* $\tau \le t \le T(\tau, a)$.

Proof. Let $0 < a < a_s(\overline{K}) \le a_0$, $N_3(a) = \max\{\bar{N}_1(a), \hat{N}_1(a), N_2(a)\}$ and $\tau < -N_3(a)$, where $\bar{N}_1(a)$, $\hat{N}_1(a)$ and $N_2(a)$ are as in Remarks 62, 69, and Lemma

68, respectively. Let $\tilde{\boldsymbol{Q}} = (\tilde{Q}_x, \tilde{Q}_y) \in \tilde{\Xi}_W(\tau, a)$, and consider the trajectory $\boldsymbol{\phi}(t; \tau, \tilde{\boldsymbol{Q}})$. Observe that by construction $\dot{x}(\tau; \tau, \tilde{\boldsymbol{Q}}) < 0 < \dot{y}(\tau; \tau, \tilde{\boldsymbol{Q}})$ and define

$$\hat{T} := \sup\{T > \tau \mid \dot{x}(t; \tau, \tilde{\boldsymbol{Q}}) < 0 < \dot{y}(t; \tau, \tilde{\boldsymbol{Q}}) \,, \text{ and } x(t; \tau, \tilde{\boldsymbol{Q}}) > 0 \text{ for every } t \in [\tau, T]\}.$$

If $\hat{T} = +\infty$, then $\boldsymbol{\phi}(t; \tau, \tilde{\boldsymbol{Q}})$ converges to the origin as $t \to +\infty$, so $\lim_{t\to+\infty} \overline{\mathcal{H}}^s(\boldsymbol{\phi}(t; \tau, \tilde{\boldsymbol{Q}})) = \lim_{t\to+\infty} \underline{\mathcal{H}}^s(\boldsymbol{\phi}(t; \tau, \tilde{\boldsymbol{Q}})) = 0$. Hence, from (70) we get $\boldsymbol{\phi}(t; \tau, \tilde{\boldsymbol{Q}}) \in \mathcal{F}_s$ for any $t \geq \tau$ (see Figure 7 on the left), i.e., $\tilde{\boldsymbol{Q}} \in \tilde{M}^s(\tau)$. However, from Lemma 68 it follows that $\mathcal{H}(\tilde{\boldsymbol{Q}}; \tau) < 0$, which contradicts (76), so $\hat{T} < +\infty$.

Notice that $\dot{y}(t) > 0$ along $\mathcal{L}(a)$ for any $t \in \mathbb{R}$, since $a < a_s(\overline{K}) \leq |\overline{E}_{y,s}|$. Further, $\tilde{\boldsymbol{Q}} \in \mathcal{L}(a)$, so, from an analysis of the phase portrait, we see that either $x(\hat{T}; \tau, \tilde{\boldsymbol{Q}}) = 0$ or $\dot{x}(\hat{T}; \tau, \tilde{\boldsymbol{Q}}) = 0$. If we assume the former, we are done: the trajectory $\boldsymbol{\phi}(t; \tau, \tilde{\boldsymbol{Q}})$ corresponds to a crossing solution $u(r; d)$ of (5). So, assume the latter by contradiction.

Let us set $\mathcal{Q}_4 := \{(x, y) \mid y < 0 < x\}$. We introduce an auxiliary system which is C^1 in $\mathcal{Q}_4$, coincides with the original one in $\Lambda(a)$, and is negatively invariant in $\mathcal{Q}_4 \setminus \Lambda(a)$. We denote by $\boldsymbol{\phi}_m(t; \tau, \boldsymbol{Q})$ the trajectories of such a system.

Consider the compact connected set $\hat{W}^s(\hat{T}, a)$, and define its evolution via the modified system at $t \leq \hat{T}$, by setting

$$W^s_m(t, \hat{T}; a) := \{\boldsymbol{\phi}_m(t; \hat{T}, \boldsymbol{Q}) \mid \boldsymbol{Q} \in \hat{W}^s(\hat{T}, a)\}.$$

Clearly, if $\boldsymbol{Q} \in \hat{W}^s(\hat{T}, a)$, then there is $T_o \leq \hat{T}$ such that $\boldsymbol{\phi}_m(t; \hat{T}, \boldsymbol{Q}) \in \Lambda(a)$ for any $t \geq T_o$, so that

$$\boldsymbol{\phi}_m(t; \hat{T}, \boldsymbol{Q}) \equiv \boldsymbol{\phi}(t; \hat{T}, \boldsymbol{Q}) \qquad \text{for any } t \geq T_o.$$

Further, we can assume without loss of generality that $\boldsymbol{\phi}_m(t; \hat{T}, \boldsymbol{Q}) \in \mathcal{Q}_4 \setminus \Lambda(a)$ for any $t < T_o$. Then, it is easy to check that

$$\hat{W}^s(t, a) = W^s_m(t, \hat{T}; a) \cap \Lambda(a) \qquad \text{for any } t \leq \hat{T}. \tag{77}$$

Let us consider the compact set $\mathcal{C} = \mathcal{C}(\tau, a)$ delimited by $\hat{W}^s(\tau, a)$, the y negative semi-axis and the line $\mathcal{L}(a)$ for any $\tau \in \mathbb{R}$. We see that $\mathcal{C}(t, a) \subset \Lambda(a)$ for any $t \in [\tau, \hat{T}]$. From (76) and Lemma 68, we get $\tilde{\boldsymbol{Q}} \in \mathcal{C}(\tau, a)$.

So, there is $T_e = T_e(\tau, a) \in \,]\tau, \hat{T}]$ such that $\boldsymbol{\phi}(t; \tau, \tilde{\boldsymbol{Q}}) \in \mathcal{C}(t, a)$ for any $\tau \leq t \leq T_e$ and it leaves $\mathcal{C}(t, a)$ when $t > T_e$. Thus, either $\boldsymbol{\phi}(t; \tau, \tilde{\boldsymbol{Q}})$ crosses the y negative semi-axis at $t = T_e$, and the lemma is proved, or it crosses $\hat{W}^s(T_e, a) \subset \tilde{M}^s(T_e)$ at $t = T_e$ in $\boldsymbol{R}(T_e)$. Then, it follows that $\boldsymbol{\phi}(t; \tau, \tilde{\boldsymbol{Q}}) \equiv \boldsymbol{\phi}(t; T_e, \boldsymbol{R}(T_e)) \in \mathcal{F}_s$ for any $t \geq T_e$ and it converges to the origin as $t \to +\infty$, which contradicts $\hat{T} < +\infty$, so the lemma is proved. □

Now assume $\mathbf{(B_l)}$, i.e. there is $\rho_1 > 0$ such that $\mathcal{K}(r) \geq \mathcal{K}(\rho_1)$ when $r \geq \rho_1$, and define $\tau_1 := \ln(\rho_1)$. Then, setting

$$K_m(t) := \begin{cases} K(t) & \text{if } t \leq \tau_1\,, \\ K(\tau_1) & \text{if } t \geq \tau_1\,, \end{cases}$$

we get $K(t) \ge K_m(t)$ for any $t \in \mathbb{R}$. Set $\overline{K}_m := \sup\{K_m(t) \mid t \in \mathbb{R}\}$, repeating the truncation argument in [12, Lemma 51], we find the following.

Lemma 71. *Assume* $(\mathbf{H}_\ell)$ *with* $\ell \le \frac{n+\delta}{p-1}$ *and* $(\mathbf{B}_l)$. *Then, for every* $0 < a < a_s(\overline{K}_m)$ *the same conclusion as in Lemma 70 holds.*

Proof of Lemma 11. We fix $a \in]0, a_s(\overline{K})[$ if $(\mathbf{W}_s)$ holds, and $a \in]0, a_s(\overline{K}_m)[$ if $(\mathbf{B}_l)$ holds; consider the segment $\mathcal{L}(a)$ defined as in (43). From Lemma 48, there is $N_1(a) > 0$ such that, for every $\tau < -N_1(a)$, $\tilde{W}^u(\tau)$ intersects $\mathcal{L}(a)$ in the compact non-empty set $\tilde{\Xi}_W(\tau, a)$. Now, from Lemmas 70 and 71, we see that there is $N_3(a) > N_1(a)$ such that the trajectory $\boldsymbol{\phi}(t; \tau, \boldsymbol{Q})$ is a crossing trajectory for any $\boldsymbol{Q} \in \tilde{\Xi}_W(\tau, a)$ when $\tau < -N_3(a)$, i.e. it crosses the y negative semi-axis.

Further, according to Lemma 52, for any $d \ge \hat{D}_W(-N_3(a))$ there is $\tilde{\boldsymbol{Q}}(d) = \boldsymbol{\phi}(\tau_W(d); d) \in \tilde{\Xi}_W(\tau_W(d), a)$ and $\tau_W(d) \le -N_3(a)$, so the trajectory $\boldsymbol{\phi}(t; \tau_W(d), \tilde{\boldsymbol{Q}}(d))$ of (14) corresponds to the regular solution $u(r; d)$ of (5). Hence, $u(r; d)$ is a crossing solution for any $d \ge \hat{D}_W(-N_3(a))$, and the lemma is proved. □

Proof of Proposition 12. Fix $a \in]0, a_s(\overline{K})[$ if $(\mathbf{W}_s)$ holds and $a \in]0, a_s(\overline{K}_m)[$ if $(\mathbf{B}_l)$ holds. From the proof of Lemma 11, we see that $[\hat{D}_W(-N_3(a)), +\infty[\subset J$. Then, the case $\ell = \ell_p^*$ follows from Proposition 8. The case $\ell > \ell_p^*$ is proved using Proposition 51, and following Propositions 42, 46 in [12]. □

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