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Letter　　Editors' Suggestion

Harnessing nonadiabatic excitations promoted by a quantum critical point: Quantum battery and spin squeezing

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Crossing a quantum critical point in finite time challenges the adiabatic condition due to the closing of the energy gap, which ultimately results in the formation of excitations. Such nonadiabatic excitations are typically deemed detrimental in many scenarios, and consequently several strategies have been put forward to circumvent their formation. Here, however, we show how these nonadiabatic excitations—originated from the failure to meet the adiabatic condition due to the presence of a quantum critical point—can be controlled and thus harnessed to perform certain tasks advantageously. We focus on closed cycles reaching the quantum critical point of fully connected models analyzing two examples. First, a quantum battery that is loaded by approaching a quantum critical point, whose stored and extractable work increases exponentially via repeating cycles. Second, a scheme for the fast preparation of spin squeezed states containing multipartite entanglement that offer a metrological advantage, analogous to a two-axis twisting scheme. The corresponding figure of merit in both examples crucially depends on the universal critical exponents and the scaling of the protocol in the vicinity of the transition. Our results highlight the rich interplay between quantum thermodynamics and metrology with critical nonequilibrium dynamics.

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Introduction. Nonequilibrium dynamics triggered by a quantum phase transition (QPT) is one of the most fascinating aspects in the area of quantum many-body systems [1,2]. The understanding of nonequilibrium scaling behavior near a quantum critical point has attracted significant attention in condensed matter and statistical physics [1–4]. The ongoing theoretical efforts are benefiting from the extraordinary experimental progress made during the last decades in ultracold atomic and molecular gases, trapped ions, and solid state systems, where quantum many-body systems can now be realized and controlled to a very high degree of isolation [5–12]. These advances are opening exciting new directions aiming at exploiting and harnessing quantum effects to perform different tasks [13,14], for instance in the fields of quantum computation and information processing [15,16], quantum sensing [17,18], and quantum thermodynamics [19,20]. In this regard, a finite-time modulation of a control parameter in a quantum system may be essential to carry out the desired goal as, for example, in quantum thermodynamic engines [21,22] or in quantum annealing and adiabatic quantum computation [23]. However, a finite-time evolution may entail nonadiabatic excitations, which are typically detrimental to the performance and, therefore, different strategies have been proposed to circumvent their formation [24–28]. This is particularly relevant when the evolution traverses or reaches a QPT, since the vanishing energy gap at the critical point leads to the breakdown of the adiabatic condition [29], and the formation of excitations, as described within the Kibble-Zurek mechanism [4,30–32].

Among the distinct directions within the flourishing field of quantum thermodynamics [19,20], quantum many-body systems acting as working medium of heat engines or as quantum batteries are attracting considerable attention in the quest to enhance thermodynamic performance thanks to quantum effects [33–43]. Yet, the resulting nonadiabatic excitations due to existence of a critical point can impair their performance [44]. Hence, underpinning scenarios where the existence of a QPT can be advantageous is of primarily importance to devise strategies to scale up quantum machines and disclose novel settings to achieve quantum enhancement.

Conversely, it is well known that the ground state at a QPT can serve as a resource for quantum information processing tasks [45–47] due their large degree of entanglement, which can also be beneficial in quantum metrology [43,48–52] and quantum thermometry [53]. Nevertheless, the preparation of such critical ground states is challenged by the breakdown of the adiabatic condition, and thus either long evolution times and/or sophisticated protocols are typically required to exploit their critical features. In this manner, new strategies capable of yielding resourceful quantum states while minimizing the time and complex protocols are very desirable.

In this Letter we tackle these problems and show that nonadiabatic excitations, formed as a consequence of the vanishing energy gap at a critical point, can be harnessed and controlled in critical fully connected models featuring a QPT [54], whose effective description reduces to a driven quantum harmonic oscillator, such as the quantum Rabi model [55–57] or Lipkin-Meshkov-Glick (LMG) model [58–60], among others. We consider closed cycles in these systems reaching the critical point, and show the advantage that such nonadiabatic excitations entail for two different scenarios.

On the one hand, promoting excitations in a critical system can be considered as a resource to produce and store work. Besides a full characterization of the accumulated work and fluctuations thereof, we demonstrate that the consecutive repetition of M cycles can lead to an exponential increase of the stored and extractable work from such quantum critical battery. On the other hand, we show how to harness these nonadiabatic excitations to yield a large degree of spin squeezing in the LMG model, which describes the long-range interaction of N spin-1/2 particles. We show that such spin squeezed states contain multipartite entanglement and are therefore useful for quantum metrology tasks [18,49]. Owing to the inherently nonadiabatic nature of the protocol, such states are obtained in a fast fashion. Our results, which can be readily applied to different experimental platforms, highlight the rich interplay between quantum thermodynamics, quantum metrology, and critical nonequilibrium dynamics.

Preliminaries. We shall start our analysis considering a driven quantum harmonic oscillator ($\hbar = 1$),

$$H(t) = \omega a^\dagger a - \frac{g^2(t)\omega}{4}(a + a^\dagger)^2, \tag{1}$$

which effectively captures the critical features of different fully connected models such the LMG model [58–60], the critical quantum Rabi model, and related systems displaying a superradiant phase transition [55,56,61–65], as well as different realizations of Bose-Einstein condensates [66–69]. It is worth mentioning that these models have been realized experimentally [11,66–70]. The Hamiltonian of Eq. (1) is a valid description of these models for $|g| \leqslant g_c = 1$, where g_c denotes the critical point at which the energy gap $\epsilon(g) = \omega\sqrt{1-g^2}$ vanishes as $\epsilon(g) \approx |g - g_c|^{z\nu}$ with $z\nu = 1/2$ dependent on the critical exponents of the QPT [3]. The harmonic oscillator is described in terms of the standard creation and annihilation operators, $[a, a^\dagger] = 1$, and is driven according to a cyclic transformation reaching the critical point g_c. In particular, the time-dependent protocol reads as

$$g(t) = \begin{cases} g_c\left(1 - \frac{(\tau - t)^r}{\tau^r}\right), & 0 \leqslant t \leqslant \tau, \\ g_c\left(1 - \frac{(t-\tau)^r}{\tau^r}\right), & \tau < t \leqslant 2\tau, \end{cases} \tag{2}$$

where we have assumed, without loss of generality, that $g(0) = g(2\tau) = 0$. The nonlinear exponent $r > 0$ controls how the system approaches the critical region [71], namely, $|g_c - g(t)| \propto |t - \tau|^r$, and the rate at which the system is driven, $|\dot{g}(t)| = 2g_c r|t - \tau|^{r-1}\tau^{-r}$ (cf. Fig. 1).

Under these general considerations, one can show that the evolved state of the system at time t reads $|\psi(t)\rangle \propto e^{b(t)(a^\dagger)^2}|0\rangle$ where $|0\rangle$ is the initial state and the parameter $b(t)$ changes

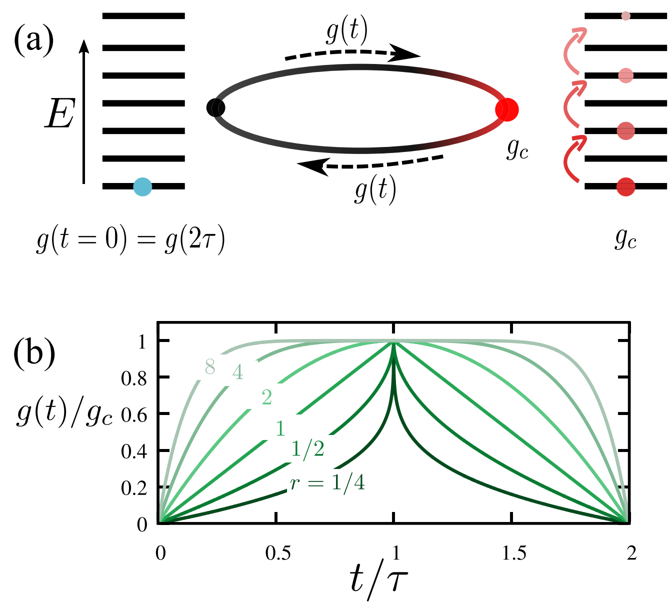

FIG. 1. (a) Sketch of the cyclic transformation to harness nonadiabatic excitations formed in the vicinity of the critical point g_c. In light of the Hamiltonian in Eq. (1), the state becomes squeezed. (b) Distinct profiles of $g(t)$ for different nonlinear exponents r [cf. Eq. (2)].

according to the equation of motion [72]

$$\dot{b}(t) = -i\omega\left(2b(t) - \frac{g^2(t)}{4}[1 + 2b(t)]^2\right), \tag{3}$$

with $b(0) = 0$. Hence, the state is simply a squeezed state $|\psi(t)\rangle = S(s)|0\rangle$ with $|s| = \text{artanh}[2|b(t)|]$ and $S(s) = e^{s(a^\dagger)^2/2 - s^* a^2/2}$ the squeezing operator [72–75]. By setting $\dot{b}(t) = 0$ one recovers the ground state squeezing of Eq. (1), that is, $|s| = -\log(1 - g^2)/4$ [55].

As a consequence of the breakdown of the adiabatic condition due to a vanishing energy gap $\epsilon(g_c) = 0$ [29], the initial state is never retrieved upon a slow cycle, regardless of how slow it is performed, that is, independently of how large $\omega\tau$ is [72,76]. In particular, the resulting final state after one cycle for $\omega\tau \gtrsim 1$ acquires a squeezing [72]

$$|s| = \text{arcosh}\left[\csc\left(\frac{\pi}{2 + 2z\nu r}\right)\right] \tag{4}$$

that solely depends on the critical exponents $z\nu$ and the nonlinear exponent r. Moreover, since nonadiabatic excitations are formed in the vicinity of the critical point, only the nonlinear behavior of $g(t)$ close to g_c is relevant [77].

Quantum critical battery. Let us consider a driven quantum harmonic oscillator as a quantum battery, which is loaded via the controllable creation of nonadiabatic excitations boosted at the quantum critical point. Through the cyclic transformation in Eq. (2) we leave the battery in a state $\rho = |\psi(2\tau)\rangle\langle\psi(2\tau)\rangle|$ and stored a certain amount of work $\langle W\rangle$. Then the maximum extractable work from the battery in a state ρ under any cyclic unitary transformation is given by the so-called ergotropy [78–81], given by $\mathcal{E} = \sum_n \epsilon_n(\rho_{nn} - r_n)$ where $H = \sum_n \epsilon_n |n\rangle\langle n|$ with $\epsilon_n = n\omega$ in our case, and the state is $\rho = \sum_n r_n |r_n\rangle\langle r_n|$ with r_n the eigenvalues of ρ sorted

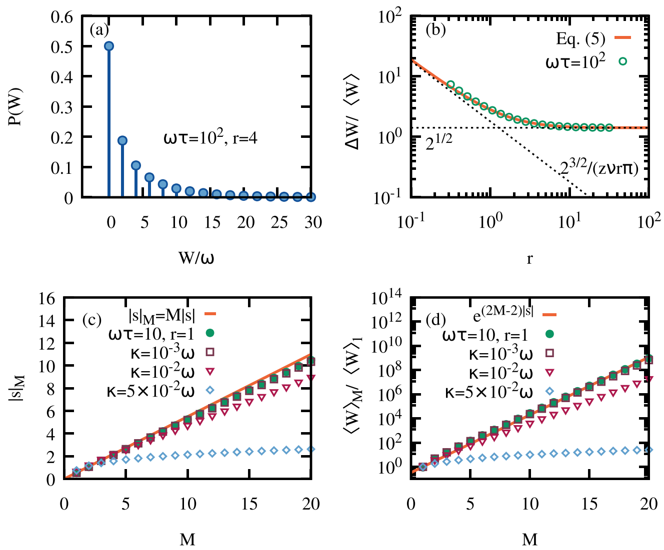

FIG. 2. (a) An example for the work probability distribution $P(W)$ for $\omega\tau = 10^2$ and $r = 4$. The resulting fluctuations of the work distribution $\Delta W/\langle W\rangle$ match the theoretical prediction (solid line), shown in (b) for $\omega\tau = 10^2$. The dashed lines are guide to eyes for the asymptotic behavior, $\Delta W/\langle W\rangle \sim 2^{1/2}$ for $z\nu r \gg 1$, and $2^{3/2}/(z\nu r\pi)$ for $z\nu r \ll 1$. As illustrated in (c), repeating M cycles can lead to an amplification of the squeezing parameter $|s|_M = M|s|$. This leads to an exponential increase of the stored and extractable work [cf. Eq. (6)] $\langle W\rangle_M/\langle W\rangle_1 \sim e^{(2M-2)|s|}$ as shown in (d). The open points in (c) and (d) show the decoherence effect on the battery with $N_{\rm th} = 2$.

in descending order, and $|r_n\rangle$ the corresponding eigenstate, and $\rho_{nn} = \sum_{n'} r_{n'} |\langle r_{n'}|n\rangle|^2$.

Upon completion of the protocol, the work performed on the battery follows a probability distribution given by $P(W) = \sum_{n=0} |c_n|^2 \delta(W - n\omega)$, where $c_n = \langle n|\psi(2\tau)\rangle$. Since $|\psi(2\tau)\rangle = S(s)|0\rangle$ only even Fock states are populated, and thus energy is transferred to the battery in units of $2n\omega$ with $n = 0, 1, \ldots$. Since the amount of squeezing can be controlled by shaping the protocol $g(t)$, i.e., tuning the nonlinear exponent r, so are the average work and its variance, whose expressions are given by $\langle W\rangle = \omega \sinh^2(|s|) = \omega \tan^{-2}[\pi/(2 + 2z\nu r)]$ and $\Delta^2 W = \langle W^2\rangle - \langle W\rangle^2 = 2\omega^2 \cos^2[\pi/(2 + 2z\nu r)] \sin^{-4}[\pi/(2 + 2z\nu r)]$, respectively. Moreover, the ergotropy equals the stored work $\mathcal{E} = \langle W\rangle$ [72]. In particular, the fluctuations of the work distribution are given by

$$\frac{\Delta W}{\langle W\rangle} = \sqrt{2} \cos^{-1}[\pi/(2 + 2z\nu r)]. \tag{5}$$

The fluctuations are dominant for $z\nu r \ll 1$, where $\Delta W/\langle W\rangle \sim 2^{3/2}(z\nu r\pi)^{-1}$, while they become constant in the opposite limit $\Delta W/\langle W\rangle \sim 2^{1/2}$. The resulting work distribution and fluctuations are illustrated in Figs. 2(a) and 2(b), respectively, which clearly shows how ΔW and $\langle W\rangle$, and thus $\mathcal{E}$, can be controlled by tailoring the nonlinear protocol. Note that the results plotted in Figs. 2(a) and 2(b) are independent of the specific duration τ provided $\omega\tau \gtrsim 1$, i.e., excluding the regime of sudden quenches, which leave the state trivially unaltered.

So far we have focused on the realization of a single cycle. Yet, performing M cycles can significantly boost the generated squeezing, and consequently the stored and extractable work beyond the linear scenario $\langle W\rangle_M \gg M\langle W\rangle_1$ and $\mathcal{E}_M \gg M\mathcal{E}_1$. Indeed, the state after M cycles under $g(t)$ can result in a M-fold squeezing, i.e., $|s|_M = M|s|$ with $|s|$ given in Eq. (4). The total duration is then equal to $2M\tau$ and $g(2m\tau + t) \equiv g(t)$ with $m = 0, 1, \ldots, M-1$. Remarkably, such proportional increase in $|s|_M$ translates into an exponential rise of the stored and extractable work [72],

$$\frac{\langle W\rangle_M}{\langle W\rangle_1} = \frac{\mathcal{E}_M}{\mathcal{E}_1} \sim e^{(2M-2)|s|}, \tag{6}$$

while the fluctuations enter the constant regime $\Delta W_M/\langle W\rangle_M \sim 2^{1/2}$. This is reported in Figs. 2(c) and 2(d). The solid points show the numerically computed squeezing after the Mth cycle, which follows well the predicted value $|s|_M = M|s|$. In this manner, the stored work $\langle W\rangle_M$ (and $\mathcal{E}_M$) grows exponentially with M, as exemplified in Fig. 2(d) for $r = 1$ and $\omega\tau = 10$. Similar results can be found for other choices of r and $\omega\tau$.

At this moment, a note is in order. Although for $M = 1$ the amount of squeezing is τ independent [cf. Eq (4)], its phase naturally depends on the total evolution time. This accumulated phase becomes however relevant when $M > 1$. As in any quantum engine or battery involving several cycles, the performance becomes phase sensitive [82,83]. In our case, we find that constructive interference of the accumulated phase leads to a sustained squeezing amplification across M cycles when $\arg\{b(2m\tau)\} = (2n+1)\pi/2$ with $n = 0, 1$ and $m = 1, 2, \ldots, M$ [72]. To the contrary, if $\arg\{b(2\tau)\} = 2n\pi$ with $n = 0, 1$, the subsequent cycle ($M = 2$) counteracts leading to a suppression of squeezing so that $|s|_{M=2} \approx 0$ and $|s|_M \lesssim |s|$. In particular, we find $\arg\{b(2\tau)\} = \mathrm{mod}\{-\pi(1 + \omega\tau)/2, 2\pi\}$ for $r = 1$ so that $\omega\tau = 10$ leads to $\pi/2$ and holds for increasing M. This case corresponds to Figs. 2(c) and 2(d) (see [72] for further details). Note that the presented results are robust against moderate imperfections in the control [72]. Finally, we comment that decoherence impairs the performance, although an exponential advantage can still be achieved if $2\tau\kappa \ll 1$ where κ denotes the noise rate [cf. Figs. 2(c) and 2(d)]. For that, we consider a reservoir at temperature T interacting with the system in a standard Lindblad form [84] $\dot{\rho} = -i[H(t), \rho] + \kappa(N_{\rm th} + 1)/2(2a\rho a^\dagger - \{a^\dagger a, \rho\}) + \kappa N_{\rm th}/2(2a^\dagger \rho a - \{aa^\dagger, \rho\})$ [72] with $N_{\rm th} = (e^{\omega/k_B T} - 1)^{-1}$.

Finally, it is worth noting that a finite-size fully connected model will contain corrections to Eq. (1) that will limit the exponential growth of $\langle W\rangle_M$ and $\mathcal{E}_M$. The leading-order correction appears as $(a + a^\dagger)^4/N$ where N is the system size [57]. For a fixed N, one can estimate that the exponential advantage holds until $\mathrm{Tr}[\rho(a + a^\dagger)^4] \sim N$, moment at which the finite-size correction spoils the Gaussian nature of the state. This allows us to set a limit for number of cycles, $M_l \sim \log N$, after which the exponential advantage breaks down.

Spin squeezing and metrological gain. Let us now consider N spin-1/2 particles collectively coupled and interacting as described by the LMG Hamiltonian [58], which can be written as

$$H_{\text{LMG}}(t) = -\omega J_z - \frac{g^2(t)\omega}{N} J_x^2, \tag{7}$$

where the pseudospin operators are defined as $J_\alpha = \frac{1}{2}\sum_{i=1}^{N} \sigma_i^\alpha$ for $\alpha \in \{x, y, z\}$, where σ_i^α refer to the Pauli matrices for the ith spin. Since $[J^2, H_{\text{LMG}}] = 0$ one can restrict to the highest pseudospin subspace, where the dimension of the Hilbert space grows linearly with N.

In the thermodynamic limit, $N \to \infty$, the LMG exhibits a QPT at $g_c = 1$ [59,60]. In this limit, the LMG can be effectively described by a single bosonic mode. This is achieved by performing the Holstein-Primakoff transformation $J_z = J - a^\dagger a$ and $J_+ = \sqrt{2J}\sqrt{1 - a^\dagger a/(2J)}a$ and $J_x = (J_+ + J_-)/2$. Taking the $N \to \infty$ limit, H_{LMG} reduces to Eq. (1) up to a constant energy shift, and therefore $z\nu = 1/2$ [59,60]. Thus, the bosonic squeezed state $S(s)|0\rangle$ translates into a spin squeezing of the form

$$|\xi\rangle = S_{\text{spin}}(\xi)|J, m_J = J\rangle, \tag{8}$$

where $S_{\text{spin}}(\xi) = e^{\xi^* J_+^2/2 - \xi J_-^2/2}$ corresponds to the spin squeezing operator. This is similar to generating spin squeezing via a two-axis twisting Hamiltonian [85,86]. From the previous considerations, one can immediately see that $|\xi| = |s|/N$ in the thermodynamic limit, with $|s|$ given in Eq. (4). On the one hand, as the system size increases $N \to \infty$, i.e., as finite-size effects become negligible, the evolved state upon a cycle $g(t)$, $|\psi(2\tau)\rangle = U(2\tau)|J, m_J = J\rangle$ with $U(t) = \mathcal{T}e^{-i\int_0^t ds H_{\text{LMG}}(s)}$, becomes closer to $|\xi\rangle$ with $|\xi| = |s|/N$.

On the other hand, since the energy gap $\epsilon(g_c)$ of H_{LMG} is nonzero for any finite N, the duration of the protocol τ must be such that the dynamics lie in the quasiadiabatic regime. That is, $\omega\tau \gtrsim 1$ as aforementioned (to exclude sudden quench dynamics), together with $\epsilon(g_c)\tau \lesssim 1$, which prevents true adiabatic cycles in a finite system. Since $\epsilon(g_c) \propto N^{-z}$ with $z = 1/3$ [87,88], it follows that $1 \lesssim \omega\tau \lesssim N^{1/3}$. In addition, it is worth mentioning that for a ramp toward the critical point, Kibble-Zurek scaling laws emerge in this driving regime [55,57,89,90], which further highlights that nonadiabatic excitations are caused due to the existence of a critical point. In order to exemplify the generation of spin squeezed states via a cyclic protocol $g(t)$, we compute the state of the form in Eq. (8) which maximizes $F_\xi = \langle\xi|\rho|\xi\rangle$ for different system sizes N and where ρ denotes the state after the cycle according to $\dot{\rho} = -i[H_{\text{LMG}}(t), \rho] + \kappa/2(J_+\rho J_- - \{J_+J_-, \rho\})$. For $\kappa = 0$, the resulting parameter $|\xi|$ closely follows the expected relation $|\xi| = |s|/N$, as illustrated in Fig. 3(a) for $\omega\tau = 2$ and $r = 2$, which also holds for reasonably small noise rates $\tau\kappa\sqrt{N} \lesssim 10^{-2}$. Moreover, the fidelity $F_\xi \to 1$ as $N \to \infty$, which is shown in Fig. 3(b). For $N = 10^3$ we find already $F_\xi \approx 0.9999$, which corroborates the ability to generate spin squeezing by harnessing controllable nonadiabatic excitations generated by a QPT. Note that slower cycles impair the generation of spin squeezing [72].

Such spin squeezed states may contain multipartite entanglement shared by its N spins and thus, they may offer a metrological advantage for sub shot-noise phase sensitivity. In order to quantify such an advantage, we rely on the quantity χ^2_{min} related to the quantum Fisher information [49], which

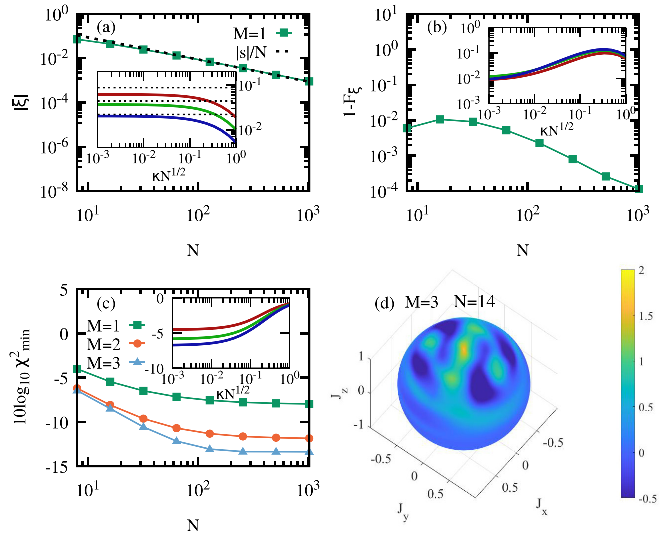

FIG. 3. (a) The resulting squeezing parameter $|\xi|$ for $\kappa = 0$ as a function of the number of spins N for one cycle. The dashed line shows the predicted theoretical value $|s|/N = \text{arcosh}\{\csc[\pi/(2 + 2z\nu r)]\}/N$ with $z\nu = 1/2$, while the inset shows $|\xi|$ as κ increases for $N = 10$, 20, and 40 spins (lines from top to bottom) with their corresponding $|s|/N$ (dashed lines). The fidelity of the generated state is very close to the expected spin squeezed state, as shown in (b) for $\kappa = 0$, i.e., $F_\xi \to 1$ as $N \to \infty$. The inset shows F_ξ for $\kappa \neq 0$ as in (a). In (c) we show χ^2_{min} [cf. Eq. (9)] for $M = 1$, 2, and 3 cycles which demonstrates the amount of useful multipartite entanglement, and survives for $\kappa \neq 0$ (see inset for $M = 1$ cycle). (d) The Wigner function for the state achieved upon $M = 3$ cycles, $N = 14$ spins, and $\kappa = 0$. All the results correspond to $\omega\tau = 2$ and $r = 2$.

witnesses multipartite entanglement and metrological advantage (sub shot-noise sensitivity) when $\chi^2_{\text{min}} < 1$, and reads

$$\chi^2_{\text{min}} = \min_{\vec{n}} \frac{N}{4(\Delta R_{\vec{n}})^2}, \tag{9}$$

with the operator $R_{\vec{n}}$ defined as $\{R_{\vec{n}}, \rho\} = i[J_{\vec{n}}, \rho]$, $(\Delta R_{\vec{n}})^2 = \langle R_{\vec{n}}\rangle^2 - \langle R_{\vec{n}}\rangle^2$ its variance on the state ρ, and $J_{\vec{n}} = (J_x \sin\theta\cos\phi, J_y \sin\theta\sin\phi, J_z\cos\theta)$ being the pseudospin operator in an arbitrary direction $\vec{n}$. Note that $\Delta R_{\vec{n}} = \Delta J_{\vec{n}}$ for a pure state ρ. From Eq. (8) it is clear that χ^2_{min} is achieved when $\theta = \pi/2$, while the angle ϕ in the xy plane depends on the phase of ξ. We compute χ^2_{min} minimizing over any possible pair of angles $\{\theta, \phi\}$ for each state in Fig. 3(a) and find that the produced states always contain multipartite entanglement, provided $\tau\kappa\sqrt{N} \lesssim 1$, and as expected, are thus useful for sub shot-noise sensitivity since $\chi^2_{\text{min}} < 1$. Moreover, as shown above in the bosonic case, realizing more cycles may lead to more squeezing. In this context, this is reflected in a reduction of χ^2_{min}, as shown in Fig. 3(c), where $\chi^2_{\text{min}} \approx 0.05$ after $M = 3$ cycles for $N = 10^3$ spins. However, further cycles do not improve significantly χ^2_{min}. In order to illustrate the features of the resulting spin state, we show as an example the Wigner function $\mathcal{W}(\theta, \phi)$ for $N = 14$ spins upon $M = 3$ cycles, which is calculated following the standard procedure [72,91], while the examples for $M = 1$ and 2 are provided in [72].

Finally, we compare our method with the standard one- and two-axis twisting Hamiltonians [86]. For that we compute the squeezing $\xi_S^2 = 4\min_{\vec{n}}(\Delta J_{\vec{n}})^2/N$ as defined in [85], which is known to scale as $\xi_S^2 \propto N^{-2/3}$ and N^{-1} for one- and two-axis twisting, respectively. As our method produces a state that is equivalent to a two-axis twisting [cf. Eq. (8)], we find that $\xi_S^2 \propto N^{-1}$ (see [72] for further details), and thus approaching Heisenberg limit for phase sensitivity [86].

Conclusions. In this Letter we have investigated the production of nonadiabatic excitations as a result of cyclic protocols reaching a quantum critical point, showing how they can be controlled and thus harnessed. In particular, we have focused on critical fully connected models, whose effective description reduces to a driven quantum harmonic oscillator. As a result of the breakdown of the adiabatic condition, the amount of nonadiabatic excitations depends solely on the critical exponents and the shape of the protocol close to the QPT. These nonadiabatic excitations produce squeezing in the initially prepared ground state.

We first considered an effective model, namely, a quantum harmonic oscillator as a quantum battery, where work is performed through the cyclic driving, whose average value and fluctuations obey simple relations and are dictated by the nonlinear form of the protocol in the vicinity of the critical point, as well as the maximum extractable work. Interestingly, we showed that the realization of M consecutive cycles can yield an exponential increase of the stored and extractable work.

As a second example we considered a system made of N spin-1/2 particles interacting according to the LMG model, which reduces to an effective quantum harmonic oscillator model in the thermodynamic limit. We showed that nonadiabatic excitations translate into spin squeezing. In this manner, by performing a quasiadiabatic cycle toward the critical point, one generates spin squeezing in a controllable fashion, and analogous to a two-axis twisting Hamiltonian. As expected, such spin squeezed states contain multipartite entanglement and are therefore useful for sub shot-noise phase sensitivity.

Finally, since different and experimentally realizable fully connected quantum many-body systems can be effectively described in terms of a quantum harmonic oscillator where the control parameter can be tuned in time [11,66–70], our findings are relevant to distinct platforms. Our results will motivate further research in the exciting arena of nonequilibrium phenomena and critical dynamics with applications in quantum thermodynamics and metrology.

ACKNOWLEDGMENTS

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