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Lie Symmetries of Differential Equations: A Computational Approach to Optimal Systems of Lie Subalgebras

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Declaration of Authorship

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- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
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Abstract

Department of Mathematics and Computer Sciences

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**Lie Symmetries of Differential Equations:
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by Emanuele SGROI

Lie groups of symmetries of differential equations constitute a fundamental tool for constructing group-invariant solutions. The number of subgroups is potentially infinite and so the number of invariant solutions; thus, it is crucial to obtain a classification of subgroups in order to have an *optimal system* of inequivalent solutions from which all other solutions can be derived by action of the group itself. Since Lie groups are intimately connected to Lie algebras, a classification of inequivalent subgroups induces a classification of inequivalent Lie subalgebras, and vice versa. A general method for classifying the Lie subalgebras of a finite–dimensional Lie algebra uses inner automorphisms that are obtained by exponentiating the adjoint groups. In this thesis, after shortly reviewing the basic notions about Lie algebras and Lie groups of transformations of differential equations, we present an effective algorithm able to automatically determine optimal systems of Lie subalgebras of a generic finite–dimensional Lie algebra abstractly assigned by means of its structure constants, or realized in terms of matrices or vector fields, or defined by a basis and the set of non-zero Lie brackets. The algorithm is implemented in the computer algebra system *Wolfram Mathematica*™. Various meaningful and non-trivial examples are considered. In particular, we classify the optimal systems of Lie subalgebras of all real Lie algebra of dimension 3, 4 and 5. Also, we analyze the optimal systems of Lie subalgebras of Noether symmetries of some geodesic equations associated to special metrics in a four–dimensional space, as well as the optimal systems of Lie symmetries admitted by some well known PDEs (linear heat equation, Burgers' equation, Korteweg-deVries equation).

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Contents

Introduction

The theory developed by Sophus Lie [1–3] had their starting point in his intuition that the concept of symmetry provides a unified framework able to connect the various methods commonly used to solve special classes of ordinary differential equations with infinitesimal transformations, that are strictly related to one–parameter groups of transformations. Lie's original idea has been inspired by N. Abel and É. Galois works on algebraic equations (see [4]), where some symmetry properties of equations determine whether they are solvable or not by radicals. In the study of differential equations, Lie added a geometric dimension, which has been developed into a general integration procedure. Such methods were based on the invariance of a differential equation with respect to a group of continuous transformations. Many authors have expanded Lie's theories further. In particular, Élie Cartan [5] carried Lie's theories into several mathematical fields. Nowadays, the symmetry method developed by Lie represents a milestone in the investigation of algebraic structures known as Lie groups. In turn, Lie groups are intimately connected to Lie algebras [6–9]. In particular, Lie groups of symmetries of partial differential equations, besides representing the main ingredient for constructing group-invariant solutions [10–13], are also used for mapping differential equations into equivalent ones [14], or constructing conservation laws [15], to quote few applications.

When using Lie symmetries for characterizing group-invariant solutions, we need to consider particular Lie subgroups of continuous transformations admitted by a differential equation. Due to the potential infinite number of subgroups, that reflects on the number of group-invariant solutions, it is desirable to classify these solutions in order to have an optimal system of inequivalent group-invariant solutions from which all other solutions can be derived by action of the group itself [13, Ch. IV, Sects. 7–9]. A classification of inequivalent invariant solutions can be done by using some special automorphisms of the Lie group.

It is well known that for a group $(G, \cdot)$ an automorphism of G is a bijective map $\phi : G \to G$ such that

$$\phi(a \cdot b) = \phi(a) \cdot \phi(b) \quad \forall\, a, b \in G.$$

In the group of all automorphisms of G, there are the automorphisms

$$\phi_a : G \to G, \qquad \phi_a(b) = a^{-1} \cdot b \cdot a,$$

a^{-1} denoting the inverse of $a \in G$. called *inner automorphisms* of G. The set of all inner automorphisms of G is denoted by $\mathrm{Int}(G)$.

Since $a^{-1} \cdot b \cdot a = b$ is equivalent to saying $b \cdot a = a \cdot b$, the existence and number of inner automorphisms different from the identity is a sort of measure of the failure of the commutative law in the group.

A subgroup $H \subseteq G$ is *similar* to a subgroup $H' \subseteq G$ if there exists $a \in G$ such that $H' = aHa^{-1}$, *i.e.*, the subgroups H and H' are connected by inner automorphisms of the group. This similarity is a relation of equivalence and the corresponding equivalence classes are said *conjugacy classes*.

Since non–essentially different invariant solutions are found from similar Lie subgroups, the problem of the classification of H–invariant solutions (see [11, Ch. 3, Sect. 3.3] and [13, Ch. IV, Sects. 7–9]) is reduced to the classification of subgroups of a Lie group G, up to similarity. In the literature, the similar subgroups (subalgebras) Lie [10] are also known as conjugate subgroups (subalgebras) Lie [11]. This problem, for connected Lie groups of symmetries, in turn, is reduced to the corresponding problem of classification of Lie subalgebras, that can be approached more easily from an algorithmic perspective. The implementation of algorithms for investigating Lie algebras is a well established and promising area (see, for instance, [6, 16–18]).

Moreover, the explicit construction of optimal systems of Lie subalgebras of the Lie algebra of symmetries of partial differential equations, leading to optimal systems of subgroups, is of particular relevance in the applications (classification of inequivalent group-invariant solutions); in fact, there is a rapidly increasing literature about the optimal systems of Lie subalgebras connected to the symmetries of partial differential equations (see, for instance, [19–29]).

Determining the similar Lie subalgebras, and then associating to them similar Lie subgroups, is more effective algorithmically because the group of inner automorphisms of a Lie algebra $\mathcal{L}$, $\mathrm{Int}(\mathcal{L})$, is always a group of linear transformations on the main space, whereas the group $\mathrm{Int}(G)$ is necessarily not.

Therefore, by defining the similarity between Lie subalgebras, all subalgebras of a fixed dimension of a Lie algebra $\mathcal{L}$ are partitioned into classes of similar subalgebras. The set of the representatives of each class is called an *optimal system of subalgebras* and [11, Ch. 3, Sect. 3.3]) and ([13, Ch. IV, Sects. 7–9]. In the following, we will be concerned with finite–dimensional Lie algebras of dimension r, $\mathcal{L}_r$.

Thus, the optimal system of subalgebras of a Lie algebra $\mathcal{L}_r$ with inner automorphisms $\mathcal{A} = \mathrm{Int}(\mathcal{L}_r)$ is a set of subalgebras $\Theta_{\mathcal{A}}(\mathcal{L}_r)$ such that:

1. there are no two elements of this set which can be transformed into each other by inner automorphisms of the Lie algebra $\mathcal{L}_r$;

2. any subalgebra of the Lie algebra $\mathcal{L}_r$ can be transformed into one of subalgebras of the set $\Theta_{\mathcal{A}}(\mathcal{L}_r)$.

The union of the elements of the optimal system of given dimensionality d is called *optimal system of order d*, and denoted by the symbol $\Theta^d_{\mathcal{A}}$; since the dimension of an algebra is invariant under automorphisms, the solution of the classification problem for a finite-dimensional Lie algebra $\mathcal{L}_r$ yields tables of optimal systems for every $d = 1, \ldots, r-1$.

Although not strictly connected to the determination of optimal systems of Lie subalgebras faced in the sequel, it is worth noticing that many physically relevant differential equations may admit discrete symmetries (see, for instance [30] where non-classical reductions of Boussinesq equation are also considered). Moreover, in [31] a method for determining the most general equation admitting a given discrete symmetry is given. A special mention is due to Hydon [32, 33] (see also the book [34, Ch. 11]) who gave effective algorithms for computing discrete symmetries of both ordinary and partial differential equations.

In this thesis, after introducing the basic notions about Lie algebras and Lie groups of transformations admitted by differential equations, we face the problem of classifying Lie subalgebras from a computational viewpoint, and present an effective algorithm that can automatically determine the optimal systems of Lie subalgebras of a generic finite–dimensional Lie algebra [35]. Actually, we consider the set of families of Lie subalgebras that we define in Chapter 4, and introduce a more general relation that is both reflexive and transitive but could not be symmetric.

Some authors [36, 37] claimed the implementation of computer algebra algorithms for determining optimal systems of Lie subalgebras; nevertheless, these algorithms are not systematic, and the results are obtained using the computer algebra system interactively as a symbolic calculator. On the contrary, our algorithm, implemented in a package, `SymboLie` [38], written in the computer algebra system *Wolfram Mathematica*™ [39] is able to recover optimal systems of Lie subalgebras almost automatically. The name `SymboLie` (due to Lucia Margheriti, a PhD student at the University of Messina who in 2008 started to work on this problem from a computational viewpoint [40]) merges the word *Symbol* with *Lie*: the reason is that, in Sophus Lie's notation, the infinitesimal generator of a Lie group of transformations was denoted as the *symbol*. Preliminary results about the `SymboLie` program, with special emphasis on one–dimensional Lie subalgebras, are contained in [41].

The structure of the thesis is as follows. In Chapters 1 and 2, also devoted to fix the notation, we briefly recall some basic notions about Lie algebras, the similarity relation among Lie subalgebras, and Lie groups of transformations of differential equations. In Chapter 3, we shortly describe the use of Lie symmetries for finding group–invariant solutions, and face the problem of separating them into inequivalent classes. Chapter 4 contains the definition of families of Lie subalgebras allowing us to state the problem of the determination of optimal systems of Lie subalgebras in a suitable way from a computational point of view. Moreover, we give a detailed description of the methods and algorithms for deriving optimal systems of families of Lie subalgebras, both one–dimensional and multi–dimensional. In Chapter 5, we describe how to use the program, and present some non–trivial case studies, and compare the results obtained with `SymboLie` with those available in the literature. In Chapter 6 we present a hierarchy of coupled Burgers–like equations possessing Lie algebras of symmetries that are isomorphic to the Lie algebra of symmetries of classical Burgers' equation. The hierarchy arises repeatedly looking for conditional symmetries starting from classical Burgers' equation. Finally, Chapter 7 gives some not yet concluding remarks, outlines possible developments, and reports some preliminar results currently under investigation.

The original results presented in this thesis are contained in some coauthored papers. In particular, the paper [35] contains the theoretical formulation underlying the p–families of Lie subalgebras and the relation (which in general is a preorder) among them; in this paper, the algorithms used in the package `SymboLie` (see [38] for the source code and a series of applications) are also described and tested. The characterization of the optimal systems of all the real three– and four–dimensional Lie algebra with `Symbolie` is contained in a paper submitted for publication [42], and compared with the results obtained by Patera and Winternitz [26]. The results discussed in Chapter 6 are contained in a paper submitted for publication [43]. Finally, in the conclusive Chapter, some original preliminar results currently under investigation are presented, concerned with the optimal systems of the five–dimensional real Lie algebra characterized in [44] and of the Noether symmetries of geodesic equations associated to special metrics.

Chapter 1

Short review about Lie algebras

In the study of differential equations, symmetries are crucial for simplifying problems and finding solutions. Among the most powerful tools for understanding symmetries are Lie groups and their associated Lie algebras. Before delving into the theory of Lie symmetries admitted by differential equations, it is essential to first establish a foundation about the basic concepts of Lie algebras [6–9, 18].

Lie algebras provide a linearized framework to study the local properties of Lie groups. They allow us to explore the infinitesimal structure of continuous symmetries, which is fundamental when analyzing the behavior of differential equations under these symmetries. The algebraic structure of Lie algebras, characterized by the Lie bracket, reveals how symmetries interact, forming an essential part of the theory of Lie group actions.

This chapter introduces the basic notions of Lie algebras, including definitions, examples, and key properties. By understanding these concepts, we build the necessary basics for the subsequent chapters, where we will explore how these algebraic structures are used to solve differential equations through group of symmetries.

1.1 Definitions and examples

Definition 1. *Let $\mathbb{F}$ be a field. A Lie algebra over $\mathbb{F}$ is an $\mathbb{F}$–vector space $\mathcal{L}$, equipped with a bilinear map*

$$\begin{aligned} [\cdot,\cdot]:\ & \mathcal{L}\times\mathcal{L}\to\mathcal{L} \\ & (x,y)\mapsto[x,y], \end{aligned}$$

such that, for all $x,y,z\in\mathcal{L}$, the following axioms are satisfied:

(a_1) $[x,x]=0,$

(a_2) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ *(Jacobi identity).*

The map $[\cdot,\cdot]$ is called the Lie bracket.

Let $\mathcal{L}$ be a Lie algebra over $\mathbb{F}$, and let $x,y\in\mathcal{L}$. Then, by the bilinearity of the Lie bracket, we have:

$$0=[x+y,x+y]=[x,x]+[x,y]+[y,x]+[y,y]=[x,y]+[y,x].$$

Thus, it follows that the Lie bracket satisfies the property of antisymmetry:

$$[x,y]=-[y,x] \text{ for all } x,y\in\mathcal{L}. \tag{1.1}$$

On the other hand, from (1.1), we have $[x, x] = -[x, x]$, that is, $2[x, x] = 0$ for every $x \in \mathcal{L}$. Therefore, if $\mathbb{F}$ has characteristic different from 2, the axiom (a_1) is equivalent to (1.1).

Starting from these few axioms, we can easily prove the following simple property.

Property 1. *Let $\mathcal{L}$ be a Lie algebra over $\mathbb{F}$, and let $x, y \in \mathcal{L}$. We have:*

1. $[x, \mathbf{0}] = \mathbf{0} = [\mathbf{0}, x]$;
2. *if $[x, y] \neq 0$, then x and y are linearly independent over $\mathbb{F}$.*

Proof.

1. We can write $\mathbf{0} = 0 \cdot z$, for some $z \in \mathcal{L}$ and with $0 \in \mathbb{F}$. Then we have:

$$[x, \mathbf{0}] = [x, 0 \cdot y] = 0 \cdot [x, y] = \mathbf{0}.$$

   A similar argument applies to $[\mathbf{0}, x]$.

2. Suppose that x and y are not linearly independent. Then there exists $\alpha \in \mathbb{F}$ such that $y = \alpha x$. It follows that:

$$[x, y] = [x, \alpha x] = \alpha[x, x] = 0,$$

   a contradiction.

□

Example 1. *Let V be an $\mathbb{F}$–vector space. If we define the Lie bracket*

$$[\cdot, \cdot] : V \times V \to V$$
$$(x, y) \mapsto 0$$

then V trivially becomes a Lie algebra over $\mathbb{F}$.

Example 2. *Let V be an $\mathbb{F}$–vector space, and denote by $End(V)$ the set of endomorphisms of V. The space V, equipped with the following Lie bracket*

$$[\cdot, \cdot] : End\ V \times End\ V \to End\ V$$
$$(x, y) \mapsto [x, y] = x \circ y - y \circ x$$

*becomes a Lie algebra over $\mathbb{F}$ called the **general linear algebra**, denoted by $\mathfrak{gl}(V)$.*

Example 3. *Denote by $\mathfrak{gl}(n, \mathbb{F})$ the vector space of all $n \times n$ matrices over $\mathbb{F}$. By introducing the Lie bracket*

$$[\cdot, \cdot] : \mathfrak{gl}(n, \mathbb{F}) \times \mathfrak{gl}(n, \mathbb{F}) \to \mathfrak{gl}(n, \mathbb{F})$$
$$(x, y) \mapsto [x, y] = xy - yx$$

for all $x, y \in \mathfrak{gl}(n, \mathbb{F})$, where xy is the usual row-column product of matrices, we obtain a Lie algebra over $\mathbb{F}$.

Example 4. *Let $\mathfrak{sl}(n, \mathbb{F})$ be the subspace of $\mathfrak{gl}(n, \mathbb{F})$ consisting of all matrices with trace zero. Since for all $x, y \in \mathfrak{sl}(n, \mathbb{F})$, the matrix $xy - yx$ is traceless, the commutator $[x, y] = xy - yx$ defines a Lie algebra structure on $\mathfrak{sl}(n, \mathbb{F})$. This Lie algebra is known as the **special linear algebra**.*

Example 5. *Let $\mathfrak{b}(n,\mathbb{F})$ be the subspace of upper-triangular matrices in $\mathfrak{gl}(n,\mathbb{F})$; equipped with the Lie bracket of $\mathfrak{gl}(n,\mathbb{F})$, it becomes a Lie algebra.*

1.2 Subalgebras and ideals

Subsequently, we define the concepts of subalgebra and ideal of a Lie algebra, and also introduce operations between ideals and various important ideals in the analysis of Lie algebras.

Definition 2. *Let $\mathcal{L}$ be a Lie algebra over $\mathbb{F}$. We say that $\mathcal{K} \subseteq \mathcal{L}$ is a subalgebra of $\mathcal{L}$ if it is a vector subspace such that*

$$[x,y] \in \mathcal{K} \quad \text{for every } x,y \in \mathcal{K}.$$

Therefore, $\mathcal{K}$ is a vector subspace of $\mathcal{L}$ that is closed under the Lie bracket. Note that $\mathcal{K}$, equipped with the Lie bracket of $\mathcal{L}$ restricted to $\mathcal{K}$, is a Lie algebra.

Definition 3. *Let $\mathcal{L}$ be a Lie algebra over $\mathbb{F}$ and let $\mathcal{I} \subseteq \mathcal{L}$. We say that $\mathcal{I}$ is an ideal of $\mathcal{L}$ if*

$$[x,y] \in \mathcal{I} \quad \text{for every } x \in \mathcal{L}, y \in \mathcal{I}.$$

Ideals in Lie algebra theory play a role analogous to that of normal subgroups in group theory and two-sided ideals in ring theory. Note that, due to the antisymmetry property, there is no need to distinguish between left and right ideals as in the case of non-commutative rings.

Remark 1. *From the definition, it follows that an ideal is also a subalgebra, but the converse is not necessarily true.*
For example, $\mathfrak{b}(2,\mathbb{R})$ is a subalgebra of $\mathfrak{gl}(2,\mathbb{R})$ but not an ideal. Indeed, considering

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{gl}(2,\mathbb{R}), \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{b}(2,\mathbb{R}),$$

we have:

$$\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \notin \mathfrak{b}(2,\mathbb{R}).$$

The Lie algebra $\mathcal{L}$ and $\{0_{\mathcal{L}}\}$ are ideals of $\mathcal{L}$, called **trivial ideals**. Any ideal different from $\mathcal{L}$ is called a **proper ideal**.
We can also introduce operations on ideals to construct new ideals from two ideals $\mathcal{I}$ and $\mathcal{J}$ of a Lie algebra $\mathcal{L}$.
First, we observe that $\mathcal{I} \cap \mathcal{J}$ is an ideal of $\mathcal{L}$. Indeed, $\mathcal{I} \cap \mathcal{J}$ is a vector subspace of $\mathcal{L}$, and for $x \in \mathcal{L}$ and $y \in \mathcal{I} \cap \mathcal{J}$, we have $[x,y] \in \mathcal{I}$ since $\mathcal{I}$ is an ideal, and similarly $[x,y] \in \mathcal{J}$. Hence, $[x,y] \in \mathcal{I} \cap \mathcal{J}$.

The union of ideals, however, is not generally an ideal. However, the following result holds:

Proposition 1. *Let $\mathcal{L}$ be a Lie algebra, and let $\mathcal{I}$ and $\mathcal{J}$ be ideals of $\mathcal{L}$. Then*

$$\mathcal{I} + \mathcal{J} = \{x + y : x \in \mathcal{I}, y \in \mathcal{J}\}$$

is an ideal.

Proof. We know that $\mathcal{I} + \mathcal{J}$ is a vector subspace of $\mathcal{L}$. We need to show that it is an ideal. Let $x \in \mathcal{L}$ and $z \in \mathcal{I} + \mathcal{J}$. Then there exist $y_1 \in \mathcal{I}$ and $y_2 \in \mathcal{J}$ such that

$z = y_1 + y_2$. Thus:

$$[x, z] = [x, y_1 + y_2] = [x, y_1] + [x, y_2] \in \mathcal{I} + \mathcal{J},$$

therefore, the result follows. □

The ideal $\mathcal{I} + \mathcal{J}$ is called the **sum** of the two ideals $\mathcal{I}$ and $\mathcal{J}$.

1.3 Center and derived algebra

We can define another operation between ideals: given two ideals $\mathcal{I}$ and $\mathcal{J}$ of a Lie algebra $\mathcal{L}$ over the field $\mathbb{F}$, we denote by $[\mathcal{I}, \mathcal{J}]$ the set

$$[\mathcal{I}, \mathcal{J}] = \left\{ \sum_{i=1}^{n} \alpha_i [x_i, y_i] : \alpha_i \in \mathbb{F}, x_i \in \mathcal{I}, y_i \in \mathcal{J}, n \in \mathbb{N} \right\},$$

which we call the **product of the ideals** $\mathcal{I}$ and $\mathcal{J}$.

Proposition 2. *Let $\mathcal{L}$ be a Lie algebra over $\mathbb{F}$, and let $\mathcal{I}$ and $\mathcal{J}$ be ideals of $\mathcal{L}$. Then $[\mathcal{I}, \mathcal{J}]$ is an ideal.*

Proof. Clearly, $[\mathcal{I}, \mathcal{J}]$ is a vector subspace of $\mathcal{L}$, so we need to show that it is an ideal. Let $x \in \mathcal{L}$ and $z \in [\mathcal{I}, \mathcal{J}]$. Then there exist $\alpha_1, \dots, \alpha_n \in \mathbb{F}$, $x_1, \dots, x_n \in \mathcal{I}$, and $y_1, \dots, y_n \in \mathcal{J}$ such that

$$z = \sum_{i=1}^{n} \alpha_i [x_i, y_i].$$

We need to show that $[x, z] \in [\mathcal{I}, \mathcal{J}]$:

$$\begin{aligned} [x, z] &= \left[x, \sum_{i=1}^{n} \alpha_i [x_i, y_i] \right] = \\ &= \sum_{i=1}^{n} \alpha_i [x, [x_i, y_i]] = \\ &= \sum_{i=1}^{n} \alpha_i \left(-[x_i, [y_i, x]] - [y_i, [x, x_i]] \right) = \\ &= \sum_{i=1}^{n} \alpha_i \left(-[x_i, [y_i, x]] + [[x, x_i], y_i] \right) \in [\mathcal{I}, \mathcal{J}], \end{aligned}$$

since $x_i \in \mathcal{I}$, $[y_i, x] \in \mathcal{J}$, $[x, x_i] \in \mathcal{I}$, and $y_i \in \mathcal{J}$ for each $i = 1, \dots, n$. □

From this proposition, it follows that $[\mathcal{L}, \mathcal{L}]$ is an ideal of $\mathcal{L}$, so we can give the following definition.

Definition 4. *Let $\mathcal{L}$ be a Lie algebra. The ideal $\mathcal{L}' = [\mathcal{L}, \mathcal{L}]$ is called the **derived algebra** of $\mathcal{L}$.*

Given a Lie algebra $\mathcal{L}$ and a subset $S \subseteq \mathcal{L}$, we can define the set of elements of $\mathcal{L}$ that commute with all elements of S:

$$Z_{\mathcal{L}}(S) = \{x \in \mathcal{L} : [x, s] = 0 \text{ for every } s \in S\}.$$

The set $Z_{\mathcal{L}}(S)$ is called the **centralizer** of S in $\mathcal{L}$. It is easy to verify that $Z_{\mathcal{L}}(S)$ is a subalgebra of $\mathcal{L}$. Indeed, given $x, y \in Z_{\mathcal{L}}(S)$ and $s \in S$, the Jacobi identity gives:

$$[[x,y],s] = -[[y,s],x] - [[s,x],y] = [0,x] - [0,y] = 0,$$

so $[x,y] \in Z_{\mathcal{L}}(S)$.

Definition 5. *Let $\mathcal{L}$ be a Lie algebra. The set of elements of $\mathcal{L}$ that commute with all elements of $\mathcal{L}$,*

$$Z(\mathcal{L}) = \{x \in \mathcal{L} : [x,y] = 0 \text{ for every } y \in \mathcal{L}\},$$

is called the center of $\mathcal{L}$.

Note that the center of $\mathcal{L}$ is the centralizer of $\mathcal{L}$ in itself, i.e., $Z(\mathcal{L}) = Z_{\mathcal{L}}(\mathcal{L})$. The following property holds.

Property 2. *Let $\mathcal{L}$ be a Lie algebra over a field $\mathbb{F}$. Then $Z(\mathcal{L})$ is an ideal.*

Proof. First, we show that $Z(\mathcal{L})$ is a vector subspace of $\mathcal{L}$: let $\alpha, \beta \in \mathbb{F}$, $x, y \in Z(\mathcal{L})$, and $z \in \mathcal{L}$. Then:

$$[\alpha x + \beta y, z] = \alpha[x,z] + \beta[y,z] = 0,$$

so $\alpha x + \beta y \in Z(\mathcal{L})$, and thus $Z(\mathcal{L})$ is a subspace of $\mathcal{L}$. Now, we prove the second property: let $x \in \mathcal{L}$ and $y \in Z(\mathcal{L})$, and we verify that $[x,y] \in Z(\mathcal{L})$. Let $z \in \mathcal{L}$; using the Jacobi identity, we have:

$$[[x,y],z] = -[[y,z],x] - [[z,x],y] = [0,x] - 0 = 0,$$

so the claim follows. □

Definition 6. *A Lie algebra $\mathcal{L}$ is called commutative (or abelian) if $[x,y] = 0$ for all $x, y \in \mathcal{L}$.*

Proposition 3. *Let $\mathcal{L}$ be a Lie algebra. $\mathcal{L}$ is abelian if and only if $Z(\mathcal{L}) = \mathcal{L}$.*

Proof. If $\mathcal{L}$ is an abelian Lie algebra, for any $x \in \mathcal{L}$, we have $[x,y] = 0$ for every $y \in \mathcal{L}$, so $x \in Z(\mathcal{L})$ and thus $\mathcal{L} = Z(\mathcal{L})$.
Conversely, if $\mathcal{L} = Z(\mathcal{L})$, then $[x,y] = 0$ for all $x, y \in \mathcal{L}$, i.e., $\mathcal{L}$ is abelian. □

Given a vector space V and a subspace $W \subseteq V$, we know that we can construct the quotient vector space V/W.
We now want to define the quotient Lie algebra. Let $\mathcal{L}$ be a Lie algebra over $\mathbb{F}$ and let $\mathcal{I}$ be an ideal of $\mathcal{L}$. We know that $\mathcal{I}$ is a particular subspace of $\mathcal{L}$, so we can consider the quotient space $\mathcal{L}/\mathcal{L}$. On this space, we define the following operation:

$$\begin{aligned} &[\cdot,\cdot] : \mathcal{L}/\mathcal{I} \times \mathcal{L}/\mathcal{I} \to \mathcal{L}/\mathcal{I} \\ &(x+\mathcal{I}, y+\mathcal{I}) \to [x+\mathcal{I}, y+\mathcal{I}] = [x,y] + \mathcal{I} \end{aligned}$$

We will verify the following facts.

1. $[\cdot,\cdot]$ is well-defined:

   Let $x, y \in \mathcal{L}$, we know that $[x+\mathcal{I}, y+\mathcal{I}] = [x,y] + \mathcal{I}$. Let $x', y' \in \mathcal{L}$ such that $x + \mathcal{I} = x' + \mathcal{I}$ and $y + \mathcal{I} = y' + \mathcal{I}$. We need to show that

   $$[x+\mathcal{I}, y+\mathcal{I}] = [x'+\mathcal{I}, y'+\mathcal{I}] \iff [x,y] - [x',y'] \in \mathcal{I}.$$

Since x is equivalent to x', there exists $z_1 \in \mathcal{I}$ such that $x = x' + z_1$. Similarly, there exists $z_2 \in \mathcal{I}$ such that $y = y' + z_2$. Then

$$[x,y] - [x',y'] = [x'+z_1, y'+z_2] - [x',y'] = [x',z_2] + [z_1,y'] + [z_1,z_2] \in \mathcal{I}.$$

2. $[\cdot,\cdot]$ is bilinear:

   We prove linearity in the first argument; linearity in the second argument can be proven similarly. Let $\alpha, \beta \in \mathbb{F}$ and let $x,y,z \in \mathcal{L}$. We have:

$$\begin{aligned}[\alpha(x+\mathcal{I}) + \beta(y+\mathcal{I}), z+\mathcal{I}] &= [(\alpha x+\mathcal{I}) + (\beta y+\mathcal{I}), z+\mathcal{I}] = \\ &= [(\alpha x+\beta y)+\mathcal{I}, z+\mathcal{I}] = \\ &= [\alpha x+\beta y, z] + \mathcal{I} = \\ &= (\alpha[x,z] + \beta[y,z]) + \mathcal{I} = \\ &= \alpha([x,z]+\mathcal{I}) + \beta([y,z]+\mathcal{I}) = \\ &= \alpha[x+\mathcal{I}, z+\mathcal{I}] + \beta[y+\mathcal{I}, z+\mathcal{I}].\end{aligned}$$

3. $[\cdot,\cdot]$ satisfies (a_1) and (a_2):

   Let $x \in \mathcal{L}$,

$$[x+\mathcal{I}, x+\mathcal{I}] = [x,x] + \mathcal{I} = 0 + \mathcal{I} = \mathcal{I}.$$

   Thus (a_1) is trivially satisfied.
   Let $x,y,z \in L$, we verify (a_2):

$$\begin{aligned}&[x+\mathcal{I},[y+\mathcal{I},z+\mathcal{I}]] + [y+\mathcal{I},[z+\mathcal{I},x+\mathcal{I}]] + [z+\mathcal{I},[x+\mathcal{I},y+\mathcal{I}]] = \\ &\quad= [x+\mathcal{I},[y,z]+\mathcal{I}] + [y+\mathcal{I},[z,x]+\mathcal{I}] + [z+\mathcal{I},[x,y]+\mathcal{I}] = \\ &\quad= ([x,[y,z]]+\mathcal{I}) + ([y,[z,x]]+\mathcal{I}) + ([z,[x,y]]+\mathcal{I}) = \\ &\quad= ([x,[y,z]] + [y,[z,x]] + [z,[x,y]]) + \mathcal{I} = \\ &\quad= 0 + \mathcal{I} = \mathcal{I}.\end{aligned}$$

Thus $\mathcal{L}/\mathcal{I}$ with the Lie bracket defined as above is a Lie algebra over $\mathbb{F}$ and is called the *quotient Lie algebra*.

1.4 Structure constants and homomorphisms

Let $\mathcal{L}$ be an r-dimensional Lie algebra over a field $\mathbb{F}$ with basis $\{x_1, \ldots, x_r\}$. For each pair x_i, x_j with $i \neq j$, we can express the Lie bracket as a linear combination of the basis elements of $\mathcal{L}$, that is, there exist $c_{ij}^k \in \mathbb{F}$, with $k = 1, \ldots, r$, such that:

$$[x_i, x_j] = \sum_{k=1}^{r} c_{ij}^k x_k.$$

Then, the constants c_{ij}^k completely determine the operation. Indeed, given $x, y \in L$, we can express them as linear combinations of the basis:

$$x = \sum_{i=1}^{r} \alpha_i x_i, \quad y = \sum_{j=1}^{r} \beta_j x_j,$$

and we have:

$$[x,y] = \left[\sum_{i=1}^{r} \alpha_i x_i, \sum_{j=1}^{r} \beta_j x_j\right] = \sum_{i,j=1}^{r} \alpha_i \beta_j [x_i, x_j] = \sum_{i,j,k=1}^{r} \alpha_i \beta_j c_{ij}^k x_k.$$

The constants c_{ij}^k are called **structure constants**, and the following result holds.

Proposition 4. *Let $\mathcal{L}$ be a Lie algebra over a field $\mathbb{F}$ with basis $\{x_1, \ldots, x_r\}$. Then the structure constants satisfy the following relations:*

1. $c_{ii}^k = 0$,
2. $c_{ij}^k = -c_{ji}^k$,
3. $\sum_{\ell=1}^{r} (c_{il}^m c_{jk}^\ell + c_{j\ell}^m c_{ki}^\ell + c_{k\ell}^m c_{ij}^\ell) = 0$,

for every $1 \leq i, j, k, m \leq r$.

Proof.

1. Fix $i \in \{1, \ldots, r\}$.
   By the axiom (a_1), we have $[x_i, x_i] = 0$. Thus:

   $$[x_i, x_i] = \sum_{k=1}^{r} c_{ii}^k x_k = 0,$$

   and since $\{x_1, \ldots, x_r\}$ is a basis of $\mathcal{L}$, it follows that $c_{ii}^k = 0$ for every $k = 1, \ldots, r$.

2. Fix $i, j \in \{1, \ldots, r\}$.
   By property (1.1), we have $[x_i, x_j] = -[x_j, x_i]$. Thus:

   $$\sum_{k=1}^{r} c_{ij}^k x_k = -\sum_{k=1}^{r} c_{ji}^k x_k \quad \Longrightarrow \quad \sum_{k=1}^{r} (c_{ij}^k + c_{ji}^k) x_k = 0.$$

   Since $\{x_1, \ldots, x_r\}$ is a basis, it follows that $c_{ij}^k = -c_{ji}^k$ for every $k = 1, \ldots, r$.

3. Fix $i, j, k \in \{1, \ldots, n\}$.
   By the axiom (a_2), we have $[x_i, [x_j, x_k]] + [x_j, [x_k, x_i]] + [x_k, [x_i, x_j]] = 0$. Thus:

   $$\begin{aligned} 0 &= \left[x_i, \sum_{\ell=1}^{r} c_{jk}^\ell x_\ell\right] + \left[x_j, \sum_{\ell=1}^{r} c_{ki}^\ell x_\ell\right] + \left[x_k, \sum_{\ell=1}^{r} c_{ij}^\ell x_\ell\right] = \\ &= \sum_{\ell=1}^{r} \left(c_{j\ell}^\ell [x_i, x_\ell] + c_{ki}^\ell [x_j, x_\ell] + c_{ij}^\ell [x_k, x_\ell]\right) = \\ &= \sum_{\ell=1}^{r} \sum_{m=1}^{r} (c_{jk}^\ell c_{i\ell}^m x_m + c_{ki}^\ell c_{j\ell}^m x_m + c_{ij}^\ell c_{k\ell}^m x_m) = \\ &= \sum_{m=1}^{r} \left(\sum_{\ell=1}^{r} c_{i\ell}^m c_{jk}^\ell + c_{j\ell}^m c_{ki}^\ell + c_{k\ell}^m c_{ij}^\ell\right) x_m. \end{aligned}$$

   Since $\{x_1, \ldots, x_r\}$ is a basis, it follows that

   $$\sum_{\ell=1}^{r} c_{i\ell}^m c_{jk}^\ell + c_{j\ell}^m c_{ki}^\ell + c_{k\ell}^m c_{ij}^\ell = 0,$$

for every $m = 1, \dots, r$.

□

Finally, we introduce the definition of homomorphism between Lie algebras.

Definition 7. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be Lie algebras over a field $\mathbb{F}$. A linear map $\varphi : \mathcal{L}_1 \to \mathcal{L}_2$ is called a Lie algebra homomorphism if*

$$\varphi([x,y]) = [\varphi(x), \varphi(y)] \quad \text{for all } x, y \in \mathcal{L}_1,$$

where $[\cdot,\cdot]$ denotes the Lie bracket of each respective algebra.

We say that φ is a *monomorphism* if it is injective, an *epimorphism* if it is surjective, and an *isomorphism* if it is bijective. An isomorphism from a Lie algebra $\mathcal{L}$ to itself is called an *automorphism*.

Finally, two Lie algebras $\mathcal{L}_1$ and $\mathcal{L}_2$ are said to be *isomorphic* if there exists an isomorphism $\varphi : \mathcal{L}_1 \to \mathcal{L}_2$.

1.5 Solvable, nilpotent and simple Lie algebras

The ideals constitute an important ingredient for classifying Lie algebras. Previously, we introduced a special ideal of a Lie algebra $\mathcal{L}$ known as the derived algebra: $\mathcal{L}' = [\mathcal{L}, \mathcal{L}]$. The following lemma holds.

Lemma 1. *Let $\mathcal{L}$ be a Lie algebra and $\mathcal{I} \subseteq \mathcal{L}$ an ideal. Then the quotient $\mathcal{L}/\mathcal{I}$ is abelian if and only if $\mathcal{I}$ contains the derived algebra $\mathcal{L}'$.*

Proof. The quotient $\mathcal{L}/\mathcal{I}$ is abelian if and only if for every $x, y \in \mathcal{L}$ we have:

$$[x + \mathcal{I}, y + \mathcal{I}] = [x, y] + \mathcal{I} = \mathcal{I},$$

i.e., for every $x, y \in \mathcal{L}$, we have $[x, y] \in \mathcal{I}$. This holds if and only if $\mathcal{L}' \subseteq \mathcal{I}$, since $\mathcal{I}$ is a vector subspace of $\mathcal{L}$ and $\mathcal{L}'$ is generated by the set $\{[x, y] : x, y \in \mathcal{L}\}$. □

This lemma suggests that the derived algebra $\mathcal{L}'$ is the smallest ideal of $\mathcal{L}$ such that the quotient is abelian. By a similar reasoning, we can say that the derived algebra of $\mathcal{L}'$, denoted by $\mathcal{L}^{(2)}$, is the smallest ideal of $\mathcal{L}'$ such that the quotient $\mathcal{L}'/\mathcal{L}^{(2)}$ is abelian. We can define the **derived series** of $\mathcal{L}$ as follows:

Definition 8. *Let $\mathcal{L}$ be a Lie algebra. Consider the ideals:*

$$\mathcal{L}^{(0)} = \mathcal{L}, \quad \mathcal{L}^{(1)} = \mathcal{L}' = [\mathcal{L}, \mathcal{L}], \quad \mathcal{L}^{(2)} = [\mathcal{L}^{(1)}, \mathcal{L}^{(1)}], \dots, \quad \mathcal{L}^{(k)} = [\mathcal{L}^{(k-1)}, \mathcal{L}^{(k-1)}].$$

The sequence $\mathcal{L} \supseteq \mathcal{L}^{(1)} \supseteq \mathcal{L}^{(2)} \supseteq \cdots$ is called the derived series of $\mathcal{L}$.

Definition 9. *Let $\mathcal{L}$ be a Lie algebra. It is said to be* solvable *if there exists $m \geq 1$ such that $\mathcal{L}^{(m)} = 0$.*

If $\mathcal{L}$ is solvable, then the derived series of $\mathcal{L}$ provides a "approximation" of $\mathcal{L}$ via a finite sequence of ideals whose quotients are abelian. The converse also holds.

Lemma 2. *Let $\mathcal{L}$ be a Lie algebra and let $\mathcal{I}_0, \mathcal{I}_1, \dots, \mathcal{I}_m$, with $m \in \mathbb{N}$, be ideals of $\mathcal{L}$ such that*

$$L = \mathcal{I}_0 \supseteq \mathcal{I}_1 \supseteq \cdots \supseteq \mathcal{I}_{m-1} \supseteq \mathcal{I}_m = 0$$

and the quotients $\mathcal{I}_{k-1}/\mathcal{I}_k$ are abelian for every $1 \leq k \leq m$. Then $\mathcal{L}$ is solvable.

Proof. We will show that $\mathcal{L}^{(m)} = 0$.
We know by assumption that the quotient $\mathcal{L}/\mathcal{I}_1$ is abelian, so by Lemma 1, we have $\mathcal{L}' \subseteq \mathcal{I}_1$. Suppose this inclusion holds for $k-1$, i.e., $\mathcal{L}^{(k-1)} \subseteq \mathcal{I}_{k-1}$, and prove it for k. By assumption, the quotient $\mathcal{I}_{k-1}/\mathcal{I}_k$ is abelian, so by Lemma 1, we have $[\mathcal{I}_{k-1}, \mathcal{I}_{k-1}] \subseteq \mathcal{I}_k$. By the inductive hypothesis, we have $\mathcal{L}^{(k-1)} \subseteq \mathcal{I}_{k-1}$, so that

$$\mathcal{L}^{(k)} = [\mathcal{L}^{(k-1)}, \mathcal{L}^{(k-1)}] \subseteq [\mathcal{I}_{k-1}, \mathcal{I}_{k-1}] \subseteq \mathcal{I}_k,$$

thus the claim follows. □

We now introduce another fundamental class of Lie algebras, namely *nilpotent* Lie algebras. To do this, we define another sequence of ideals.

Definition 10. *Let $\mathcal{L}$ be a Lie algebra. Consider the ideals:*

$$\mathcal{L}^0 = \mathcal{L}, \quad \mathcal{L}^1 = \mathcal{L}', \quad \mathcal{L}^2 = [\mathcal{L}, \mathcal{L}'], \ldots, \quad \mathcal{L}^k = [\mathcal{L}, \mathcal{L}^{k-1}].$$

The sequence $\mathcal{L} \supseteq \mathcal{L}^1 \supseteq \mathcal{L}^2 \supseteq \cdots$ is called the central descending series.

Definition 11. *Let $\mathcal{L}$ be a Lie algebra. It is said to be* nilpotent *if there exists $m \geq 1$ such that $\mathcal{L}^m = 0$.*

Remark 2. *Every nilpotent Lie algebra is solvable. The converse is not generally true. For example, the Lie algebra $\mathfrak{b}(n, \mathbb{F})$ of upper-triangular $n \times n$ matrices is solvable, but it is not nilpotent for $n \geq 2$.*

Definition 12. *Let $\mathcal{L}$ be a Lie algebra. It is said to be* simple *if $\mathcal{L}$ is non-abelian and has only trivial ideals.*

Example 6. *Let $\mathcal{L}$ be the three-dimensional Lie algebra with basis $\{x_1, x_2, x_3\}$ with Lie brackets*

$$[x_1, x_2] = \mathbf{x}_3, \qquad [x_2, x_3] = x_1, \qquad [x_3, x_1] = x_2.$$

It is easy to verify that this Lie algebra has no proper ideals.

Remark 3. *In this Chapter, the basis of a generic Lie algebra has been denoted as $\{x_1, \ldots, x_r\}$, and the structure constants as c_{ij}^k. In the following, since we are mainly interested to finite dimensional Lie algebras spanned by infinitesimal generators of symmetries of differential equations, we will denote the basis as $\{\Xi_1, \ldots, \Xi_r\}$ (this is the notation used in SymboLie [38]), and the corresponding structure constants as C_{ij}^k.*

Chapter 2

Lie groups of differential equations

For solving ordinary or partial differential equations a general theory of Lie groups is unnecessary if transformations are limited to scalings, translations, or rotations. However, there may be other classes of transformations that leave some differential equations invariant. The infinitesimal characterization of a Lie group of transformations is essential for discovering such transformations.

This chapter provides an overview of the fundamental concepts about Lie group analysis of differential equations [9–11, 13, 34, 45–51].

2.1 Lie groups of transformations

Definition 13 (Group). *A group G is a non–empty set of elements with a binary operation $\gamma : G \times G \to G$ satisfying the following axioms:*

1. *$\gamma(a, \gamma(b, c)) = \gamma(\gamma(a, b), c)$, for any $a, b, c \in G$ (associative property);*
2. *there exists a unique identity element $e \in G$ such that $\gamma(a, e) = \gamma(e, a) = a$, for all $a \in G$ (identity element);*
3. *For any element $a \in G$ there exists a unique inverse element a^{-1} such that $\gamma(a, a^{-1}) = \gamma(a^{-1}, a) = e$ (inverse element).*

Definition 14 (Group of Transformations). *Let $D \subseteq \mathbb{R}^N$ and $S \subseteq \mathbb{R}$ be open subsets. The set of transformations*

$$\mathbf{Z} : D \times S \to D, \quad \mathbf{z} \to \mathbf{z}^\star = \mathbf{Z}(\mathbf{z}, a),$$

depending on the parameter a, forms a group of transformations on D if:

1. *For each $a \in S$, $\mathbf{Z}$ is bijective;*
2. *(S, γ), with $\gamma : S \times S \to S$ being the composition law, is a group with identity e;*
3. *$\mathbf{z}^\star = \mathbf{z}$ when $a = e$, that is, $\mathbf{Z}(\mathbf{z}, e) = \mathbf{z}$;*
4. *if $\mathbf{z}^\star = Z(\mathbf{z}, a)$ and $\mathbf{z}^{\star\star} = \mathbf{Z}(\mathbf{z}^\star, b)$, then $\mathbf{z}^{\star\star} = \mathbf{Z}(\mathbf{z}, \gamma(a, b))$ for all $a, b \in S$.*

Definition 15 (Lie Group of Transformations). *A group of transformations defines a one–parameter Lie group of transformations if it also satisfies the following axioms:*

1. *a is a continuous parameter, that is, S is connected;*
2. *$\mathbf{Z}$ is C^∞ with respect to $\mathbf{z}$ in D, and an analytic function of a in S;*
3. *for each $a, b \in S$, $\gamma(a, b)$ and a^{-1} are analytic functions.*

The definitions provided above can be naturally extended to general manifolds, as discussed in [11]. However, to avoid unnecessary technicalities, we have chosen to present these definitions in their current, more streamlined form (see [9, 10, 34, 46]). This approach ensures clarity while preserving the essential structure of the intended mathematical framework. However, let us recall for completeness the definition of a Lie group [11].

Definition 16 (Lie group). *A Lie group is a group G which also carries the structure of an smooth manifold in such a way that both the group operation*

$$\gamma : G \times G \to G, \quad \gamma(g,h) = g \cdot h, \quad g,h \in G,$$

and the inversion

$$i : G \to G, \quad i(g) = g^{-1}, \quad g \in G,$$

are smooth maps between manifolds.

2.1.1 Examples of one–parameter Lie group of transformations

Example 7 (Translation Group in the Plane $\mathbb{R}^2$). *Let $D = \mathbb{R}^2$ and $\mathbf{z_0} \in \mathbb{R}^2$ be a fixed point in the plane. The set of transformations*

$$\mathbf{z}^\star = \mathbf{z} + a\mathbf{z_0}, \tag{2.1}$$

($a \in \mathbb{R}$ and $\gamma(a,b) = a + b$) defines a one-parameter Lie group of transformations, called the translation group.

Example 8 (Scaling Group). *Let $D = \mathbb{R}^2$. The set of transformations*

$$\mathbf{z}^\star = a\mathbf{z}, \tag{2.2}$$

($a \in \mathbb{R}^+$ and $\gamma(a,b) = a \cdot b$) defines a one-parameter Lie group of transformations, called the scaling group.

Example 9 (Rotation Group in the Plane). *Let $D = \mathbb{R}^2$ and let*

$$A = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}. \tag{2.3}$$

The set of transformations

$$\mathbf{z}^\star = A\mathbf{z}, \tag{2.4}$$

($a \in \mathbb{R}$ and $\gamma(a,b) = a + b$) defines the group of clockwise rotations by an angle a in the plane.

Definition 17 (Homomorphism between Lie Groups). *A homomorphism between Lie groups is a differentiable function*

$$\phi : G \to G',$$

between two Lie groups G (with operation γ) and G' (with operation γ') such that

$$\phi(\gamma(g,h)) = \gamma'(\phi(g), \phi(h)), \qquad \forall\, g,h \in G.$$

If ϕ is bijective, then it is called an isomorphism between G and G'.

Lemma 3. *Let S be an open interval in $\mathbb{R}$ and let $\gamma : S \times S \to S$ be an operation that makes S a group. If $\gamma(a,b)$ and a^{-1} are analytic, then there exists a group homomorphism $\phi : S \to \mathbb{R}$, that is,*

$$\phi(\gamma(a,b)) = \phi(a) + \phi(b).$$

Proof. Consider the function

$$\Gamma(a) = \left.\frac{\partial \gamma(p,q)}{\partial q}\right|_{(p,q)=(a^{-1},a)},$$

and consider the following Cauchy problem

$$\phi'(a) = \Gamma(a), \quad \phi(e) = 0, \tag{2.5}$$

where $e \in S$ is the identity element of the group (S,γ). Since γ is an analytic function, thus of class C^1, the Cauchy problem (2.5) has a unique solution. Integrating (2.5), we get:

$$\phi(a) = \int_e^a \Gamma(x)\, dx\,.$$

By the assumptions, the function ϕ is analytic. It remains to prove that it represents a group homomorphism, thus we must prove that

$$\phi(\gamma(a,b)) = \phi(a) + \phi(b),$$

that is,

$$\int_e^{\gamma(a,b)} \Gamma(x)\, dx = \int_e^a \Gamma(x)\, dx + \int_e^b \Gamma(x)\, dx\,. \tag{2.6}$$

Differentiating both sides with respect to b, we obtain

$$\Gamma(\gamma(a,b))\frac{\partial \gamma(a,b)}{\partial b} = \Gamma(b), \tag{2.7}$$

whose integration between e and b gives

$$\int_e^{\gamma(a,b)} \Gamma(x)\, dx = \int_e^b \Gamma(x)\, dx + \varphi(a).$$

In particular, choosing $b = e$, we get

$$\varphi(a) = \int_e^a \Gamma(x)\, dx\,.$$

This implies that (2.6) and (2.7) are equivalent, thus it is enough to verify (2.7) to prove the theorem. By the associative property of the operation γ, we know that

$$\gamma(c,\gamma(a,b)) = \gamma(\gamma(c,a),b), \quad \forall\, a,b,c \in S.$$

Then, differentiating with respect to b, we obtain:

$$\frac{\partial \gamma(c,\gamma(a,b))}{\partial \gamma(a,b)}\frac{\partial \gamma(a,b)}{\partial b} = \frac{\partial \gamma(\gamma(c,a),b)}{\partial b}.$$

In particular, choosing $c = (\gamma(a,b))^{-1} = \gamma(b^{-1},a^{-1})$, we have:

$$\begin{aligned}
\Gamma(\gamma(a,b))\frac{\partial\gamma(a,b)}{\partial b} &= \frac{\partial\gamma(\gamma(\gamma(a,b)^{-1},a),b)}{\partial b} = \\
&= \frac{\partial\gamma(\gamma(\gamma(b^{-1},a^{-1}),a),b)}{\partial b} = \\
&= \frac{\partial\gamma(\gamma(b^{-1},\gamma(a^{-1},a)),b)}{\partial b} = \\
&= \frac{\partial\gamma(\gamma(b^{-1},e),b)}{\partial b} = \\
&= \frac{\partial\gamma(b^{-1},b)}{\partial b} = \\
&= \Gamma(b).
\end{aligned}$$

□

Remark 4. *Thanks to the previous lemma, in the case of a one-parameter Lie group of transformations, we can assume, without loss of generality, the group operation to be the usual addition in* $\mathbb{R}$.

Example 10. *Consider the scaling group*

$$z^\star = az, \qquad z \in \mathbb{R},$$

with $a \in S =]0,+\infty[$*, where the group operation, as we have already seen, is*

$$\gamma(a,b) = a \cdot b.$$

Then, we have:

$$\Gamma(a) = \left.\frac{\partial\gamma(a,b)}{\partial b}\right|_{(a^{-1},a)} = \frac{1}{a},$$

thus, the function ϕ *is defined by the integral*

$$\phi(a) = \int_1^a \frac{1}{x}\,dx = \ln a,$$

which is indeed a group homomorphism since

$$\phi(\gamma(a,b)) = \phi(ab) = \ln(ab) = \ln a + \ln b = \phi(a) + \phi(b).$$

2.2 Infinitesimal transformations

Consider a one-parameter Lie group of transformations

$$\mathbf{z}^\star = \mathbf{Z}(\mathbf{z},a). \tag{2.8}$$

Since $\mathbf{Z}$ is analytic with respect to a, we can expand it in a power series around $a = 0$:

$$\begin{aligned}
\mathbf{z}^\star &= \mathbf{z} + a\left.\frac{\partial\mathbf{Z}(\mathbf{z},a)}{\partial a}\right|_{a=0} + \frac{a^2}{2}\left.\frac{\partial^2\mathbf{Z}(\mathbf{z},a)}{\partial a^2}\right|_{a=0} + \cdots \\
&= \mathbf{z} + a\left.\frac{\partial\mathbf{Z}(\mathbf{z},a)}{\partial a}\right|_{a=0} + O(a^2).
\end{aligned}$$

Setting

$$\boldsymbol{\zeta}(\mathbf{z}) = \left.\frac{\partial \mathbf{Z}(\mathbf{z},a)}{\partial a}\right|_{a=0},$$

the transformation

$$\mathbf{z}^\star = \mathbf{z} + a\,\boldsymbol{\zeta}(\mathbf{z}) + O(a^2)$$

is called the *infinitesimal transformation* of the Lie group of transformations.

The first fundamental theorem of Lie shows that the infinitesimal transformation contains all the essential information for characterizing a one-parameter Lie group of transformations.

Theorem 1 (First Fundamental Theorem of Lie). *The Lie group of transformations (2.8) is equivalent to the solution of the initial value problem*

$$\frac{d\mathbf{z}^\star}{da} = \boldsymbol{\zeta}(\mathbf{z}^\star), \qquad \mathbf{z}^\star(0) = \mathbf{z}. \tag{2.9}$$

Proof. Given $\epsilon \in \mathbb{R}$, we know that

$$\mathbf{Z}(\mathbf{z}, a+\epsilon) = \mathbf{Z}(\mathbf{z}^\star, \epsilon).$$

Expanding both sides in a power series around $\epsilon = 0$, we get:

$$\begin{aligned}\mathbf{Z}(\mathbf{z}, a+\epsilon) &= \mathbf{Z}(\mathbf{z},a) + \epsilon\,\frac{\partial \mathbf{Z}(\mathbf{z},a)}{\partial a} + O(\epsilon^2) = \\ &= \mathbf{z}^\star + \epsilon\,\frac{d\mathbf{z}^\star}{da} + O(\epsilon^2),\end{aligned}$$

$$\begin{aligned}\mathbf{Z}(\mathbf{z}^\star, \epsilon) &= \mathbf{Z}(\mathbf{z}^\star, 0) + \epsilon\left.\frac{\partial \mathbf{Z}(\mathbf{z}^\star,\epsilon)}{\partial a}\right|_{\epsilon=0} + O(\epsilon^2) = \\ &= \mathbf{z}^\star + \epsilon\,\boldsymbol{\zeta}(\mathbf{z}^\star) + O(\epsilon^2),\end{aligned}$$

where $\mathbf{z}^\star$ is given by (2.8).

Comparing the two expressions, it follows that $\mathbf{z}^\star = \mathbf{Z}(\mathbf{z},a)$ satisfies the initial value problem

$$\frac{d\mathbf{z}^\star}{da} = \boldsymbol{\zeta}(\mathbf{z}^\star), \qquad \mathbf{z}^\star(0) = \mathbf{z}.$$

On the other hand, since $\boldsymbol{\zeta}(\mathbf{z}^\star)$ is of class C^1, the Cauchy theorem guarantees the existence and uniqueness of the solution to the problem (2.9), and it can only be (2.8). □

Theorem 1 establishes a one-to-one correspondence between the Lie group of transformations and $\boldsymbol{\zeta}(\mathbf{z})$ and, for this reason, we can call it the *infinitesimal generator of the group*.

Example 11. *Consider in* $\mathbb{R}^2$ *the infinitesimal generator* $\boldsymbol{\zeta} = (z_1, z_2)$. *Integrating the Lie equations*

$$\begin{aligned}\frac{dz_1^\star}{da} &= z_1^\star, \qquad z_1^\star(0) = z_1,\\ \frac{dz_2^\star}{da} &= z_2^\star, \qquad z_2^\star(0) = z_2,\end{aligned}$$

we obtain

$$z_1^\star = \exp(a)z_1, \qquad z_2^\star = \exp(a)z_2,$$

which corresponds to the scaling group in the plane.

2.3 Infinitesimal generator and invariance

The infinitesimal generator $\boldsymbol{\zeta}(\mathbf{z})$ of the one-parameter Lie group of transformations (2.8) allows us to introduce a first-order differential operator

$$\Xi = \boldsymbol{\zeta}(\mathbf{z}) \cdot \boldsymbol{\nabla} = \zeta_1(\mathbf{z})\frac{\partial}{\partial z_1} + \cdots + \zeta_N(\mathbf{z})\frac{\partial}{\partial z_N}, \tag{2.10}$$

which is also called the *infinitesimal generator* of the Lie group.

For any differentiable function $F(\mathbf{z})$ we have

$$\Xi(F) = \boldsymbol{\zeta}(\mathbf{z}) \cdot \boldsymbol{\nabla} F = \zeta_1(\mathbf{z})\frac{\partial F}{\partial z_1} + \cdots + \zeta_N(\mathbf{z})\frac{\partial F}{\partial z_N},$$

and, in particular,

$$\Xi(\mathbf{z}) = \boldsymbol{\zeta}(\mathbf{z}).$$

A one-parameter Lie group of transformations, which by Theorem 1 is equivalent to its infinitesimal transformation, is also equivalent to the infinitesimal generator Ξ.

The following theorem shows that the use of the infinitesimal generator allows us to obtain an algorithm to find the explicit solution of the initial value problem (2.9).

Theorem 2. *The one-parameter Lie group of transformations (2.8) is equivalent to*

$$\mathbf{z}^\star = \exp(a\Xi)(\mathbf{z}) = z + a\,\Xi(\mathbf{z}) + \frac{a^2}{2}\Xi^2(\mathbf{z}) + \cdots = \sum_{k=0}^{\infty} \frac{a^k}{k!}\Xi^k(\mathbf{z}), \tag{2.11}$$

where the operator Ξ is defined by (2.10) and $\Xi^k(\mathbf{z}) = \Xi(\Xi^{k-1}(\mathbf{z}))$. In particular $\Xi^0(\mathbf{z}) = \mathbf{z}$.

Proof. Consider

$$\Xi = \zeta_1(\mathbf{z})\frac{\partial}{\partial z_1} + \cdots + \zeta_N(\mathbf{z})\frac{\partial}{\partial z_N},$$

and

$$\Xi^\star = \zeta_1(\mathbf{z}^\star)\frac{\partial}{\partial z_1^\star} + \cdots + \zeta_N(\mathbf{z}^\star)\frac{\partial}{\partial z_N^\star},$$

where $\mathbf{z}^\star = \mathbf{Z}(\mathbf{z}, a)$. Expanding the latter in a Taylor series around $a = 0$, we get:

$$\mathbf{z}^\star = \sum_{k=0}^{\infty} \frac{a^k}{k!} \frac{\partial^k \mathbf{Z}(\mathbf{z}, a)}{\partial a^k}\bigg|_{a=0} = \sum_{k=0}^{\infty} \frac{a^k}{k!} \frac{d^k \mathbf{z}^\star}{da^k}\bigg|_{a=0}.$$

Since for any differentiable function $F(\mathbf{z})$ we have

$$\frac{dF(\mathbf{z}^\star)}{da} = \sum_{i=1}^{N} \frac{\partial F(\mathbf{z}^\star)}{\partial z_i^\star}\frac{dz_i^\star}{da} = \sum_{i=1}^{N} \zeta_i(\mathbf{z}^\star)\frac{\partial F(\mathbf{z}^\star)}{\partial z_i^\star} = \Xi^\star(F(\mathbf{z}^\star)),$$

it follows in particular that

$$\frac{d\mathbf{z}^\star}{da} = \Xi^\star(\mathbf{z}^\star),$$

and

$$\frac{d^2\mathbf{z}^\star}{da^2} = \frac{d}{da}\left(\frac{d\mathbf{z}^\star}{da}\right) = \Xi^\star(\Xi^\star(\mathbf{z}^\star)) = \Xi^{\star 2}(\mathbf{z}^\star).$$

In general, we have:

$$\frac{d^k\mathbf{z}^\star}{da^k} = \Xi^{\star k}(\mathbf{z}^\star) \qquad (k \in \mathbb{N}). \tag{2.12}$$

From (2.12) it follows

$$\left.\frac{d^k\mathbf{z}^\star}{da^k}\right|_{a=0} = \Xi^k(\mathbf{z}) \qquad (k \in \mathbb{N}),$$

from which (2.11) is obtained. □

The (2.11) is called the *Lie series*.

Corollary 1. *If $F(\mathbf{z})$ is of class C^∞, then:*

$$F(\mathbf{z}^\star) = F(\exp(a\Xi)(\mathbf{z})) = \exp(a\Xi)(F(\mathbf{z})).$$

Proof.

$$\begin{aligned} F(\mathbf{z}^\star) &= F(\mathbf{z}) + a\left.\frac{dF(\mathbf{z}^\star)}{da}\right|_{a=0} + \frac{a^2}{2!}\left.\frac{d^2F(\mathbf{z}^\star)}{da^2}\right|_{a=0} + \cdots = \\ &= F(\mathbf{z}) + a\,\Xi(F(\mathbf{z})) + \frac{a^2}{2!}\Xi^2(F(\mathbf{z})) + \cdots = \\ &= \exp(a\Xi)(F(\mathbf{z})). \end{aligned}$$

□

Now we can introduce the concept of invariance of a function with respect to a Lie group of transformations and prove the related invariance criterion.

Definition 18 (Invariant Function)**.** *A function $F(\mathbf{z})$ of class C^∞ is said to be an invariant function of the Lie group of transformations (2.8) if and only if for every transformation of the group (2.8) the condition*

$$F(\mathbf{z}^\star) = F(\mathbf{z})$$

holds true.

The invariance of a function is characterized by the use of the infinitesimal generator, as shown by the following theorem.

Theorem 3. *$F(\mathbf{z})$ is invariant with respect to (2.8) if and only if*

$$\Xi(F(\mathbf{z})) = 0.$$

Proof. From Corollary 1, we have:

$$\begin{aligned} F(\mathbf{z}^\star) &= F(\exp(a\Xi)(\mathbf{z})) = \exp(a\Xi)(F(\mathbf{z})) = \sum_{k=0}^{\infty}\frac{a^k}{k!}\Xi^k(F(\mathbf{z})) = \\ &= F(\mathbf{z}) + a\,\Xi(F(\mathbf{z})) + \frac{a^2}{2!}\Xi^2(F(\mathbf{z})) + \cdots \end{aligned} \tag{2.13}$$

If we assume that $F(\mathbf{z}^\star) = F(\mathbf{z})$, then (2.13) implies that $\Xi(F(\mathbf{z})) = 0$. Conversely, if $\Xi(F(\mathbf{z})) = 0$, then $\Xi^k(F(\mathbf{z})) = 0$ for every $k > 0$, thus from (2.13) we have that $F(\mathbf{z}^\star) = F(\mathbf{z})$. □

It is also possible to define the invariance of a surface in $\mathbb{R}^N$ with respect to a one-parameter Lie group of transformations.

Definition 19 (Invariant Surface). *A surface $F(\mathbf{z}) = 0$ is said to be an invariant surface with respect to the one-parameter Lie group of transformations (2.8) if $F(\mathbf{z}^\star) = 0$ when $F(\mathbf{z}) = 0$.*

As a consequence of Theorem 3, we can give the following criterion of invariance for a surface.

Theorem 4. *A surface $F(\mathbf{z}) = 0$ is invariant with respect to the group (2.8) if and only if*

$$\Xi(F(\mathbf{z})) = 0 \quad \textit{when} \quad F(\mathbf{z}) = 0,$$

that is,

$$\Xi(F(\mathbf{z})) = \mathbf{\Lambda}(\mathbf{z})F(\mathbf{z})$$

for some function $\mathbf{\Lambda}(\mathbf{z})$.

2.4 Canonical coordinates

Given in $\mathbb{R}^N$ the one-parameter Lie group of transformations (2.8), let us suppose to make the change of variables defined by the one-to-one and (at least) C^1 transformation:

$$\mathbf{y} = \mathbf{Y}(\mathbf{z}), \quad \mathbf{y} = (y_1, \dots, y_N).$$

If

$$\Xi = \sum_{i=1}^{N} \zeta_i(\mathbf{z}) \frac{\partial}{\partial z_i}$$

is the infinitesimal generator in terms of the coordinates $\mathbf{z}$, the corresponding generator in terms of $\mathbf{y}$ has a different representation

$$\widetilde{\Xi} = \sum_{i=1}^{N} \widetilde{\zeta}_i(\mathbf{y}) \frac{\partial}{\partial y_i}.$$

We should have the same group action, even if the infinitesimals have different expressions, whence the following theorem is stated.

Theorem 5. *Given*

$$\Xi = \sum_{i=1}^{N} \zeta_i(\mathbf{z}) \frac{\partial}{\partial z_i}$$

and a one-to-one and (at least) C^1 transformation:

$$\mathbf{y} = \mathbf{g}(\mathbf{z}), \quad \mathbf{y} = (y_1, \dots, y_N),$$

it is

$$\widetilde{\Xi} = \sum_{i=1}^{N} \widetilde{\zeta}_i(\mathbf{y}) \frac{\partial}{\partial y_i},$$

where

$$\widetilde{\zeta}_i(\mathbf{y}) = \Xi\,(y_i) \quad (i = 1, \dots, N).$$

Proof. Let us consider a smooth function $F(\mathbf{z})$, and the invertible transformation $\mathbf{y} = \mathbf{g}(\mathbf{z})$ together with its inverse $\mathbf{z} = \mathbf{h}(\mathbf{y})$; we have

$$F(\mathbf{z}) = F(\mathbf{h}(\mathbf{y})) = \widetilde{F}(\mathbf{y}).$$

Therefore, by using the chain rule,

$$\begin{aligned}
\Xi(F(\mathbf{z})) &= \sum_{i=1}^{N} \zeta_i \frac{\partial}{\partial z_i}(F(\mathbf{h}(\mathbf{y})) = \sum_{i=1}^{N} \zeta_i \frac{\partial}{\partial z_i}(\widetilde{F}(\mathbf{y})) = \\
&= \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \zeta_i \frac{\partial \widetilde{F}(\mathbf{y})}{\partial y_j} \frac{\partial y_j}{\partial z_i} \right) = \sum_{j=1}^{N} \left(\sum_{i=1}^{N} \zeta_i \frac{\partial y_j}{\partial z_i} \right) \frac{\partial \widetilde{F}(\mathbf{y})}{\partial y_j} = \\
&= \sum_{j=1}^{N} \Xi\,(y_j) \frac{\partial \widetilde{F}(\mathbf{y})}{\partial y_j} = \sum_{j=1}^{N} \widetilde{\zeta}_j \frac{\partial \widetilde{F}(\mathbf{y})}{\partial y_j} = \widetilde{\Xi}(\widetilde{F}(\mathbf{y})),
\end{aligned}$$

where

$$\widetilde{\zeta}_j = \Xi\,(y_j) = \sum_{i=1}^{N} \zeta_i(\mathbf{z}) \frac{\partial y_j}{\partial z_i} \quad (j = 1, \dots, N).$$

□

Definition 20 (Canonical coordinates)**.** *A change of coordinates (1.89) defines a set of canonical coordinates for the one-parameter Lie group of transformations (1.39) if, in terms of such coordinates, the group (1.39) becomes*

$$\begin{aligned}
&y_i^\star = y_i, \quad i = 1, \dots, N-1, \\
&y_N^\star = y_N + a,
\end{aligned}$$

i.e., a translation of only one component, say the N-th one.

Theorem 6. *For any Lie group of transformations (2.8) there exists a set of canonical coordinates $\mathbf{y} = (y_1(\mathbf{z}), \dots, y_N(\mathbf{z}))$ such that (1.39) is equivalent to (1.97).*

Proof. From Theorem 2, we have

$$y_i^\star = y_i \quad \text{if and only if } \Xi\,(y_i) = 0 \quad (i = 1, \dots, N-1),$$

and, from Theorem 5

$$y_N^\star = y_N + a \text{ if and only if } \Xi\,(y_N) = 1.$$

Thus, the first order linear partial differential equations

$$\begin{aligned}
&\zeta_1(\mathbf{z}) \frac{\partial y_i}{\partial z_1} + \dots + \zeta_N(\mathbf{z}) \frac{\partial y_i}{\partial z_N} = 0 \quad (i = 1, \dots, N-1), \\
&\zeta_1(\mathbf{z}) \frac{\partial y_N}{\partial z_1} + \dots + \zeta_N(\mathbf{z}) \frac{\partial y_N}{\partial z_N} = 1,
\end{aligned} \tag{2.14}$$

characterize the N canonical variables. In fact, the equations $(2.14)_1$ have $N-1$ functionally independent solutions: these solutions are the $N-1$ essential constants appearing in the general solution of the system of N first order ordinary differential equations

$$\frac{d\mathbf{z}}{da} = \boldsymbol{\zeta}(\mathbf{z})$$

resulting from the characteristic equations of the system (2.14)$_1$ (because of the autonomous form of the system (2.14), the N-th constant is nonessential). The N-th canonical variable is a particular solution of the nonhomogeneous first order linear partial differential equation (2.14)$_2$. □

Remark 5. *In terms of the canonical coordinates the infinitesimal operator of the Lie group writes in the simplest form:*

$$\widetilde{\Xi} = \frac{\partial}{\partial y_N}.$$

Example 12. *Let the Lie group in $\mathbb{R}^2$ be generated by*

$$\Xi = z_1^2 \frac{\partial}{\partial z_1} + z_1 z_2 \frac{\partial}{\partial z_2}.$$

The canonical variables (y_1, y_2) are such that

$$\Xi\,(y_1) = z_1^2 \frac{\partial y_1}{\partial z_1} + z_1 z_2 \frac{\partial y_1}{\partial z_2} = 0,$$
$$\Xi\,(y_2) = z_1^2 \frac{\partial y_2}{\partial z_1} + z_1 z_2 \frac{\partial y_2}{\partial z_2} = 1.$$

A possible choice is:

$$y_1 = \frac{z_2}{z_1}, \quad y_2 = -\frac{1}{z_1},$$

whereupon the generator writes

$$\widetilde{\Xi} = \frac{\partial}{\partial y_2}.$$

Example 13. *Let us consider the group of rotations in the plane $\mathbb{R}^2$*

$$x^\star = x\cos(a) - y\sin(a),$$
$$y^\star = x\sin(a) + y\cos(a),$$

whose infinitesimal generator is given by

$$\Xi = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

As before, the canonical variables (ρ, θ) are such that

$$\Xi\,(\rho) = -y\frac{\partial \rho}{\partial x} + x\frac{\partial \rho}{\partial y} = 0,$$
$$\Xi\,(\theta) = -y\frac{\partial \theta}{\partial x} + x\frac{\partial \theta}{\partial y} = 1.$$

By integrating these equations, we can get

$$\rho = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

Thus, the canonical coordinates are exactly the polar coordinates. The infinitesimal generator in terms of the new variables is written as

$$\widetilde{\Xi} = \frac{\partial}{\partial\theta},$$

and the rotation group is expressed as follows

$$\begin{aligned} \rho^\star &= \rho, \\ \theta^\star &= \theta + a. \end{aligned}$$

2.5 Lie group of transformations of differential equations

Here we want to determine one-parameter Lie transformation groups that are admitted by a given system of differential equations $\mathcal{S}$ involving n independent variables

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{D}_n \subseteq \mathbb{R}^n,$$

and m dependent variables

$$\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{D}_m \subseteq \mathbb{R}^m,$$

where $\mathbb{D}_n$ and $\mathbb{D}_m$ are open domains in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

For a clearer exposition, it is convenient to split the variable $\mathbf{z}$ in order to distinguish the transformations involving the independent variables $\mathbf{x}$ from those involving the dependent variables $\mathbf{u}$. Thus, the one-parameter Lie transformation group

$$\mathbf{z}^\star = \mathbf{Z}(\mathbf{z}, a),$$

can be rewritten as

$$\begin{aligned} \mathbf{x}^\star &= \mathbf{X}(\mathbf{x}, \mathbf{u}, a), \\ \mathbf{u}^\star &= \mathbf{U}(\mathbf{x}, \mathbf{u}, a), \end{aligned} \tag{2.15}$$

which acts on the space $\mathbb{D} = \mathbb{D}_n \times \mathbb{D}_m \subseteq \mathbb{R}^{n+m}$ of the variables $(\mathbf{x}, \mathbf{u})$.

Let

$$\mathbf{u} = \mathbf{\Theta}(\mathbf{x}) \equiv (\Theta_1(\mathbf{x}), \ldots, \Theta_m(\mathbf{x}))$$

be a solution of the system $\mathcal{S}$.

A Lie transformation group of the form (2.15) admitted by $\mathcal{S}$ has two equivalent properties:

1. a transformation of the group maps every solution of $\mathcal{S}$ into another solution of $\mathcal{S}$ (in some cases the same solution);

2. a transformation of the group leaves the system $\mathcal{S}$ invariant, in the sense that the system $\mathcal{S}$ reads the same way, both in terms of the variables $(\mathbf{x}, \mathbf{u})$, and in terms of the transformed variables $(\mathbf{x}^\star, \mathbf{u}^\star)$.

Since a system of differential equations involves, in addition to the variables $\mathbf{x}$ and $\mathbf{u}$, also derivatives up to a certain finite order, we need to determine the transformations of such derivatives.

Let $\mathbf{u}^{(1)}$ be the vector whose $n \cdot m$ components are all the first-order partial derivatives of $\mathbf{u}$ with respect to $\mathbf{x}$,

$$\mathbf{u}^{(1)} \equiv \left(\frac{\partial u_1}{\partial x_1}, \ldots, \frac{\partial u_1}{\partial x_n}, \ldots, \frac{\partial u_m}{\partial x_1}, \ldots, \frac{\partial u_m}{\partial x_n}\right),$$

and, in general, let $\mathbf{u}^{(k)}$ denote the vector whose components are the $m \cdot \binom{n+k-1}{k}$ k-th order partial derivatives of $\mathbf{u}$ with respect to $\mathbf{x}$.

The infinitesimal transformations of the derivatives are obtained from the infinitesimal transformations corresponding to the group (2.15):

$$\begin{aligned} x_i^\star &= x_i + a\,\xi_i(\mathbf{x},\mathbf{u}) + O(a^2), \qquad (i = 1,2,\dots,n) \\ u_\alpha^\star &= u_\alpha + a\,\eta_\alpha(\mathbf{x},\mathbf{u}) + O(a^2), \qquad (\alpha = 1,2,\dots,m). \end{aligned} \tag{2.16}$$

Differentiating $(2.16)_1$ with respect to $x_j^\star$, we obtain

$$\frac{\partial x_i}{\partial x_j^\star} = \delta_{ij} - a\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_i}{\partial u_\alpha}\frac{\partial u_\alpha}{\partial x_j}\right) + O(a^2)$$

which is used to determine the transformation relating to the first-order partial derivatives:

$$\begin{aligned} \frac{\partial u_\alpha^\star}{\partial x_i^\star} &= \frac{\partial u_\alpha}{\partial x_i} + a\left(\left(\frac{\partial \eta_\alpha}{\partial x_i} + \frac{\partial \eta_\alpha}{\partial u_\beta}\frac{\partial u_\beta}{\partial x_i}\right) - \left(\frac{\partial \xi_j}{\partial x_j} + \frac{\partial \xi_j}{\partial u_\beta}\frac{\partial u_\beta}{\partial x_j}\right)\frac{\partial u_\alpha}{\partial x_j}\right) + O(a^2) \\ &= \frac{\partial u_\alpha}{\partial x_i} + a\,\eta_{[\alpha,i]}(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)}) + O(a^2). \end{aligned}$$

Introducing the total derivative operator

$$\frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + \frac{\partial u_\alpha}{\partial x_i}\frac{\partial}{\partial u_\alpha},$$

we can write

$$\eta_{[\alpha,i]} = \frac{D\eta_\alpha}{Dx_i} - \frac{D\xi_j}{Dx_i}\frac{\partial u_\alpha}{\partial x_j}.$$

The infinitesimal transformations define the action of the prolonged group on the space $(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)})$.

Similarly, we can obtain the transformations relating to the k-th order derivatives for $k \geq 2$.

The prolongations of the infinitesimal transformations correspond to the prolongations of the respective infinitesimal generators. If the infinitesimal generator is given by

$$\Xi = \xi_i(\mathbf{x},\mathbf{u})\frac{\partial}{\partial x_i} + \eta_\alpha(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u_\alpha},$$

where $i = 1,\dots,n$, $\alpha = 1,\dots,m$, the first prolongation of the generator is given by

$$\Xi^{(1)} = \Xi + \eta_{[\alpha,i]}\frac{\partial}{\partial u_{\alpha,i}},$$

where $u_{\alpha,i} = \frac{\partial u_\alpha}{\partial x_i}$. In general, the k-th prolongation of the infinitesimal generator is defined by

$$\Xi^{(k)} = \Xi^{(k-1)} + \eta_{[\alpha,i_1\dots i_k]}\frac{\partial}{\partial u_{\alpha,i_1\dots i_k}},$$

where $u_{\alpha,i_1\dots i_k} = \frac{\partial^k u_\alpha}{\partial x_{i_1}\dots\partial x_{i_k}}$, and $\eta_{[\alpha,i_1\dots i_k]}$ is defined recursively by the relation

$$\eta_{[\alpha,i_1\dots i_k]} = \frac{D\eta_{[\alpha,i_1\dots i_{k-1}]}}{Dx_{i_k}} - \frac{D\xi_j}{Dx_{i_k}}u_{\alpha,i_1\dots i_{k-1}j}.$$

A system of differential equations $\mathcal{S}$ of order $r > 0$ can be written in the form

$$\boldsymbol{\Delta}(\mathbf{x}, \mathbf{u}, \ldots, \mathbf{u}^{(r)}) = \mathbf{0}, \tag{2.17}$$

where $\boldsymbol{\Delta} \equiv (\Delta_1, \ldots, \Delta_q)$.

We can introduce the so-called *jet space* $J^r(\mathbb{R}^{n+m}, n)$, whose coordinates, for simplicity denoted by $\mathbf{z}$, are the independent variables $\mathbf{x}$, the dependent variables $\mathbf{u}$, and their derivatives up to order r; the dimension of the jet space $J^r(\mathbb{R}^{n+m}, n)$ is

$$N = n + m\binom{n+r}{r}.$$

Each Δ_j, $(j = 1, \ldots, q)$ is a function from an open subset of $J^r(\mathbb{R}^{n+m}, n)$ to $\mathbb{R}$. Letting

$$q' = \operatorname{rank}\left\|\frac{\partial \Delta_j}{\partial z_l}\right\|, \quad j = 1, \ldots, q, \quad l = 1, \ldots N,$$

the system $\mathcal{S}$ can be viewed geometrically as an $(N - q')$–dimensional manifold in the N–dimensional jet space.

From this perspective, we can study the invariance of a system of differential equations with respect to a one-parameter Lie transformation group.

We say that the one-parameter Lie transformation group (2.15) *is admitted* by (2.17) if and only if its r-th prolongation leaves invariant the manifold in jet space $(\mathbf{x}, \mathbf{u}, \ldots, \mathbf{u}^{(r)})$ defined by (2.17).

The theorem (4) allows us to prove the following important result which provides an algorithmic method for calculating the group admitted by a given system of differential equations.

Theorem 7. *Let*

$$\Xi = \xi_i(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial x_i} + \eta_\alpha(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial u_\alpha}$$

be the infinitesimal generator corresponding to the group (2.15) and let $\Xi^{(r)}$ be the r-th prolongation of the infinitesimal generator. The group (2.15) is admitted by (2.17) if and only if

$$\Xi^{(r)}\left(\boldsymbol{\Delta}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)})\right) = \mathbf{0}, \tag{2.18}$$

when $\boldsymbol{\Delta}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}) = \mathbf{0}$.

From (2.18), we obtain an overdetermined system of linear homogeneous differential equations, called *determining equations*, whose solutions are the infinitesimal generators ξ_i and η_α; these generators depend, besides on $\mathbf{x}$ and $\mathbf{u}$, also on arbitrary constants (and in some cases on arbitrary functions). For this reason, we effectively have a multi-parameter Lie group.

We show an example of calculating a Lie group admitted by a differential equation.

Example 14. *Let us consider the following nonlinear differential equation*

$$u_{,tt} - k(u)u_{,xx} = 0, \tag{2.19}$$

with $k(u) > 0$ and $k'(u) \neq 0$.
The infinitesimal generator corresponding to the group admitted by equation is

$$\Xi = \xi_t(t, x, u)\frac{\partial}{\partial t} + \xi_x(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u}.$$

Since we have a second–order equation, we need to consider the following prolongation

$$\begin{aligned}\Xi^{(2)} =&\xi_t\frac{\partial}{\partial t}+\xi_x\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial u}+\eta_{[,t]}\frac{\partial}{\partial u_{,t}}+\eta_{[,x]}\frac{\partial}{\partial u_{,x}}\\&+\eta_{[,tt]}\frac{\partial}{\partial u_{,tt}}+\eta_{[,tx]}\frac{\partial}{\partial u_{,tx}}+\eta_{[,xx]}\frac{\partial}{\partial u_{,xx}},\end{aligned}$$

where

$$\eta_{[,z]}=\frac{D\eta}{Dz}-\frac{D\xi_t}{Dz}\frac{\partial u}{\partial t}-\frac{D\xi_x}{Dz}\frac{\partial u}{\partial x},\quad z\in\{t,x\},$$

and

$$\eta_{[,zz]}=\frac{D\eta_{[,z]}}{Dz}-\frac{D\xi_t}{Dz}\frac{\partial^2 u}{\partial t^2}-\frac{D\xi_x}{Dz}\frac{\partial^2 u}{\partial x^2},\quad z\in\{t,x\}.$$

We note that equation (2.19) does not involve the derivative u_{tx}, so we may not compute the corresponding $\eta_{[,tx]}$. After simple algebra, we obtain

$$\eta_{[,z]}=\frac{\partial\eta}{\partial z}+u_{,z}\frac{\partial\eta}{\partial u}-u_{,t}\left(\frac{\partial\xi_t}{\partial z}+u_{,z}\frac{\partial\xi_t}{\partial u}\right)-u_{,x}\left(\frac{\partial\xi_x}{\partial z}+u_{,z}\frac{\partial\xi_x}{\partial u}\right),\quad z\in\{t,x\},$$

and,

$$\begin{aligned}\eta_{[,zz]}=&\frac{\partial^2\eta}{\partial z^2}+2\frac{\partial\eta}{\partial z\partial u}u_{,z}+\frac{\partial\eta}{\partial u}u_{,zz}+\frac{\partial^2\eta}{\partial u^2}(u_{,z})^2-2\frac{\partial\xi_z}{\partial z}u_{,zz}-3\frac{\partial\xi_z}{\partial u}u_{,z}u_{,zz}-\frac{\partial^2\xi_z}{\partial z^2}u_{,z}\\&-2\frac{\partial\xi_z}{\partial z\partial u}(u_{,z})^2-\frac{\partial^2\xi_z}{\partial u^2}(u_{,z})^3-2\frac{\partial\xi_{z'}}{\partial z}u_{,z'z}-2\frac{\partial\xi_{z'}}{\partial u}u_{,z}u_{,z'z}-\frac{\partial^2\xi_{z'}}{\partial z^2}u_{,z'}\\&-2\frac{\partial^2\xi_{z'}}{\partial z\partial u}u_{,z}u_{,z'}-\frac{\partial\xi_{z'}}{\partial u}u_{,z'}u_{,zz}-\frac{\partial^2\xi_{z'}}{\partial u^2}u_{,z'}(u_{,z})^2,\end{aligned}$$

where $z,z'\in\{t,x\}$ and $z\neq z'$.

By explicating condition (2.18), namely,

$$\begin{aligned}&\Xi^{(2)}(u_{,tt}-k(u)u_{,xx})\big|_{u_{,tt}-k(u)u_{,xx}=0}=\\&\left(\eta_{[,tt]}-u_{,xx}k'(u)\eta-k(u)\eta_{[,xx]}\right)\big|_{u_{,tt}-k(u)u_{,xx}=0}=0,\end{aligned}$$

we get

$$\begin{aligned}&-k(u)\frac{\partial^2\xi_x}{\partial u^2}(u_{,x})^3-k(u)\frac{\partial^2\xi_t}{\partial u^2}u_{,t}(u_{,x})^2+k(u)\left(\frac{\partial^2\eta}{\partial t\partial u}-2\xi^x_{xu}\right)(u_{,x})^2+\frac{\partial^2\xi_x}{\partial u^2}u_{,x}(u_{,t})^2\\&+2\left(\frac{\partial^2\xi_x}{\partial t\partial u}-k(u)\frac{\partial^2\xi_t}{\partial x\partial u}\right)u_{,t}u_{,x}-2k(u)\frac{\partial\xi_x}{\partial u}u_{,x}u_{,xx}-2k(u)\frac{\partial\xi_t}{\partial u}u_{,x}u_{,xt}\\&+\left(2k(u)\frac{\partial^2\eta}{\partial x\partial u}+\frac{\partial^2\xi_x}{\partial t^2}-k(u)\frac{\partial^2\xi_x}{\partial x^2}\right)u_{,x}+\frac{\partial^2\xi_t}{\partial u^2}(u_{,t})^3+\left(-\frac{\partial^2\eta}{\partial u}+2\frac{\partial^2\xi_t}{\partial t\partial u}\right)(u_{,t})^2\\&+2k(u)\frac{\partial\xi_t}{\partial u}u_{,t}u_{,xx}-\left(2\frac{\partial^2\eta}{\partial t\partial u}-\frac{\partial^2\xi_t}{\partial t^2}+k(u)\frac{\partial^2\xi_t}{\partial x^2}\right)u_{,t}\\&+\left(k'(u)\eta-2k(u)\frac{\partial\xi_x}{\partial x}+2k\frac{\partial\xi_t}{\partial t}\right)u_{,xx}+2u_{xt}\left(\xi^x_t-k\xi^t_x\right)\\&-\frac{\partial^2\eta}{\partial t^2}+k(u)\frac{\partial^2\eta}{\partial x^2}=0.\end{aligned}$$

It is a polynomial in the derivatives $u_{,x}$, $u_{,t}$, $u_{,tx}$, $u_{,tt}$ and $u_{,xx}$ whose coefficients, involving *the infinitesimals and their partial derivatives, must vanish. Hence, the following set of* determining equations *is found:*

$$
\begin{aligned}
&\frac{\partial^2 \eta}{\partial u^2} = 0,\\
&\frac{\partial \xi_x}{\partial u} = 0,\\
&\frac{\partial \xi_t}{\partial u} = 0,\\
&2k(u)\frac{\partial^2 \eta}{\partial x \partial u} + \frac{\partial^2 \xi_x}{\partial t^2} - k(u)\frac{\partial^2 \xi_x}{\partial x^2} = 0,\\
&2\frac{\partial^2 \eta}{\partial t \partial u} - \frac{\partial^2 \xi_t}{\partial t^2} + k(u)\frac{\partial^2 \xi_t}{\partial x^2} = 0,\\
&k'(u)\eta - 2k(u)\left(\frac{\partial \xi_x}{\partial x} - \frac{\partial \xi_t}{\partial t}\right) = 0,\\
&k(u)\frac{\partial^2 \eta}{\partial x^2} - \frac{\partial^2 \eta}{\partial t^2} = 0,\\
&\frac{\partial \xi_x}{\partial t} - k(u)\frac{\partial \xi_t}{\partial x} = 0.
\end{aligned}
$$

By integrating the last equation with respect to u, the relations

$$\frac{\partial \xi_t}{\partial x} = 0, \qquad \frac{\partial \xi_x}{\partial t} = 0.$$

are obtained. Since $k'(u) \neq 0$, the expression for η can be obtained from the 6th equation:

$$\eta = 2\frac{k(u)}{k'(u)}\left(\frac{\partial \xi_x}{\partial x} - \frac{\partial \xi_t}{\partial t}\right).$$

Applying the first equation, one has

$$\frac{d^2}{du^2}\left(\left(\frac{k(u)}{k'(u)}\right)\right)\left(\frac{\partial \xi_x}{\partial x} - \frac{\partial \xi_t}{\partial t}\right) = 0. \tag{2.20}$$

The latter equation is called a classifying relation *and indicates that the symmetry group admitted by equation (2.19) depends on the function $k(u)$.*
We analyze two cases. First, suppose we want to determine the admitted Lie group for all functions $k = k(u)$. Thus, from the equation (2.20) and some determining equations, can be obtained

$$\frac{\partial \xi_x}{\partial x} - \frac{\partial \xi_t}{\partial t} = 0, \quad \eta = 0, \quad \frac{\partial^2 \xi_x}{\partial x^2} = 0, \quad \frac{\partial^2 \xi_t}{\partial t^2} = 0.$$

These can be easily integrated, and we obtain the solution:

$$
\begin{aligned}
\xi_t &= c_1 t + c_2,\\
\xi_x &= c_1 x + c_3,\\
\eta &= 0,
\end{aligned}
$$

where c_1, c_2 and c_3 are arbitrary constants. Therefore, they may be seen as linear combinations of fundamental solutions, giving 3 *different symmetries (obtained by setting one* *constant egual to 1 and the remaining ones to 0), whose generators are*

$$\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x}, \qquad \Xi_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}.$$

In the second case, we can search for additional symmetries by considering specific forms of the function $k(u)$. Indeed, the relation (2.20) holds if the following condition is satisfied:

$$\frac{d^2}{du^2}\left(\frac{k(u)}{k'(u)}\right) = 0,$$

which implies that

$$k'(u) = \frac{k(u)}{\alpha u + \beta},$$

where $\alpha, \beta \in \mathbb{R}$ are arbitrary constants.
If $\alpha = 0$, the general solution to the above differential equation is $k(u) = \gamma \exp(qu)$, where $\gamma \in \mathbb{R}$ is a constant and $q = \frac{1}{\beta}$. In this case, solving the determining equations yields two additional infinitesimal generators:

$$\Xi_4 = t\frac{\partial}{\partial t} - 2\beta\frac{\partial}{\partial u}, \qquad \Xi_5 = x\frac{\partial}{\partial x} + 2\beta\frac{\partial}{\partial u}.$$

If $\alpha \neq 0$, the solution takes the form $k(u) = \gamma(\alpha u + \beta)^q$, where $q = \frac{1}{\alpha}$ and γ is an arbitrary constant. By analyzing the determining equations, three distinct cases emerge: $q = \pm 4$ and q as an arbitrary constant.
In the general case with arbitrary q, the additional infinitesimal generators are:

$$\Xi_6 = t\frac{\partial}{\partial t} - 2(\alpha u + \beta)\frac{\partial}{\partial u}, \qquad \Xi_7 = x\frac{\partial}{\partial x} + 2(\alpha u + \beta)\frac{\partial}{\partial u}.$$

Furthermore, when $q = \pm 4$, an extra infinitesimal generator is introduced. For $q = -4$, we obtain:

$$\Xi_8 = t^2\frac{\partial}{\partial t} + (u - 4\beta)t\frac{\partial}{\partial u},$$

while for $q = 4$, the corresponding generator is:

$$\Xi_9 = x^2\frac{\partial}{\partial x} + (u + 4\beta)x\frac{\partial}{\partial u}.$$

Remark 6. *Most of the lengthy though straightforward calculations needed to find the Lie symmetries of differential equations can be managed almost automatically by means of specific computer algebra packages [45, 51–65]. The computations done for the examples in this thesis have been carried out by means of the Reduce [66] package ReLie [67].*

2.6 Lie algebra of generators

A Lie group of transformations with r–parameters defined on $D \subseteq \mathbb{R}^N$ is characterized by its generators

$$\Xi_\mu = \sum_{i=1}^{N} \xi_{\mu i}(\mathbf{z})\frac{\partial}{\partial z_i}, \qquad \mu = 1, \ldots, r.$$

We define the *commutator* (or *Lie bracket*) of two generators Ξ_μ and Ξ_ν as the first-order operator

$$\begin{aligned}
[\Xi_\mu, \Xi_\nu] &= \Xi_\mu \Xi_\nu - \Xi_\nu \Xi_\mu = \\
&= \left(\sum_{i=1}^{N} \xi_{\mu i}(\mathbf{z}) \frac{\partial}{\partial z_i}\right) \left(\sum_{j=1}^{N} \xi_{\nu j}(\mathbf{z}) \frac{\partial}{\partial z_j}\right) = \\
&- \left(\sum_{i=1}^{N} \xi_{\nu i}(\mathbf{z}) \frac{\partial}{\partial z_i}\right) \left(\sum_{j=1}^{N} \xi_{\mu j}(\mathbf{z}) \frac{\partial}{\partial z_j}\right) = \\
&= \sum_{i=1}^{N} \widehat{\xi}_i(\mathbf{z}) \frac{\partial}{\partial z_j},
\end{aligned}$$

where

$$\widehat{\xi}_i(\mathbf{z}) = \sum_{j=1}^{N} \left(\xi_{\mu j}(\mathbf{z}) \frac{\partial \xi_{\nu i}(\mathbf{z})}{\partial z_j} - \xi_{\nu j}(\mathbf{z}) \frac{\partial \xi_{\mu i}(\mathbf{z})}{\partial z_j} \right).$$

As a consequence of this definition, the commutator is antisymmetric,

$$[\Xi_\mu, \Xi_\nu] = -[\Xi_\nu, \Xi_\mu],$$

bilinear,

$$[\Xi_\lambda, \alpha\Xi_\mu + \beta\Xi_\nu] = \alpha[\Xi_\lambda, \Xi_\mu] + \beta[\Xi_\lambda, \Xi_\nu],$$

and satisfies the Jacobi identity

$$[\Xi_\lambda, [\Xi_\mu, \Xi_\nu]] + [\Xi_\mu, [\Xi_\nu, \Xi_\lambda]] + [\Xi_\nu, [\Xi_\lambda, \Xi_\mu]] = 0.$$

Furthermore, the following result holds.

Theorem 8 (Second Fundamental Theorem of Lie)**.** *The commutator of two arbitrary infinitesimal generators of a Lie group with r–parameters is still an infinitesimal generator of the Lie group with r–parameters and we have:*

$$[\Xi_\mu, \Xi_\nu] = \sum_{\lambda=1}^{r} C_{\mu\nu}^{\lambda} \Xi_\lambda,$$

where the coefficients $C_{\mu\nu}^{\lambda}$ are called structure constants.

From this result, it follows that the generators of a Lie group with r–parameters generate an r–dimensional Lie algebra. Now we want to prove that the set of symmetries admitted by a system of differential equations is a Lie algebra.

Lemma 4. *The commutator of two generators is form invariant under any invertible change of variables.*

Proof. Let $\mathbf{y} = \mathbf{g}(\mathbf{z})$ be a change of variables. We have

$$\begin{aligned}
\Xi_\mu &= \sum_{i=1}^{n} \xi_{\mu i} \frac{\partial}{\partial z_i}, &\qquad \widetilde{\Xi}_\mu &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \xi_{\mu i} \frac{\partial y_j}{\partial z_j} \right) \frac{\partial}{\partial y_j}, \\
\Xi_\nu &= \sum_{i=1}^{n} \xi_{\nu i} \frac{\partial}{\partial z_i}, &\qquad \widetilde{\Xi}_\nu &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \xi_{\nu i} \frac{\partial y_j}{\partial z_j} \right) \frac{\partial}{\partial y_j}.
\end{aligned}$$

Using the Einstein summation convention for repeated indices, we have:

$$\begin{aligned}
\left[\widetilde{\Xi}_\mu, \widetilde{\Xi}_\nu\right] &= \left(\widetilde{\xi}_{\mu j}\frac{\partial \widetilde{\xi}_{\nu k}}{\partial y_j} - \widetilde{\xi}_{\nu j}\frac{\partial \widetilde{\xi}_{\mu k}}{\partial y_j}\right)\frac{\partial}{\partial y_k} = \\
&= \left(\xi_{\mu i}\frac{\partial y_j}{\partial z_i}\frac{\partial}{\partial y_j}\left(\xi_{\nu l}\frac{\partial y_k}{\partial z_l}\right) - \xi_{\nu i}\frac{\partial y_j}{\partial z_i}\frac{\partial}{\partial y_j}\left(\xi_{\mu l}\frac{\partial y_k}{\partial z_l}\right)\right)\frac{\partial}{\partial y_k} = \\
&= \left(\xi_{\mu i}\frac{\partial y_j}{\partial z_i}\frac{\partial \xi_{\nu l}}{\partial z_m}\frac{\partial z_m}{\partial y_j}\frac{\partial y_k}{\partial z_l} - \xi_{\nu i}\frac{\partial y_j}{\partial z_i}\frac{\partial \xi_{\mu l}}{\partial z_m}\frac{\partial z_m}{\partial y_j}\frac{\partial y_k}{\partial z_l}\right)\frac{\partial}{\partial y_k} = \\
&= \left(\left(\xi_{\mu i}\frac{\partial \xi_{\nu l}}{\partial z_m} - \xi_{\nu i}\frac{\partial \xi_{\mu l}}{\partial z_m}\right)\frac{\partial z_m}{\partial y_j}\frac{\partial y_j}{\partial z_i}\frac{\partial y_k}{\partial z_l}\right)\frac{\partial}{\partial y_k} = \\
&= \left(\left(\xi_{\mu i}\frac{\partial \xi_{\nu l}}{\partial z_m} - \xi_{\nu i}\frac{\partial \xi_{\mu l}}{\partial z_m}\right)\delta_{mi}\frac{\partial y_k}{\partial z_l}\right)\frac{\partial}{\partial y_k} = \\
&= \left(\left(\xi_{\mu i}\frac{\partial \xi_{\nu l}}{\partial z_i} - \xi_{\nu i}\frac{\partial \xi_{\mu l}}{\partial z_i}\right)\frac{\partial y_k}{\partial z_l}\right)\frac{\partial}{\partial y_k} = \widetilde{[\Xi_\mu, \Xi_\nu]}.
\end{aligned}$$

□

Theorem 9. *If a regular manifold $F(\mathbf{z}) = 0$ is invariant under the generators Ξ_μ and Ξ_ν, then it is also invariant under their commutator $[\Xi_\mu, \Xi_\nu]$.*

Proof. Since the manifold $F(\mathbf{z}) = 0$ is invariant under the Lie group generated by Ξ_μ and Ξ_ν, we have

$$\Xi_\mu(F(\mathbf{z})) = \Lambda_\mu(\mathbf{z})F(\mathbf{z}),$$
$$\Xi_\nu(F(\mathbf{z})) = \Lambda_\nu(\mathbf{z})F(\mathbf{z}),$$

where $\Lambda_\mu(\mathbf{z})$ and $\Lambda_\nu(\mathbf{z})$ are specific Lagrange multipliers. Then,

$$\begin{aligned}
[\Xi_\mu, \Xi_\nu](F(\mathbf{z})) &= \Xi_\mu\Xi_\nu(F(\mathbf{z})) - \Xi_\nu\Xi_\mu(F(\mathbf{z})) = \\
&= \Xi_\mu(\Lambda_\nu(\mathbf{z})F(\mathbf{z})) - \Xi_\nu(\Lambda_\mu(\mathbf{z})F(\mathbf{z})) = \\
&= \Xi_\mu(\Lambda_\nu(\mathbf{z}))F(\mathbf{z}) + \Lambda_\nu(\mathbf{z})\Xi_\mu(F(\mathbf{z})) \\
&\quad - \Xi_\nu(\Lambda_\mu(\mathbf{z}))F(\mathbf{z}) - \Lambda_\mu(\mathbf{z})\Xi_\nu(F(\mathbf{z})) = \\
&= (\Xi_\mu(\Lambda_\nu(\mathbf{z})) - \Xi_\nu(\Lambda_\mu(\mathbf{z})))F(\mathbf{z}) = \\
&= \Lambda(\mathbf{z})F(\mathbf{z}),
\end{aligned}$$

so proving that the manifold $F(\mathbf{z})$ is also invariant with respect to the commutator.

□

Theorem 10. *The prolongation of a generator commutes with the commutator, that is,*

$$[\Xi_1^{(k)}, \Xi_2^{(k)}] = [\Xi_1, \Xi_2]^{(k)}, \qquad k \geq 1.$$

Proof. To prove this theorem, since the definition of the prolongation of a generator is a recursive operation, it is sufficient to restrict the proof to the first-order prolongation.

Consider two generators Ξ_1 and Ξ_2 involving the variables $\mathbf{x}$ and $\mathbf{u}$. Since the commutator is invariant with respect to invertible transformations of variables, we can introduce the canonical variables of the generator Ξ_1, so the generators can be written as follows:

$$\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \eta_\alpha \frac{\partial}{\partial u_\alpha}.$$

Then, we have

$$[\Xi_1, \Xi_2] = \sum_{i=1}^{n} \frac{\partial \xi_i}{\partial x_1} \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \frac{\partial \eta_\alpha}{\partial x_1} \frac{\partial}{\partial u_\alpha},$$

while the first-order prolonged operator becomes

$$\begin{aligned}
\Xi_1^{(1)} &= \Xi_1, \\
\Xi_2^{(1)} &= \Xi_2 + \sum_{\alpha=1}^{m} \sum_{k=1}^{n} \eta_{[\alpha,k]} \frac{\partial}{\partial u_{\alpha,k}},
\end{aligned}$$

where

$$\eta_{[\alpha,k]} = \frac{D\eta_\alpha}{Dx_k} - \frac{D\xi_j}{Dx_k} u_{\alpha,j}.$$

Therefore,

$$\begin{aligned}
[\Xi_1^{(1)}, \Xi_2^{(1)}] &= [\Xi_1, \Xi_2] + \left[\Xi_1, \sum_{\alpha=1}^{m} \sum_{k=1}^{n} \eta_{[\alpha,k]} \frac{\partial}{\partial u_{\alpha,k}} \right] = \\
&= \sum_{i=1}^{n} \frac{\partial \xi_i}{\partial x_1} \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \frac{\partial \eta_\alpha}{\partial x_1} \frac{\partial}{\partial u_\alpha} + \sum_{\alpha=1}^{m} \sum_{k=1}^{n} \frac{\partial \eta_{[\alpha,k]}}{\partial x_1} \frac{\partial}{\partial u_{\alpha,k}}.
\end{aligned}$$

On the other hand, the prolongation of the commutator $[\Xi_1, \Xi_2]$ is

$$[\Xi_1, \Xi_2]^{(1)} = [\Xi_1, \Xi_2] + \sum_{\alpha=1}^{m} \sum_{k=1}^{n} \widehat{\eta}_{[\alpha,k]} \frac{\partial}{\partial u_{\alpha,k}},$$

where

$$\begin{aligned}
\widehat{\eta}_{[\alpha,k]} &= \frac{D}{Dx_k} \left(\frac{\partial \eta_\alpha}{\partial x_1} \right) - \frac{D}{Dx_k} \left(\frac{\partial \xi_j}{\partial x_1} \right) u_{\alpha,j} = \\
&= \frac{\partial}{\partial x_1} \left(\frac{D\eta_\alpha}{Dx_k} - \frac{D\xi_j}{Dx_k} u_{\alpha,j} \right) = \\
&= \frac{\partial \eta_{[\alpha,k]}}{\partial x_1}.
\end{aligned}$$

Thus

$$[\Xi_1, \Xi_2]^{(1)} = \left[\Xi_1^{(1)}, \Xi_2^{(1)} \right].$$

□

Because of the previous results, we can state the following theorem.

Theorem 11. *If a system of differential equations $\mathcal{S}$ admits the generators Ξ_μ and Ξ_ν, then it also admits the generator $[\Xi_\mu, \Xi_\nu]$.*

Remark 7. *The set of generators admitted by a system $\mathcal{S}$ of differential equations is a vector space, as is the space of solutions of (2.18), being solutions of a system of linear and homogeneous differential equations. The previous theorem also implies that this vector space is also* *a Lie algebra, called the* ***principal Lie algebra****, and knowledge of its subalgebras allows us to construct and classify particular solutions of the system, called* ***invariant solutions****.*

We have seen that the generators of a Lie group with r parameters generate an r–dimensional Lie algebra. In fact, the vice versa also holds true.

Theorem 12. *Let $\mathcal{L}_r$ be an r-dimensional vector space spanned by the operators*

$$\Xi_\alpha = \zeta_{\alpha i}\frac{\partial}{\partial z_i}, \quad \alpha = 1, \dots, r.$$

The composition $T_{\mathbf{a}} = T_{a_r} \cdots T_{a_1}$, with $\mathbf{a} = (a_1, \dots, a_r)$, of r one-parameter groups of transformations T_{a_α} generated individually by each of the base operators Ξ_α via the Lie equations

$$\frac{dz_i^\star}{da_\alpha} = \zeta_{\alpha i}, \quad z_i^\star\big|_{a_\alpha=0} = z_i, \quad i = 1, \dots, n, \tag{2.21}$$

is an r-parameter (local) group $\mathcal{G}_r$ if and only if $\mathcal{L}_r$ is a Lie algebra. By applying the same construction to any s-dimensional subalgebra of $\mathcal{L}_r$, one generates an s-parameter subgroup of the group $\mathcal{G}_r$.

Here are two illustrative examples of this result:

Example 15. *Consider a three-dimensional Lie algebra generated by*

$$\Xi_1 = \frac{\partial}{\partial x}, \quad \Xi_2 = \frac{\partial}{\partial y}, \quad \Xi_3 = y\frac{\partial}{\partial x}$$

The solution to the Lie equations (2.21) corresponding to these operators yields the following three one-parameter groups with parameters a_1, a_2, a_3:

$$\begin{aligned} T_{a_1} &: x^\star = x + a_1, & y^\star &= y \\ T_{a_2} &: x^\star = x, & y^\star &= y + a_2 \\ T_{a_3} &: x^\star = x + a_3 y, & y^\star &= y \end{aligned}$$

Their composition is given by

$$T_a = T_{a_3}T_{a_2}T_{a_1}$$

where $a = (a_1, a_2, a_3)$, resulting in the transformation:

$$\begin{aligned} x^\star &= x + a_3 y + a_1 + a_2 a_3 \\ y^\star &= y + a_2 \end{aligned} \tag{2.22}$$

The successive application of T_a and T_b, where $b = (b_1, b_2, b_3)$, results in the transformation T_bT_a:

$$\begin{aligned} x^{\star\star} &= x + (a_3 + b_3)y + a_2(a_3 + b_3) + b_2 b_3 + a_1 + b_1 \\ y^{\star\star} &= y + a_2 + b_2 \end{aligned}$$

The equation $T_bT_a = T_c$ implies the following equations:

$$\begin{aligned} x^{\star\star} &= x + c_3 y + c_1 + c_2 c_3 \\ y^\star &= y + c_2 \end{aligned}$$

From this, we derive:

$$c_1 = a_1 + b_1 - b_2 a_3$$
$$c_2 = a_2 + b_2$$
$$c_3 = a_3 + b_3$$

Thus, the transformation (2.22) establishes, in accordance with Theorem 12, a three-parameter group with the composition law described by the following equations:

$$\phi_1(a,b) = a_1 + b_1 - b_2 a_3$$
$$\phi_2(a,b) = a_2 + b_2$$
$$\phi_3(a,b) = a_3 + b_3$$

Example 16. *Let L_2 be the two-dimensional vector space spanned by*

$$\Xi_1 = \frac{\partial}{\partial x}, \quad \Xi_2 = x\frac{\partial}{\partial y} \tag{2.23}$$

The operators X_1 and X_2 generate the following two one-parameter groups:

$$T_{a_1} : x^\star = x + a_1, \quad y^\star = y$$
$$T_{a_2} : x^\star = x, \quad y^\star = y + a_2 x$$

Their composition is given by

$$T_a = T_{a_2} T_{a_1}$$

where $a = (a_1, a_2)$, yielding the transformation:

$$\begin{aligned} x^\star &= x + a_1 \\ y^\star &= y + x a_2 + a_1 a_2 \end{aligned} \tag{2.24}$$

The successive application of T_a and T_b, where $b = (b_1, b_2)$, results in the transformations $T_b T_a$:

$$\begin{aligned} x^{\star\star} &= x + a_1 + b_1 \\ y^{\star\star} &= y + (x + a_1)(a_2 + b_2) + b_1 b_2 \end{aligned} \tag{2.25}$$

The equation $T_b T_a = T_c$ implies:

$$\begin{aligned} x^{\star\star} &= x + c_1 \\ y^{\star\star} &= y + x c_2 + c_1 c_2 \end{aligned} \tag{2.26}$$

Equations (2.25) and (2.26) yield:

$$\begin{aligned} c_1 &= a_1 + b_1 \\ c_2 &= a_2 + b_2 \\ c_1 c_2 &= a_1 (a_2 + b_2) + b_1 b_2 \end{aligned} \tag{2.27}$$

Substituting c_1 and c_2 from the first two equations of (2.27) into the third equation results in $b_1 a_2 = 0$. Since this equation cannot hold for arbitrary a and b, the two-parameter family of transformations (2.24) does not satisfy property (4) of the group of transformations. This conclusion aligns with Theorem 12, as the vector space L_2 generated by the operators (2.23) does not form a Lie algebra.

Chapter 3

Group invariant solutions and the classification problem

Using the methods from the previous chapter, we can systematically find the Lie point symmetries of a given differential equation. Let us focus on partial differential equations. In some cases, exact methods involve converting the given PDE into one or more ODEs. For example, the general solution of a first-order quasilinear PDE can be obtained by integrating its characteristic equations. Since it is often not possible to find a *general solution* for many PDEs, we must instead turn to various ansatze. We may seek solutions such as similarity solutions, traveling waves, separable solutions, and others. Many of these approaches involve identifying solutions that remain invariant under a specific group of symmetries.

This chapter explains how to use these symmetries to obtain exact solutions. In particular, the one-parameter Lie group of transformations admitted by a given system of differential equations can be used to find exact solutions in two ways:

- find particular solutions, called *invariant solutions*;
- derive new solutions, once a specific solution is known.

3.1 Group invariant solutions

The function $\mathbf{u} = \boldsymbol{\Theta}(\mathbf{x})$ is called an *invariant solution* of the differential equations $\boldsymbol{\Delta} = \mathbf{0}$ if and only if:

1. $\mathbf{u} = \boldsymbol{\Theta}(\mathbf{x})$ is an invariant surface of (2.15);
2. $\mathbf{u} = \boldsymbol{\Theta}(\mathbf{x})$ is a solution of the system (2.17).

From Theorem 4 it follows that a solution is invariant if and only if:

$$\Xi(\mathbf{u} - \boldsymbol{\Theta}(\mathbf{x})) = 0 \quad \text{for} \quad \mathbf{u} = \boldsymbol{\Theta}(\mathbf{x}), \tag{3.1}$$

$$\boldsymbol{\Delta}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}) = \mathbf{0}. \tag{3.2}$$

The equation (3.1), called the *invariant surface condition*, has the form

$$\xi_1(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_1} + \cdots + \xi_n(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_n} = \boldsymbol{\eta}(\mathbf{x}, \mathbf{u}), \tag{3.3}$$

and is solved by introducing the corresponding characteristic equations:

$$\frac{dx_1}{\xi_1(\mathbf{x}, \mathbf{u})} = \cdots = \frac{dx_n}{\xi_n(\mathbf{x}, \mathbf{u})} = \frac{du_1}{\eta_1(\mathbf{x}, \mathbf{u})} = \cdots = \frac{du_m}{\eta_m(\mathbf{x}, \mathbf{u})}.$$

This allows us to express the solution $\mathbf{u} = \mathbf{\Theta}(\mathbf{x})$ as

$$\mathbf{u} = \mathbf{\Phi}(I_1(\mathbf{x},\mathbf{u}),\ldots,I_{n-1}(\mathbf{x},\mathbf{u})); \tag{3.4}$$

substituting (3.4) into the system (2.17), we obtain a reduced system of differential equations with $n-1$ independent variables.

If $n = 2$, that is, the equation has two independent variables, the reduced equation has only one independent variable, and so it is an ordinary differential equation.

Example 17. *We consider the viscous Burgers' equation*

$$u_t + uu_x - u_{xx} = 0$$

and let

$$\Xi = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}$$

be a generic infinitesimal generator. Finding an invariant solution means adding a constraint, i.e., the invariant surface condition:

$$\begin{cases} u_t + uu_x - u_{xx} = 0, \\ \tau\dfrac{\partial u}{\partial t} + \xi\dfrac{\partial u}{\partial x} = \eta. \end{cases}$$

Burgers' equation admits a 5–parameter Lie group of transformations with infinitesimal generators

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial t}, \quad \textit{(Time translation)}\\
&\Xi_2 = \frac{\partial}{\partial x}, \quad \textit{(Space translation)}\\
&\Xi_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, \quad \textit{(Scaling)}\\
&\Xi_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad \textit{(Galilean transformation)}\\
&\Xi_5 = t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} + (x-tu)\frac{\partial}{\partial u} \quad \textit{(Projective transformation)}
\end{aligned}$$

spanning a 5–dimensional Lie algebra.

For each one-parameter subgroup of the symmetry group, there is an associated class of solutions that remains invariant under group transformations. These solutions are obtained by solving a reduced ordinary differential equation, the form of which generally depends on the specific subgroup being analyzed.

(a) Time–Invariant Solutions. *For symmetry generator Ξ_1, the invariant surface condition becomes*

$$u_t = 0,$$

so the representation of the invariant solution is $u = U(x)$. Substituting this expression into the Burgers' equation we find the following reduced ordinary differential equation

$$UU' - U'' = 0.$$

The first integration provides

$$\frac{U^2}{2} - U' = c_1,$$

with $c_1 \in \mathbb{R}$*. Then, three cases can be considered:* $c_1 = 0$*,* $c_1 > 0$*, and* $c_1 < 0$*. For* $c_1 = 0$*, we get*

$$U(x) = -\frac{2}{x+b}, \quad b \in \mathbb{R}.$$

For $c_1 > 0$*, choosing* $c_1 = 2a^2$ *with* $a \in \mathbb{R}$*, the solution becomes*

$$U(x) = -2a \tanh\left(a(x+b)\right),$$

and finally, for $c_1 = -2a^2$*, we have*

$$U(x) = 2a \tan\left(a(x+b)\right).$$

Thus, depending on the initial conditions, we obtain one of the following three stationary invariant solutions:

$$\begin{aligned} u(t,x) &= -\frac{2}{x+b}, \\ u(t,x) &= -2a \tanh\left(a(x+b)\right), \\ u(t,x) &= 2a \tan\left(a(x+b)\right), \end{aligned}$$

with $a, b \in \mathbb{R}$*.*

(b) Space–Invariant Solutions. *The invariant solutions with respect to* Ξ_2 *are only those that are constant, i.e.,* $u = k$*, with* $k \in \mathbb{R}$*.*

(c) Scale–Invariant Solutions. *For symmetry generator* Ξ_3*, the invariant surface condition is*

$$2tu_t + xu_x + u = 0,$$

which is solved by integrating the characteristic equations

$$\frac{dt}{2t} = \frac{dx}{x} = \frac{du}{-u}.$$

On the half space $\{(t,x,u) : t > 0\}$*, the global invariants are given by the functions*

$$\omega = \frac{x^2}{t}, \quad U = \sqrt{t}u.$$

Because ω *is independent of* U*, every invariant solution is of the form*

$$U = U(\omega),$$

so that

$$u = \frac{1}{\sqrt{t}} U(\omega). \tag{3.5}$$

Computing the derivatives, we have

$$\begin{aligned} u_t &= -\frac{1}{2t^{5/2}}\left(tU + 2x^2U'\right), \qquad u_x = \frac{2x}{t^{3/2}}U', \\ u_{xx} &= \frac{2}{t^{5/2}}\left(tU' + 2x^2U''\right), \end{aligned}$$

and, substituting these expressions into the Burgers' equation, we find the reduced equation under the scaling group:

$$8\omega U'' - 4\sqrt{\omega}UU' + 2(\omega+2)U' + U = 0. \tag{3.6}$$

Therefore (3.5) is a solution of the Burgers' equation if $U(\omega)$ satisfies equation (3.6).

(d) Galilean–Invariant Solutions. *In this case, the invariant surface condition is*

$$u_x = \frac{1}{t},$$

from which we obtain

$$u = \frac{x}{t} + U(t).$$

Inserting the latter in the Burgers' equation, one has the reduced equation

$$U' + \frac{U}{t} = 0,$$

whose general solution is

$$U = \frac{U_0}{t},$$

with $U_0 \in \mathbb{R}$. So the invariant solution is

$$u = \frac{1}{t}\left(x + U_0\right).$$

(e) Projective group–Invariant Solutions. *As in the previous cases, integrating the invariant surface condition by the method of characteristics, we obtain the global invariants of the one–parameter projective group generated by Ξ_5:*

$$\omega = \frac{x}{t}, \quad U = tu - x,$$

and hence

$$u = \frac{x}{t} + \frac{U}{t}.$$

Finally, we obtain the reduced equation

$$UU' - U'' = 0.$$

Again, as in case (a), we obtain three invariant solutions with respect to projective group:

$$\begin{aligned}
u(t,x) &= \frac{x^2 + btx - 2t}{bt^2 + tx}, \\
u(t,x) &= \frac{1}{t}\left(x - 2a\tanh\left(a\frac{x+bt}{t}\right)\right), \\
u(t,x) &= \frac{1}{t}\left(x + 2a\tan\left(a\frac{x+bt}{t}\right)\right),
\end{aligned}$$

with $a, b \in \mathbb{R}$.

We found five classes of group invariant solutions; however, they are not the only ones, as we can consider invariant solutions by using various linear combinations of the admitted symmetries. For example, because the Burgers' equation is invariant with respect to time and space translations, we can consider a combination of them

$$\Xi_1 + c\Xi_2 = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}.$$

Finding invariant solutions under this symmetry gives traveling wave solutions. Indeed, in this case, by solving the condition (3.3) we find that this invariant solution takes the form

$$u = U(x - ct),$$

and the reduced equation results

$$UU' - cU' - U'' = 0.$$

By setting $V = U - c$, we obtain

$$VV' - V'' = 0,$$

that is a reduced equation having the form of the reduced equation obtained in the case of time translation.

3.2 New solutions from a known solution

A Lie transformation group admitted by a system of differential equations has the aforementioned property of mapping every solution into another solution. This suggests a method for generating new solutions from *a priori* known solutions; this is interesting when non-trivial solutions can be obtained from trivial ones.

We can summarize this procedure in the following theorem.

Theorem 13. *Let $\boldsymbol{\Delta} = \mathbf{0}$ be a system of differential equations admitting the Lie group of transformations*

$$\mathbf{x}^\star = \mathbf{X}(\mathbf{x}, \mathbf{u}; a),$$
$$\mathbf{u}^\star = \mathbf{U}(\mathbf{x}, \mathbf{u}; a),$$

and let $\mathbf{u} = \boldsymbol{\Theta}(\mathbf{x})$ be a solution of $\boldsymbol{\Delta} = \mathbf{0}$ that is not invariant with respect to the Lie group. Then,

$$\mathbf{u} = \mathbf{U}(\mathbf{X}(\mathbf{x}, \mathbf{u}; a), \boldsymbol{\Theta}(\mathbf{X}(\mathbf{x}, \mathbf{u}; a)); -a)$$

implicitly defines a one–parameter family of solutions of the given system.

Example 18. *The linear heat equation*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{3.7}$$

admits the group with the generator

$$\Xi = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - \left(\frac{x^2}{4} + \frac{t}{2}\right) u\frac{\partial}{\partial u}. \tag{3.8}$$

The related finite transformation is:

$$x^\star = \frac{x}{1-\alpha t}, \quad t^\star = \frac{t}{1-\alpha t},$$
$$u^\star = u\sqrt{1-\alpha t}\exp\left(-\frac{\alpha x^2}{4(1-\alpha t)}\right).$$

One can obtain the inverse by exchanging $(\mathrm{x}, \mathrm{t}, \mathrm{u})$ *and* $(\mathrm{x}^\star, \mathrm{t}^\star, \mathrm{u}^\star)$ *and replacing a by* $-a$ *:*

$$x = \frac{x^\star}{1+\alpha t^\star}, \quad t = \frac{t^\star}{1+\alpha t^\star},$$
$$u = u^\star\sqrt{1+\alpha t^\star}\exp\left(\frac{\alpha (x^\star)^2}{4\,(1+\alpha t^\star)}\right).$$

By applying this transformation to the trivial solution $u = A$ *(A constant), the nontrivial solution*

$$u = \frac{A}{\sqrt{1+\alpha t}}\exp\left(-\frac{\alpha x^2}{4(1+\alpha t)}\right)$$

is immediately generated.

3.3 Classification of invariant solutions

Typically, for each subgroup of the symmetry group admitted by a system of differential equations, there is a corresponding family of group invariant solutions. Because of the potentially infinite number of such subgroups, listing all possible invariant solutions is unpractical. Therefore, an effective and systematic classification method is required to derive an *optimal system* of group invariant solutions, from which every other solution can be obtained.

3.3.1 Classification of Lie subgroups

For simplicity, we focus on the problem of equivalence of solutions that are invariant under a one–parameter Lie group of transformations.

Let $\mathcal{G}$ be a multiparameter Lie group of transformations admitted by a system of differential equations $\mathbf{\Delta} = \mathbf{0}$. Moreover, let $\mathbf{u} = \mathbf{\Theta}(\mathbf{x})$ be a invariant solution with respect to a one–parameter subgroup of transformations $T^\alpha \in \mathcal{G}$. In other words, if we set

$$F(\mathbf{x}, \mathbf{u}) = \mathbf{u} - \mathbf{\Theta}(\mathbf{x}),$$

it means to require that

$$F(T^\alpha(\mathbf{x}, \mathbf{u})) = F(\mathbf{x}^\star, \mathbf{u}^\star) = 0, \quad \text{when} \quad F(\mathbf{x}, \mathbf{u}) = 0,$$

i.e.,

$$F(\mathbf{x}^\star, \mathbf{u}^\star) = \mathbf{\Lambda}(\mathbf{x}, \mathbf{u}; \alpha) F(\mathbf{x}, \mathbf{u}).$$

Let $G \in \mathcal{G}$ be another fixed transformation of the Lie group $\mathcal{G}$ and denote $\widetilde{F}(\mathbf{x}^\star, \mathbf{u}^\star) \equiv GF(\mathbf{x}, \mathbf{u})$. We want to prove that $\widetilde{F}$ is invariant with respect to the conjugate subgroup $\widetilde{T}^\alpha = GT^\alpha G^{-1}$ of the full symmetry group $\mathcal{G}$. In fact, one has

$$\widetilde{T}^\alpha \widetilde{F} = GT^\alpha G^{-1}\widetilde{F} = GT^\alpha F = G(\mathbf{\Lambda} F) = (G\mathbf{\Lambda})(GF) = \widetilde{\mathbf{\Lambda}}\widetilde{F}.$$

Since s–parameter subgroups ($s > 1$) are completely determined by their one–parameter subgroups, this remark can be generalized in the following

Proposition 5. *Let $\mathcal{G}$ be the symmetry group of a system of differential equations $\mathbf{\Delta} = \mathbf{0}$ and let $\mathcal{H} \subset \mathcal{G}$ be an s–parameter subgroup. If $\mathbf{u} = \mathbf{\Theta}(\mathbf{x})$ is an $\mathcal{H}$–invariant solution to $\mathbf{\Delta} = \mathbf{0}$ and $G \in \mathcal{G}$ is any other group element, then the transformed $\mathbf{u} = G\mathbf{\Theta}(\mathbf{x})$ is a $\widetilde{\mathcal{H}}$–invariant solution, where $\widetilde{\mathcal{H}} = G\mathcal{H}G^{-1}$ is the conjugate subgroup to $\mathcal{H}$ under G, and*

$$G\mathcal{H}G^{-1} = \{GHG^{-1} \,:\, H \in \mathcal{H}\}.$$

Example 19. *In Example 18, we have seen that the linear heat equation admits the one-parameter Lie group T^α defined by the transformations*

$$\begin{aligned} x^\star &= \frac{x}{1-\alpha t}, \quad t^\star = \frac{t}{1-\alpha t}, \\ u^\star &= u\sqrt{1-\alpha t}\exp\left(-\frac{\alpha x^2}{4(1-\alpha t)}\right), \end{aligned}$$

whose infinitesimal generator is given by (3.8). It can be verified that

$$u = \frac{1}{\sqrt{t}}\left(c_1\frac{x}{t} + c_2\right)\exp\left(-\frac{x^2}{4t}\right)$$

is a T^α–invariant solution of the equation (3.7). Indeed, denoting by

$$\mathcal{S}(t,x,u) \equiv u - \frac{1}{\sqrt{t}}\left(c_1\frac{x}{t} + c_2\right)\exp\left(-\frac{x^2}{4t}\right) = 0$$

its corresponding manifold, one has

$$\widetilde{\mathcal{S}} = T^\alpha\mathcal{S} = \mathcal{S}.$$

Now, let G be another element of the full symmetry group defined by

$$\begin{aligned} x^\star &= \frac{x}{1-t}, \quad t^\star = \frac{1}{1-t}, \\ u^\star &= u\sqrt{1-t}\exp\left(-\frac{x^2}{4(1-t)}\right), \end{aligned}$$

and let G^{-1} denote the inverse transformation, defined as:

$$\begin{aligned} x^\star &= \frac{x}{t}, \quad t^\star = \frac{t-1}{t}, \\ u^\star &= u\sqrt{t}\exp\left(\frac{x^2}{4t}\right). \end{aligned}$$

By computing the conjugate subgroup gives us:

$$\begin{aligned} \widetilde{T}^\alpha &= GT^\alpha G^{-1}(t,x,u) \\ &= (GT^\alpha)G^{-1}((t,x,u)) \\ &= G^{-1}((GT^\alpha)(t,x,u)) \\ &= G^{-1}(GT^\alpha((t,x,u))) \\ &= G^{-1}(T^\alpha(G(t,x,u))) \end{aligned}$$

$$
\begin{aligned}
&= G^{-1}(T^{\alpha}(G((t,x,u)))) \\
&= (G^{-1} \circ T^{\alpha} \circ G)(t,x,u) \\
&= (t+\alpha, x, u),
\end{aligned}
$$

whose infinitesimal generator is

$$
\widetilde{\Xi} = \frac{\partial}{\partial t}
$$

In fact, applying G to $\mathcal{S}$, we obtain

$$
\widetilde{S} = G\mathcal{S} = \exp\left(\frac{x^2}{4(t-1)}\right)\sqrt{1-t}\,(u - c_1 x - c_2)\,,
$$

namely,

$$
u = c_1 x + c_2
$$

is a time-independent solution of the heat equation, say it is invariant under time translation, as expected.

Since nonessentially different invariant solutions are found from conjugate subgroups, the problem of the classification of $\mathcal{H}$–invariant solutions is reduced to the classification of subgroups of the group $\mathcal{G}$, up to conjugation. Thus, we say that a subgroup $\mathcal{H} \subseteq \mathcal{G}$ is *equivalent* to a subgroup $\widetilde{\mathcal{H}} \subseteq \mathcal{G}$ if there exists $G \in \mathcal{G}$ such that $\widetilde{\mathcal{H}} = G\mathcal{H}G^{-1}$. It is a relation of equivalence and the corresponding equivalence classes are said *conjugacy classes*.

It can be proved that this problem, in turn, is reduced to the corresponding problem of classification of Lie subalgebras, that can be approached more easily from an algorithmic point of view.

Remark 8. *In Chapter 5, we will show that the infinitesimal generator of the transformation T^{α} from above example (known as the projective group) and the infinitesimal generator of the time translation, when considered as elements of the Lie algebra associated with the group of Lie transformations admitted by the heat equation, are in fact equivalent under the group of inner automorphisms of the Lie algebra.*

3.3.2 Equivalent Lie subgroups and corresponding Lie subalgebras

Again, for a simple approach, we begin by considering Lie subgroups of one-parameter Lie transformations.

Let us $T^{\alpha} \in \mathcal{G}$ be a one–parameter Lie subgroup of transformations generated by its infinitesimal generator Ξ, and let $G \in \mathcal{G}$ another one–parameter subgroup. We denote by $\widetilde{\Xi} = GT^{\alpha}G^{-1}$ the infinitesimal generator equivalent to $\widetilde{T}^{\alpha}$.

For any smooth function F, one has

$$
\widetilde{\Xi}F(\mathbf{z}^{\star}) = G\Xi F(\mathbf{z}) = G\Xi G^{-1}F(\mathbf{z}^{\star}).
$$

Therefore,

$$
\widetilde{\Xi}^2 F(\mathbf{z}^{\star}) = G\Xi G^{-1}G\Xi G^{-1}F(\mathbf{z}^{\star}) = G\Xi^2 G^{-1}F(\mathbf{z}^{\star}),
$$

and so on. Since the transformations are analytic function with respect to the parameter, we can assume the convergence and consider the Lie series

$$
\widetilde{T}^{\alpha}F(\mathbf{z}^{\star}) = \exp(\alpha\widetilde{\Xi})F(\mathbf{z}^{\star}) = G\exp(\alpha\Xi)G^{-1}F(\mathbf{z}^{\star}) = GT^{\alpha}G^{-1}F(\mathbf{z}^{\star}).
$$

Due to the arbitrariness of F we have that

$$\widetilde{\Xi} = G\Xi G^{-1} \tag{3.9}$$

is the infinitesimal generator of $\widetilde{T}^{\alpha}$.

Hence, two one–parameter Lie subgroups are equivalent if and only if the relation (3.9) between the respective infinitesimal generators holds true.

Remark 9. *We note that T^{α} and $\widetilde{T}^{\alpha}$, as the parameter changes, generate one–parameter subgroups of $\mathcal{G}$. On the other side, Ξ and $\widetilde{\Xi}$ are two elements of the corresponding Lie algebra, and they generate two corresponding one–dimensional subalgebras. Moreover, if T^{α} and $\widetilde{T}^{\alpha}$ are equivalent via G, then the corresponding Lie subalgebras generated by Ξ and $\widetilde{\Xi}$ are related by the same Lie group transformation G. That is, the above subalgebras are* equivalent.

Let us now see how to construct, using only the infinitesimal generator and without using finite transformations of the Lie group, the maps that establish equivalence between Lie subalgebras.

The remainder of the current chapter deals only with equivalence under Lie symmetries that are generated by a finite–dimensional Lie algebra $\mathcal{L}$ with a basis $\{\Xi_1, \ldots, \Xi_r\}$.

For each generator Ξ_i in the basis, we can consider

$$\widetilde{X} = \exp(\alpha\Xi_i) X \exp(-\alpha\Xi_i), \tag{3.10}$$

for any generator X.

Differentiating the expression (3.10), we obtain

$$\begin{aligned}\frac{d\widetilde{X}}{d\alpha} &= \Xi_i \exp(\alpha\Xi_i) X \exp(-\alpha\Xi_i) - \exp(\alpha\Xi_i) X \exp(-\alpha\Xi_i)\Xi_i \\ &= \Xi_i\widetilde{X} - \widetilde{X}\Xi_i \\ &= [\Xi_i, \widetilde{X}].\end{aligned}$$

Thus, $\widetilde{X}$ satisfies the following initial–value problem:

$$\frac{d\widetilde{X}}{d\alpha} = [\Xi_i, \widetilde{X}], \qquad \widetilde{X}|_{\alpha=0} = X. \tag{3.11}$$

Again, calculating the second derivative, we obtain

$$\frac{d^2\widetilde{X}}{d\alpha^2} = [\Xi_i, \frac{d\widetilde{X}}{d\alpha}] = [\Xi_i, [\Xi_i, \widetilde{X}]],$$

and so on. So, because Taylor's theorem (in a neighborhood of $\alpha = 0$), one has

$$\widetilde{X} = X + \alpha[\Xi_i, X] + \frac{\alpha}{2!}[\Xi_i, [\Xi_i, X]] + \cdots .$$

Considering the adjoint map $\mathrm{ad}_{\Xi_i} : \mathcal{L} \to \mathcal{L}$ defined by

$$\mathrm{ad}_X(Y) = [X, Y], \quad \text{for each } X, Y \in \mathcal{L},$$

we have

$$\widetilde{X} = X + \alpha\,\mathrm{ad}_{\Xi_i}(X) + \frac{\alpha}{2!}\,\mathrm{ad}_{\Xi_i}(\mathrm{ad}_{\Xi_i}(X)) + \cdots = \sum_{k=0}^{+\infty} \frac{\alpha\,\mathrm{ad}^k_{\Xi_i}(X)}{k!}, \tag{3.12}$$

that is, the exponential of adjoint map ad_{Ξ_i}. The convergence follows since (3.11) is a system of linear ordinary differential equations with constant coefficients.

We can construct the matrix of the linear map $\mathrm{ad}_X : \mathcal{L} \to \mathcal{L}$, relative to the basis $\{Xi_1, \dots, \Xi_r\}$ of $\mathcal{L}$. Let $X = \sum_{i=1}^r f^i \Xi_i$, then

$$\mathrm{ad}_X(\Xi_j) = \sum_{i=1}^r f^i [\Xi_i, \Xi_j] = \sum_{k=1}^r \left(\sum_{i=1}^r f^i c^k_{ij} \right).$$

It follows that the matrix associated to ad_X is $(f^i c^k_{ij})^k_j$, where the Einstein convention on sums over repeated indices has been used.

Then, in the finite-dimensional case, we can construct the invertible transformations that map equivalent Lie subalgebras by computing the exponential of these matrices. It can be shown that, given an element $X \in \mathcal{L}$, $\exp(\mathrm{ad}_X)$ is a Lie algebra automorphism, known as an *inner automorphism* of $\mathcal{L}$. The set of all inner automorphisms of $\mathcal{L}$ is denoted by $\mathrm{Int}(\mathcal{L})$.

Example 20. *Consider the non-abelian two-dimensional Lie algebra $\mathcal{L}$ with a basis $\{\Xi_1, \Xi_2\}$ such that*

$$[\Xi_1, \Xi_2] = \Xi_1$$

A generic element $X \in \mathcal{L}$ is of the form

$$X = f^1 \Xi_1 + f^2 \Xi_2.$$

In this case it is simple to compute the inner automorphisms related to the generators Ξ_1 and Ξ_2. In fact, it is sufficient to write the series (3.12); we begin with $\exp(t_1\,\mathrm{ad}_{\Xi_1})$*:*

$$\begin{aligned}
\exp(t_1\,\mathrm{ad}_{\Xi_1})(X) &= X + t_1\,[\Xi_1, X] + \frac{t_1^2}{2!}\,[\Xi_1, [\Xi_1, X]] + \cdots \\
&= f^i \left(\Xi_i + t_1\,[\Xi_1, \Xi_i] + \frac{t_1^2}{2!}\,[\Xi_1, [\Xi_1, \Xi_i]] + \cdots \right) \\
&= f^1 \Xi_1 + f^2\,(\Xi_2 + t_1 \Xi_1) \\
&= \left(f^1 + t_1 f^2 \right) \Xi_1 + f^2 \Xi_2
\end{aligned}$$

Furthermore, computing $\exp(t_2\,\mathrm{ad}_{\Xi_2})$ *yields*

$$\begin{aligned}
\exp(t_2\,\mathrm{ad}_{\Xi_2})(X) &= f^i \left(\Xi_i + t_2\,[\Xi_2, \Xi_i] + \frac{t_2^2}{2!}\,[\Xi_2, [\Xi_2, \Xi_i]] + \cdots \right) \\
&= \exp(-t_2) f^1 \Xi_1 + f^2 \Xi_2.
\end{aligned}$$

Example 21. *Consider the three-dimensional Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. We choose a basis $\{\Xi_1, \Xi_2, \Xi_3\}$ such that*

$$[\Xi_1, \Xi_2] = \Xi_1, \quad [\Xi_1, \Xi_3] = -2\Xi_2, \quad [\Xi_2, \Xi_3] = \Xi_3.$$

We can calculate the matrix representation of adjoint maps ad_{Ξ_i}, *with* $i = 1, 2, 3$*:*

$$\mathrm{ad}_{\Xi_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathrm{ad}_{\Xi_2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$\mathrm{ad}_{\Xi_3} = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

The inner automorphisms $A_i = \exp(t_i \, \mathrm{ad}_{\Xi_i})$ *are obtained by calculating the respective exponential matrices:*

$$A_1 = \begin{pmatrix} 1 & t_1 & t_1^2 \\ 0 & 1 & 2t_1 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \exp(-t_2) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(t_2) \end{pmatrix},$$

and

$$A_3 = \begin{pmatrix} 1 & 0 & 0 \\ -2t_3 & 1 & 0 \\ t_3^2 & -t_3 & 1 \end{pmatrix}.$$

We say that two subalgebras H_1 and H_2 of $\mathcal{L}$ are *equivalent* if there exists an inner automorphism A such that $H_2 = A(H_1)$.

Hence, the following proposition holds.

Proposition 6. *Let* $\mathcal{G}$ *be the symmetry group of a system of differential equations* $\mathbf{\Delta} = \mathbf{0}$*. Let* $\mathcal{H}$ *and* $\widetilde{\mathcal{H}}$ *be s–parameter subgroups of* $\mathcal{G}$ *with corresponding Lie subalgebras* H *and* $\widetilde{H}$ *of the Lie algebras* $\mathcal{L}$ *of* $\mathcal{G}$*. Then,* $\mathcal{H}$ *and* $\widetilde{\mathcal{H}}$ *are equivalent if and only if* H *and* $\widetilde{H}$ *are Lie subalgebras equivalent.*

Now, we can give the definition of *optimal system of Lie subalgebras*.

Definition 21. *The optimal system of subalgebras of a Lie algebra* $\mathcal{L}$ *with inner automorphisms* $A = \mathrm{Int}(\mathcal{L})$ *is a set of subalgebras* $\Theta_A(\mathcal{L})$ *such that:*

1. *there are no two elements of this set which can be transformed into each other by inner automorphisms of the Lie algebra* $\mathcal{L}$*;*

2. *any subalgebra of the Lie algebra* $\mathcal{L}$ *can be transformed into one of subalgebras of the set* $\Theta_A(\mathcal{L})$*.*

The union of the elements of the optimal system of given dimensionality s is called optimal system of order *s and denoted by the symbol* Θ_A^s*; the solution of the classification problem for a finite-dimensional Lie algebra* $\mathcal{L}$ *yields tables of optimal systems for every* $s = 1, \ldots, r-1$.

In summary, we have seen that the problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras, and so we concentrate on the latter.

For one–dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the inner automorphisms, since each one-dimensional subalgebra is determined by a nonzero vector in $\mathcal{L}$. Applying the inner automorphisms $\exp(\mathrm{ad}_{\Xi_i})$ (with $i = 1, \ldots, r$) in turn, each element is reduced to the simplest equivalent form.

We show an illustrative example.

Example 22. *Consider the Lie algebra $\mathcal{L}_4$ spanned by the vector fields*

$$\Xi_1 = \frac{\partial}{\partial x}, \quad \Xi_2 = \frac{\partial}{\partial t}, \quad \Xi_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad \Xi_4 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}$$

generating the Lie point symmetries of Korteweg-deVries equation [11],

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

In order to compute inner automorphisms, we first compute the moltiplication table and consequently the adjoint maps. We have

$$[\Xi_1, \Xi_4] = \Xi_1, \quad [\Xi_2, \Xi_3] = \Xi_1, \quad [\Xi_2, \Xi_4] = 3\Xi_2, \quad [\Xi_3, \Xi_4] = -2\Xi_3.$$

Thus, we can compute ad_{Ξ_i}, $i = 1, \ldots, 4$*:*

$$\mathrm{ad}_{\Xi_1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathrm{ad}_{\Xi_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{ad}_{\Xi_3} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathrm{ad}_{\Xi_4} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now, using (3.12) leads to $A_i = \exp(t_i \, \mathrm{ad}_{\Xi_i})$*:*

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & -t_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 & -t_2 & 0 \\ 0 & 1 & 0 & -3t_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & t_3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} \exp(t_4) & 0 & 0 & 0 \\ 0 & \exp(3t_4) & 0 & 0 \\ 0 & 0 & \exp(-2t_4) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let

$$X = (f^1, f^2, f^3, f^4),$$

be an element of $\mathcal{L}_4$. Our aim is to simplify the coefficients of X so as to obtain a simpler representative (i.e., with the greatest number of null components). We divide the computation into several cases.

Case 1: $f^4 \neq 0$*. For simplicity, we can assume $f_4 = 1$; applying A_3 to X, it follows*

$$X' = (f^1 + f^2 t_3, f^2, f^3 + 2f^4 t_3, 1),$$

and choosing $t_3 = -f^3/2$, we get a first simplification

$$X' = (f^1 - \frac{f^2 f^3}{2}, f^2, 0, 1).$$

Then we let A_2 act on X', yielding

$$X'' = (f^1 - \frac{f^2 f^3}{2}, f^2 - 3t_2, 0, 1).$$

Choosing $t_2 = f^2/3$, we obtain

$$X'' = (f^1 - \frac{f^2 f^3}{2}, 0, 0, 1).$$

Finally, applying A_1 to X'' and choosing $t_1 = f^1 - f^2 f^3/2$ it turns out that the first component vanishes. Therefore, any one–dimensional subalgebras spanned by a element with $f^4 \neq 0$ is equivalent to the subalgebras spanned by Ξ_4.

Case 2: *$f^4 = 0$. We can distinguish between two subcases: $f^3 \neq 0$ and $f^3 = 0$.*

Subcase 2.1: *$f^3 \neq 0$. Again, we can assume $f^3 = 1$ and apply A_2 on X. This results in*

$$X' = (f^1 - t_2, f^2, 1, 0),$$

and by choosing $t_2 = f^1$, we obtain $X' = f^2\Xi_2 + \Xi_3$. We can then apply the inner automorphism A_4 to X' and, after scaling, obtain:

$$X'' = (0, \exp(5t_4) f^2, 1, 0).$$

It occurs

$$X'' = (0, \text{sign}(f^2), 1, 0), \qquad \textit{with} \quad t_4 = \frac{\log\left(\frac{1}{|f^2|}\right)}{5}.$$

Thus, any subalgebra generated by an element with $f^4 = 0$ and $f^3 \neq 0$ is equivalent to one of the three subalgebras: $\Xi_3 \pm \Xi_2$ or Ξ_3, depending on the sign of f^2.

Subcase 2.2: *$f^3 = 0$. If $f^2 \neq 0$, we can apply A_3 to $X = f^1\Xi_1 + \Xi_2$, which can then be reduced to Ξ_2 by choosing $t_3 = -f^1$. Otherwise, if $f^2 = 0$, we have the one–dimensional subalgebra generated by Ξ_1.*

In summary, we have found the optimal system of one-dimensional subalgebras, whose representatives are:

$$\begin{aligned}
\Xi_1 &= \frac{\partial}{\partial x},\\
\Xi_2 &= \frac{\partial}{\partial t},\\
\Xi_3 &= t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},\\
\Xi_4 &= 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u},\\
\Xi_3 \pm \Xi_2 &= t\frac{\partial}{\partial x} \pm \frac{\partial}{\partial t} + \frac{\partial}{\partial u}.
\end{aligned}$$

The computations required to produce an optimal system of subalgebras are relatively simple in low-dimensional cases but become significantly more complex as the dimension increases. As illustrated in the previous example, the difficulty of managing the calculations and distinguishing between different cases becomes evident. With the increasing dimension of the Lie algebra and the subalgebras to be optimized, the problem grows in complexity. It follows from these considerations that a computational approach becomes essential. Advanced techniques and algorithms are needed to systematically analyze subalgebras, thus facilitating the analysis for finding optimal systems of high-dimensional Lie subalgebras.

In the next Chapter, we investigated the problem of finding optimal systems of families of Lie subalgebras of finite dimensional Lie algebras almost automatically by means of a program written in the computer algebra system (CAS) *Wolfram Mathematica*™ [39].

Chapter 4

Symbolic computation of optimal systems: the SymboLie package

In this Chapter, we describe in detail the package SymboLie [35] for automatically computing optimal systems of families of Lie subalgebras. The package has been written in the Wolfram Language and runs in the CAS Wofram Mathematica™[39]. The source code of the package, as well as some illustrative notebooks, are freely available [38].

4.1 Theoretical setting

In this Section, we introduce some new definitions useful for the algorithmic computation of optimal systems of Lie subalgebras, and clarify them by means of some examples. Then, in the next Section, we will describe in detail the main algorithms allowing us to automatically determine optimal systems of an r–dimensional real Lie algebra $\mathcal{L}_r$ represented by a suitable realization, or in abstract way assigning a formal basis and the set of non-zero Lie brackets (in both cases the program will compute the structure constants), or giving explicitly the structure constants.

4.1.1 One–dimensional Lie subalgebras

A one-dimensional Lie subalgebra is completely defined by the coefficients of the linear combination of the generators of the basis of $\mathcal{L}_r$, say

$$X = f^1\Xi_1 + f^2\Xi_2 + \cdots + f^r\Xi_r,$$

and actions of inner automorphisms can be transferred to the *coordinates* $(f^1, f^2, \ldots, f^r)$ of an r–dimensional vector; the method largely used in the literature for obtaining optimal systems of one-dimensional Lie subalgebras consists in using inner automorphisms to obtain from the most general r–dimensional vector the maximum possible number of zero coordinates, also using the invariants of the inner automorphisms. More in detail, the method takes a tuple $\{f^1, f^2, \ldots, f^r\}$, and, through *judicious* applications of inner automorphisms, simplifies as many of the coefficients f^α [11]. Though this approach is straightforward, it may imply technical difficulties when implemented in a computer algebra system, since one needs to solve algebraic equations and make suitable choices during the process to distinguish cases; moreover, it is a relatively easy task only for low dimensional Lie algebras, and the nature of the obtained results is not always clear. Typically, this is a top-down approach.

Our aim is to render the process of identifying similar subalgebras automatic; to achieve this result, we adopt a general algorithm with a bottom–up philosophy, and reduce to a minimum the need of solving algebraic equations. To this end, let us introduce some more definitions.

Definition 22. *For any integer $r > 1$, let $\mathcal{S}_r$ be the set of all possible tuples with r components (not all zero) chosen in $\{0,1\}$,* i.e.,

$$\mathcal{S}_r = \{0,1\}^r \setminus \{(0,0,\dots,0)\}.$$

Definition 23. *Let $\mathcal{L}_r$ be an r–dimensional Lie algebra generated by $\{\Xi_1, \Xi_2, \dots, \Xi_r\}$, and let $\mathbf{f} \equiv \{f^1, f^2, \dots, f^r\}$ be a tuple of r functions depending on some variables belonging to a set $\mathcal{P}$. Then, the family of one–dimensional subalgebras*

$$X = f^1 s_1 \Xi_1 + f^2 s_2 \Xi_2 + \cdots + f^r s_r \Xi_r,$$

*where $\mathbf{s} \equiv (s_1, \dots, s_r) \in \mathcal{S}_r$, is called a p–*family *of one–dimensional Lie subalgebras of $\mathcal{L}_r$ if the rank of the Jacobian matrix of $(f^1 s_1, f^2 s_2, \dots, f^r s_r)$ with respect to the elements of $\mathcal{P}$ is equal to $p = \sum_{i=1}^r s_i$. This is equivalent to say that the decomposition of X in terms of the elements of the basis of the Lie algebra involves p functionally independent components.*

Example 23. *Let $\mathcal{L}_4$ be a four–dimensional Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$, and consider the family of one–dimensional subalgebras*

$$X = f^1 \Xi_1 + f^2 \Xi_2,$$

with f^1 and f^2 arbitrary real numbers; in this case, it is $\mathcal{P} = \{f^1, f^2\}$, and we have a 2–family of one-dimensional Lie subalgebras. Also the family

$$X' = (f^1 \cos(t) - f^2 \sin(t))\Xi_1 + (f^1 \sin(t) + f^2 \cos(t))\Xi_2,$$

where $\mathcal{P} = \{f^1, f^2, t\}$, is a 2–family of one–dimensional Lie subalgebras which has to be considered indistinguishable from X. On the contrary, the family

$$Y = f^1 \Xi_1 + f^1 (f^2)^2 \Xi_2 + f^1 f^2 \Xi_3,$$

even if the vector $(f^1, f^1(f^2)^2, f^1 f^2, 0)$ possesses three non-zero components, does not correspond to a 3–family; in fact, we have $\mathcal{P} = \{f^1, f^2\}$, and the rank of the Jacobian matrix of its components with respect to the elements of $\mathcal{P}$ is 2.

Remark 10. *The p–family of one–dimensional Lie subalgebras of $\mathcal{L}_r$*

$$X = f^1 s_1 \Xi_1 + f^2 s_2 \Xi_2 + \cdots + f^r s_r \Xi_r,$$

where $\mathbf{s} \equiv \{s_1, s_2, \dots, s_r\} \in \mathcal{S}_r$, $p = \sum_{i=1}^r s_i$, in the following will be represented by the tuple

$$(f^1 s_1, f^2 s_2, \dots, f^r s_r)$$

which in turn can be identified by the integer $c_X = \displaystyle\sum_{k=1}^{r} s_k 2^{k-1}$. Moreover, where there is no need to specify a value for p, we will write simply family instead of p–family.

Let us introduce a relation $\mathcal{R}$ between families of one–dimensional Lie subalgebras.

Definition 24. *Let $\mathcal{L}_r$ be an r–dimensional Lie algebra generated by $\{\Xi_1, \dots, \Xi_r\}$ and let $X \equiv (f^1 s_1, \dots, f^r s_r)$ and $Y \equiv (\tilde{f}^1 \tilde{s}_1, \dots, \tilde{f}^r \tilde{s}_r)$ two different families of Lie subalgebras.*

We write

$$X \, \mathcal{R} \, Y \quad \textit{or} \quad (X, Y) \in \mathcal{R}$$

if there exists an inner automorphism $A \in Int(\mathcal{L}_r)$ such that $Z^T = A \cdot X^T$ (^{T} denotes the transpose operator) is a family of one–dimensional Lie subalgebras such that $c_Z = c_Y$.

If it is also

$$Y \, \mathcal{R} \, X \quad \textit{or} \quad (Y, X) \in \mathcal{R},$$

then we say that the two families X and Y are equivalent.

According to Definition 24, the relation $\mathcal{R}$ is clearly reflexive and transitive, but, in general, it is not necessarily a symmetric relation, as will be shown in Example 24. In such a case, $\mathcal{R}$ is a so-called *preorder*, and, it is well known that a preorder induces an equivalence relation, defined by forcing the symmetric property; nevertheless, we do not assume the symmetric property to be *a priori* satisfied by the relation $\mathcal{R}$.

Remark 11. *For an r–dimensional Lie algebra, all $2^r - 1$ possible families of one–dimensional subalgebras, according to relation $\mathcal{R}$, can be represented by means of a suitable* directed *multigraph $\mathcal{G}(\mathcal{L}_r)$ (this because more than one automorphism can be such that the two families belong to the relation $\mathcal{R}$), where the vertices correspond to the various families of one–dimensional Lie subalgebras, and the edges to the automorphisms connecting couples of subalgebras belonging to $\mathcal{R}$. Indeed, we represent this multigraph as a graph by means of its adjacency matrix whose (i, j)-th entry is 1 if the i–th family is mapped by some automorphism to j–th family, and zero otherwise.*

Example 24. *Let $\mathcal{L}_4$ be a 4–dimensional Lie algebra,*

$$\mathcal{L}_4 = \langle \Xi_1, \Xi_2, \Xi_3, \Xi_4 \rangle,$$

with non-zero Lie brackets

$$[\Xi_2, \Xi_4] = \Xi_1, \qquad [\Xi_3, \Xi_4] = \Xi_2.$$

Since the inner automorphism $\exp(t_1 \, \mathrm{ad}_{\Xi_1})$ *is the identity morphism, in the following we write the matrices associated to inner automorphisms* $A_i = \exp(t_i \, \mathrm{ad}_{\Xi_i})$ $(i = 2, 3, 4)$*:*

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & -t_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -t_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; A_4 = \begin{pmatrix} 1 & t_4 & \frac{t_4^2}{2} & 0 \\ 0 & 1 & t_4 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Acting with the automorphism A_4 on the 2*–family* $(0, f^2, f^3, 0)$*, we get*

$$(f^2 t_4 + \frac{f^3 t_4^2}{2}, f^2 + f^3 t_4, f^3, 0).$$

which is a 3–family according to Definition 23; thus, we can say

$$(0, f^2, f^3, 0) \; \mathcal{R} \; (f^1, f^2, f^3, 0).$$

Vice versa, it is

$$(f^1, f^2, f^3, 0) \overset{A_4}{\mapsto} (f^1 + f^2 t_4 + \frac{f^3 t_4^2}{2}, f^2 + f^3 t_4, f^3, 0),$$

and the image cannot be reduced to $(0, f^2, f^3, 0)$ *for all choices of* f^1, f^2 *and* f^3, *whence*

$$(f^1, f^2, f^3, 0) \not\mathcal{R} (0, f^2, f^3, 0).$$

Moreover, choosing $t_4 = -\dfrac{f^2}{f^3}$, *the 2–family* $(0, f^2, f^3, 0)$ *is mapped by* A_4 *to* $\left(-\dfrac{(f^2)^2}{2f^3}, 0, f^3, 0\right)$, *which is a 2-family. Hence, we can write*

$$(0, f^2, f^3, 0) \mathcal{R} (f^1, 0, f^3, 0).$$

Finally, since

$$(f^1, 0, f^3, 0) \overset{A_4}{\mapsto} \left(f^1 + \frac{f^3 t_4^2}{2}, f^3 t_4, f^3, 0\right),$$

we have

$$(f^1, 0, f^3, 0) \mathcal{R} (f^1, f^2, f^3, 0),$$

but

$$(f^1, 0, f^3, 0) \not\mathcal{R} (0, f^2, f^3, 0).$$

Definition 25 (Ordering of p–families of one–dimensional Lie subalgebras)**.** *The families of one-dimensional Lie subalgebras are sorted according to the short-lexicographical ordering (slex). In more detail, let* $\mathcal{L}_r$ *be an r–dimensional Lie algebra and let*

$$X = (f^1 s_1, f^2 s_2, \ldots, f^r s_r),$$
$$Y = (\tilde{f}^1 \tilde{s}_1, \tilde{f}^2 \tilde{s}_2, \ldots, \tilde{f}^r \tilde{s}_r)$$

be two families. It is

$$X >_{slex} Y \quad if \quad \begin{cases} \sum_{i=1}^{r} s_i > \sum_{i=1}^{r} \tilde{s}_i \\ or \\ \sum_{i=1}^{r} s_i = \sum_{i=1}^{r} \tilde{s}_i \quad and \quad c_X > c_Y. \end{cases}$$

Remark 12. *When we represent the graph corresponding to the set of p–families of one-dimensional Lie subalgebras, the label we append to each vertex X is the position* ind_X *in the list of ordered families.*

4.1.2 Multi–dimensional Lie subalgebras

As in the case of one–dimensional Lie subalgebras, let us define a p–family of multi–dimensional Lie subalgebras.

Definition 26. *Let* $\mathcal{L}_r$ *be an r–dimensional Lie algebra spanned by* $\{\Xi_1, \ldots, \Xi_r\}$, *and let* f_k^α $(k = 1, \ldots, d, \alpha = 1, \ldots, r)$ *be* $d \cdot r$ *functions depending on some variables belonging to a set* $\mathcal{P}$. *A tuple of d elements* $(X_1, \ldots, X_d)$, *where*

$$X_k = \sum_{\alpha=1}^{r} f_k^\alpha s_{k,\alpha} \Xi_\alpha, \qquad k = 1, \ldots, d,$$

and $\mathbf{s}_k \equiv (s_{k,1}, \ldots, s_{k,r}) \in \mathcal{S}_r$ *for* $k = 1, \ldots, d$, *is called a* p*–family of* d*–dimensional Lie subalgebras of* $\mathcal{L}_r$ *if*

(1) X_k $(k = 1, \ldots, d)$ *is a* p_k*–family of one–dimensional Lie subalgebras, and* $p = \sum_{k=1}^{d} p_k$;

(2) *the matrix* $\|f_k^\alpha s_{k,\alpha}\|$ *has rank* d;

(3) *the rank of the Jacobian matrix of* $\{f_k^\alpha s_{k,\alpha},\ k = 1, \ldots, d,\ \alpha = 1, \ldots, r\}$ *with respect to the elements of* $\mathcal{P}$ *is equal to* p;

(4) *the conditions*

$$\sum_{\alpha,\beta=1}^{r} f_i^\alpha s_{i,\alpha} f_j^\beta s_{j,\beta} C_{\alpha\beta}^\gamma = \sum_{k=1}^{s} \lambda_{ij}^k f_k^\gamma s_{k,\gamma}, \quad (i,j = 1, \ldots, d, \gamma = 1, \ldots, r)$$

are satisfied for suitable constants λ_{ij}^k *whatever the functions* f_k^α *are.*

Remark 13. *Definition 26 (in particular, item 4) implies a deep simplification of the conditions for the check that the set* $\{X_1, \ldots, X_d\}$ *is closed with respect to the Lie bracket; in fact, we need to determine the unknowns* λ_{ij}^k *whatever the values of* f_k^α *are, and this requires only simple elementary linear algebra tools.*

Remark 14. *We are aware that Definition 26 in some cases may leave out some Lie subalgebras. For instance, consider the six–dimensional Lie algebra spanned by* $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5, \Xi_6\}$ *[25] with the following non–zero Lie brackets:*

$$\begin{array}{lll} [\Xi_1, \Xi_5] = -\Xi_2, & [\Xi_2, \Xi_5] = \Xi_1, & [\Xi_3, \Xi_5] = -\Xi_4, \\ [\Xi_3, \Xi_6] = -\Xi_1, & [\Xi_4, \Xi_5] = \Xi_3, & [\Xi_4, \Xi_6] = -\Xi_2. \end{array}$$

The three–dimensional vector space spanned by

$$\{f_1^1\Xi_1 + f_1^4\Xi_4, f_2^2\Xi_2 + f_2^3\Xi_3, f_3^5\Xi_5\}$$

is not a 5*–family of three–dimensional Lie subalgebras because the closure with respect to the commutator is not ensured unless* $f_1^1 f_2^3 + f_2^2 f_1^4 = 0$, *which reduces the number of arbitrary parameters to 4. Consequently, it does not satisfy point (4) of Definition 26.*

In our approach, such instances of Lie subalgebras are not considered, since this would require some more steps that may be cumbersome to tackle automatically. We plan to face this problem in a new version of our program. Nevertheless, we observe that in most cases (for instance, almost all real three– and four–dimensional Lie algebras [26]) our approach works correctly without losing anything.

Remark 15. *Of course, because the basis of a* d*–dimensional Lie subalgebras of a Lie algebra* $\mathcal{L}_r$ *is not unique, we choose to adopt as the basis of a* p*–family of* d*–dimensional Lie subalgebras the one such that the matrix*

$$\left\| \begin{array}{cccc} f_1^1 s_{1,1} & f_1^2 s_{1,2} & \cdots & f_1^r s_{1,r} \\ \vdots & \vdots & \vdots & \vdots \\ f_d^1 s_{d,1} & f_d^2 s_{d,2} & \cdots & f_d^r s_{d,r} \end{array} \right\|$$

is in row reduced echelon form (RREF); once the basis for $\mathcal{L}_r$ *has been assigned, this matrix represents the Lie subalgebra.*

A relation $\mathcal{R}$ linking two different families of d–dimensional Lie subalgebras can be defined as well.

Definition 27. *Let $\mathcal{L}_r$ be an r–dimensional Lie algebra generated by $\{\Xi_1, \ldots, \Xi_r\}$ and let $\mathbf{X} \equiv (X_1, \ldots, X_d)$ and $\mathbf{Y} \equiv (Y_1, \ldots, Y_d)$ be two different families of d–dimensional Lie subalgebras. We write*

$$\mathbf{X}\,\mathcal{R}\,\mathbf{Y} \quad \textit{or} \quad (\mathbf{X}, \mathbf{Y}) \in \mathcal{R}$$

if there exists an inner automorphism $A \in Int(\mathcal{L}_r)$ such that

$$\mathbf{Z}^T = (A \cdot X_1^T, \ldots, A \cdot X_d^T),$$

possibly reduced to row echelon form, is a family of d–dimensional Lie subalgebras such that $c_{\mathbf{Z}} = c_{\mathbf{Y}}$.

If it is also

$$\mathbf{Y}\,\mathcal{R}\,\mathbf{X}$$

then we say that the two families are equivalent.

Example 25. *Consider the finite-dimensional Lie algebra of symmetries for the equation $u_t - u_{xx} = 0$ (linear heat equation), spanned by (we use the same basis as the one considered in [19]):*

$$\Xi_1 = \frac{\partial}{\partial t}, \quad \Xi_2 = \frac{\partial}{\partial x}, \quad \Xi_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \quad \Xi_4 = u\frac{\partial}{\partial u},$$
$$\Xi_5 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}, \quad \Xi_6 = 4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2+2t)u\frac{\partial}{\partial u}.$$

Below, we list the matrices associated with the inner automorphisms $A_i(t) = \exp(t\,\mathrm{ad}_{\Xi_i})$ for $i = 1, \ldots, 6$, omitting A_4 as it is the identity matrix:

$$A_1(t) = \begin{pmatrix} 1 & 0 & 2t & 0 & 0 & 4t^2 \\ 0 & 1 & 0 & 0 & 2t & 0 \\ 0 & 0 & 1 & 0 & 0 & 4t \\ 0 & 0 & 0 & 1 & 0 & -2t \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_2(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -t & -t^2 \\ 0 & 0 & 0 & 0 & 1 & 2t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_3(t) = \begin{pmatrix} \exp(-2t) & 0 & 0 & 0 & 0 & 0 \\ 0 & \exp(-t) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \exp(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & \exp(2t) \end{pmatrix},$$

$$A_5(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2t & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -t^2 & t & 0 & 1 & 0 & 0 \\ 0 & 0 & -t & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_6(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -4t & 0 & 1 & 0 & 0 & 0 \\ 2t & 0 & 0 & 1 & 0 & 0 \\ 0 & -2t & 0 & 0 & 1 & 0 \\ 4t^2 & 0 & -2t & 0 & 0 & 1 \end{pmatrix}.$$

Define a fixed inner automorphism $A = A_6(1/4)\, A_1(1)\, A_6(1/4)$ *and consider the bidimensional* 2*–family of Lie subalgebras* $\mathbf{X} = \left(f_1^1 \Xi_1, f_2^2 \Xi_2\right)$. *Mapping* $\mathbf{X}$ *via* A *yields*

$$\mathbf{Y} = \left(-\frac{f_2^2}{2}\,\Xi_5, \frac{f_1^1}{4}\,\Xi_6\right),$$

which remains a 2*–family. Similarly, for* $\widetilde{\mathbf{X}} = \left(f_1^3\,\Xi_3, f_2^5\,\Xi_5\right)$, *mapping it via* A *gives*

$$\widetilde{\mathbf{Y}} = \left(2 f_2^5\,\Xi_2, -f_1^3\,\Xi_3 + f_1^3\,\Xi_4\right),$$

which is not a 3*–family of Lie subalgebras. For instance, it suffices to check that condition (1) in Definition 26 does not hold for* $\widetilde{\mathbf{Y}}_2$.

Note that $(f_1^2\,\Xi_2, f_2^3\,\Xi_3 + f_2^4\,\Xi_4)$ *satisfies all the conditions of Definition 26, yielding it a* 3*–family. Consequently,* $\mathbf{X}\,\mathcal{R}\,\mathbf{Y}$ *and* $\widetilde{\mathbf{X}}\,\not{\mathcal{R}}\,\widetilde{\mathbf{Y}}$.

As in the case of one–dimensional families of Lie subalgebras, relation $\mathcal{R}$ is reflexive and transitive but in general not symmetric.

Definition 28 (Ordering of p–families of d–dimensional Lie subalgebras)**.** *The* p*–families of* d*–dimensional Lie subalgebras are sorted according to the short-lexicographical ordering. In more detail, let* $\mathcal{L}_r$ *be an* r*–dimensional Lie algebra and let* $\mathbf{X} \equiv \{X_1, \ldots, X_d\}$ *and* $\mathbf{Y} \equiv \{Y_1, \ldots, Y_d\}$ *be a* p*–family and* q*–family of different* d*–dimensional Lie subalgebras, respectively. It is*

$$\mathbf{X} >_{slex} \mathbf{Y} \quad \text{if} \quad \begin{cases} p > q \\ \text{or} \\ p = q \quad \text{and} \quad X_k >_{slex} Y_k, \text{ where } k = \min\{i : c_{X_i} \neq c_{Y_i}\}. \end{cases}$$

Remark 16. *A family* $\mathbf{X} \equiv (X_1, \ldots, X_d)$ *where*

$$X_k = \sum_{\alpha=1}^{r} f_k^\alpha s_{k,\alpha} \Xi_\alpha, \qquad k = 1, \ldots, d,$$

of d*–dimensional Lie subalgebras of an* r*–dimensional Lie algebra* $\mathcal{L}_r$ *can be represented as well by the tuple* $c_{\mathbf{X}} \equiv (c_{X_1}, \ldots, c_{X_d})$.

Moreover, when we represent the graph corresponding to the set of families of d*–dimensional Lie subalgebras, also in this case the label we append to each vertex is the position,* $ind_{\mathbf{X}}$*, in the list of ordered families.*

Now we have all the elements for clarifying the details of the algorithm leading to the determination of optimal systems of families of Lie subalgebras.

4.2 Algorithms for optimal systems of Lie subalgebras

Let $\mathcal{L}_r$ be an r–dimensional real Lie algebra assigned through its structure constants; the latter are computed if the Lie algebra is realized in terms of vector fields or matrices, or if the list of non–zero Lie brackets is provided. The program `SymboLie`, written in the Wolfram Language™ [39], provides a set of functions devoted primarily to the construction of optimal systems of families of Lie subalgebras. If the Lie algebra is assigned by means of a suitable realization, or if the list of non–zero Lie brackets is assigned, then the method `StructureConstants[]` provides the structure constants that will be used in all the remaining functions.

Example 26. *Take the three–dimensional Lie algebra of real 2×2 traceless matrices, generated by*

$$\Xi_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \Xi_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \Xi_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

After loading the package `SymboLie`, the following steps lead to the computation of the structure constants:

- *define the generators `gens` of the basis:*

```
gens={{{1,0},{0,-1}},{{0,1},{0,0}},{{0,0},{1,0}}};
```

- *compute the structure constants:*

```
cs = StructureConstants[gens];
```

The structure constants could be used for instance to display the commutator table:

```
CommutatorTable[cs] //MatrixForm
```

$$\begin{bmatrix} 0 & 2\,\Xi_2 & -2\,\Xi_3 \\ -2\,\Xi_2 & 0 & \Xi_1 \\ 2\,\Xi_3 & -\Xi_1 & 0 \end{bmatrix}.$$

The computation of the structure constants through the list of non–zero Lie brackets using a set of unassigned symbols is done as follows:

```
basis = {x1, x2, x3};
bracket = {{{x1, x2}, 2*x2},
          {{x1, x3}, -2*x3},
          {{x2, x3}, x1}
          };
cs = StructureConstants[bracket, basis];
```

We note that this Lie algebra is isomorphic to the one discussed abstractly in Example 21.

Example 27. *Consider the Lie algebra spanned by the vector fields*

$$\Xi_1 = \frac{\partial}{\partial t}, \quad \Xi_2 = \frac{\partial}{\partial x}, \quad \Xi_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad \Xi_4 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}$$

generating the Lie point symmetries of Korteweg-deVries equation [11],

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

The structure constants are computed as follows

```
vars = {t,x,u};
gens = {{1,0,0}, {0,1,0}, {0,t,1}, {3t,x,-2u}};
cs = StructureConstants[gens,vars];
```

The main function devoted to the determination of optimal systems of families of Lie subalgebras is the method `SubAlgebra[]` (see Algorithm 1) receiving as inputs the structure constants `cs` of the Lie algebra, a list `pars` made of two lists specifying arbitrary parameters and their constraints, respectively (this parameter is `{{},{}}` if no arbitrary parameter is involved in the structure constants), and the dimension `dim` of the required subalgebras.

Example 28. *Let $\mathcal{L}_4$ be a real 4–dimensional Lie algebra with the non-zero Lie brackets* $[\Xi_1, \Xi_3] = \Xi_1$, $[\Xi_2, \Xi_3] = a\Xi_2$, *where* $0 < |a| < 1$ *[26]. In this case, the argument* `pars` *of* `SubAlgebra[]` *has to be* $\{\{a\}, \{-1 < a, a < 1, a \neq 0\}\}$.

If the argument `pars` is not `{{},{}}`, the `SubAlgebra[]` method evaluates the structure constants on an instance of the parameters satisfying the assigned constraints.

Moreover, in the `SymboLie` package, there are several global variables, one of which is `PrintDebug`. By default, it is set to 0. When the user sets it to 1, a debug file is generated in the same directory as the Mathematica notebook file. The user can set the global string variable `logfile` to choose the name (the dimension of subalgebras is appended) of the text file where the trace of the computation is recorded; if `logfile` is not set by the user, then the trace of the computation is saved on the file *Debug-dim.log*.

Algorithm 1: `SubAlgebra(cs,pars,dim)`
Returns the `dim`-dimensional optimal subalgebras of the Lie algebra $\mathcal{L}_r$ with structure constants `cs` (involving parameters `pars`)

Input: structure constants `cs`, parameters `pars`, dimension `dim`
Output: adjacency matrix, candidates, structure constants `cs`

```
begin
    if pars is not empty then
        cs ← evaluation of cs on an instance of pars;
    end
    if PrintDebug then
        file.log ← printout all the algorithm's operations;
    end
    if 1 ≤ dim < r then
        return FindAdjacency(cs,dim);
    end
end
```

The function `FindAdjacency[]` is the method doing all the work. Inside this function, first of all, the list $\mathcal{C}$ of possible candidates for families of d–dimensional Lie subalgebras is computed: for $d = 1$ we have $2^r - 1$ candidates, whereas for $d > 1$ the cardinality of the set of candidates depends on the structure constants of the Lie algebra. In other words, for $1 < d < r$, we build all the families of d–dimensional subspaces whose representative matrices are in RREF, and then select those verifying condition (4) of Definition 26. This is done by the method `FindCandidates[]`, accepting as input the structure constants as well as the dimension of the subalgebras, and returning the list of candidates.

For $d > 1$, to check whether a d–dimensional family of vector subspaces (a candidate) is a p–family of d–dimensional Lie subalgebras of $\mathcal{L}_r$ (Definition 26), the Algorithm 2 is used.

Algorithm 2: `CheckAlgebra(alg,cs,dim)`
Whether `alg` generate a p–family of subalgebras of $\mathcal{L}_r$.

```
Input: alg, structure constants cs, dimension dim
Output: isSub boolean value
begin
    isSub ← true;
    foreach (s1, s2) in 2-element combinations of {1,...,dim} do
        X ← set[s1];
        Y ← set[s2];
        eqnI ← a_{dim+1} Σ_{α,β,γ=1}^{r} cs^γ_{αβ} X_α Y_β;
        eqnC ← eqnI − Σ_{i=1}^{dim} b_i set[i];
        matI ← JacobianMatrix(eqnI, {a_{dim+1}});
        matC ← JacobianMatrix(eqnC, {a_{dim+1}, b_i});
        if Rank(matI) ≠ Rank(matC) then
            isSub ← false;
            break foreach;
        end
    end
    return isSub;
end
```

According to Definition 26, this algorithm requires to solve only linear equations. As a second step, the inner automorphisms

$$\exp\left(t_\alpha\, \mathrm{ad}_{\Xi_\alpha}\right), \qquad \alpha = 1, \dots, r,$$

where $\{\Xi_1, \dots, \Xi_r\}$ denote the generators of a basis of $\mathcal{L}_r$, are computed.

Let A_0 be the inner automorphism made of the composition of the inner automorphisms represented by matrices in diagonal form; A_0 is the $r \times r$ identity matrix $\mathcal{I}_r$ if all inner automorphisms are not diagonal matrices. Moreover, let $\mathcal{A} = \{A_k,\ k = 1, \dots r' \leq r\}$ be the set of inner automorphisms not in diagonal form; if $r' = 0$, we set $\mathcal{A} = \{\mathcal{I}_r\}$.

Then, using the algorithm `FindCompositions[]`, the set made of the compositions of the permutations of all the inner automorphisms belonging to the nonempty subsets of $\mathcal{A}$ is computed. For the computation of this set, it is possible to use a flag variable, `FastRun`, within the package. By default this optional parameter is zero, and automorphisms are computed as previously described. If the user sets `FastRun` to 1, instead of considering permutations, we consider the compositions of all the inner automorphisms belonging to the nonempty proper subsets of $\mathcal{A}$ joined with the set of the compositions of the permutations of all inner automorphisms. Moreover, if `FastRun` is set to 2, then we only consider the compositions of the inner automorphisms belonging to the nonempty subsets of $\mathcal{A}$. The higher is `FastRun` the faster is the execution; anyway, we should be aware that there are cases where setting `FastRun` to 1 or 2 does not guarantee the completeness of the results.

Now, the process of identifying the pairs of candidates belonging to the relation $\mathcal{R}$ can start: the result is achieved by performing a double scan of the candidates.

Before analyzing the families, the adjacency matrix is set equal to the identity matrix of dimension equal to the number of candidates (this because every family is trivially equivalent to itself). In the first scan, we act on every p–family X (starting from the simplest ones, *i.e.*, those with smallest values of p) with each of the automorphisms previously characterized: let A be one of these automorphisms acting on X. We need to check if the result is a family Y of subalgebras according to the definition. This process is quite fast and, when the check is successful, allows us to link a p–family with a q–family with $p < q$. Consequently the adjacency matrix is updated setting its (ind_X, ind_Y)–th entry to 1. If this occurs, the program checks if, acting on the family Y with the automorphism A^{-1}, there are suitable values of the parameters involved in the automorphism such that a family that can be represented by X can be obtained.

This first scan is in general not able to recognize all possible pairs of families belonging to the relation $\mathcal{R}$. Thus, the program performs a second scan on couples of families of subalgebras for which the corresponding entries in the adjacency matrix are zero. The used automorphisms are those belonging to the set of the compositions of the permutations of all inner automorphisms, and it is exploited the possibility that suitable values of the parameters involved in the automorphisms allow for recognizing other couples of Lie subalgebras belonging to the relation $\mathcal{R}$.

We observe that there are cases where the first scan is sufficient to find all possible relations between the subalgebras. Furthermore, during the process, the algorithm repeatedly uses the transitivity property to update the adjacency matrix. The pseudocode of this method is shown in the Algorithm 3.

Algorithm 3: `FindAdjacency(cs,dim)`
Computation of the adjacency matrix of the multigraph $\mathcal{G}(\mathcal{L}_r)$.

```
Input: structure constants cs of L_r
Output: adj adjacency matrix of the multigraph
begin
    families ← FindCandidates(cs,dim);
    auto ← InnerAutomorphisms(cs);
    𝒜 ← FindAutomorphisms(auto);
    adj ← identity matrix of order #(families);
    foreach X in families do                                // first scan
        foreach A in 𝒜 do
            Y ← A · X;
            if CheckPFamily(Y) then
                adj[ind_X, ind_Y] ← 1;
                Y ← families(ind_Y);
                Z ← A^{-1} · Y;
                Z_0 ← an evaluation of Z such that c_{Z_0} = c_X;
                if CheckRelation(Z, X) and CheckPFamily(Z_0) then
                    adj[ind_Y, ind_X] ← 1;
                end
            end
        end
    end
    foreach pair (X, Y) of families do                      // second scan
        foreach A in 𝒜 do
            if adj[ind_X, ind_Y] = 0 then
                X ← A · X;
                X_0 ← an evaluation of X such that c_{X_0} = c_Y;
                if CheckRelation(X, Y) and CheckPFamily(X_0) then
                    adj[ind_X, ind_Y] ← 1;
                end
            end
        end
    end
    return adj;
end
```

The output returned by `SubAlgebra[]` is the input for other methods that display the results:

- `PrintGraph[]`, displaying the graph corresponding to the relation $\mathcal{R}$; each connected component of this graph may be represented by the simplest family whose indegree is equal to the number of nodes in the component;
- `PrintOptimal[]`, listing the families of Lie subalgebras of the optimal system; when possible, the optimal system which is returned is simplified by a suitable rescaling;
- `PrintClasses[]`, listing the families in each connected component of the graph.

Several methods are employed to characterize the representatives of the optimal system of Lie subalgebras. The main method is `FindClasses[]` (Algorithm 5), where the `Adj2Classes[]` method (Algorithm 4) is initially invoked. It takes two arguments as input: the candidates (*i.e.*, the p–families) and the adjacency matrix. First, a list is created made of lists, each containing one of the candidates. Then, the adjacency matrix is scanned and the candidates corresponding to the entries of the i–th row of the adjacency matrix not vanishing are joined to the i–th list. Finally, the `Compact[]` method is called, which is responsible for merging classes with common elements.

This process yields classes whose elements represent families of Lie subalgebras belonging to the relation $\mathcal{R}$; representing the adjacency matrix as a graph, each class corresponds to one of its connected component. The final step consists in determining for each connected component its representative, *i.e.*, if there exists at least one node whose indegree is equal to n, where n is the number of nodes in the connected component. The smallest representative with respect to the slex ordering gives the family able to represent the connected component. In some cases it could be necessary to have more than one representative family for some connected component.

Algorithm 4: `Adj2Classes(pfam,adj)`
Returns the connected components of the p-families in `pfam` according to the adjacency matrix `adj`

Input: list of p–families `pfam`, adjacency matrix `adj`
Output: list of connected components
begin
  connected $\leftarrow$ list of singletons of `pfam`;
  foreach (k_1, k_2) in 2-element combinations of $\{1, \dots, \#\texttt{pfam}\}$ **do**
    if `adj`$[k_1, k_2] \neq 0$ **then**
      connected$[k_1] \leftarrow$ *connected*$[k_1] \cup \{\texttt{pfam}[k_2]\}$;
    end
  end
  return `Compact`(*connected*);
end

Algorithm 5: `FindClasses(pfam,adj)`
Returns the classes of the p-families in `pfam` according to the adjacency matrix `adj`

Input: list of p–families `pfam`, adjacency matrix `adj`
Output: list of representatives `reps`, list of classes `classes`

```
begin
    connected ← Adj2Classes(pfam,adj);
    foreach component comp in connected do
        tmp ← {};
        foreach p-family X in comp do
            Y ← the first family in comp with maximum indegree;
            ind_X ← index of X in pfam;
            ind_Y ← index of Y in pfam;
            if adj[ind_X, ind_Y] = 1 then
                tmp ← tmp ∪ {Y};
            end
        end
        classes ← classes ∪ {tmp};
        reps ← reps ∪ {{ family in tmp with maximum indegree}};
    end
    return {reps, classes};
end
```

After the optimal system of families of Lie subalgebras has been obtained, the method `PrintOptimal[]` displays the result; this process is mediated by the `RescaleAlgebra[]` method (Algorithm 6) that rescales and writes the representatives of the optimal systems properly; as a result, the coefficients entering the expression of the optimal system are labeled with latin letters if they can assume arbitrary non-vanishing values, with greek letters if they can assume the values ± 1 only.

Algorithm 6: `RescaleAlgebra(X,cs,dim)`
Computation of the appropriate constants for a representative

Input: p-family `X`, structure constants `cs`, dimension `dim`
Output: appropriate representative

```
begin
    r ← dimension of the Lie algebra;
    auto ← InnnerAutomorphisms(cs);
    A ← InvariantAutomorphisms(X, auto);
    Y ← RREF(A · X);
    S ← scaling parameters involved in Y;
    foreach entry Y_ij ∉ {0,1} of Y do
        P ← parameters in S involved in the (i,j)-th entry of Y;
        if P ≠ ∅ then
            X_ij ← a suitable greek letter;
            S ← S \ {P[1]};
        else
            X_ij ← a suitable latin letter;
        end
    end
    return { ∑_{j=1}^{r} X_ij Ξ_j : i = 1,...,dim };
end
```

In particular, this method uses the `InvariantAutomorphisms[]` method (Algorithm 7) which returns a list of inner automorphisms that leave the p–family invariant.

Algorithm 7: `InvariantAutomorphisms(X,auto)`
Find A in `auto` leaving invariant the p–family X, up to RREF

Input: p–family `X`, list of automorphisms `auto`
Output: list of automorphisms `invauto`
begin
  foreach A in `auto` **do**
    $Y \leftarrow A \cdot X$;
    if $c_Y = c_X$ or $c_{RREF(Y)} = c_X$ **then**
      `invauto` $\leftarrow$ `invauto` $\cup \{A\}$;
    end
  end
  return `invauto`;
end

Chapter 5

Case Studies with SymboLie

In this Chapter, we present some relevant examples of Lie algebras corresponding to point symmetries of differential equations, and show how the use of `SymboLie` package, developed in the CAS *Wolfram Mathematica*™ [39], allows us to determine optimal systems of Lie subalgebras. Relevant examples, used as a test in the development of the program SymboLie are the real three– and four–dimensional Lie algebras. In fact, Patera and Winternitz [26] classified and listed all these algebras and exhibited all the optimal systems. Using the SymboLie package we recovered almost completely the results reported in [26]; the minor discrepancies between the results in [26] and those obtained in few minutes with SymboLie [42] (see also [38] for the complete notebooks reporting the results) are clarified. We also determined the optimal systems of the forty five–dimensional real algebras characterized in [44]. Using SymboLie, the optimal systems of all real three– and four–dimensional Lie algebras are derived in few minutes [42]. We also analyze, and compare the results obtained by using our package with those available in the literature about optimal systems of Lie symmetries of some partial differential equations relevant in the applications.

5.1 An example showing the use of SymboLie

To use `SymboLie`, open a Mathematica notebook and load the package by issuing

```
<< "SymboLie.wl"
```

So doing, we can specify the Lie algebra we want to analyze and start the computation.

First, we present a *non–critical* example in order to clarify the notation used in `SymboLie`.

Example 29. Let $\mathcal{L}_4$ be the 4D Lie algebra $A_2 \oplus 2A_1$ (see [26, Table II]) spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$ with only one non–zero commutator, say

$$[\Xi_1, \Xi_2] = \Xi_2.$$

Let us write the matrices A_1 and A_2 associated to the non–trivial inner automorphisms $\exp(t_1\,\mathrm{ad}_{\Xi_1})$ and $\exp(t_2\,\mathrm{ad}_{\Xi_2})$:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \exp(t_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -t_2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Patera and Winternitz in [26] have obtained the following 1D optimal system:

$$\begin{aligned}\Theta_A^1 \equiv\ & \{\{\Xi_2\}, \{\cos\phi\Xi_3 + \sin\phi\Xi_4\}, \{\Xi_1 + x(\cos\phi\Xi_3 + \sin\phi\Xi_4)\},\\ & \{\Xi_2 + \epsilon(\cos\phi\Xi_3 + \sin\phi\Xi_4)\}\},\end{aligned} \tag{5.1}$$

with $0 \leq \phi < \pi$, $x \in \mathbb{R}$ and $\epsilon = \pm 1$. We want to read the notation used in (5.1) to compare our results. Firstly, we analyze $\{\cos\phi\Xi_3 + \sin\phi\Xi_4\}$. If $\sin\phi = 0$ then we obtain the subalgebra $\{\Xi_3\}$; if $\cos\phi = 0$ then we obtain $\{\Xi_4\}$; otherwise, we can write $\{\Xi_3 + \tan\phi\Xi_4\}$ with $\tan\phi \in \mathbb{R}$. Similarly, regarding $\{\Xi_1 + x(\cos\phi\Xi_3 + \sin\phi\Xi_4)\}$, $x\cos\phi$ and $x\sin\phi$ assume every real values. Hence, if $x = 0$ we get $\{\Xi_1\}$, else $\{\Xi_1 + x\Xi_3\}$, $\{\Xi_1 + x\Xi_4\}$ and $\{\Xi_1 + x\Xi_3 + y\Xi_4\}$ with $x, y \in \mathbb{R}^* = \mathbb{R}\setminus\{0\}$. Finally, from $\{\Xi_2 + \epsilon(\cos\phi\Xi_3 + \sin\phi\Xi_4)\}$ we obtain $\{\Xi_2 + \epsilon\Xi_3\}$, $\{\Xi_2 + \epsilon\Xi_4\}$ and the last $\{\Xi_2 + \epsilon(\cos\phi\Xi_3 + \sin\phi\Xi_4)\}$ when $\phi \in]0, \pi/2[\cup]\pi/2, \pi[$.

Then, the 1D optimal system of Patera and Winternitz written in extended form is as follows:

$$\begin{aligned}\Theta_A^1 \equiv\ & \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_3 + x\Xi_4\}, \{\Xi_1 + x\Xi_3\}, \{\Xi_1 + x\Xi_4\},\\ & \{\Xi_1 + x\Xi_3 + y\Xi_4\}, \{\Xi_2 + \epsilon\Xi_3\}, \{\Xi_2 + \epsilon\Xi_4\}, \{\Xi_2 + \epsilon(\cos\phi\Xi_3 + \sin\phi\Xi_4)\}\},\end{aligned}$$

with $\phi \in]0, \pi[\setminus\{\pi/2\}$, $x, y \in \mathbb{R}^*$ and $\epsilon = \pm 1$.

The program `SymboLie` returns the following result:

$$\begin{aligned}\Psi_A^1 \equiv\ & \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_3 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_3\}, \{\Xi_1 + a_1\Xi_4\},\\ & \{\Xi_1 + a_1\Xi_3 + a_2\Xi_4\}, \{\Xi_2 + \alpha_1\Xi_3\}, \{\Xi_2 + \alpha_1\Xi_4\}, \{\Xi_2 + \alpha_1\Xi_3 + a_1\Xi_4\}\},\end{aligned}$$

with $a_1, a_2 \in \mathbb{R}^*$ and $\alpha_1 = \pm 1$.

The cardinalities of the 1D optimal system reported in [26] and the one computed by `SymboLie` coincide [42]. The two sets are equal except for their last element. Anyway, they are equivalent. Indeed, let us consider

$$\mathcal{K}_1 = \{\Xi_2 + \alpha_1(\cos\phi\Xi_3 + \sin\phi\Xi_4)\}, \text{ with } \alpha_1 = \pm 1 \text{ and } \phi \in]0, \pi[\setminus\{\pi/2\}.$$

Applying the inner automorphism A_1 to the generator of $\mathcal{K}_1$, we obtain

$$A_1 \cdot (0, 1, \alpha_1\cos\phi, \alpha_1\sin\phi)^T = (0, \exp(-t_1), \alpha_1\cos\phi, \alpha_1\sin\phi\Xi_4)^T.$$

Let us $\mathcal{K}_2$ the subalgebra generated by it, with $t_1 = \log(1/\cos\phi)$. Thus, one has

$$\mathcal{K}_2 = \{\Xi_2 + \alpha_1\Xi_3 + \alpha_1\tan\phi\Xi_4\},$$

and assuming $a_1 = \alpha_1\tan\phi \in \mathbb{R}^*$, we prove the equivalence.

A similar argument applies in the case of the 2D–optimal system of Patera and Winternitz, which in extended form results:

$$\begin{aligned}\Theta_A^2 \equiv\ & \{\{\Xi_1, \Xi_2\}, \{\Xi_1 + x\Xi_3, \Xi_2\}, \{\Xi_1 + x\Xi_4, \Xi_2\}, \{\Xi_1 + x\cos\phi\Xi_3 + x\sin\phi\Xi_4, \Xi_2\},\\ & \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \sin\phi\Xi_3 - \cos\phi\Xi_4\}, \{\Xi_1 + x\Xi_3, \Xi_4\}, \{\Xi_1 + x\Xi_4, \Xi_3\},\\ & \{\Xi_2 + \epsilon\Xi_3, \Xi_4\}, \{\Xi_2 + \epsilon\Xi_4, \Xi_3\}, \{\Xi_3, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\},\\ & \{\Xi_2, \sin\phi\Xi_3 - \cos\phi\Xi_4\}, \{\Xi_1 + x\cos\phi\Xi_3 + x\sin\phi\Xi_4, \sin\phi\Xi_3 - \cos\phi\Xi_4\},\\ & \{\Xi_2 + \epsilon(\cos\phi\Xi_3 + \sin\phi\Xi_4), \sin\phi\Xi_3 - \cos\phi\Xi_4\}\},\end{aligned}$$

with $\phi \in]0, \pi[\setminus\{\pi/2\}$, $x \in \mathbb{R}^*$ and $\epsilon = \pm 1$.

Using `SymboLie`, we obtain the following result:

$$\begin{aligned}\Psi_A^2 \equiv\ & \{\{\Xi_1,\Xi_2\},\{\Xi_1+a_1\Xi_3,\Xi_2\},\{\Xi_1+a_1\Xi_4,\Xi_2\},\{\Xi_1+a_1\Xi_3+a_2\Xi_4,\Xi_2\},\\ & \{\Xi_1,\Xi_3\},\{\Xi_1,\Xi_4\},\{\Xi_1,\Xi_3+a_1\Xi_4\},\{\Xi_1+a_1\Xi_3,\Xi_4\},\{\Xi_1+a_1\Xi_4,\Xi_3\},\\ & \{\Xi_2+\alpha_1\Xi_3,\Xi_4\},\{\Xi_2+\alpha_1\Xi_4,\Xi_3\},\{\Xi_3,\Xi_4\},\{\Xi_2,\Xi_3\},\{\Xi_2,\Xi_4\},\\ & \{\Xi_2,\Xi_3+a_1\Xi_4\},\{\Xi_1+a_1\Xi_4,\Xi_3+a_2\Xi_4\},\{\Xi_2+\alpha_1\Xi_4,\Xi_3+a_1\Xi_4\}\},\end{aligned}$$

with $a_1, a_2 \in \mathbb{R}^*$ and $\alpha_1 = \pm 1$.

Also in this case the cardinalities of both 2D–optimal systems coincide and the representatives of Θ_A^2 and Ψ_A^2 are all the same except the last two.

Concerning $\{\Xi_1 + x\cos\phi\Xi_3 + x\sin\phi\Xi_4, \sin\phi\Xi_3 - \cos\phi\Xi_4\}$, we observe that after a row reduction it can be written as

$$\{\Xi_1 + x\csc\phi\Xi_4, \Xi_3 - \cot\phi\Xi_4\},$$

and since x and ϕ are arbitrary, we can set $a_1 = x\csc\phi$ and $a_2 = -\cot\phi$, obtaining

$$\{\Xi_1 + a_1\Xi_4, \Xi_3 + a_2\Xi_4\}.$$

Similarly, for $\{\Xi_2 + \epsilon(\cos\phi\Xi_3 + \sin\phi\Xi_4), \sin\phi\Xi_3 - \cos\phi\Xi_4\}$ after a row reduction we obtain

$$\{\Xi_2 + \epsilon\csc\phi\Xi_4, \Xi_3 - \cot\phi\Xi_4\}.$$

Moreover, applying A_1 with $t_1 = \log(\sin\phi)$, we get

$$\{\Xi_2 + \epsilon\Xi_4, \Xi_3 - \cot\phi\Xi_4\},$$

and assuming $\alpha_1 = \epsilon$ and $a_1 = -\cot\phi$ we obtain $\{\Xi_2 + \alpha_1\Xi_4, \Xi_3 + a_1\Xi_4\}$.

Finally, the 3D–optimal systems of Patera and Winternitz and `SymboLie` are

$$\begin{aligned}\Theta_A^3 \equiv\ & \{\{\Xi_1,\Xi_3,\Xi_4\},\{\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_3,\Xi_2\},\{\Xi_1,\Xi_4,\Xi_2\},\\ & \{\Xi_1,\sin\phi\Xi_3-\cos\phi\Xi_4,\Xi_2\},\{\Xi_1+x\Xi_3,\Xi_4,\Xi_2\},\{\Xi_1+x\Xi_4,\Xi_3,\Xi_2\},\\ & \{\Xi_1+x\cos\phi\Xi_3+x\sin\phi\Xi_4,\sin\phi\Xi_3-\cos\phi\Xi_4,\Xi_2\}\},\end{aligned}$$

with $\phi \in]0,\pi[\setminus\{\pi/2\}$ and $x \in \mathbb{R}^*$, and

$$\begin{aligned}\Psi_A^3 \equiv\ & \{\{\Xi_1,\Xi_3,\Xi_4\},\{\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\\ & \{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_4\},\{\Xi_1+a_1\Xi_3,\Xi_2,\Xi_4\},\{\Xi_1+a_1\Xi_4,\Xi_2,\Xi_3\},\\ & \{\Xi_1+a_1\Xi_4,\Xi_2,\Xi_3+a_2\Xi_4\}\},\end{aligned}$$

with $a_1, a_2 \in \mathbb{R}^*$, respectively.

The representatives of the both 3D–optimal systems coincide. In particular, after the row reduction the last representative of Θ^3 becomes

$$\{\Xi_2, \Xi_1 + x\csc\phi\Xi_4, \Xi_3 - \cot\phi\Xi_4\},$$

and, assuming $a_1 = x\csc\phi$ and $a_2 = -\cot\phi$, we obtain $\{\Xi_2, \Xi_1 + a_1\Xi_4, \Xi_3 + a_2\Xi_4\}$ and so the 3D–optimal systems are the same.

5.2 Optimal systems of subalgebras of all real 3D Lie algebras

Optimal systems of Lie subalgebras of all real 3D Lie algebras have been listed in [26, Table I]. All such optimal systems coincide with those ones computed by `SymboLie` except the algebra $A_{3,8}$ (that is $\mathfrak{su}(1,1)$). Here, we analyze such a case.

Algebra ($\mathbf{A_{3,8}}$). Let $\mathcal{L}_3$ be the 3D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3\}$ with the non–zero commutators:

$$[\Xi_1, \Xi_2] = \Xi_1, \quad [\Xi_2, \Xi_3] = \Xi_3, \quad [\Xi_3, \Xi_1] = 2\Xi_2.$$

In the following, we write the matrices associated to the inner automorphisms $\exp(t\,\mathrm{ad}_{\Xi_1})$, $\exp(t\,\mathrm{ad}_{\Xi_2})$ and $\exp(t\,\mathrm{ad}_{\Xi_3})$:

$$A_1 = \begin{pmatrix} 1 & -t_1 & -t_1^2 \\ 0 & 1 & 2t_1 \\ 0 & 0 & 1 \end{pmatrix}, \; A_2 = \begin{pmatrix} \exp(t_2) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(-t_2) \end{pmatrix}, \; A_3 = \begin{pmatrix} 1 & 0 & 0 \\ -2t_3 & 1 & 0 \\ -t_3^2 & t_3 & 1 \end{pmatrix}.$$

Let us denote with A the group generated by $\{A_1, A_2, A_3\}$.

Patera and Winternitz [26] have obtained the following optimal systems:

$$\Theta_A^1 \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_1 + \Xi_3\}\}, \quad \Theta_A^2 \equiv \{\{\Xi_1, \Xi_2\}\}.$$

Using `SymboLie`, we obtain the following results:

$$\Psi_A^1 \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_1 + \alpha_1\Xi_3\}\}, \quad \Psi_A^2 \equiv \{\{\Xi_1, \Xi_2\}\}.$$

Regarding the 2D–optimal system, we can immediately see that the family of subalgebras $\{\Xi_1, \Xi_2\}$ is the only representative in both results. Hence, $\Theta_A^2 = \Psi_A^2$.

On the other hand, with regard to Θ_A^1 and Ψ_A^1, the only difference being the third representative. The subalgebra $\{\Xi_1 + \Xi_3\}$ cannot be treated computationally by `SymboLie`, so we must refer to the corresponding 2–family $X = \{\Xi_1 + a_1\Xi_3\}$. Furthermore, since the coefficient a_1 in the 2-family X can be rescaled, it is represented by the Greek letter α, as described in Algorithm 6. `SymboLie` is not capable of doing further specific analysis on the p-families. The one–dimensional optimal system computed by `SymboLie` is represented graphically in Figure 5.1.

In Table 5.1, the optimal systems of all eleven three–dimensional Lie algebras, as found by `SymboLie`, are reported.

5.3 Optimal systems of subalgebras of all real 4D Lie algebras

Optimal systems of Lie subalgebras of real 4D Lie algebras have been listed in [26, Table II]. Similarly to the three–dimensional case, we analyze those algebras where there are some differences between the optimal systems computed by `SymboLie` and the ones by Patera and Winternitz.

Algebra ($\mathbf{2A_2}$). Let $\mathcal{L}_4$ be the 4D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$ with the non–zero commutators:

$$[\Xi_1, \Xi_2] = \Xi_2, \quad [\Xi_3, \Xi_4] = \Xi_4.$$

TABLE 5.1: Optimal systems of real Lie algebras of dimension 3 (the algebras are listed in the same order used in [26]). `SymboLie` completes the computation in less than one minute. $a_1, a_2, a_3, a_4 \in \mathbb{R}$ are non vanishing, and $\alpha_1 = \pm 1$.

Non-zero Lie brackets	1D Optimal System	2D Optimal System
	$\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\},$ $\{\Xi_1 + a_1\Xi_2\}, \{\Xi_1 + a_1\Xi_3\},$ $\{\Xi_2 + a_1\Xi_3\},$ $\{\Xi_1 + a_1\Xi_2 + a_2\Xi_3\}$	$\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_2, \Xi_3\},$ $\{\Xi_1, a_1\Xi_2 + a_2\Xi_3\},$ $\{a_1\Xi_1 + a_2\Xi_2, \Xi_3\},$ $\{a_1\Xi_1 + a_2\Xi_3, \Xi_2\},$ $\{a_1\Xi_1 + a_2\Xi_3, a_3\Xi_2 + a_4\Xi_3\}$
$[\Xi_1, \Xi_2] = \Xi_2$	$\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\},$ $\{\Xi_1 + a_1\Xi_3\}, \{\Xi_2 + \alpha_1\Xi_3\}$	$\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\},$ $\{\Xi_2, \Xi_3\}, \{\Xi_1 + a_1\Xi_3, \Xi_2\}$
$[\Xi_1, \Xi_2] = \Xi_2$, $[\Xi_2, \Xi_3] = \Xi_1$	$\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_2 + a_1\Xi_3\}$	$\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\},$ $\{\Xi_1, \Xi_2 + a_1\Xi_3\}$
$[\Xi_1, \Xi_3] = \Xi_1$, $[\Xi_2, \Xi_3] = \Xi_1 + \Xi_2$	$\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}$	$\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}$
$[\Xi_1, \Xi_3] = \Xi_1$, $[\Xi_2, \Xi_3] = \Xi_2$	$\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\},$ $\{\Xi_1 + a_1\Xi_2\}$	$\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\},$ $\{\Xi_2, \Xi_3\}, \{\Xi_1 + a_1\Xi_2, \Xi_3\}$
$[\Xi_1, \Xi_3] = \Xi_1$, $[\Xi_2, \Xi_3] = -\Xi_2$	$\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\},$ $\{\Xi_1 + \alpha_1\Xi_2\}$	$\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\},$ $\{\Xi_2, \Xi_3\}$
$[\Xi_1, \Xi_3] = \Xi_1$, $[\Xi_2, \Xi_3] = a\Xi_2$ $(0 < \|a\| < 1)$	$\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\},$ $\{\Xi_1 + \alpha_1\Xi_2\}$	$\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_2, \Xi_3\}$
$[\Xi_1, \Xi_3] = -\Xi_2$, $[\Xi_2, \Xi_3] = \Xi_1$	$\{\Xi_1\}, \{\Xi_3\}$	$\{\Xi_1, \Xi_2\}$
$[\Xi_1, \Xi_3] = a\Xi_1 - \Xi_2$, $[\Xi_2, \Xi_3] = \Xi_1 + a\Xi_2$, $(a > 0)$	$\{\Xi_1\}, \{\Xi_3\}$	$\{\Xi_1, \Xi_2\}$
$[\Xi_1, \Xi_2] = \Xi_1$, $[\Xi_2, \Xi_3] = \Xi_3$, $[\Xi_3, \Xi_1] = 2\Xi_2$	$\{\Xi_1\}, \{\Xi_2\}, \{\Xi_1 + \alpha_1\Xi_3\}$	$\{\Xi_1, \Xi_2\}$
$[\Xi_1, \Xi_2] = \Xi_3$, $[\Xi_3, \Xi_1] = \Xi_2$, $[\Xi_2, \Xi_3] = \Xi_1$	$\{\Xi_1\}$	

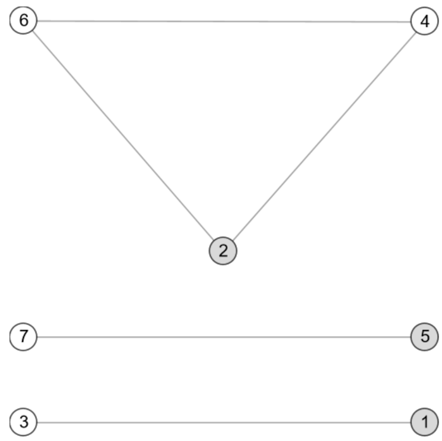

FIGURE 5.1: $1 \to \{\Xi_1\}$, $2 \to \{\Xi_2\}$, $3 \to \{\Xi_3\}$, $4 \to \{\Xi_1 + a_1\Xi_2\}$, $5 \to \{\Xi_1 + a_1\Xi_3\}$, $6 \to \{\Xi_2 + a_1\Xi_3\}$, $7 \to \{\Xi_1 + a_1\Xi_2 + a_2\Xi_3\}$

The 1D optimal system computed by `SymboLie` coincide with Patera and Winternitz system. It can be represented graphically in Figure 5.2.

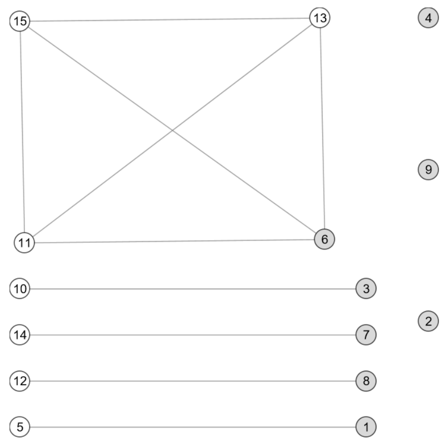

FIGURE 5.2: $1 \to \{\Xi_1\}$, $2 \to \{\Xi_2\}$,$3 \to \{\Xi_3\}$,$4 \to \{\Xi_4\}$,$5 \to \{\Xi_1 + a_1\Xi_2\}$,$6 \to \{\Xi_1 + a_1\Xi_3\}$,$7 \to \{\Xi_2 + a_1\Xi_3\}$,$8 \to \{\Xi_1 + a_1\Xi_4\}$,$9 \to \{\Xi_2 + a_1\Xi_4\}$,$10 \to \{\Xi_3 + a_1\Xi_4\}$,$11 \to \{\Xi_1 + a_1\Xi_2 + a_2\Xi_3\}$,$12 \to \{\Xi_1 + a_1\Xi_2 + a_2\Xi_4\}$,$13 \to \{\Xi_1 + a_1\Xi_3 + a_2\Xi_4\}$,$14 \to \{\Xi_2 + a_1\Xi_3 + a_2\Xi_4\}$,$15 \to \{\Xi_1 + a_1\Xi_2 + a_2\Xi_3 + a_3\Xi_4\}$

Instead, concerning the 2D optimal systems we have

$$\Theta^2_A = \Psi^2_A \cup \{\{\Xi_1 + \Xi_3, \Xi_2 + \epsilon\Xi_4\}\}, \quad \text{with } \epsilon = \pm 1.$$

We observe that the representative in the results by Patera and Winternitz that is absent in the results provided by SymboLie identify two particular subalgebras. In such a case, in `SymboLie`, we have to refer to the 4–family related to such subalgebras to make computation. Anyway, $\{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_4\}$ is not a 4–family, because it is not closed with respect to the Lie bracket for arbitrary a_1 and a_2. Therefore, it cannot be analyzed during the calculation of the optimal system.

The 3D optimal system computed by `SymboLie` is as follows:

$$\Theta^3_A \equiv \{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1 + a_1\Xi_3, \Xi_2, \Xi_4\}\},$$

with $a_1 \in \mathbb{R}^*$. The discrepancy with respect to the optimal system reported in [26] consists in the 4–family $\{\Xi_1 + a_1\Xi_3, \Xi_2, \Xi_4\}$. In [26], the optimal system contains the special subalgebras $A_{3,3}$, $A_{3,4}$ and $A^a_{3,5}$ (under particular conditions) that represent the aforementioned 4–family.

Algebra ($\mathbf{A_{3,6} \oplus A_1}$). Let $\mathcal{L}_4$ be the 4D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$ with the non–zero commutators:

$$[\Xi_1, \Xi_3] = -\Xi_2, \quad [\Xi_2, \Xi_3] = \Xi_1.$$

Let us write the matrices associated to inner automorphisms $\exp(t_1\,\mathrm{ad}_{\Xi_1})$, $\exp(t_2\,\mathrm{ad}_{\Xi_2})$ and $\exp(t_3\,\mathrm{ad}_{\Xi_3})$:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 0 & -t_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ A_3 = \begin{pmatrix} \cos t_3 & \sin t_3 & 0 & 0 \\ -\sin t_3 & \cos t_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The optimal systems of order 1 and 3 in [26] coincide with those computed by `SymboLie`. The graph of the one-dimensional optimal system is provided in [38].

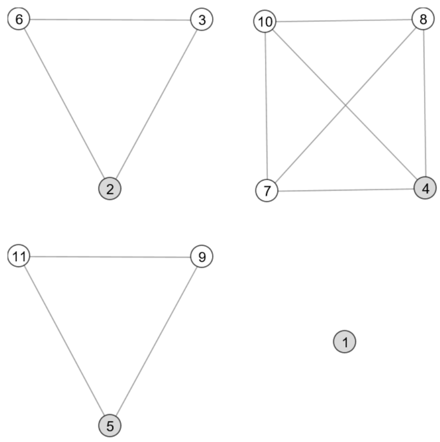

FIGURE 5.3: $1 \to \{\Xi_1, \Xi_2\}$, $2 \to \{\Xi_1, \Xi_4\}$, $3 \to \{\Xi_2, \Xi_4\}$, $4 \to \{\Xi_3, \Xi_4\}$, $5 \to \{\Xi_1, \Xi_2 + a_1\Xi_4\}$, $6 \to \{\Xi_1 + a_1\Xi_2, \Xi_4\}$, $7 \to \{\Xi_1 + a_1\Xi_3, \Xi_4\}$, $8 \to \{\Xi_2 + a_1\Xi_3, \Xi_4\}$, $9 \to \{\Xi_1 + a_1\Xi_4, \Xi_2\}$, $10 \to \{\Xi_1 + a_1\Xi_2 + a_2\Xi_3, \Xi_4\}$, $11 \to \{\Xi_1 + a_1\Xi_4, a_2\Xi_2 + a_3\Xi_4\}$

From the graph in Figure 5.3, it can be seen that two–dimensional optimal systems have different representatives of the same connected component. Indeed, the 2D optimal system obtained by Patera and Winternitz and `SymboLie` are

$$\Theta^2_A \equiv \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_4\}, \{\Xi_1 + x\Xi_4, \Xi_2\}, \{\Xi_3, \Xi_4\}\},$$
and
$$\Psi^2_A \equiv \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1\Xi_4\}, \{\Xi_3, \Xi_4\}\},$$

respectively.

Let us show that the family of subalgebras $\{\Xi_1, \Xi_2 + a_1\Xi_4\}$ is equivalent to the family of subalgebras $\{\Xi_1 + x\Xi_4, \Xi_2\}$ via the inner automorphism A_3. In fact, applying it we have

$$A_3 \cdot (1, 0, 0, 0)^T = (\cos t_3, -\sin t_3, 0, 0)^T \quad \text{and} \quad A_3 \cdot (0, 1, 0, a_1)^T = (\sin t_3, \cos t_3, 0, a_1)^T.$$

Choosing $t_3 = \pi/2$, we immediately obtain the representative of Θ^2_A, and thus the optimal systems coincide.

Algebra ($\mathbf{A^a_{3,7}} \oplus \mathbf{A_1}$). Let $\mathcal{L}_4$ be the 4D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$ with the non–zero commutators:

$$[\Xi_1, \Xi_3] = a\Xi_1 - \Xi_2, \quad [\Xi_2, \Xi_3] = \Xi_1 + a\Xi_2,$$

with $a > 0$.

In the following, we write the matrices associated to the inner automorphisms $\exp(t_1\,\mathrm{ad}_{\Xi_1})$, $\exp(t_2\,\mathrm{ad}_{\Xi_2})$ and $\exp(t_3\,\mathrm{ad}_{\Xi_3})$:

$$A_1 = \begin{pmatrix} 1 & 0 & -at_1 & 0 \\ 0 & 1 & t_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; A_2 = \begin{pmatrix} 1 & 0 & -t_2 & 0 \\ 0 & 1 & -t_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} \exp(at_3)\cos t_3 & \exp(at_3)\sin t_3 & 0 & 0 \\ -\exp(at_3)\sin t_3 & \exp(at_3)\cos t_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The optimal systems with dimensions 1 and 3 coincide. As in the previous case, regarding the two–dimensional optimal system, `SymboLie` returns another representative than the one found by Patera and Winternitz. The 2D–optimal system found by Patera and Winternitz is

$$\Theta^2_A \equiv \; \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_4\}, \{\Xi_1 + x\Xi_4, \Xi_2\}, \{\Xi_3, \Xi_4\}\},$$

and that computed by `SymboLie` is

$$\Psi^2_A \equiv \; \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1\Xi_4\}, \{\Xi_3, \Xi_4\}\}.$$

The graph of the 2D–optimal system computed by `SymboLie` is shown in the following figure:

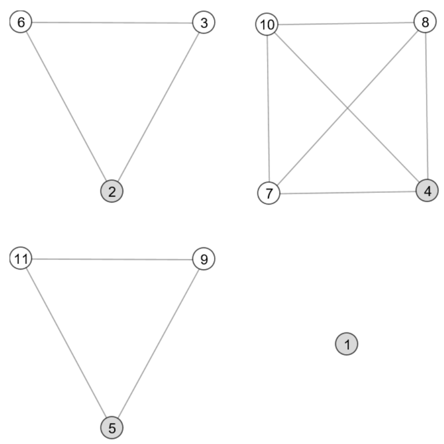

FIGURE 5.4: $1 \to \{\Xi_1, \Xi_2\}$, $2 \to \{\Xi_1, \Xi_4\}$, $3 \to \{\Xi_2, \Xi_4\}$, $4 \to \{\Xi_3, \Xi_4\}$, $5 \to \{\Xi_1, \Xi_2 + a_1\Xi_4\}$, $6 \to \{\Xi_1 + a_1\Xi_2, \Xi_4\}$, $7 \to \{\Xi_1 + a_1\Xi_3, \Xi_4\}$, $8 \to \{\Xi_2 + a_1\Xi_3, \Xi_4\}$, $9 \to \{\Xi_1 + a_1\Xi_4, \Xi_2\}$, $10 \to \{\Xi_1 + a_1\Xi_2 + a_2\Xi_3, \Xi_4\}$, $11 \to \{\Xi_1 + a_1\Xi_4, \Xi_2 + a_2\Xi_4\}$

Applying the inner automorphism A_3 to $\{\Xi_1, \Xi_2 + a_1\Xi_4\}$, we obtain

$$\{\exp(at_3)(\cos t_3\Xi_1 - \sin t_3\Xi_2), \exp(at_3)(\sin t_3\Xi_1 + \cos t_3\Xi_2 + a_1\Xi_4)\},$$

and choosing the parameter $t = \pi/2$ we have

$$\{-\exp(a\pi/2)\Xi_2, \exp(a\pi/2)\Xi_1 + a_1\Xi_4\}.$$

Finally, rescaling by the factor $-\exp(a\pi/2)$ we obtain the family of subalgebras $\{\Xi_1 + x\Xi_4, \Xi_2\}$.

Algebra ($\mathbf{A_{4,5}^{a,b}}$). Let $\mathcal{L}_4$ be the 4D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$ with the non–zero commutators:

$$[\Xi_1, \Xi_4] = \Xi_1, \quad [\Xi_2, \Xi_4] = a\Xi_2, \quad [\Xi_3, \Xi_4] = b\Xi_3,$$

with $-1 \leq a < b < 1, ab \neq 0$. In the following we write the matrices associated to inner automorphisms $\exp(t_1 \operatorname{ad}_{\Xi_1})$, $\exp(t_2 \operatorname{ad}_{\Xi_2})$, $\exp(t_3 \operatorname{ad}_{\Xi_3})$ and $\exp(t_4 \operatorname{ad}_{\Xi_4})$:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & -t_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -at_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -bt_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; A_4 = \begin{pmatrix} \exp(t_4) & 0 & 0 & 0 \\ 0 & \exp(at_4) & 0 & 0 \\ 0 & 0 & \exp(bt_4) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The 1D optimal system computed by `SymboLie` is

$$\Psi_A^1 = \Theta_A^1 \cup \{\Xi_1 + \alpha_1\Xi_2\},$$

with $\alpha_1 = \pm 1$ and its graph is shown in Figure 5.5.

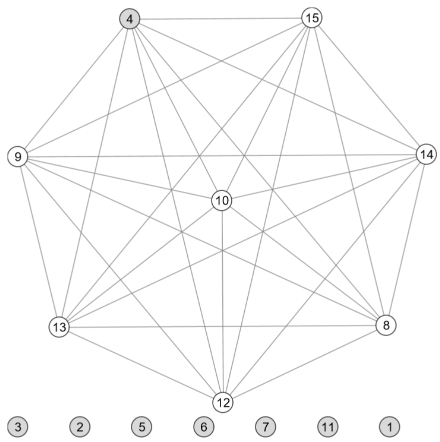

FIGURE 5.5

In [26, Table II] there are all representatives with the exception of $\{\Xi_1 + \alpha_1\Xi_2\}$. We can observe that such family of subalgebras is invariant with respect to the action of the inner automorphisms, so $\{\Xi_1 + \alpha_1\Xi_2\}$ has to belong to the optimal system.

Algebra ($\mathbf{A_{4,6}^{a,b}}$). Let $\mathcal{L}_4$ be the 4D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$ with the non–zero commutators:

$$[\Xi_1, \Xi_4] = a\Xi_1, \quad [\Xi_2, \Xi_4] = b\Xi_2 - \Xi_3, \quad [\Xi_3, \Xi_4] = \Xi_2 + b\Xi_3,$$

with $a \neq 0$ and $b \geq 0$. The matrices associated to inner automorphisms $\exp(t_1 \, \mathrm{ad}_{\Xi_1})$, $\exp(t_2 \, \mathrm{ad}_{\Xi_2})$, $\exp(t_3 \, \mathrm{ad}_{\Xi_3})$ and $\exp(t_4 \, \mathrm{ad}_{\Xi_4})$ are the following:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & -at_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -bt_2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -t_3 \\ 0 & 0 & 1 & -bt_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; A_4 = \begin{pmatrix} \exp(at_4) & 0 & 0 & 0 \\ 0 & \exp(bt_4)\cos t_4 & \exp(bt_4)\sin t_4 & 0 \\ 0 & -\exp(bt_4)\sin t_4 & \exp(bt_4)\cos t_4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Patera and Winternitz [26] have obtained the following 1D optimal system:

$$\Theta_A^1 \equiv \{\{\Xi_1\}, \{\Xi_3\}, \{\Xi_1 + x\Xi_3\}, \{\Xi_4\}\},$$

with $x > 0$. Using `SymboLie`, we obtain the following one dimensional optimal system of subalgebras:

$$\Psi_A^1 \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_1 + a_1\Xi_2\}, \{\Xi_4\}\},$$

with $a_1 \in \mathbb{R}^*$. The corresponding graph is displayed in Figure 5.6.

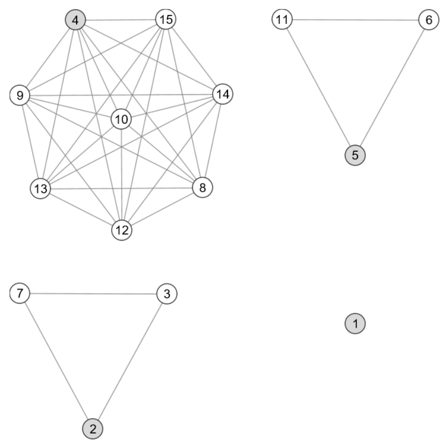

FIGURE 5.6

Let us show that there is an inner automorphism that maps $\{\Xi_2\}$ into $\{\Xi_3\}$, and then $\{\Xi_1 + a_1\Xi_2\}$ into $\{\Xi_1 + x\Xi_3\}$:

$$A_4 \cdot (0,1,0,0) = (0, \exp(bt)\cos t, -\exp(bt)\sin t, 0) = (0,0,-\exp(b\pi/2),0) \quad \text{with } t = \frac{\pi}{2},$$

from which follows that the subalgebra $\{\Xi_2\}$ is equivalent to $\{\Xi_3\}$.

Similarly, we have

$$A_4 \cdot (1,x,0,0)^T = (\exp(a\pi/2), 0, -\exp(b\pi/2)x, 0)^T, \quad \text{with } t = \frac{\pi}{2},$$

and, since $\{\exp(a\pi/2)\Xi_1 - \exp(b\pi/2)x\Xi_3\} = \{\Xi_1 + a_1\Xi_3\}$, with $a_1 = -\exp(\pi(b-a)/2)x$ arbitrary in $\mathbb{R}^*$, we obtain the assertion.

Moreover, the 2D optimal system Θ_A^2 and the 2D optimal system Ψ_A^2 are

$$\Theta_A^2 \equiv \{\{\Xi_1, \Xi_2\}, \{\Xi_2, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1 + x\Xi_3, \Xi_2\}\},$$

with $x > 0$, and

$$\Psi_A^2 \equiv \{\{\Xi_1, \Xi_2\}, \{\Xi_2, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1 + a_1\Xi_2, \Xi_3\}\},$$

with $a_1 \in \mathbb{R}^*$. The two optimal systems coincide, once we note from the graph below that the 3–families $\{\Xi_1 + x\Xi_3, \Xi_2\}$ and $\{\Xi_1 + a_1\Xi_2, \Xi_3\}$ belong to the same class.

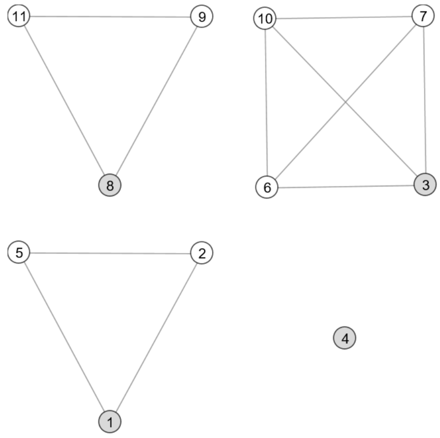

FIGURE 5.7: $1 \to \{\Xi_1, \Xi_2\}$, $2 \to \{\Xi_1, \Xi_3\}$, $3 \to \{\Xi_1, \Xi_4\}$, $4 \to \{\Xi_2, \Xi_3\}$, $5 \to \{\Xi_1, a_1\Xi_2 + a_2\Xi_3\}$, $6 \to \{\Xi_1, \Xi_2 + a_1\Xi_4\}$, $7 \to \{\Xi_1, \Xi_3 + a_1\Xi_4\}$, $8 \to \{\Xi_1 + a_1\Xi_2, \Xi_3\}$, $9 \to \{\Xi_1 + a_1\Xi_3, \Xi_2\}$, $10 \to \{\Xi_1, \Xi_2 + a_1\Xi_3 + a_2\Xi_4\}$, $11 \to \{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_3\}$

In particular, the two families under consideration are labeled with 8 and 9, and we can see from the graph that are equivalent. Indeed, using the inner automorphism A_4 and setting $t = \pi/2$, as we have already seen, we obtain the equivalence.

In Table 5.2, the optimal systems of all four–dimensional real Lie algebras, as found by `SymboLie`, are reported.

5.4 Optimal system of symmetries of KdV equation

Let us consider the four–dimensional Lie algebra of point symmetries (see Example 27). We observe that this Lie algebra corresponds in the classification contained in [26] to the algebra $A_{4,9}^b$ with $b = -2/3$.

The required code in Mathematica is the following one:

```
vars = {x, t, u};
gens = {{1, 0, 0}, {0, 1, 0}, {t, 0, 1}, {x, 3 t, -2 u}};
pars = {{},{}};
cs = StructureConstants[gens,vars];
alg1 = SubAlgebra[cs,pars,1];
alg2 = SubAlgebra[cs,pars,2];
alg3 = SubAlgebra[cs,pars,3];
```

Using `PrintOptimal[]` method it is possible to see the optimal systems of families of Lie subalgebras. Therefore, `PrintOptimal[alg1]` displays the 5 optimal families of 1D Lie subalgebras:

$$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_2 + \alpha_1\Xi_3\}\}.$$

TABLE 5.2: Optimal systems of real Lie algebras of dimension 4 (the algebras are listed in the same order used in [26]). `SymboLie` completes the computation in about 25 minutes. $a_1, a_2, a_3, a_4 \in \mathbb{R}$ are non vanishing, and $\alpha_1 = \pm 1$.

Non-zero Lie brackets	1D Optimal System	2D Optimal System	3D Optimal System
	All the 15 1D families of Lie subalgebras.	All the 35 2D families of Lie subalgebras.	All the 15 3D families of Lie subalgebras.
$[\Xi_1, \Xi_2] = \Xi_2$	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_3\}$, $\{\Xi_4\}$, $\{\Xi_1 + a_1\Xi_3\}$, $\{\Xi_2 + \alpha_1\Xi_3\}$, $\{\Xi_1 + a_1\Xi_4\}$, $\{\Xi_2 + \alpha_1\Xi_4\}$, $\{\Xi_3 + a_1\Xi_4\}$, $\{\Xi_1 + a_1\Xi_3 + a_2\Xi_4\}$, $\{\Xi_2 + \alpha_1\Xi_3 + a_1\Xi_4\}$	$\{\Xi_1, \Xi_2\}$, $\{\Xi_1, \Xi_3\}$, $\{\Xi_1, \Xi_4\}$, $\{\Xi_2, \Xi_3\}$, $\{\Xi_2, \Xi_4\}$, $\{\Xi_3, \Xi_4\}$, $\{\Xi_1, \Xi_3 + a_1\Xi_4\}$, $\{\Xi_2, \Xi_3 + a_1\Xi_4\}$, $\{\Xi_1 + a_1\Xi_3, \Xi_2\}$, $\{\Xi_1 + a_1\Xi_3, \Xi_4\}$, $\{\Xi_2 + \alpha_1\Xi_3, \Xi_4\}$, $\{\Xi_1 + a_1\Xi_4, \Xi_2\}$, $\{\Xi_1 + a_1\Xi_4, \Xi_3\}$, $\{\Xi_2 + \alpha_1\Xi_4, \Xi_3\}$, $\{\Xi_1 + a_1\Xi_4, \Xi_3 + a_2\Xi_4\}$, $\{\Xi_2 + \alpha_1\Xi_4, \Xi_3 + a_1\Xi_4\}$, $\{\Xi_1 + a_1\Xi_3 + a_2\Xi_4, \Xi_2\}$	$\{\Xi_1, \Xi_2, \Xi_3\}$, $\{\Xi_1, \Xi_2, \Xi_4\}$, $\{\Xi_1, \Xi_3, \Xi_4\}$, $\{\Xi_2, \Xi_3, \Xi_4\}$, $\{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4\}$, $\{\Xi_1 + a_1\Xi_3, \Xi_2, \Xi_4\}$, $\{\Xi_1 + a_1\Xi_4, \Xi_2, \Xi_3\}$, $\{\Xi_1 + a_1\Xi_4, \Xi_2, \Xi_3 + a_2\Xi_4\}$,
$[\Xi_1, \Xi_2] = \Xi_2$, $[\Xi_3, \Xi_4] = \Xi_4$,	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_3\}$, $\{\Xi_4\}$, $\{\Xi_1 + a_1\Xi_3\}$, $\{\Xi_2 + \alpha_1\Xi_3\}$, $\{\Xi_1 + \alpha_1\Xi_4\}$, $\{\Xi_2 + \alpha_1\Xi_4\}$	$\{\Xi_1, \Xi_2\}$, $\{\Xi_1, \Xi_3\}$, $\{\Xi_1, \Xi_4\}$, $\{\Xi_2, \Xi_3\}$, $\{\Xi_2, \Xi_4\}$, $\{\Xi_3, \Xi_4\}$, $\{\Xi_1 + a_1\Xi_3, \Xi_2\}$, $\{\Xi_1 + a_1\Xi_3, \Xi_4\}$, $\{\Xi_2 + \alpha_1\Xi_3, \Xi_4\}$, $\{\Xi_1 + \alpha_1\Xi_4, \Xi_2\}$	$\{\Xi_1, \Xi_2, \Xi_3\}$, $\{\Xi_1, \Xi_2, \Xi_4\}$, $\{\Xi_1, \Xi_3, \Xi_4\}$, $\{\Xi_2, \Xi_3, \Xi_4\}$, $\{\Xi_1 + a_1\Xi_3, \Xi_2, \Xi_4\}$
$[\Xi_2, \Xi_3] = \Xi_1$	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_3\}$, $\{\Xi_4\}$, $\{\Xi_2 + a_1\Xi_3\}$, $\{\Xi_1 + a_1\Xi_4\}$, $\{\Xi_2 + a_1\Xi_4\}$, $\{\Xi_3 + a_1\Xi_4\}$, $\{\Xi_2 + a_1\Xi_3 + a_2\Xi_4\}$	$\{\Xi_1, \Xi_2\}$, $\{\Xi_1, \Xi_3\}$, $\{\Xi_1, \Xi_4\}$, $\{\Xi_2, \Xi_4\}$, $\{\Xi_3, \Xi_4\}$, $\{\Xi_1, \Xi_2 + a_1\Xi_3\}$, $\{\Xi_1, \Xi_2 + a_1\Xi_4\}$, $\{\Xi_1, \Xi_3 + a_1\Xi_4\}$, $\{\Xi_1 + a_1\Xi_3, \Xi_4\}$, $\{\Xi_1 + a_1\Xi_4, \Xi_2\}$, $\{\Xi_1 + a_1\Xi_4, \Xi_3\}$, $\{\Xi_1, \Xi_2 + a_1\Xi_3 + a_2\Xi_4\}$, $\{\Xi_1 + a_1\Xi_4, \Xi_2 + a_2\Xi_3\}$	$\{\Xi_1, \Xi_2, \Xi_3\}$, $\{\Xi_1, \Xi_2, \Xi_4\}$, $\{\Xi_1, \Xi_3, \Xi_4\}$, $\{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4\}$, $\{\Xi_1, \Xi_2 + a_1\Xi_3, \Xi_4\}$, $\{\Xi_1, \Xi_2 + a_1\Xi_4, \Xi_3\}$, $\{\Xi_1, \Xi_2 + a_1\Xi_4, \Xi_3 + a_2\Xi_4\}$

Table 5.2: continued.

Non-zero Lie brackets	1D Optimal System	2D Optimal System	3D Optimal System
$[\Xi_2, \Xi_3] = \Xi_1 + \Xi_2$	$\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + \alpha_1\Xi_4\}, \{\Xi_2 + a_1\Xi_4\}, \{\Xi_3 + a_1\Xi_4\}$	$\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_4\}, \{\Xi_1, \Xi_3 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_4, \Xi_2\}$	$\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4\}$
$[\Xi_1, \Xi_3] = \Xi_1$, $[\Xi_2, \Xi_3] = \Xi_2$	$\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + a_1\Xi_2\}, \{\Xi_1 + \alpha_1\Xi_4\}, \{\Xi_2 + \alpha_1\Xi_4\}, \{\Xi_3 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_2 + \alpha_1\Xi_4\}$	$\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_4\}, \{\Xi_1, \Xi_3 + a_1\Xi_4\}, \{\Xi_2, \Xi_3 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_2, \Xi_3\}, \{\Xi_1 + a_1\Xi_2, \Xi_4\}, \{\Xi_1 + \alpha_1\Xi_4, \Xi_2\}, \{\Xi_1 + a_1\Xi_2, \Xi_3 + a_2\Xi_4\}, \{\Xi_1 + \alpha_1\Xi_4, \Xi_2 + a_1\Xi_4\}$	$\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_2, \Xi_3, \Xi_4\}$
$[\Xi_1, \Xi_3] = \Xi_1$,$[\Xi_2, \Xi_3] = -\Xi_2$	$\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + \alpha_1\Xi_2\}, \{\Xi_1 + \alpha_1\Xi_4\}, \{\Xi_2 + \alpha_1\Xi_4\}, \{\Xi_3 + a_1\Xi_4\}, \{\Xi_1 + \alpha_1\Xi_2 + a_1\Xi_4\}$	$\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_4\}, \{\Xi_1, \Xi_3 + a_1\Xi_4\}, \{\Xi_2, \Xi_3 + a_1\Xi_4\}, \{\Xi_1 + \alpha_1\Xi_2, \Xi_4\}, \{\Xi_1 + \alpha_1\Xi_4, \Xi_2\}, \{\Xi_1 + \alpha_1\Xi_4, \Xi_2 + a_1\Xi_4\}$	$\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4\}$
$[\Xi_1, \Xi_3] = \Xi_1$, $[\Xi_2, \Xi_3] = a\Xi_2$, with $0 < \|a\| < 1$	$\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + \alpha_1\Xi_2\}, \{\Xi_1 + \alpha_1\Xi_4\}, \{\Xi_2 + \alpha_1\Xi_4\}, \{\Xi_3 + a_1\Xi_4\}, \{\Xi_1 + \alpha_1\Xi_2 + a_1\Xi_4\}$	$\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_4\}, \{\Xi_1, \Xi_3 + a_1\Xi_4\}, \{\Xi_2, \Xi_3 + a_1\Xi_4\}, \{\Xi_1 + \alpha_1\Xi_2, \Xi_4\}, \{\Xi_1 + \alpha_1\Xi_4, \Xi_2\}, \{\Xi_1 + \alpha_1\Xi_4, \Xi_2 + a_1\Xi_4\}$	$\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4\}$
$[\Xi_1, \Xi_3] = -\Xi_2$, $[\Xi_2, \Xi_3] = \Xi_1$	$\{\Xi_1\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + a_1\Xi_4\}, \{\Xi_3 + a_1\Xi_4\}$	$\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1\Xi_4\}$	$\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4\}$

Table 5.2: continued.

Non-zero Lie brackets	1D Optimal System	2D Optimal System	3D Optimal System
$[\Xi_1, \Xi_3] = a\Xi_1 - \Xi_2$, $[\Xi_2, \Xi_3] = \Xi_1 + a\Xi_2$, with $a > 0$	$\{\Xi_1\}$, $\{\Xi_3\}$, $\{\Xi_4\}$, $\{\Xi_1 + a_1\Xi_4\}$, $\{\Xi_3 + a_1\Xi_4\}$	$\{\Xi_1, \Xi_2\}$, $\{\Xi_1, \Xi_4\}$, $\{\Xi_3, \Xi_4\}$, $\{\Xi_1, \Xi_2 + a_1\Xi_4\}$	$\{\Xi_1, \Xi_2, \Xi_3\}$, $\{\Xi_1, \Xi_2, \Xi_4\}$, $\{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4\}$
$[\Xi_1, \Xi_3] = -2\Xi_2$, $[\Xi_1, \Xi_2] = \Xi_1$, $[\Xi_2, \Xi_3] = \Xi_3$,	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_4\}$, $\{\Xi_1 + \alpha_1\Xi_3\}$, $\{\Xi_1 + \alpha_1\Xi_4\}$, $\{\Xi_2 + a_1\Xi_4\}$, $\{\Xi_1 + \alpha_1\Xi_3 + a_1\Xi_4\}$	$\{\Xi_1, \Xi_2\}$, $\{\Xi_1, \Xi_4\}$, $\{\Xi_2, \Xi_4\}$, $\{\Xi_1, \Xi_2 + a_1\Xi_4\}$, $\{\Xi_1 + \alpha_1\Xi_3, \Xi_4\}$	$\{\Xi_1, \Xi_2, \Xi_3\}$, $\{\Xi_1, \Xi_2, \Xi_4\}$
$[\Xi_1, \Xi_2] = \Xi_3$, $[\Xi_2, \Xi_3] = \Xi_1$, $[\Xi_1, \Xi_3] = -\Xi_2$	$\{\Xi_1\}$, $\{\Xi_4\}$, $\{\Xi_1 + a_1\Xi_4\}$	$\{\Xi_1, \Xi_4\}$	$\{\Xi_1, \Xi_2, \Xi_3\}$
$[\Xi_2, \Xi_4] = \Xi_1$, $[\Xi_3, \Xi_4] = \Xi_2$	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_3\}$, $\{\Xi_4\}$, $\{\Xi_1 + a_1\Xi_3\}$, $\{\Xi_3 + a_1\Xi_4\}$	$\{\Xi_1, \Xi_2\}$, $\{\Xi_1, \Xi_3\}$, $\{\Xi_1, \Xi_4\}$, $\{\Xi_2, \Xi_3\}$, $\{\Xi_1, \Xi_3 + a_1\Xi_4\}$, $\{\Xi_1 + a_1\Xi_3, \Xi_2\}$	$\{\Xi_1, \Xi_2, \Xi_3\}$, $\{\Xi_1, \Xi_2, \Xi_4\}$, $\{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4\}$
$[\Xi_1, \Xi_4] = a\Xi_1$, $[\Xi_2, \Xi_4] = \Xi_2$, $[\Xi_3, \Xi_4] = \Xi_2 + \Xi_3$, with $a \neq 0, 1$	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_3\}$, $\{\Xi_4\}$, $\{\Xi_1 + \alpha_1\Xi_2\}$, $\{\Xi_1 + a_1\Xi_3\}$	$\{\Xi_1, \Xi_2\}$, $\{\Xi_1, \Xi_3\}$, $\{\Xi_1, \Xi_4\}$, $\{\Xi_2, \Xi_3\}$, $\{\Xi_2, \Xi_4\}$, $\{\Xi_1 + a_1\Xi_2, \Xi_3\}$, $\{\Xi_1 + \alpha_1\Xi_3, \Xi_2\}$	$\{\Xi_1, \Xi_2, \Xi_3\}$, $\{\Xi_1, \Xi_2, \Xi_4\}$, $\{\Xi_2, \Xi_3, \Xi_4\}$
$[\Xi_1, \Xi_4] = \Xi_1$, $[\Xi_2, \Xi_4] = \Xi_2$, $[\Xi_3, \Xi_4] = \Xi_2 + \Xi_3$	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_3\}$, $\{\Xi_4\}$, $\{\Xi_1 + a_1\Xi_2\}$, $\{\Xi_1 + a_1\Xi_3\}$	$\{\Xi_1, \Xi_2\}$, $\{\Xi_1, \Xi_3\}$, $\{\Xi_1, \Xi_4\}$, $\{\Xi_2, \Xi_3\}$, $\{\Xi_2, \Xi_4\}$, $\{\Xi_1 + a_1\Xi_2, \Xi_3\}$, $\{\Xi_1 + a_1\Xi_2, \Xi_4\}$, $\{\Xi_1 + a_1\Xi_3, \Xi_2\}$	$\{\Xi_1, \Xi_2, \Xi_3\}$, $\{\Xi_1, \Xi_2, \Xi_4\}$, $\{\Xi_2, \Xi_3, \Xi_4\}$, $\{\Xi_1 + a_1\Xi_3, \Xi_2, \Xi_4\}$
$[\Xi_1, \Xi_4] = \Xi_1$, $[\Xi_3, \Xi_4] = \Xi_2$	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_3\}$, $\{\Xi_4\}$, $\{\Xi_1 + \alpha_1\Xi_2\}$, $\{\Xi_1 + a_1\Xi_3\}$, $\{\Xi_3 + a_1\Xi_4\}$	$\{\Xi_1, \Xi_2\}$, $\{\Xi_1, \Xi_3\}$, $\{\Xi_1, \Xi_4\}$, $\{\Xi_2, \Xi_3\}$, $\{\Xi_2, \Xi_4\}$, $\{\Xi_1, \Xi_3 + a_1\Xi_4\}$, $\{\Xi_2, \Xi_3 + a_1\Xi_4\}$, $\{\Xi_1 + a_1\Xi_2, \Xi_3\}$, $\{\Xi_1 + \alpha_1\Xi_3, \Xi_2\}$	$\{\Xi_1, \Xi_2, \Xi_3\}$, $\{\Xi_1, \Xi_2, \Xi_4\}$, $\{\Xi_2, \Xi_3, \Xi_4\}$, $\{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4\}$

Table 5.2: continued.

Non-zero Lie brackets	1D Optimal System	2D Optimal System	3D Optimal System
$[\Xi_1, \Xi_4] = \Xi_1$, $[\Xi_2, \Xi_4] = \Xi_1 + \Xi_2$, $[\Xi_3, \Xi_4] = \Xi_2 + \Xi_3$	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_3\}$, $\{\Xi_4\}$, $\{\Xi_1 + a_1\Xi_3\}$	$\{\Xi_1, \Xi_2\}$, $\{\Xi_1, \Xi_3\}$, $\{\Xi_1, \Xi_4\}$, $\{\Xi_2, \Xi_3\}$, $\{\Xi_1 + a_1\Xi_3, \Xi_2\}$	$\{\Xi_1, \Xi_2, \Xi_3\}$, $\{\Xi_1, \Xi_2, \Xi_4\}$
$[\Xi_1, \Xi_4] = \Xi_1$, $[\Xi_2, \Xi_4] = a\Xi_2$, $[\Xi_3, \Xi_4] = b\Xi_3$, with $-1 \leq a < b < 1$, $ab \neq 0$	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_3\}$, $\{\Xi_4\}$, $\{\Xi_1 + \alpha_1\Xi_2\}$, $\{\Xi_1 + \alpha_1\Xi_3\}$, $\{\Xi_2 + \alpha_1\Xi_3\}$, $\{\Xi_1 + \alpha_1\Xi_2 + a_1\Xi_3\}$	$\{\Xi_1, \Xi_2\}$, $\{\Xi_1, \Xi_3\}$, $\{\Xi_1, \Xi_4\}$, $\{\Xi_2, \Xi_3\}$, $\{\Xi_2, \Xi_4\}$, $\{\Xi_3, \Xi_4\}$, $\{\Xi_1, \Xi_2 + \alpha_1\Xi_3\}$, $\{\Xi_1 + \alpha_1\Xi_2, \Xi_3\}$, $\{\Xi_1 + \alpha_1\Xi_3, \Xi_2\}$, $\{\Xi_1 + \alpha_1\Xi_3, \Xi_2 + \alpha_2\Xi_3\}$	$\{\Xi_1, \Xi_2, \Xi_3\}$, $\{\Xi_1, \Xi_2, \Xi_4\}$, $\{\Xi_1, \Xi_3, \Xi_4\}$, $\{\Xi_2, \Xi_3, \Xi_4\}$
$[\Xi_1, \Xi_4] = \Xi_1$, $[\Xi_2, \Xi_4] = a\Xi_2$, $[\Xi_3, \Xi_4] = a\Xi_3$, with $-1 \leq a < 1, a \neq 0$	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_3\}$, $\{\Xi_4\}$, $\{\Xi_1 + \alpha_1\Xi_2\}$, $\{\Xi_1 + \alpha_1\Xi_3\}$, $\{\Xi_2 + a_1\Xi_3\}$, $\{\Xi_1 + \alpha_1\Xi_2 + a_1\Xi_3\}$	$\{\Xi_1, \Xi_2\}$, $\{\Xi_1, \Xi_3\}$, $\{\Xi_1, \Xi_4\}$, $\{\Xi_2, \Xi_3\}$, $\{\Xi_2, \Xi_4\}$, $\{\Xi_3, \Xi_4\}$, $\{\Xi_1, \Xi_2 + a_1\Xi_3\}$, $\{\Xi_1 + \alpha_1\Xi_2, \Xi_3\}$, $\{\Xi_1 + \alpha_1\Xi_3, \Xi_2\}$, $\{\Xi_2 + a_1\Xi_3, \Xi_4\}$, $\{\Xi_1 + \alpha_1\Xi_3, \Xi_2 + a_1\Xi_3\}$	$\{\Xi_1, \Xi_2, \Xi_3\}$, $\{\Xi_1, \Xi_2, \Xi_4\}$, $\{\Xi_1, \Xi_3, \Xi_4\}$, $\{\Xi_2, \Xi_3, \Xi_4\}$, $\{\Xi_1, \Xi_2 + a_1\Xi_3, \Xi_4\}$
$[\Xi_1, \Xi_4] = \Xi_1$, $[\Xi_2, \Xi_4] = a\Xi_2$, $[\Xi_3, \Xi_4] = \Xi_3$, with $-1 \leq a < 1, a \neq 0$	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_3\}$, $\{\Xi_4\}$, $\{\Xi_1 + \alpha_1\Xi_2\}$, $\{\Xi_1 + a_1\Xi_3\}$, $\{\Xi_2 + \alpha_1\Xi_3\}$, $\{\Xi_1 + \alpha_1\Xi_2 + a_1\Xi_3\}$	$\{\Xi_1, \Xi_2\}$, $\{\Xi_1, \Xi_3\}$, $\{\Xi_1, \Xi_4\}$, $\{\Xi_2, \Xi_3\}$, $\{\Xi_2, \Xi_4\}$, $\{\Xi_3, \Xi_4\}$, $\{\Xi_1, \Xi_2 + \alpha_1\Xi_3\}$, $\{\Xi_1 + \alpha_1\Xi_2, \Xi_3\}$, $\{\Xi_1 + a_1\Xi_3, \Xi_2\}$, $\{\Xi_1 + a_1\Xi_3, \Xi_4\}$, $\{\Xi_1 + a_1\Xi_3, \Xi_2 + \alpha_1\Xi_3\}$	$\{\Xi_1, \Xi_2, \Xi_3\}$, $\{\Xi_1, \Xi_2, \Xi_4\}$, $\{\Xi_1, \Xi_3, \Xi_4\}$, $\{\Xi_2, \Xi_3, \Xi_4\}$, $\{\Xi_1 + a_1\Xi_3, \Xi_2, \Xi_4\}$
$[\Xi_1, \Xi_4] = \Xi_1$, $[\Xi_2, \Xi_4] = \Xi_2$, $[\Xi_3, \Xi_4] = \Xi_3$, with $-1 \leq a < 1, a \neq 0$	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_3\}$, $\{\Xi_4\}$, $\{\Xi_1 + a_1\Xi_2\}$, $\{\Xi_1 + a_1\Xi_3\}$, $\{\Xi_2 + a_1\Xi_3\}$, $\{\Xi_1 + a_1\Xi_2 + a_2\Xi_3\}$	$\{\Xi_1, \Xi_2\}$, $\{\Xi_1, \Xi_3\}$, $\{\Xi_1, \Xi_4\}$, $\{\Xi_2, \Xi_3\}$, $\{\Xi_2, \Xi_4\}$, $\{\Xi_3, \Xi_4\}$, $\{\Xi_1, \Xi_2 + a_1\Xi_3\}$, $\{\Xi_1 + a_1\Xi_2, \Xi_3\}$, $\{\Xi_1 + a_1\Xi_2, \Xi_4\}$, $\{\Xi_1 + a_1\Xi_3, \Xi_2\}$, $\{\Xi_1 + a_1\Xi_3, \Xi_4\}$, $\{\Xi_2 + a_1\Xi_3, \Xi_4\}$, $\{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_3\}$, $\{\Xi_1 + a_1\Xi_2 + a_2\Xi_3, \Xi_4\}$	$\{\Xi_1, \Xi_2, \Xi_3\}$, $\{\Xi_1, \Xi_2, \Xi_4\}$, $\{\Xi_1, \Xi_3, \Xi_4\}$, $\{\Xi_2, \Xi_3, \Xi_4\}$, $\{\Xi_1, \Xi_2 + a_1\Xi_3, \Xi_4\}$, $\{\Xi_1 + a_1\Xi_2, \Xi_3, \Xi_4\}$, $\{\Xi_1 + a_1\Xi_3, \Xi_2, \Xi_4\}$, $\{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_3, \Xi_4\}$

Table 5.2: continued.

Non-zero Lie brackets	1D Optimal System	2D Optimal System	3D Optimal System
$[\Xi_1,\Xi_4]=a\Xi_1$, $[\Xi_2,\Xi_4]=b\Xi_2-\Xi_3$, $[\Xi_3,\Xi_4]=\Xi_2+b\Xi_3$, with $a\neq 0, b\geq 0$	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_4\}$, $\{\Xi_1+a_1\Xi_2\}$	$\{\Xi_1,\Xi_2\}$, $\{\Xi_1,\Xi_4\}$, $\{\Xi_2,\Xi_3\}$, $\{\Xi_1+a_1\Xi_2,\Xi_3\}$	$\{\Xi_1,\Xi_2,\Xi_3\}$, $\{\Xi_2,\Xi_3,\Xi_4\}$
$[\Xi_1,\Xi_4]=2\Xi_1$, $[\Xi_2,\Xi_4]=\Xi_2$, $[\Xi_3,\Xi_4]=\Xi_2+\Xi_3$, $[\Xi_2,\Xi_3]=\Xi_1$	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_3\}$, $\{\Xi_4\}$	$\{\Xi_1,\Xi_2\}$, $\{\Xi_1,\Xi_3\}$, $\{\Xi_1,\Xi_4\}$, $\{\Xi_2,\Xi_4\}$	$\{\Xi_1,\Xi_2,\Xi_3\}$, $\{\Xi_1,\Xi_2,\Xi_4\}$
$[\Xi_2,\Xi_3]=\Xi_1$, $[\Xi_2,\Xi_4]=\Xi_2$, $[\Xi_3,\Xi_4]=-\Xi_3$,	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_3\}$, $\{\Xi_4\}$, $\{\Xi_2+\alpha_1\Xi_3\}$, $\{\Xi_1+a_1\Xi_4\}$	$\{\Xi_1,\Xi_2\}$, $\{\Xi_1,\Xi_3\}$, $\{\Xi_1,\Xi_4\}$, $\{\Xi_2,\Xi_4\}$, $\{\Xi_3,\Xi_4\}$, $\{\Xi_1,\Xi_2+\alpha_1\Xi_3\}$, $\{\Xi_1+a_1\Xi_4,\Xi_2\}$, $\{\Xi_1+a_1\Xi_4,\Xi_3\}$	$\{\Xi_1,\Xi_2,\Xi_3\}$, $\{\Xi_1,\Xi_2,\Xi_4\}$, $\{\Xi_1,\Xi_3,\Xi_4\}$
$[\Xi_2,\Xi_3]=\Xi_1$, $[\Xi_1,\Xi_4]=(1+b)\Xi_1$, $[\Xi_2,\Xi_4]=\Xi_2$, $[\Xi_3,\Xi_4]=b\Xi_3$, with $0<\lvert b\rvert<1$	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_3\}$, $\{\Xi_4\}$, $\{\Xi_2+\alpha_1\Xi_3\}$	$\{\Xi_1,\Xi_2\}$, $\{\Xi_1,\Xi_3\}$, $\{\Xi_1,\Xi_4\}$, $\{\Xi_2,\Xi_4\}$, $\{\Xi_3,\Xi_4\}$, $\{\Xi_1,\Xi_2+\alpha_1\Xi_3\}$	$\{\Xi_1,\Xi_2,\Xi_3\}$, $\{\Xi_1,\Xi_2,\Xi_4\}$, $\{\Xi_1,\Xi_3,\Xi_4\}$
$[\Xi_2,\Xi_3]=\Xi_1$, $[\Xi_1,\Xi_4]=2\Xi_1$, $[\Xi_2,\Xi_4]=\Xi_2$, $[\Xi_3,\Xi_4]=\Xi_3$	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_3\}$, $\{\Xi_4\}$, $\{\Xi_2+a_1\Xi_3\}$	$\{\Xi_1,\Xi_2\}$, $\{\Xi_1,\Xi_3\}$, $\{\Xi_1,\Xi_4\}$, $\{\Xi_2,\Xi_4\}$, $\{\Xi_3,\Xi_4\}$, $\{\Xi_1,\Xi_2+a_1\Xi_3\}$, $\{\Xi_2+a_1\Xi_3,\Xi_4\}$	$\{\Xi_1,\Xi_2,\Xi_3\}$, $\{\Xi_1,\Xi_2,\Xi_4\}$, $\{\Xi_1,\Xi_3,\Xi_4\}$, $\{\Xi_1,\Xi_2+a_1\Xi_3,\Xi_4\}$
$[\Xi_2,\Xi_3]=\Xi_1$, $[\Xi_1,\Xi_4]=\Xi_1$, $[\Xi_2,\Xi_4]=\Xi_2$, $[\Xi_2,\Xi_4]=\Xi_2$	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_3\}$, $\{\Xi_4\}$, $\{\Xi_2+\alpha_1\Xi_3\}$, $\{\Xi_3+a_1\Xi_4\}$	$\{\Xi_1,\Xi_2\}$, $\{\Xi_1,\Xi_3\}$, $\{\Xi_1,\Xi_4\}$, $\{\Xi_2,\Xi_4\}$, $\{\Xi_3,\Xi_4\}$, $\{\Xi_1,\Xi_2+\alpha_1\Xi_3\}$, $\{\Xi_1,\Xi_3+a_1\Xi_4\}$	$\{\Xi_1,\Xi_2,\Xi_3\}$, $\{\Xi_1,\Xi_2,\Xi_4\}$, $\{\Xi_1,\Xi_3,\Xi_4\}$, $\{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_4\}$
$[\Xi_2,\Xi_3]=\Xi_1$, $[\Xi_2,\Xi_4]=-\Xi_3$, $[\Xi_3,\Xi_4]=\Xi_2$	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_4\}$, $\{\Xi_1+a_1\Xi_4\}$	$\{\Xi_1,\Xi_2\}$, $\{\Xi_1,\Xi_4\}$	$\{\Xi_1,\Xi_2,\Xi_3\}$

Table 5.2: continued.

Non-zero Lie brackets	1D Optimal System	2D Optimal System	3D Optimal System
$[\Xi_2, \Xi_3] = \Xi_1$, $[\Xi_1, \Xi_4] = 2a\Xi_1$, $[\Xi_2, \Xi_4] = a\Xi_2 - \Xi_3$, $[\Xi_3, \Xi_4] = \Xi_2 + a\Xi_3$, with $a > 0$	$\{\Xi_1\}$, $\{\Xi_2\}$, $\{\Xi_4\}$	$\{\Xi_1, \Xi_2\}$, $\{\Xi_1, \Xi_4\}$	$\{\Xi_1, \Xi_2, \Xi_3\}$
$[\Xi_1, \Xi_3] = \Xi_1$, $[\Xi_2, \Xi_3] = \Xi_2$, $[\Xi_1, \Xi_4] = -\Xi_2$, $[\Xi_2, \Xi_4] = \Xi_1$	$\{\Xi_1\}$, $\{\Xi_3\}$, $\{\Xi_4\}$, $\{\Xi_3 + a_1\Xi_4\}$	$\{\Xi_1, \Xi_2\}$, $\{\Xi_1, \Xi_3\}$, $\{\Xi_3, \Xi_4\}$	$\{\Xi_1, \Xi_2, \Xi_3\}$, $\{\Xi_1, \Xi_2, \Xi_4\}$, $\{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4\}$

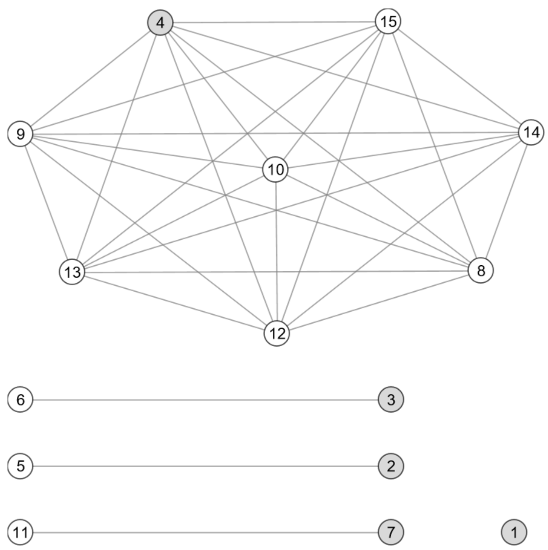

FIGURE 5.8: Algebra of Lie symmetries of KdV equation: graph of families of 1D subalgebras, where $1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 7 \to \{\Xi_2 + \alpha_1\Xi_3\}$.

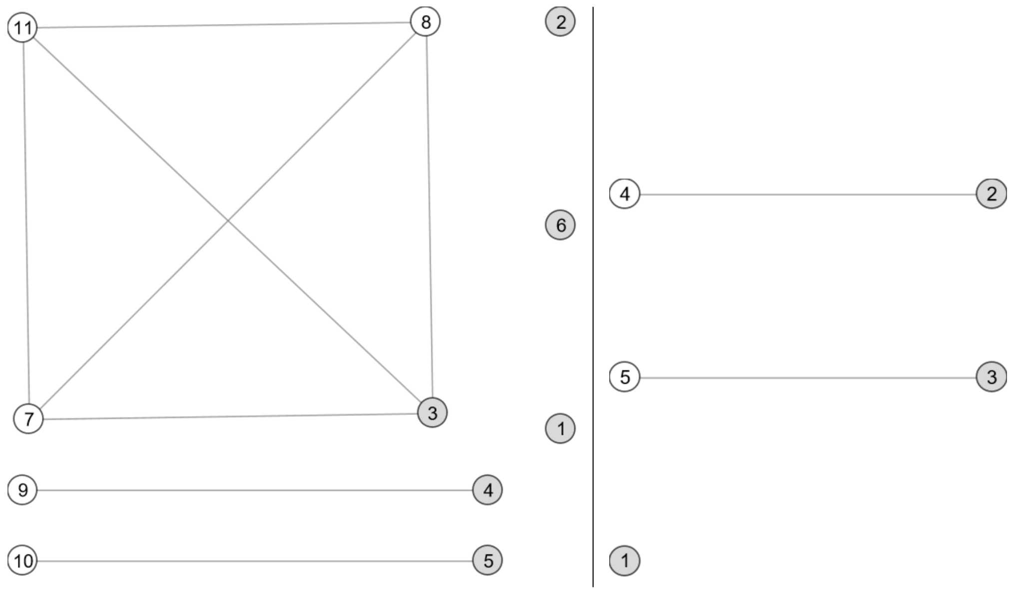

FIGURE 5.9: Lie algebra of symmetries of KdV equation: graph of families of 2D subalgebras (left), where $1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_4\}, 5 \to \{\Xi_3, \Xi_4\}, 6 \to \{\Xi_1, \Xi_2 + \alpha_1 \Xi_3\}$, and graph of families of 3D subalgebras (right), where $\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_3, \Xi_4\}$.

Remark 17. *The one-dimensional optimal system of subalgebras computed with pencil and paper in Example 22 matches the one computed by* `SymboLie` *and is the same as the one in [11].*

Similarly, the calls `PrintOptimal[alg2]` and `PrintOptimal[alg3]` show the 6 optimal families of 2D Lie subalgebras, say

$$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + \alpha_1 \Xi_3\}\},$$

and the 3 optimal families of 3D Lie subalgebras, namely

$$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}\},$$

respectively.

Finally, the method `PrintGraph[]` (with argument `alg1`, `alg2` or `alg3`), displays the graph representing the relation $\mathcal{R}$ between the p–families (see Figures 5.8 and 5.9, where the gray nodes denote the representatives of each class of equivalent families). The legend showing the correspondence between the labels with the representatives families is also displayed. In addition, by setting the second optional argument of the `PrintGraph[]` method equal to 1, it is possible to display the labels of all the families of Lie subalgebras analyzed by `SymboLie`.

5.5 Optimal system of symmetries of Burgers' equation

The five–dimensional Lie algebra of symmetries of viscous Burgers' equation,

$$u_t + u u_x - u_{xx} = 0,$$

is spanned by

$$\begin{aligned} &\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x}, \qquad \Xi_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \\ &\Xi_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, \quad \Xi_5 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + (x - tu)\frac{\partial}{\partial u}. \end{aligned} \tag{5.2}$$

SymboLie determines a set of 5 optimal 1D families of Lie subalgebras, 5 families in the 2D case, 5 families in the 3D case, and only 1 family in the 4D case:

$$\begin{aligned} \Theta_A^1 &\equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_4\}, \{\Xi_1 + \alpha_1\Xi_3\}, \{\Xi_1 + \alpha_1\Xi_5\}\}, \\ \Theta_A^2 &\equiv \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_1 + \alpha_1\Xi_3, \Xi_2\}\}, \\ \Theta_A^3 &\equiv \{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_4, \Xi_5\}, \{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1 + \alpha_1\Xi_5, \Xi_2, \Xi_3\}\}, \\ \Theta_A^4 &\equiv \{\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}\}. \end{aligned}$$

with $\alpha_1 = \pm 1$. The corresponding graphs are displayed in Figures 5.10 and 5.11.

From the graph on the left of Figure 5.10, we can see that Ξ_1 and Ξ_5 belong to the same connected component; namely, the time translation Lie group and the projective Lie group are equivalent. Indeed, let us consider the inner automorphisms $A_1 = \exp(t_1 \operatorname{ad}_{\Xi_1})$ and $A_5 = \exp(t_5 \operatorname{ad}_{\Xi_5})$, and let

$$A = A_1 A_5 = \begin{pmatrix} t_1^2 t_5^2 - 2t_1 t_5 + 1 & 0 & 0 & 2t_1^2 t_5 - 2t_1 & t_1^2 \\ 0 & 1 - t_1 t_5 & -t_1 & 0 & 0 \\ 0 & t_5 & 1 & 0 & 0 \\ t_5 - t_1 t_5^2 & 0 & 0 & 1 - 2t_1 t_5 & -t_1 \\ t_5^2 & 0 & 0 & 2t_5 & 1 \end{pmatrix}.$$

Applying A to Ξ_1, we get

$$A\,\Xi_1 = (1 - 2t_1 t_5 + t_1^2 t_5^2)\Xi_1 + (t_5 - t_1 t_5^2)\Xi_4 + t_5^2 \Xi_5,$$

whence, choosing $t_1 = t_5 = 1$, it is obtained Ξ_5.

Then, the solutions of Burgers' equation found in Example 17 left invariant with respect to the infinitesimal generators Ξ_1 and Ξ_5 are, in fact, equivalent. Let us verify this.

The integration of the Lie equations for the infinitesimal generators Ξ_1 and Ξ_5 provides the corresponding one-parameter Lie subgroups of transformations of the full Lie group admitted by Burgers' equation:

$$T_1^\alpha : \begin{cases} t^\star = t + \alpha \\ x^\star = x \\ u^\star = u \end{cases}, \qquad T_5^\beta : \begin{cases} t^\star = \dfrac{t}{1 - \beta t} \\ x^\star = \dfrac{x}{1 - \beta t} \\ u^\star = u + \beta(x - tu) \end{cases}.$$

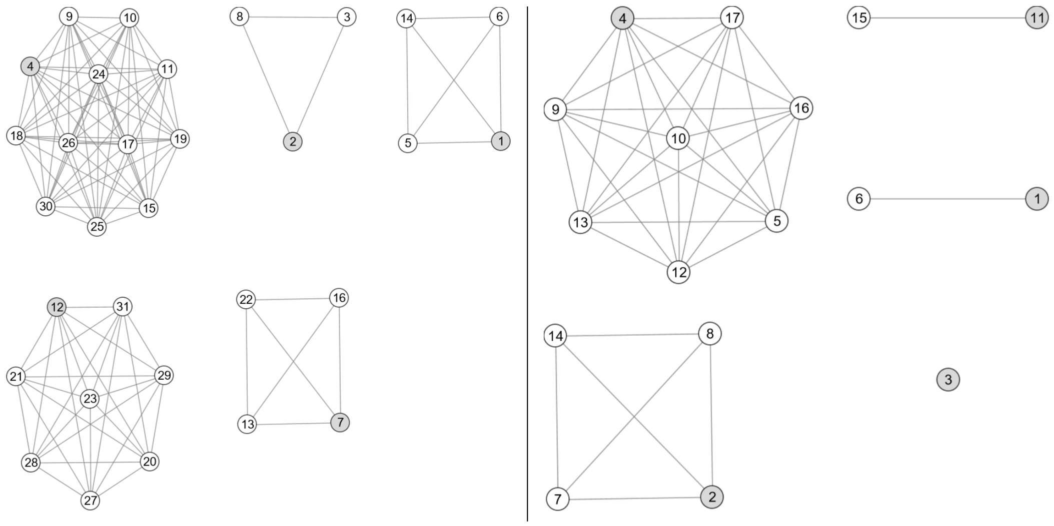

FIGURE 5.10: Algebra of symmetries of viscous Burgers' equation: graph of families of 1D Lie subalgebras (left), where $1 \to \{\Xi_1\}$, $2 \to \{\Xi_2\}$, $4 \to \{\Xi_4\}$, $7 \to \{\Xi_1 + \alpha_1\Xi_3\}$, $12 \to \{\Xi_1 + \alpha_1\Xi_5\}$, and of 2D Lie subalgebras (right), where $1 \to \{\Xi_1, \Xi_2\}$, $2 \to \{\Xi_1, \Xi_4\}$, $3 \to \{\Xi_2, \Xi_3\}$, $4 \to \{\Xi_2, \Xi_4\}$, $11 \to \{\Xi_1 + \alpha_1\Xi_3, \Xi_2\}$.

By choosing the values of the parameters $\alpha = 1$ and $\beta = 1$, we can consider $G = T_1^1\, T_5^1$, *i.e.*,

$$G: \begin{cases} t^\star = \dfrac{1}{1-t} \\ x^\star = \dfrac{1}{1-t} \\ u^\star = u(1-t) + x \end{cases}.$$

For instance, let us consider the first invariant solution from Example 17 with respect to the projective group

$$u(t,x) = \frac{x^2 + btx - 2t}{bt^2 + tx},$$

and its corresponding manifold

$$\mathcal{S} \equiv u - \frac{x^2 + btx - 2t}{bt^2 + tx} = 0.$$

After some straightforward algebraic manipulation, we obtain:

$$\widetilde{\mathcal{S}} = G\mathcal{S} \equiv \frac{t-1}{x+b}\left(2 + u(x+b)\right) = 0,$$

Thus, the invariant solution with respect to the time translation group

$$u(t,x) = -\frac{2}{x+b}$$

is obtained.

In the same way, applying similar calculations to the other two solutions, yields their corresponding invariant forms with respect to the time translation group.

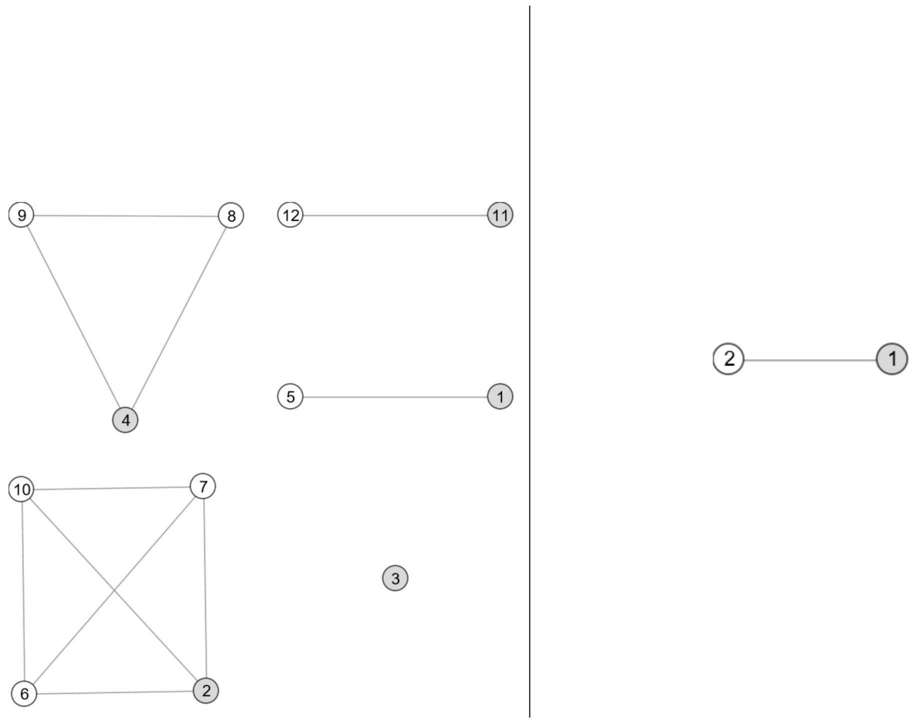

FIGURE 5.11: Algebra of symmetries of viscous Burgers' equation: graph of families of 3D Lie subalgebras (left), where $1 \to \{\Xi_1, \Xi_2, \Xi_3\}$, $2 \to \{\Xi_1, \Xi_2, \Xi_4\}$, $3 \to \{\Xi_1, \Xi_4, \Xi_5\}$, $4 \to \{\Xi_2, \Xi_3, \Xi_4\}$, $11 \to \{\Xi_1 + \alpha_1\Xi_5, \Xi_2, \Xi_3\}$, and of 4D Lie subalgebras (right), where $1 \to \{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$, $2 \to \{\Xi_2, \Xi_3, \Xi_4, \Xi_5\}$.

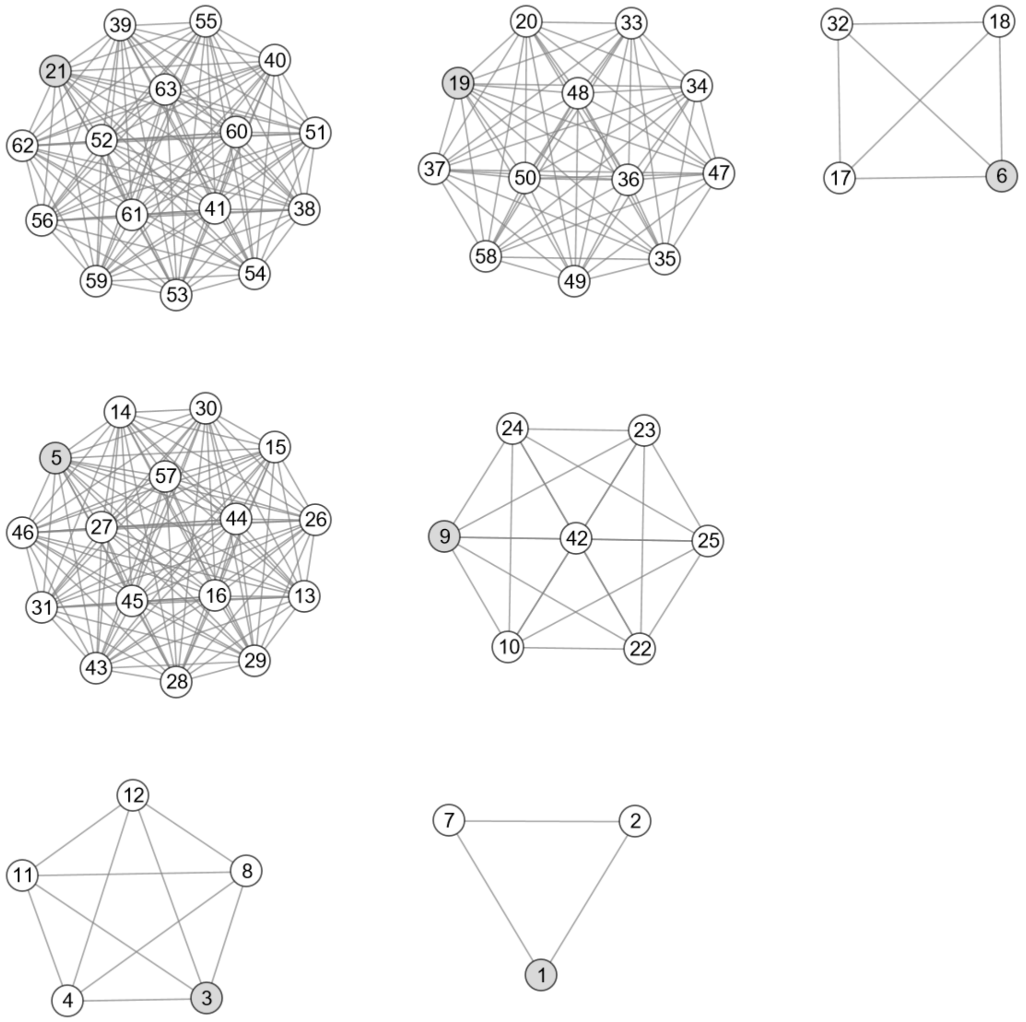

FIGURE 5.12: Graph of the families of 1D Lie subalgebras of the algebra described in Section 5.6, where $1 \to \{\Xi_1\}$, $3 \to \{\Xi_3\}$, $5 \to \{\Xi_5\}$, $6 \to \{\Xi_6\}$, $9 \to \{\Xi_2 + a_1\Xi_3\}$, $19 \to \{\Xi_3 + a_1\Xi_6\}$, $21 \to \{\Xi_5 + a_1\Xi_6\}$.

5.6 Optimal system of symmetries of a six–dimensional Lie algebra

Let $\mathcal{L}_6$ be a Lie algebra of the plane Galilei group spanned by the vector fields

$$\Xi_1 = \frac{\partial}{\partial x},\ \Xi_2 = \frac{\partial}{\partial y},\ \Xi_3 = t\frac{\partial}{\partial x},\ \Xi_4 = t\frac{\partial}{\partial y},\ \Xi_5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},\ \Xi_6 = \frac{\partial}{\partial t}.$$

The complete optimal system of Lie subalgebras has been studied in [25]. Using `SymboLie`, we obtain the following optimal system of families of Lie subalgebras:

$$\begin{aligned}
\Theta_A^1 \equiv\ & \{\{\Xi_1\}, \{\Xi_3\}, \{\Xi_5\}, \{\Xi_6\}, \{\Xi_2 + a_1\Xi_3\}, \{\Xi_3 + a_1\Xi_6\}, \{\Xi_5 + a_1\Xi_6\}\},\\
\Theta_A^2 \equiv\ & \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_6\}, \{\Xi_3, \Xi_4\}, \{\Xi_5, \Xi_6\},\\
& \{\Xi_1, \Xi_2 + a_1\Xi_3\}, \{\Xi_1, \Xi_3 + a_1\Xi_4\}, \{\Xi_1, \Xi_3 + a_1\Xi_6\}, \{\Xi_1, \Xi_4 + a_1\Xi_6\},\\
& \{\Xi_1 + a_1\Xi_3, \Xi_4\}, \{\Xi_2 + a_1\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3 + a_1\Xi_4 + a_2\Xi_6\},\\
& \{\Xi_1 + a_1\Xi_4, \Xi_2 + a_2\Xi_3\}\},
\end{aligned}$$

$$\begin{aligned}\Theta_A^3 \equiv \{&\{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_6\},\{\Xi_1,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_3,\Xi_6\},\\ &\{\Xi_3,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_6\},\{\Xi_1,\Xi_2,\Xi_5+a_1\Xi_6\},\\ &\{\Xi_1,\Xi_3,\Xi_4+a_1\Xi_6\},\{\Xi_1,\Xi_2+a_1\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2+a_1\Xi_3,\Xi_6\},\\ &\{\Xi_1,\Xi_2+a_1\Xi_3,\Xi_4+a_2\Xi_6\}\},\\ \Theta_A^4 \equiv \{&\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_6\},\{\Xi_1,\Xi_2,\Xi_5,\Xi_6\},\\ &\{\Xi_1,\Xi_2,\Xi_3+\alpha_1\Xi_6,\Xi_4+a_1\Xi_6\}\},\\ \Theta_A^5 \equiv \{&\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_6\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5+a_1\Xi_6\}\},\end{aligned}$$

with $a_1, a_2 \neq 0$. This result is in agreement with the one given by Ovsiannikov [25], except for the choice of some representatives of the equivalence classes.

The graphs of the optimal systems computed by `SymboLie` are shown in Figures 5.12–5.14.

As far as the graph of 5D families of optimal Lie subalgebras are concerned, we observe that it is the trivial graph made of three isolated vertices.

5.7 Optimal system of symmetries of linear heat equation

Consider the finite–dimensional Lie algebra of symmetries of $u_t - u_{xx} = 0$, generated by the vector fields, as discussed in Example 25:

$$\begin{aligned}&\Xi_1 = \frac{\partial}{\partial t}, \quad \Xi_2 = \frac{\partial}{\partial x}, \quad \Xi_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \quad \Xi_4 = u\frac{\partial}{\partial u},\\ &\Xi_5 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}, \quad \Xi_6 = 4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2+2t)u\frac{\partial}{\partial u}.\end{aligned}$$

`SymboLie` recovers a system of 9 optimal 1D families of Lie subalgebras, 13 families in the 2D case, 8 families in the 3D case, 5 families in the 4D case and only 1 5D family:

$$\begin{aligned}\Theta_A^1 \equiv \{&\{\Xi_1\},\{\Xi_2\},\{\Xi_3\},\{\Xi_4\},\{\Xi_1+\alpha_1\Xi_4\},\{\Xi_3+a_1\Xi_4\},\{\Xi_1+\alpha_1\Xi_5\},\\ &\{\Xi_1+\alpha_1\Xi_6\},\{\Xi_1+\alpha_1\Xi_3+a_1\Xi_6\}\},\\ \Theta_A^2 \equiv \{&\{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_3\},\{\Xi_1,\Xi_4\},\{\Xi_2,\Xi_3\},\{\Xi_2,\Xi_4\},\{\Xi_3,\Xi_4\},\{\Xi_3,\Xi_5\},\\ &\{\Xi_3,\Xi_6\},\{\Xi_1+\alpha_1\Xi_4,\Xi_2\},\{\Xi_1,\Xi_3+a_1\Xi_4\},\{\Xi_2,\Xi_3+a_1\Xi_4\},\\ &\{\Xi_1+\alpha_1\Xi_5,\Xi_4\},\{\Xi_1+\alpha_1\Xi_6,\Xi_4\}\},\\ \Theta_A^3 \equiv \{&\{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\{\Xi_1,\Xi_3,\Xi_4\},\{\Xi_2,\Xi_3,\Xi_4\},\{\Xi_2,\Xi_4,\Xi_5\},\\ &\{\Xi_3,\Xi_5,\Xi_6\},\{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_4\},\{\Xi_1+\alpha_1\Xi_5,\Xi_2,\Xi_4\}\},\\ \Theta_A^4 \equiv \{&\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4,\Xi_6\},\\ &\{\Xi_2,\Xi_3,\Xi_4,\Xi_5\},\{\Xi_1+\alpha_1\Xi_6,\Xi_2,\Xi_4,\Xi_5\}\},\\ \Theta_A^5 \equiv \{&\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5\}\}.\end{aligned}$$

with $a_1 \neq 0$ and $\alpha_1 = \pm 1$. The corresponding non–trivial graphs are displayed in Figures 5.15 and 5.16.

Remark 18. *As anticipated in Remark 8, the graph in Figures 5.15 shows that the subalgebra labeled* 1 *is in the same connected component as the one labeled* 6*, corresponding to the infinitesimal generators of the time translation and the projective group, respectively. This explains the existence, in Example 19, of a group transformation that maps the invariant solution with respect to the projective group into a time-independent solution.*

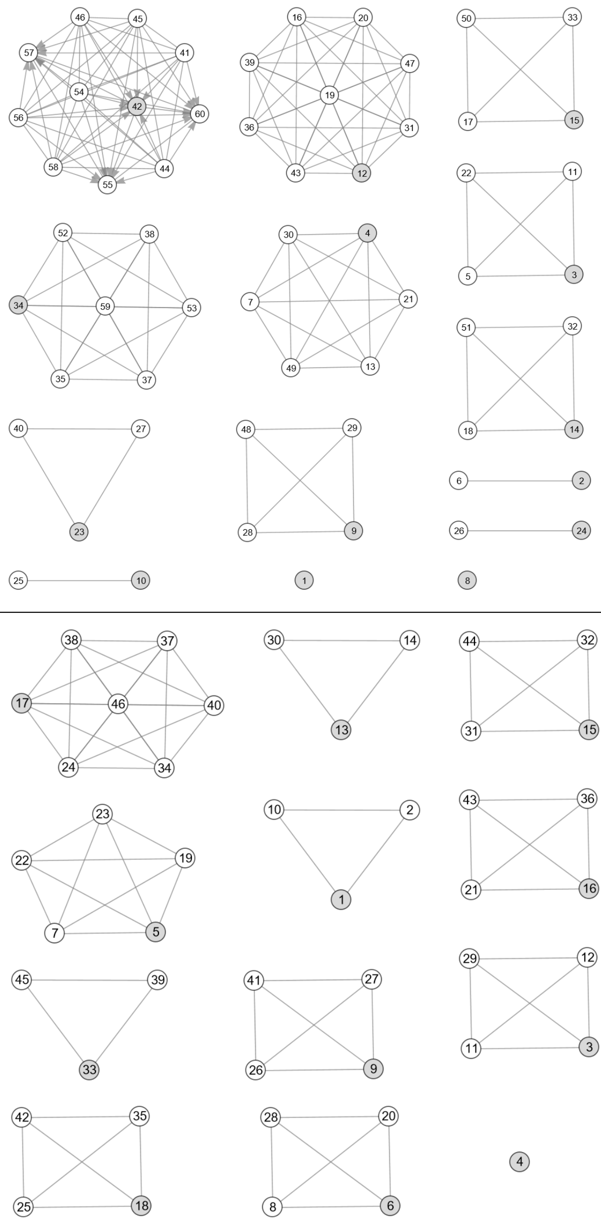

FIGURE 5.13: Algebra described in Section 5.6: graph of families of 2D Lie subalgebras (top), where $1 \to \{\Xi_1, \Xi_2\}$, $2 \to \{\Xi_1, \Xi_3\}$, $3 \to \{\Xi_1, \Xi_4\}$, $4 \to \{\Xi_1, \Xi_6\}$, $10 \to \{\Xi_1, \Xi_2 + a_1\Xi_3\}$, $12 \to \{\Xi_1, \Xi_3 + a_1\Xi_4\}$, $14 \to \{\Xi_1, \Xi_3 + a_1\Xi_6\}$, $15 \to \{\Xi_1, \Xi_4 + a_1\Xi_6\}$, $23 \to \{\Xi_1 + a_1\Xi_3, \Xi_4\}$, $24 \to \{\Xi_2 + a_1\Xi_3, \Xi_4\}$, $34 \to \{\Xi_1, \Xi_3 + a_1\Xi_4 + a_2\Xi_6\}$, $42 \to \{\Xi_1 + a_1\Xi_4, \Xi_2 + a_2\Xi_3\}$, and of 3D Lie subalgebras (bottom), where $1 \to \{\Xi_1, \Xi_2\}$, $2 \to \{\Xi_1, \Xi_3\}$, $3 \to \{\Xi_1, \Xi_4\}$, $4 \to \{\Xi_1, \Xi_6\}$, $10 \to \{\Xi_1, \Xi_2 + a_1\Xi_3\}$, $12 \to \{\Xi_1, \Xi_3 + a_1\Xi_4\}$, $14 \to \{\Xi_1, \Xi_3 + a_1\Xi_6\}$, $15 \to \{\Xi_1, \Xi_4 + a_1\Xi_6\}$, $23 \to \{\Xi_1 + a_1\Xi_3, \Xi_4\}$, $24 \to \{\Xi_2 + a_1\Xi_3, \Xi_4\}$, $34 \to \{\Xi_1, \Xi_3 + a_1\Xi_4 + a_2\Xi_6\}$, $42 \to \{\Xi_1 + a_1\Xi_4, \Xi_2 + a_2\Xi_3\}$.

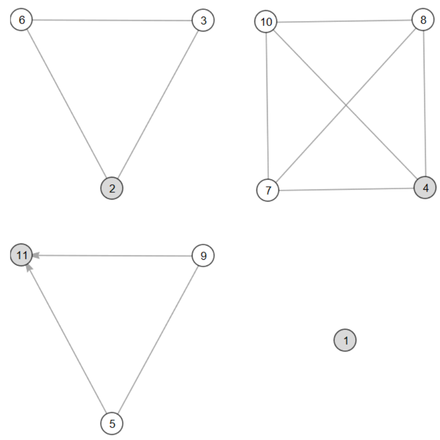

FIGURE 5.14: Graph of families of 4D Lie subalgebras of the algebra described in Section 5.6, where $1 \to \{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$, $2 \to \{\Xi_1, \Xi_2, \Xi_3, \Xi_6\}$, $4 \to \{\Xi_1, \Xi_2, \Xi_5, \Xi_6\}$, $11 \to \{\Xi_1, \Xi_2, \Xi_3 + \alpha_1\Xi_6, \Xi_4 + a_1\Xi_6\}$.

The `SymboLie` *package also allows the classes of different connected components to be displayed using the* `PrintClasses[]` *method. This method returns a list containing two elements: the first is the set of the optimal system, and the second represents the classes of the connected components of the graph generated from the adjacency matrix. For the heat equation, we obtain the following classes:*

$$
\begin{aligned}
&\{\{\{\Xi_1\}, \{\Xi_6\}\}, \{\{\Xi_2\}, \{\Xi_2 + \alpha_1\Xi_4\}, \{\Xi_5\}, \{\Xi_2 + \alpha_1\Xi_5\}, \{\Xi_4 + \alpha_1\Xi_5\},\\
&\ \{\Xi_2 + \alpha_1\Xi_4 + \alpha_2\Xi_5\}\}, \{\{\Xi_3\}, \{\Xi_1 + \alpha_1\Xi_3\}, \{\Xi_2 + \alpha_1\Xi_3\}, \{\Xi_1 + \alpha_1\Xi_2 + \alpha_2\Xi_3\},\\
&\ \{\Xi_3 + \alpha_1\Xi_5\}, \{\Xi_3 + \alpha_1\Xi_6\}, \{\Xi_3 + \alpha_1\Xi_5 + \alpha_2\Xi_6\}\}, \{\{\Xi_4\}\}, \{\{\Xi_1 + \alpha_1\Xi_2\},\\
&\ \{\Xi_1 + \alpha_1\Xi_4\}, \{\Xi_1 + \alpha_1\Xi_2 + \alpha_2\Xi_4\}, \{\Xi_4 + \alpha_1\Xi_6\}, \{\Xi_5 + \alpha_1\Xi_6\},\\
&\ \{\Xi_4 + \alpha_1\Xi_5 + \alpha_2\Xi_6\}\}, \{\{\Xi_3 + a_1\Xi_4\}, \{\Xi_1 + \alpha_1\Xi_3 + \alpha_2\Xi_4\}, \{\Xi_2 + \alpha_1\Xi_3 + \alpha_2\Xi_4\},\\
&\ \{\Xi_1 + \alpha_1\Xi_2 + \alpha_2\Xi_3 + a_1\Xi_4\}, \{\Xi_1 + \alpha_1\Xi_3 + a_1\Xi_5\}, \{\Xi_2 + \alpha_1\Xi_3 + a_1\Xi_5\},\\
&\ \{\Xi_1 + \alpha_1\Xi_2 + \alpha_2\Xi_3 + a_1\Xi_5\}, \{\Xi_3 + \alpha_1\Xi_5 + a_1\Xi_4\}, \{\Xi_1 + \alpha_1\Xi_3 + a_1\Xi_4 + a_2\Xi_5\},\\
&\ \{\Xi_2 + \alpha_1\Xi_3 + \alpha_2\Xi_4 + \alpha_3\Xi_5\}, \{\Xi_1 + \alpha_1\Xi_2 + \alpha_2\Xi_3 + \alpha_3\Xi_4 + \alpha_4\Xi_5\},\\
&\ \{\Xi_2 + \alpha_1\Xi_3 + a_1\Xi_6\}, \{\Xi_3 + \alpha_1\Xi_6 + a_1\Xi_4\}, \{\Xi_2 + \alpha_1\Xi_3 + a_1\Xi_4 + a_2\Xi_6\},\\
&\ \{\Xi_2 + \alpha_1\Xi_3 + \alpha_2\Xi_5 + a_1\Xi_6\}, \{\Xi_3 + \alpha_1\Xi_5 + \alpha_2\Xi_6 + a_1\Xi_4\},\\
&\ \{\Xi_2 + \alpha_1\Xi_3 + \alpha_2\Xi_4 + \alpha_3\Xi_5 + \alpha_4\Xi_6\}\}, \{\{\Xi_1 + \alpha_1\Xi_5\}, \{\Xi_1 + \alpha_1\Xi_2 + a_1\Xi_5\},\\
&\ \{\Xi_1 + \alpha_1\Xi_4 + a_1\Xi_5\}, \{\Xi_1 + \alpha_1\Xi_2 + \alpha_2\Xi_4 + a_1\Xi_5\}, \{\Xi_2 + \alpha_1\Xi_6\}, \{\Xi_2 + \alpha_1\Xi_4 + a_1\Xi_6\},\\
&\ \{\Xi_2 + \alpha_1\Xi_5 + a_1\Xi_6\}, \{\Xi_2 + \alpha_1\Xi_4 + \alpha_2\Xi_5 + a_1\Xi_6\}\}, \{\{\Xi_1 + \alpha_1\Xi_6\}\},\\
&\ \{\{\Xi_1 + \alpha_1\Xi_2 + a_1\Xi_6\}, \{\Xi_1 + \alpha_1\Xi_3 + a_1\Xi_6\}, \{\Xi_1 + \alpha_1\Xi_2 + a_1\Xi_3 + a_2\Xi_6\},\\
&\ \{\Xi_1 + \alpha_1\Xi_4 + a_1\Xi_6\}, \{\Xi_1 + \alpha_1\Xi_2 + \alpha_2\Xi_4 + a_1\Xi_6\},\\
&\ \{\Xi_1 + \alpha_1\Xi_3 + \alpha_2\Xi_4 + \alpha_3\Xi_6\}, \{\Xi_1 + \alpha_1\Xi_2 + \alpha_2\Xi_3 + a_1\Xi_4 + a_2\Xi_6\},\\
&\ \{\Xi_1 + \alpha_1\Xi_5 + a_1\Xi_6\}, \{\Xi_1 + \alpha_1\Xi_2 + a_1\Xi_5 + a_2\Xi_6\}, \{\Xi_1 + \alpha_1\Xi_3 + a_1\Xi_5 + a_2\Xi_6\},\\
&\ \{\Xi_1 + \alpha_1\Xi_2 + a_1\Xi_3 + a_2\Xi_5 + a_3\Xi_6\}, \{\Xi_1 + \alpha_1\Xi_4 + \alpha_2\Xi_5 + a_1\Xi_6\},\\
&\ \{\Xi_1 + \alpha_1\Xi_2 + \alpha_2\Xi_4 + a_1\Xi_5 + a_2\Xi_6\}, \{\Xi_1 + \alpha_1\Xi_3 + \alpha_2\Xi_4 + a_1\Xi_5 + a_2\Xi_6\},\\
&\ \{\Xi_1 + \alpha_1\Xi_2 + \alpha_2\Xi_3 + \alpha_3\Xi_4 + \alpha_4\Xi_5 + \alpha_5\Xi_6\}\}\}.
\end{aligned}
$$

In fact, it can be observed that the operators Ξ_1 *and* Ξ_6 *are in the same class of equivalence.*

Comparing the results recovered by `SymboLie` with the ones found in the literature, some comments are in order:

- in [11, 19, 36, 37] is exhibited a system of optimal 1D subalgebras with the same cardinality, but with different representatives;
- in the graphs representing the relation $\mathcal{R}$ for families of 1D and 2D subalgebras, there are unidirectional edges: this occurs because some families cannot be *entirely* mapped to others. For example, the 1D 2–family $X = (f_1, f_2, 0, 0, 0, 0)$ (label 7 in the graph) is mapped via $A = \exp(t\, ad_{\Xi_5})$ into $(f_1, 0, 0, \frac{f_2^2}{4f_1}, 0, 0)$, which satisfies Definition 23, so that we identify it with the 2–family $Y = (f_1, 0, 0, f_4, 0, 0)$ (label 10). Vice versa, acting on Y with the inner automorphisms A gives $(f_1, 2f_1 t, 0, f_4 - f_1 t^2, 0, 0)$, that cannot be reduced to the family X for all choices of f_1 and f_4;
- compared to the 2D–dimensional optimal system found in [19], `SymboLie` also returns the 2-families $\{\Xi_3, \Xi_5\}$ and $\{\Xi_3, \Xi_6\}$. It can be seen that they can be mapped into a subfamily of $\{\Xi_2, \Xi_3 + a_1\Xi_4\}$ and $\{\Xi_1, \Xi_3 + a_1\Xi_4\}$, respectively, so that it does not satisfy Definition 26;
- the 3D optimal system discussed in [19] includes the Lie subalgebra $\{\Xi_1, \Xi_3 - \frac{1}{2}\Xi_4, \Xi_6\}$. Since `SymboLie` deals with p–familes of subalgebras, it does not include $\{\Xi_1, \Xi_3 + a_1\Xi_4, \Xi_6\}$, with $a_1 \neq 0$, among the candidates, since for a_1 arbitrary the closure with respect to the Lie bracket is not guaranteed. Furthermore, the family $\{\Xi_3, \Xi_5, \Xi_6\}$ does not appear in the optimal system of [19]. In fact, it is mapped into a subfamily of $\{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4\}$, and so `SymboLie` does not consider them belonging to the relation;
- the optimal systems of 4D and 5D subalgebras computed by `SymboLie` coincide with those recovered in [19].

5.8 Optimal systems of symmetries of $(2+1)$-dimensional ZK-BBM equation

Let $\mathcal{L}_5$ be the Lie algebra of continuous symmetries of $(2+1)$-dimensional Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZK-BBM) equation

$$u_t + u_x - a(u^2)_x - (bu_{xt} + ku_{yt})_x = 0$$

spanned by the vector fields:

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial x}, \quad \Xi_2 = \frac{\partial}{\partial y}, \quad \Xi_3 = \frac{\partial}{\partial t}, \\
&\Xi_4 = \left(-x + \frac{2b}{k}y\right)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \left(\frac{1}{2a} - u\right)\frac{\partial}{\partial u}, \\
&\Xi_5 = t\frac{\partial}{\partial t} + \left(\frac{1}{2a} - u\right)\frac{\partial}{\partial u},
\end{aligned}$$

where a, b and k are real constants.

Recently, Tanwar [29] computed the optimal one-dimensional Lie subalgebras of the Lie algebra of symmetries of $(2+1)$-dimensional ZK-BBM equation with the

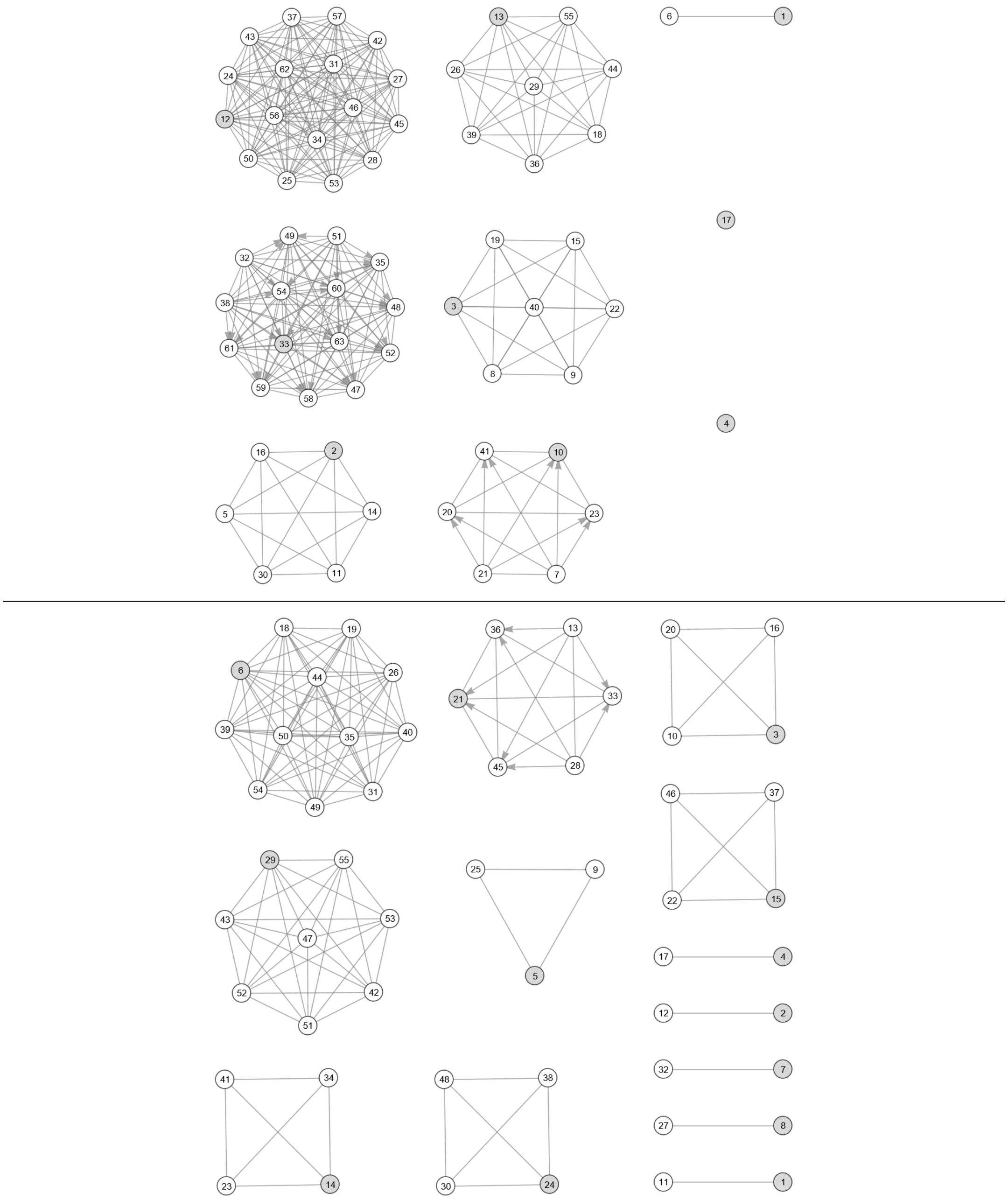

FIGURE 5.15: Algebra of symmetries of Heat equation: graph of families of 1D Lie subalgebras (top), where $1 \to \{\Xi_1\}$, $2 \to \{\Xi_2\}$, $3 \to \{\Xi_3\}$, $4 \to \{\Xi_4\}$, $10 \to \{\Xi_1 + \alpha_1\Xi_4\}$, $12 \to \{\Xi_3 + a_1\Xi_4\}$, $13 \to \{\Xi_1 + \alpha_1\Xi_5\}$, $17 \to \{\Xi_1 + \alpha_1\Xi_6\}$, $33 \to \{\Xi_1 + \alpha_1\Xi_3 + a_1\Xi_6\}$, and of 2D Lie subalgebras (bottom), where $1 \to \{\Xi_1, \Xi_2\}$, $2 \to \{\Xi_1, \Xi_3\}$, $3 \to \{\Xi_1, \Xi_4\}$, $4 \to \{\Xi_2, \Xi_3\}$, $5 \to \{\Xi_2, \Xi_4\}$, $6 \to \{\Xi_3, \Xi_4\}$, $7 \to \{\Xi_3, \Xi_5\}$, $8 \to \{\Xi_3, \Xi_6\}$, $21 \to \{\Xi_1 + \alpha_1\Xi_4, \Xi_2\}$, $14 \to \{\Xi_1, \Xi_3 + a_1\Xi_4\}$, $15 \to \{\Xi_2, \Xi_3 + a_1\Xi_4\}$, $24 \to \{\Xi_1 + \alpha_1\Xi_5, \Xi_4\}$, $29 \to \{\Xi_1 + \alpha_1\Xi_6, \Xi_4\}$

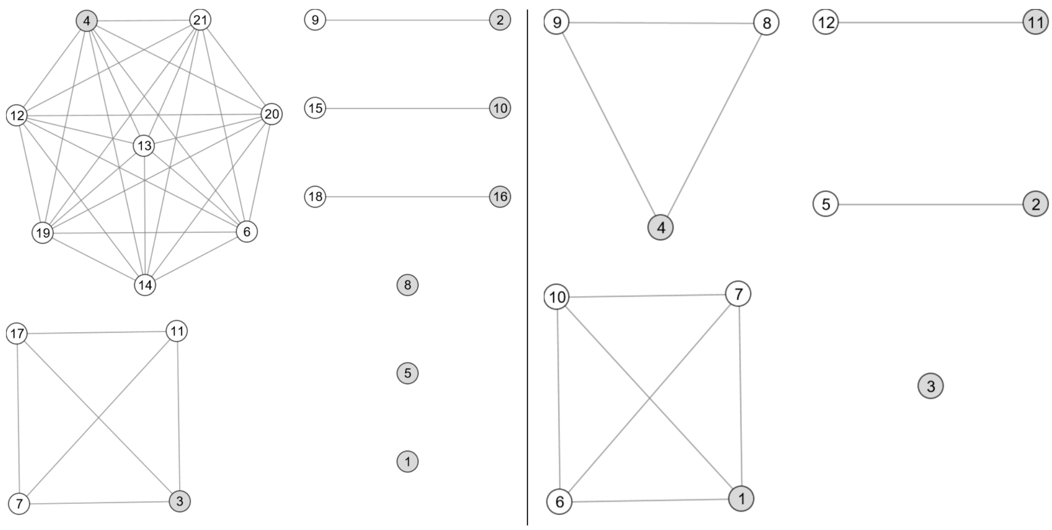

FIGURE 5.16: Algebra of symmetries of Heat equation: graph of families of 3D Lie subalgebras (left), where $1 \to \{\Xi_1, \Xi_2, \Xi_3\}$, $2 \to \{\Xi_1, \Xi_2, \Xi_4\}$, $3 \to \{\Xi_1, \Xi_3, \Xi_4\}$, $4 \to \{\Xi_2, \Xi_3, \Xi_4\}$, $5 \to \{\Xi_2, \Xi_4, \Xi_5\}$, $8 \to \{\Xi_3, \Xi_5, \Xi_6\}$, $10 \to \{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4\}$, $16 \to \{\Xi_1 + \alpha_1\Xi_5, \Xi_2, \Xi_4\}$, and of 4D Lie subalgebras (right), where $1 \to \{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$, $2 \to \{\Xi_1, \Xi_2, \Xi_4, \Xi_5\}$, $3 \to \{\Xi_1, \Xi_3, \Xi_4, \Xi_6\}$, $4 \to \{\Xi_2, \Xi_3, \Xi_4, \Xi_5\}$, $11 \to \{\Xi_1 + \alpha_1\Xi_6, \Xi_2, \Xi_4, \Xi_5\}$.

aim of determining group-invariant solutions. The use of `SymboLie` gives the classification of one-dimensional Lie subalgebras in few seconds, and the results, though slightly different in the choice of some representatives of the equivalence classes from those reported in [29], coincide. Moreover, our program is able to compute the complete system of optimal Lie subalgebras in about two minutes. The results are listed below:

$$
\begin{aligned}
\Theta_A^1 \equiv \{ & \{\Xi_1\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_5\}, \{\Xi_1 + \alpha_1\Xi_2\}, \{\Xi_1 + \alpha_1\Xi_3\}, \\
& \{\Xi_1 + \alpha_1\Xi_2 + \alpha_2\Xi_3\}, \{\Xi_3 + \alpha_1\Xi_4\}, \{\Xi_1 + \alpha_1\Xi_5\}, \\
& \{\Xi_1 + \alpha_1\Xi_2 + a_1\Xi_5\}, \{\Xi_4 + a_1\Xi_5\}\}, \\
\Theta_A^2 \equiv \{ & \{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_5\}, \{\Xi_1 + \alpha_1\Xi_2, \Xi_3\}, \\
& \{\Xi_1 + \alpha_1\Xi_2, \Xi_5\}, \{\Xi_3, \Xi_4\}, \{\Xi_3, \Xi_5\}, \{\Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_3\}, \\
& \{\Xi_1, \Xi_3 + \alpha_1\Xi_4\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_5\}, \{\Xi_1, \Xi_4 + a_1\Xi_5\}, \{\Xi_3, \Xi_4 + a_1\Xi_5\}, \\
& \{\Xi_1 + \alpha_1\Xi_3, \Xi_2 + \alpha_2\Xi_3\}, \{\Xi_1 + \alpha_1\Xi_5, \Xi_2 + a_1\Xi_5\}, \{\Xi_1 + \alpha_1\Xi_5, \Xi_3\}, \\
& \{\Xi_1 + \alpha_1\Xi_2 + a_1\Xi_5, \Xi_3\}\}, \\
\Theta_A^3 \equiv \{ & \{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_5\}, \\
& \{\Xi_1, \Xi_4, \Xi_5\}, \{\Xi_1 + \alpha_1\Xi_2, \Xi_3, \Xi_5\}, \{\Xi_3, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_3 + \alpha_1\Xi_4\}, \\
& \{\Xi_1, \Xi_2, \Xi_4 + a_1\Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4 + a_1\Xi_5\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_5, \Xi_3\}, \\
& \{\Xi_1 + \alpha_1\Xi_5, \Xi_2 + a_1\Xi_5, \Xi_3\}\}, \\
\Theta_A^4 \equiv \{ & \{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4, \Xi_5\}, \\
& \{\Xi_1, \Xi_2, \Xi_3, \Xi_4 + a_1\Xi_5\}\},
\end{aligned}
$$

where $a_1 \neq 0$, and $\alpha_1, \alpha_2 \in \{-1, 1\}$.

Chapter 6

Hierarchy of coupled Burgers–like equations

In this Chapter, we consider a hierarchy of systems of coupled Burgers–like equations possessing a Lie algebra of point symmetries isomorphic to the Lie algebra of symmetries admitted by classical Burgers' equation. The hierarchy has its starting point in the investigation of nonclassical symmetries [68] of Burgers' equation [69].

6.1 Hierarchy of coupled Burgers–like equations

We have seen that the Lie algebra of classical symmetries of Burgers' equation, generated by the operators (5.2), is a five–dimensional. The corresponding commutator table is displayed in Table 6.1.

	Ξ_1	Ξ_2	Ξ_3	Ξ_4	Ξ_5
Ξ_1	0	0	Ξ_2	$2\Xi_1$	Ξ_4
Ξ_2	0	0	0	Ξ_2	Ξ_3
Ξ_3	$-\Xi_2$	0	0	$-\Xi_3$	0
Ξ_4	$-2\Xi_1$	$-\Xi_2$	Ξ_3	0	$2\Xi_5$
Ξ_5	$-\Xi_4$	$-\Xi_3$	0	$-2\Xi_5$	0

TABLE 6.1: Commutator table of the Lie algebra of symmetries of viscous Burgers' equation.

Now, let us introduce the following hierarchy of systems of coupled Burgers–like equations. These systems are indexed by two integers: $m \in \mathbb{N}$ and $k = \lceil m/2 \rceil$. For $m = 1$, we obtain the classical Burgers' equation

$$\mathbf{\Delta}_1 \equiv u^{(1)}_{1,t} + u^{(1)}_1 u^{(1)}_{1,x} - u^{(1)}_{1,xx} = 0.$$

For $m = 2$, whereupon $k = 1$, we obtain a system of two coupled Burgers–like equations:

$$\mathbf{\Delta}_2 \equiv \begin{cases} u^{(1)}_{1,t} + u^{(1)}_1 u^{(1)}_{1,x} - u^{(1)}_{1,xx} + u^{(1)}_{2,x} = 0, \\ u^{(1)}_{2,t} + u^{(1)}_2 u^{(1)}_{1,x} - u^{(1)}_{2,xx} = 0. \end{cases}$$

For $m = 3$, whereupon $k = 2$, we obtain a system of three coupled Burgers–like equations:

$$\mathbf{\Delta}_3 \equiv \begin{cases} u^{(2)}_{1,t} + u^{(2)}_1 u^{(2)}_{1,x} - u^{(2)}_{1,xx} + u^{(2)}_{2,x} = 0, \\ u^{(2)}_{2,t} + u^{(2)}_2 u^{(2)}_{1,x} - u^{(2)}_{2,xx} + u^{(2)}_{3,x} = 0, \\ u^{(2)}_{3,t} + u^{(2)}_3 u^{(2)}_{1,x} - u^{(2)}_{3,xx} = 0. \end{cases}$$

And so on. In general, we have a system as follows:

$$\boldsymbol{\Delta}_m \equiv \begin{cases} u^{(k)}_{\alpha,t} + u^{(k)}_{\alpha} u^{(k)}_{1,x} - u^{(k)}_{\alpha,xx} + u^{(k)}_{\alpha+1,x} = 0, \\ u^{(k)}_{m,t} + u^{(k)}_{m} u^{(k)}_{1,x} - u^{(k)}_{m,xx} = 0, \end{cases} \tag{6.1}$$

where $\alpha = 1, \dots, m-1$.

We can observe that the classical Lie point symmetries of a generic element of this infinite hierarchy span a five-dimensional Lie algebra, and the following proposition holds true.

Proposition 7. *Let m be a positive integer, and let $k = \lceil m/2 \rceil$. The system of Burgers-like equations (6.1) for $m = 1$ (classical Burgers' equation) admits the Lie point symmetries generated by:*

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x}, \\
&\Xi_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u^{(1)}_1 \frac{\partial}{\partial u^{(1)}_1}, \\
&\Xi_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u^{(1)}_1}, \\
&\Xi_5 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + (x - tu^{(1)}_1)\frac{\partial}{\partial u^{(1)}_1};
\end{aligned}$$

for $m = 2$ the Lie point symmetries generated by:

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x}, \\
&\Xi_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u^{(1)}_1 \frac{\partial}{\partial u^{(1)}_1} - 2u^{(1)}_2 \frac{\partial}{\partial u^{(1)}_2}, \\
&\Xi_4 = t\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u^{(1)}_1} - u^{(1)}_1 \frac{\partial}{\partial u^{(1)}_2}, \\
&\Xi_5 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + (2x - tu^{(1)}_1)\frac{\partial}{\partial u^{(1)}_1} - (xu^{(1)}_1 + 2tu^{(1)}_2 + 2)\frac{\partial}{\partial u^{(1)}_2};
\end{aligned}$$

for $m \geq 3$ the Lie point symmetries generated by:

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x}, \\
&\Xi_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \sum_{\alpha=1}^{m} \alpha u^{(k)}_\alpha \frac{\partial}{\partial u^{(k)}_\alpha}, \\
&\Xi_4 = t\frac{\partial}{\partial x} + m\frac{\partial}{\partial u^{(k)}_1} + \sum_{\alpha=2}^{m} (\alpha - m - 1) u^{(k)}_{\alpha-1} \frac{\partial}{\partial u^{(k)}_\alpha}, \\
&\Xi_5 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + \left(mx - tu^{(k)}_1\right)\frac{\partial}{\partial u^{(k)}_1} - \left((m-1)(xu^{(k)}_1 + m) + 2tu^{(k)}_2\right)\frac{\partial}{\partial u^{(k)}_2} \\
&\qquad - \sum_{\alpha=3}^{m} \left(\alpha t u^{(k)}_\alpha + (m - \alpha + 1)\left(xu^{(k)}_{\alpha-1} - (m - \alpha + 2)u^{(k)}_{\alpha-2}\right)\right)\frac{\partial}{\partial u^{(k)}_\alpha}.
\end{aligned}$$

Whatever the number m of coupled equations is, we have always a five-dimensional Lie algebra (time and space translation, scaling, Galilean and projective transformation, respectively); these Lie algebras, although realized in terms of vector fields on manifolds with different dimensionality, share the same structure constants and so they are all isomorphic.

Actually, this hierarchy originates from the analysis of the conditional symmetries of the Burgers' equation and the recursive investigation of their conditional symmetries as they are successively applied to the resulting systems.

6.2 Brief review on conditional symmetries

Let us consider an rth order differential equation, say

$$\Delta\left(x_i, u_\alpha, u_{\alpha,i}, \ldots, u_{\alpha,i_1,\ldots,i_r}\right) = 0, \tag{6.2}$$

where x_i $(i = 1, \ldots, n)$ are the independent variables, u_α $(\alpha = 1, \ldots, m)$ the dependent variables, and $u_{\alpha,i_1,\ldots,i_k} = \dfrac{\partial^k u_\alpha}{\partial x_{i_1} \ldots \partial x_{i_k}}$ $(k = 1, \ldots, r)$.

As pointed out in Chapter 2, Lie point symmetry of (6.2) is characterized by the infinitesimal operator

$$\Xi = \sum_{i=1}^{n} \xi_i(x_j, u_\beta) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \eta_\alpha(x_j, u_\beta) \frac{\partial}{\partial u_\alpha} \tag{6.3}$$

such that

$$\Xi^{(r)}\left(\Delta\right)\Big|_{\Delta=0} = 0, \tag{6.4}$$

where $\Xi^{(r)}$ is the rth prolongation of (6.3) [11, 13, 48]. Condition (6.4) leads to a system of linear partial differential equations (*determining equations*) whose integration provides the *infinitesimals* ξ_i and η_α. Invariant solutions corresponding to a given Lie point symmetry are found by solving the invariant surface conditions

$$Q_\alpha \equiv \sum_{i=1}^{n} \xi_i(x_j, u_\beta) u_{\alpha,i} - \eta_\alpha(x_j, u_\beta) = 0, \qquad \alpha = 1, \ldots, m, \tag{6.5}$$

and inserting their solutions in (6.2).

In 1969, Bluman and Cole [68] introduced a generalization of classical Lie symmetries, and applied their method (called *nonclassical*) to the linear heat equation. The basic idea was that of imposing the invariance to a system made by the differential equation at hand, the invariance surface condition together with the differential consequences of the latter. This method requires to solve a set of *nonlinear* determining equations whose general integration is usually difficult. Nonclassical symmetries are now part of conditional symmetries, *i.e.*, symmetries of differential equations where some additional differential conditions are imposed to restrict the set of solutions. This method revealed useful in many applied problems modeled by differential equations (for instance, reaction-diffusion equations [70–72]) possessing very few Lie point symmetries; consequently, more rich reductions leading to wide classes of exact solutions are possible.

The nonclassical symmetries introduced by Bluman and Cole are now referred to as Q-conditional symmetries [72]. In such a case, Q-conditional symmetries are expressed by vector fields Ξ such that

$$\Xi^{(r)}(\Delta)\Big|_{\mathcal{M}} = 0, \tag{6.6}$$

where $\mathcal{M}$ is the manifold of the jet space defined by

$$\Delta = 0, \qquad Q_\alpha = 0, \qquad \frac{D}{Dx_{j_1}}\frac{D}{Dx_{j_2}}\cdots\frac{D}{Dx_{j_k}}Q_\alpha = 0, \tag{6.7}$$

with $1 \leq j_1, j_2, \dots, j_k \leq n$, $1 \leq k \leq r-1$, and $\alpha = 1, \dots, m$, along with the Lie derivative

$$\frac{D}{Dx_k} = \frac{\partial}{\partial x_k} + u_{\alpha,k}\frac{\partial}{\partial u_\alpha} + u_{\alpha,ik}\frac{\partial}{\partial u_{\alpha,i}} + u_{\alpha,ijk}\frac{\partial}{\partial u_{\alpha,ij}} + \cdots, \tag{6.8}$$

where the Einstein convention on sums over repeated indices has been used.

Trivially, a (classical) Lie symmetry is a Q-conditional symmetry. However, differently from Lie symmetries, all possible conditional symmetries of a differential equation form a set which is neither a Lie algebra nor a linear space in the general case. Furthermore, if the vector field of a Q-conditional symmetry is multiplied by an arbitrary nonvanishing smooth function of dependent and independent variables, we have still a Q-conditional symmetry.

In the following, we will be concerned with second order partial differential equations ruling the evolution of m unknown functions depending on t and x, and consider Q-conditional symmetries corresponding to the vector field

$$\Xi = \frac{\partial}{\partial t} + \xi(t, x, u_\beta)\frac{\partial}{\partial x} + \sum_{\alpha=1}^{m} \eta_\alpha(t, x, u_\beta)\frac{\partial}{\partial u_\alpha}. \tag{6.9}$$

Below it will be shown that, starting with the classical Burgers' equation or with a special pair of coupled Burgers-like equations, and looking for Q-conditional symmetries, an infinite hierarchy of systems of Burgers-like equations is recovered.

6.3 Conditional symmetries of Burgers-like equations

In this Section, we start considering the classical Burgers' equation [73]. In [69], it was proved that the Q-conditional symmetries of Burgers' equation are expressed in terms of three functions that are solutions of a system of coupled Burgers-like equations. In what follows, we prove that the latter system of coupled Burgers-like equations admits Q-conditional symmetries expressed in terms of five functions satisfying a new system of coupled Burgers-like equations. This process can be repeatedly used, and a hierarchy of systems involving an odd number of unknowns arises. Moreover, we prove also that, starting with a pair of coupled Burgers-like equations, another hierarchy of systems involving an even number of coupled Burgers-like equations is generated.

6.3.1 Hierarchy originating from Burgers' equation

Let us consider the Burgers' equation

$$\Delta_1 \equiv u^{(1)}_{,t} + u^{(1)}u^{(1)}_{,x} - u^{(1)}_{,xx} = 0 \tag{6.10}$$

for the unknown $u^{(1)}(t,x)$, and take the vector field

$$\Xi = \frac{\partial}{\partial t} + \xi(t,x,u^{(1)})\frac{\partial}{\partial x} + \eta(t,x,u^{(1)})\frac{\partial}{\partial u^{(1)}}. \tag{6.11}$$

In order to compute the Q-conditional symmetries of (6.10) associated to (6.11), let us define the manifold $\mathcal{M}_1$ as

$$\begin{cases} \Delta_1 = 0, \\ Q_1 \equiv \Xi\left(u^{(1)} - u^{(1)}(t,x)\right) = 0, \\ \dfrac{DQ_1}{Dt} = \dfrac{DQ_1}{Dx} = 0, \end{cases} \tag{6.12}$$

whence the conditional symmetries are found by requiring

$$\Xi^{(2)}(\Delta_1)\Big|_{\mathcal{M}_1} = 0.$$

The latter provides the following polynomial of third degree in the derivative $u^{(1)}_{,x}$

$$\begin{aligned}
&\frac{\partial^2\xi}{\partial u^{(1)2}}\left(u^{(1)}_{,x}\right)^3 + \left(2\frac{\partial^2\xi}{\partial x\partial u^{(1)}} - \frac{\partial^2\eta}{\partial u^{(1)2}} + 2\frac{\partial\xi}{\partial u^{(1)}}u^{(1)} - 2\xi\frac{\partial\xi}{\partial u^{(1)}}\right)\left(u^{(1)}_{,x}\right)^2 \\
&+ \left(\frac{\partial^2\xi}{\partial x^2} - 2\frac{\partial^2\eta}{\partial x\partial u^{(1)}} - \frac{\partial\xi}{\partial t} + \frac{\partial\xi}{\partial x}u^{(1)} - 2\xi\frac{\partial\xi}{\partial x} + 2\frac{\partial\xi}{\partial u^{(1)}}\eta + \eta\right)u^{(1)}_{,x} \\
&- \frac{\partial^2\eta}{\partial x^2} + 2\frac{\partial\xi}{\partial x}\eta + \frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial x}u^{(1)} = 0.
\end{aligned}$$

Annihilating the coefficients of this polynomial, after simple algebra, we get

$$\begin{aligned}
\xi &= \kappa u^{(1)} + \frac{1}{2}u^{(2)}_1, \\
\eta &= \frac{\kappa(1-\kappa)}{3}\left(u^{(1)}\right)^3 - \frac{\kappa}{2}u^{(2)}_1\left(u^{(1)}\right)^2 + \frac{1}{4}u^{(1)}u^{(2)}_2 + \frac{1}{4}u^{(2)}_3,
\end{aligned} \tag{6.13}$$

where κ is a constant such that $\kappa(\kappa-1)(2\kappa+1) = 0$, whereas $u^{(2)}_1(t,x)$, $u^{(2)}_2(t,x)$ and $u^{(2)}_3(t,x)$ are functions depending on the indicated arguments.

Three cases must be distinguished, say $\kappa = 0$, $\kappa = 1$ and $\kappa = -1/2$, the latter being the most interesting one. In fact, if $\kappa = 0$ then it is simple to prove that the functions $u^{(2)}_1$, $u^{(2)}_2$ and $u^{(2)}_3$ involved in (6.13) exhibit the following form:

$$\begin{aligned}
u^{(2)}_1(t,x) &= \frac{(\alpha t+\beta)x + \gamma t + \delta}{\alpha t^2 + 2\beta t + \gamma}, \\
u^{(2)}_2(t) &= -\frac{\alpha t + \beta}{\alpha t^2 + 2\beta t + \gamma}, \\
u^{(2)}_3(t,x) &= \frac{\alpha x + \gamma}{\alpha t^2 + 2\beta t + \gamma},
\end{aligned}$$

α, β, γ and δ being arbitrary constants. In this case, it is not difficult to recognize that the symmetry reductions are those provided by classical symmetries of Burgers' equation. Moreover, if $\kappa = 1$, then it is easily obtained $u^{(2)}_1 = u^{(2)}_2 = u^{(2)}_3 = 0$, and also in this case it the corresponding symmetry reductions provide results that can be recovered within the context of classical symmetries.

On the contrary, when $\kappa = -1/2$, the functions $u_1^{(2)}(t,x)$, $u_2^{(2)}(t,x)$ and $u_3^{(2)}(t,x)$ satisfy the system

$$\boldsymbol{\Delta}_3 \equiv \begin{cases} u^{(2)}_{1,t} + u^{(2)}_1 u^{(2)}_{1,x} - u^{(2)}_{1,xx} + u^{(2)}_{2,x} = 0, \\ u^{(2)}_{2,t} + u^{(2)}_2 u^{(2)}_{1,x} - u^{(2)}_{2,xx} + u^{(2)}_{3,x} = 0, \\ u^{(2)}_{3,t} + u^{(2)}_3 u^{(2)}_{1,x} - u^{(2)}_{3,xx} = 0. \end{cases} \tag{6.14}$$

As already remarked, this result has been obtained in [69].

Then, it can be interesting to explore the Q-conditional symmetries admitted by the system (6.14). As a result, the following Proposition is proved.

Proposition 8. *The vector field*

$$\Xi = \frac{\partial}{\partial t} + \xi(t,x,u^{(2)}_\beta)\frac{\partial}{\partial x} + \sum_{\alpha=1}^{3} \eta_\alpha(t,x,u^{(2)}_\beta)\frac{\partial}{\partial u^{(2)}_\alpha}, \tag{6.15}$$

gives a Q-conditional symmetry of the system (6.14) *provided that*

$$\begin{aligned}
\xi &= \frac{1}{2}\left(-u^{(2)}_1 + u^{(3)}_1\right), \\
\eta_1 &= \frac{1}{4}\left(-\left(u^{(2)}_1\right)^3 - 2u^{(2)}_1 u^{(2)}_2 + u^{(3)}_1\left(u^{(2)}_1\right)^2 + u^{(3)}_2 u^{(2)}_1 + u^{(3)}_1 u^{(2)}_2 - u^{(2)}_3 + u^{(3)}_3\right), \\
\eta_2 &= \frac{1}{4}\left(-\left(u^{(2)}_1\right)^2 u^{(2)}_2 - u^{(2)}_1 u^{(2)}_3 - \left(u^{(2)}_2\right)^2 + u^{(3)}_1 u^{(2)}_1 u^{(2)}_2 + u^{(3)}_2 u^{(2)}_2 + u^{(3)}_1 u^{(2)}_3 + u^{(3)}_4\right), \\
\eta_3 &= \frac{1}{4}\left(-\left(u^{(2)}_1\right)^2 u^{(2)}_3 - u^{(2)}_2 u^{(2)}_3 + u^{(3)}_1 u^{(2)}_1 u^{(2)}_3 + u^{(3)}_2 u^{(2)}_3 + u^{(3)}_5\right),
\end{aligned} \tag{6.16}$$

where the functions $u^{(3)}_\alpha(t,x)$ ($\alpha = 1,\dots,5$) satisfy the constraints

$$\boldsymbol{\Delta}_5 \equiv \begin{cases} u^{(3)}_{1,t} + u^{(3)}_1 u^{(3)}_{1,x} - u^{(3)}_{1,xx} + u^{(3)}_{2,x} = 0, \\ u^{(3)}_{2,t} + u^{(3)}_2 u^{(3)}_{1,x} - u^{(3)}_{2,xx} + u^{(3)}_{3,x} = 0, \\ u^{(3)}_{3,t} + u^{(3)}_3 u^{(3)}_{1,x} - u^{(3)}_{3,xx} + u^{(3)}_{4,x} = 0, \\ u^{(3)}_{4,t} + u^{(3)}_4 u^{(3)}_{1,x} - u^{(3)}_{4,xx} + u^{(3)}_{5,x} = 0, \\ u^{(3)}_{5,t} + u^{(3)}_5 u^{(3)}_{1,x} - u^{(3)}_{5,xx} = 0. \end{cases} \tag{6.17}$$

Proof. The proof immediately follows by requiring

$$\Xi^{(2)}(\boldsymbol{\Delta}_3)\Big|_{\mathcal{M}_3} = \mathbf{0}, \tag{6.18}$$

where $\mathcal{M}_3$ is the manifold of the jet space defined by the system (6.14) together with the invariant surface conditions and their differential consequences, say

$$\begin{cases} \boldsymbol{\Delta}_3 = \mathbf{0}, \\ Q_\alpha \equiv \Xi\left(u^{(2)}_\alpha - u^{(2)}_\alpha(t,x)\right) = 0, \qquad \alpha = 1,2,3, \\ \dfrac{DQ_\alpha}{Dt} = \dfrac{DQ_\alpha}{Dx} = 0. \end{cases} \tag{6.19}$$

The lengthy computations can be done by using the program `ReLie` [67] written in the Computer Algebra System *Reduce* [66]. As a result, we obtain three polynomials of third degree in the derivatives $u^{(2)}_{\alpha,x}$. We immediately obtain

$$\xi = \kappa u_1^{(2)} + \frac{1}{2}u_1^{(3)}, \tag{6.20}$$

$u_1^{(3)}(t,x)$ being a function of the indicated arguments, and κ is a constant that has to satisfy the condition

$$\kappa(2\kappa+1) = 0. \tag{6.21}$$

The most interesting case again corresponds to the choice $\kappa = -1/2$. After straightforward though tedious computations, all the determining equations can be solved, and the vector field (6.15) assumes the form (6.16), along with the functions $u_\alpha^{(3)}(t,x)$ $(\alpha = 1,\dots,5)$ that satisfy the system of differential equations (6.17). □

Remark 19. *We note that the system (6.17) has the same structure as the system (6.14), even if it involves two more unknowns.*

Nothing prevents us to repeat the procedure looking for Q-conditional symmetries of system (6.17). The result we obtain is stated with the following Proposition.

Proposition 9. *There exist Q-conditional symmetries of (6.17) in correspondence to the vector field*

$$\Xi = \frac{\partial}{\partial t} + \xi(t,x,u_\beta^{(3)})\frac{\partial}{\partial x} + \sum_{\alpha=1}^{5}\eta_\alpha(t,x,u_\beta^{(3)})\frac{\partial}{\partial u_\alpha^{(3)}}, \tag{6.22}$$

where

$$\begin{aligned}
\xi &= \frac{1}{2}\left(-u_1^{(3)} + u_1^{(4)}\right),\\
\eta_1 &= \frac{1}{4}\left(-\left(u_1^{(3)}\right)^3 - 2u_1^{(3)}u_2^{(3)} + u_1^{(4)}\left(u_1^{(3)}\right)^2 + u_2^{(4)}u_1^{(3)} + u_1^{(4)}u_2^{(3)} - u_3^{(3)} + u_3^{(4)}\right),\\
\eta_2 &= \frac{1}{4}\left(-\left(u_1^{(3)}\right)^2u_2^{(3)} - u_1^{(3)}u_3^{(3)} - \left(u_2^{(3)}\right)^2 + u_1^{(4)}u_1^{(3)}u_2^{(3)} + u_2^{(4)}u_2^{(3)} + u_1^{(4)}u_3^{(3)} - u_4^{(3)} + u_4^{(4)}\right),\\
\eta_3 &= \frac{1}{4}\left(-\left(u_1^{(3)}\right)^2u_3^{(3)} - u_1^{(3)}u_4^{(3)} - u_2^{(3)}u_3^{(3)} + u_1^{(4)}u_1^{(3)}u_3^{(3)} + u_2^{(4)}u_3^{(3)} + u_1^{(4)}u_4^{(3)} - u_5^{(3)} + u_5^{(4)}\right),\\
\eta_4 &= \frac{1}{4}\left(-\left(u_1^{(3)}\right)^2u_4^{(3)} - u_1^{(3)}u_5^{(3)} - u_2^{(3)}u_4^{(3)} + u_1^{(4)}u_1^{(3)}u_4^{(3)} + u_2^{(4)}u_4^{(3)} + u_1^{(4)}u_5^{(3)} + u_6^{(4)}\right),\\
\eta_5 &= \frac{1}{4}\left(-\left(u_1^{(3)}\right)^2u_5^{(3)} - u_2^{(3)}u_5^{(3)} + u_1^{(4)}u_1^{(3)}u_5^{(3)} + u_2^{(4)}u_5^{(3)} + u_7^{(4)}\right),
\end{aligned} \tag{6.23}$$

and the functions $u_\alpha^{(4)}(t,x)$ $(\alpha = 1,\dots,7)$ *satisfy the system*

$$\mathbf{\Delta}_7 \equiv \begin{cases} u_{1,t}^{(4)} + u_1^{(4)}u_{1,x}^{(4)} - u_{1,xx}^{(4)} + u_{2,x}^{(4)} = 0, \\ u_{2,t}^{(4)} + u_2^{(4)}u_{1,x}^{(4)} - u_{2,xx}^{(4)} + u_{3,x}^{(4)} = 0, \\ u_{3,t}^{(4)} + u_3^{(4)}u_{1,x}^{(4)} - u_{3,xx}^{(4)} + u_{4,x}^{(4)} = 0, \\ u_{4,t}^{(4)} + u_4^{(4)}u_{1,x}^{(4)} - u_{4,xx}^{(4)} + u_{5,x}^{(4)} = 0, \\ u_{5,t}^{(4)} + u_5^{(4)}u_{1,x}^{(4)} - u_{5,xx}^{(4)} + u_{6,x}^{(4)} = 0, \\ u_{6,t}^{(4)} + u_6^{(4)}u_{1,x}^{(4)} - u_{6,xx}^{(4)} + u_{7,x}^{(4)} = 0, \\ u_{7,t}^{(4)} + u_7^{(4)}u_{1,x}^{(4)} - u_{7,xx}^{(4)} = 0. \end{cases} \tag{6.24}$$

Proof. The proof requires only straightforward though lengthy computations. Also in this case the Reduce program `ReLie` has been used. □

The results heretofore obtained can be summarized as follows:

- there are Q-conditional symmetries of the Burgers' equation expressed in terms of three functions representing arbitrary solutions of the system $\mathbf{\Delta}_3$ made of three coupled Burgers-like equations;
- there are Q-conditional symmetries of $\mathbf{\Delta}_3$ expressed in terms of five functions representing arbitrary solutions of the system $\mathbf{\Delta}_5$ made of five coupled Burgers-like equations;
- there are Q-conditional symmetries of $\mathbf{\Delta}_5$ expressed in terms of seven functions representing arbitrary solutions of the system $\mathbf{\Delta}_7$ made of seven coupled Burgers-like equations.

It seems natural to conjecture that repeatedly searching for Q-conditional symmetries, and starting from the classical Burgers' equation, a hierarchy of systems made of an odd number of Burgers-like equations may arise.

In the next Subsection, we shall consider the case of a coupled system made of an even number of Burgers-like equations. In particular, the starting point will be the system of two Burgers-like equations whose structure is deduced from (6.14) where we set $u_3^{(2)} \equiv 0$.

6.3.2 Hierarchy originating from a pair of coupled Burgers-like equations

Let us consider the following system made of two coupled Burgers-like equations

$$\mathbf{\Delta}_2 \equiv \begin{cases} u_{1,t}^{(1)} + u_1^{(1)}u_{1,x}^{(1)} - u_{1,xx}^{(1)} + u_{2,x}^{(1)} = 0, \\ u_{2,t}^{(1)} + u_2^{(1)}u_{1,x}^{(1)} - u_{2,xx}^{(1)} = 0. \end{cases} \tag{6.25}$$

By looking for Q-conditional symmetries of (6.25) in correspondence to the vector field

$$\Xi = \frac{\partial}{\partial t} + \xi(t,x,u_\beta^{(1)})\frac{\partial}{\partial x} + \eta_1(t,x,u_\beta^{(1)})\frac{\partial}{\partial u_1^{(1)}} + \eta_2(t,x,u_\beta^{(1)})\frac{\partial}{\partial u_2^{(1)}}, \tag{6.26}$$

and requiring that

$$\Xi^{(2)}(\mathbf{\Delta}_2)\Big|_{\mathcal{M}_2} = \mathbf{0}, \tag{6.27}$$

where $\mathcal{M}_2$ is the manifold of the jet space defined by the system (6.25) together with the invariant surface conditions and their differential consequences, say

$$\begin{cases} \mathbf{\Delta}_2 = \mathbf{0}, \\ Q_\alpha \equiv \Xi\left(u_\alpha^{(1)} - u_\alpha^{(1)}(t,x)\right) = 0, \qquad \alpha = 1,2, \\ \dfrac{DQ_\alpha}{Dt} = \dfrac{DQ_\alpha}{Dx} = 0, \end{cases} \tag{6.28}$$

we obtain the following invariance conditions:

$$\begin{aligned}
&\frac{\partial^2\xi}{\partial u_1^{(1)2}}\left(u_{1,x}^{(1)}\right)^3 + 2\frac{\partial^2\xi}{\partial u_1^{(1)}\partial u_2^{(1)}}\left(u_{1,x}^{(1)}\right)^2 u_{2,x}^{(1)} + \frac{\partial^2\xi}{\partial u_2^{(1)2}}u_{1,x}^{(1)}\left(u_{2,x}^{(1)}\right)^2 \\
&+\left(2\frac{\partial^2\xi}{\partial x\partial u_1^{(1)}} - \frac{\partial^2\eta_1}{\partial u_1^{(1)2}} + 2\frac{\partial\xi}{\partial u_1^{(1)}}u_1^{(1)} - 2\xi\frac{\partial\xi}{\partial u_1^{(1)}} + \frac{\partial\xi}{\partial u_2^{(1)}}u_2^{(1)}\right)\left(u_{1,x}^{(1)}\right)^2 \\
&+\left(2\frac{\partial^2\xi}{\partial x\partial u_2^{(1)}} - 2\frac{\partial^2\eta_1}{\partial u_1^{(1)}\partial u_2^{(1)}} + 2\frac{\partial\xi}{\partial u_1^{(1)}} - 2\xi\frac{\partial\xi}{\partial u_2^{(1)}} + \frac{\partial\xi}{\partial u_2^{(1)}}u_1^{(1)}\right)u_{1,x}^{(1)}u_{2,x}^{(1)} \\
&+\left(\frac{\partial\xi}{\partial u_2^{(1)}} - \frac{\partial^2\eta_1}{\partial u_2^{(1)2}}\right)\left(u_{2,x}^{(1)}\right)^2 \\
&+\left(\frac{\partial^2\xi}{\partial x^2} - 2\frac{\partial^2\eta_1}{\partial x\partial u_1^{(1)}} - \frac{\partial\xi}{\partial t} + \frac{\partial\xi}{\partial x}u_1^{(1)} - 2\xi\frac{\partial\xi}{\partial x} + 2\eta_1\frac{\partial\xi}{\partial u_1^{(1)}} - \frac{\partial\eta_1}{\partial u_2^{(1)}}u_2^{(1)} + \frac{\partial\eta_2}{\partial u_1^{(1)}} + \eta_1\right)u_{1,x}^{(1)} \\
&+\left(-2\frac{\partial\eta_1}{\partial x\partial u_2^{(1)}} + \frac{\partial\xi}{\partial x} + 2\eta_1\frac{\partial\xi}{\partial u_2^{(1)}} - \frac{\partial\eta_1}{\partial u_1^{(1)}} + \frac{\partial\eta_1}{\partial u_2^{(1)}}u_1^{(1)} + \frac{\partial\eta_2}{\partial u_2^{(1)}}\right)u_{2,x}^{(1)} \\
&-\frac{\partial^2\eta_1}{\partial x^2} + 2\eta_1\frac{\partial\xi}{\partial x} + \frac{\partial\eta_1}{\partial t} + \frac{\partial\eta_1}{\partial x}u_1^{(1)} + \frac{\partial\eta_2}{\partial x} = 0, \\
&\frac{\partial^2\xi}{\partial u_1^{(1)2}}\left(u_{1,x}^{(1)}\right)^2 u_{2,x}^{(1)} + 2\frac{\partial^2\xi}{\partial u_1^{(1)}\partial u_2^{(1)}}u_{1,x}^{(1)}\left(u_{2,x}^{(1)}\right)^2 + \frac{\partial^2\xi}{\partial u_2^{(1)2}}\left(u_{2,x}^{(1)}\right)^3 + \left(\frac{\partial\xi}{\partial u_1^{(1)}}u_2^{(1)} - \frac{\partial^2\eta_2}{\partial u_1^{(1)2}}\right)\left(u_{1,x}^{(1)}\right)^2 \\
&+\left(2\frac{\partial^2\xi}{\partial x\partial u_1^{(1)}} - 2\frac{\partial^2\eta_2}{\partial u_1^{(1)}\partial u_2^{(1)}} + \frac{\partial\xi}{\partial u_1^{(1)}}u_1^{(1)} - 2\xi\frac{\partial\xi}{\partial u_1^{(1)}} + 2\frac{\partial\xi}{\partial u_2^{(1)}}u_2^{(1)}\right)u_{1,x}^{(1)}u_{2,x}^{(1)} \\
&+\left(2\frac{\partial^2\xi}{\partial x\partial u_2^{(1)}} - \frac{\partial^2\eta_2}{\partial u_2^{(1)2}} + \frac{\partial\xi}{\partial u_1^{(1)}} - 2\xi\frac{\partial\xi}{\partial u_2^{(1)}}\right)\left(u_{2,x}^{(1)}\right)^2 \\
&+\left(-2\frac{\partial^2\eta_2}{\partial x\partial u_1^{(1)}} + \frac{\partial\xi}{\partial x}u_2^{(1)} + 2\eta_2\frac{\partial\xi}{\partial u_1^{(1)}} + \frac{\partial\eta_1}{\partial u_1^{(1)}}u_2^{(1)} - \frac{\partial\eta_2}{\partial u_1^{(1)}}u_1^{(1)} - \frac{\partial\eta_2}{\partial u_2^{(1)}}u_2^{(1)} + \eta_2\right)u_{1,x}^{(1)} \\
&+\left(\frac{\partial^2\xi}{\partial x^2} - 2\frac{\partial^2\eta_2}{\partial x\partial u_2^{(1)}} - \frac{\partial\xi}{\partial t} - 2\xi\frac{\partial\xi}{\partial x} + 2\eta_2\frac{\partial\xi}{\partial u_2^{(1)}} + \frac{\partial\eta_1}{\partial u_2^{(1)}}u_2^{(1)} - \frac{\partial\eta_2}{\partial u_1^{(1)}}\right)u_{2,x}^{(1)} \\
&-\frac{\partial^2\eta_2}{\partial x^2} + 2\eta_2\frac{\partial\xi}{\partial x} + \frac{\partial\eta_2}{\partial t} + \frac{\partial\eta_1}{\partial x}u_2^{(1)} = 0;
\end{aligned}$$

the latter are polynomials of third degree in the derivatives $u_{1,x}^{(1)}$ and $u_{2,x}^{(1)}$, whose coefficients must be vanishing. After simple algebra, we get

$$\xi = \kappa u_1^{(1)} + \frac{1}{2}u_1^{(2)}, \tag{6.29}$$

where $u_1^{(2)}(t,x)$ is a function of the indicated arguments, and κ is a constant that has to satisfy the constraint $\kappa(2\kappa+1)=0$. Again, looking for the Q-conditional symmetries of system (6.25), we choose $\kappa=-1/2$. Thence, integrating the determining equations, we obtain the Q-conditional symmetries characterized by the vector field (6.26), with

$$\begin{aligned}
\xi &= \frac{1}{2}\left(-u_1^{(1)}+u_1^{(2)}\right),\\
\eta_1 &= \frac{1}{4}\left(-\left(u_1^{(1)}\right)^3-2u_1^{(1)}u_2^{(1)}+u_1^{(2)}\left(u_1^{(1)}\right)^2+u_2^{(2)}u_1^{(1)}+u_1^{(2)}u_2^{(1)}+u_3^{(2)}\right),\\
\eta_2 &= \frac{1}{4}\left(-\left(u_1^{(1)}\right)^2u_2^{(1)}-\left(u_2^{(1)}\right)^2+u_1^{(2)}u_1^{(1)}u_2^{(1)}+u_2^{(2)}u_2^{(1)}+u_4^{(2)}\right),
\end{aligned}\tag{6.30}$$

and the functions $u_\alpha^{(2)}(t,x)$ $(\alpha=1,\dots,4)$ satisfying the constraints

$$\mathbf{\Delta}_4 \equiv \begin{cases}
u_{1,t}^{(2)}+u_1^{(2)}u_{1,x}^{(2)}-u_{1,xx}^{(2)}+u_{2,x}^{(2)}=0,\\
u_{2,t}^{(2)}+u_2^{(2)}u_{1,x}^{(2)}-u_{2,xx}^{(2)}+u_{3,x}^{(2)}=0,\\
u_{3,t}^{(2)}+u_3^{(2)}u_{1,x}^{(2)}-u_{3,xx}^{(2)}+u_{4,x}^{(2)}=0,\\
u_{4,t}^{(2)}+u_4^{(2)}u_{1,x}^{(2)}-u_{4,xx}^{(2)}=0.
\end{cases}\tag{6.31}$$

Repeating the same algorithm for the latter system of four coupled Burgers-like equations, the admitted Q-conditional symmetries are expressed in terms of six arbitrary functions depending on t and x. We write this result in the following Proposition.

Proposition 10. *The system (6.31) admits the vector field Ξ of the Q-conditional symmetries, say*

$$\Xi = \frac{\partial}{\partial t}+\xi(t,x,u_\beta^{(2)})\frac{\partial}{\partial x}+\sum_{\alpha=1}^{4}\eta_\alpha(t,x,u_\beta^{(2)})\frac{\partial}{\partial u_\alpha^{(2)}},\tag{6.32}$$

where

$$\begin{aligned}
\xi &= \frac{1}{2}\left(-u_1^{(2)}+u_1^{(3)}\right),\\
\eta_1 &= \frac{1}{4}\left(-\left(u_1^{(2)}\right)^3-2u_1^{(2)}u_2^{(2)}+u_1^{(3)}\left(u_1^{(2)}\right)^2+u_2^{(3)}u_1^{(2)}+u_1^{(3)}u_2^{(2)}-u_3^{(2)}+u_3^{(3)}\right),\\
\eta_2 &= \frac{1}{4}\left(-\left(u_1^{(2)}\right)^2u_2^{(2)}-u_1^{(2)}u_3^{(2)}-\left(u_2^{(2)}\right)^2+u_1^{(3)}u_1^{(2)}u_2^{(2)}+u_2^{(3)}u_2^{(2)}+u_1^{(3)}u_3^{(2)}-u_4^{(2)}+u_4^{(3)}\right),\\
\eta_3 &= \frac{1}{4}\left(-\left(u_1^{(2)}\right)^2u_3^{(2)}-u_1^{(2)}u_4^{(2)}-u_2^{(2)}u_3^{(2)}+u_1^{(3)}u_1^{(2)}u_3^{(2)}+u_2^{(3)}u_3^{(2)}+u_1^{(3)}u_4^{(2)}+u_5^{(3)}\right),\\
\eta_4 &= \frac{1}{4}\left(-\left(u_1^{(2)}\right)^2u_4^{(2)}-u_2^{(2)}u_4^{(2)}+u_1^{(3)}u_1^{(2)}u_4^{(2)}+u_2^{(3)}u_4^{(2)}+u_6^{(3)}\right),
\end{aligned}\tag{6.33}$$

and the functions $u_\alpha^{(3)}(t,x)$ $(\alpha = 1,\dots,6)$ *satisfy the system*

$$\boldsymbol{\Delta}_6 \equiv \begin{cases} u^{(3)}_{1,t} + u^{(3)}_1 u^{(3)}_{1,x} - u^{(3)}_{1,xx} + u^{(3)}_{2,x} = 0, \\ u^{(3)}_{2,t} + u^{(3)}_2 u^{(3)}_{1,x} - u^{(3)}_{2,xx} + u^{(3)}_{3,x} = 0, \\ u^{(3)}_{3,t} + u^{(3)}_3 u^{(3)}_{1,x} - u^{(3)}_{3,xx} + u^{(3)}_{4,x} = 0, \\ u^{(3)}_{4,t} + u^{(3)}_4 u^{(3)}_{1,x} - u^{(3)}_{4,xx} + u^{(3)}_{5,x} = 0, \\ u^{(3)}_{5,t} + u^{(3)}_5 u^{(3)}_{1,x} - u^{(3)}_{5,xx} + u^{(3)}_{6,x} = 0, \\ u^{(3)}_{6,t} + u^{(3)}_6 u^{(3)}_{1,x} - u^{(3)}_{6,xx} = 0. \end{cases} \tag{6.34}$$

Proof. Straightforward, by direct computation. □

We can repeat the same procedure for the system (6.34) made of six coupled Burgers-like equations, and the results are exhibited in the following Proposition.

Proposition 11. *The system* (6.34) *admits the vector field* Ξ *of the Q-conditional symmetries, say*

$$\Xi = \frac{\partial}{\partial t} + \xi(t,x,u^{(3)}_\beta)\frac{\partial}{\partial x} + \sum_{\alpha=1}^{6} \eta_\alpha(t,x,u^{(3)}_\beta)\frac{\partial}{\partial u^{(3)}_\alpha}, \tag{6.35}$$

where

$$\begin{aligned}
\xi &= \frac{1}{2}\left(-u^{(3)}_1 + u^{(4)}_1\right), \\
\eta_1 &= \frac{1}{4}\left(-\left(u^{(3)}_1\right)^3 - 2u^{(3)}_1 u^{(3)}_2 + u^{(4)}_1\left(u^{(3)}_1\right)^2 + u^{(4)}_2 u^{(3)}_1 + u^{(4)}_1 u^{(3)}_2 - u^{(3)}_3 + u^{(4)}_3\right), \\
\eta_2 &= \frac{1}{4}\left(-\left(u^{(3)}_1\right)^2 u^{(3)}_2 - u^{(3)}_1 u^{(3)}_3 - \left(u^{(3)}_2\right)^2 + u^{(4)}_1 u^{(3)}_1 u^{(3)}_2 + u^{(4)}_2 u^{(3)}_2 + u^{(4)}_1 u^{(3)}_3 - u^{(3)}_4 + u^{(4)}_4\right), \\
\eta_3 &= \frac{1}{4}\left(-\left(u^{(3)}_1\right)^2 u^{(3)}_3 - u^{(3)}_1 u^{(3)}_4 - u^{(3)}_2 u^{(3)}_3 + u^{(4)}_1 u^{(3)}_1 u^{(3)}_3 + u^{(4)}_2 u^{(3)}_3 + u^{(4)}_1 u^{(3)}_4 - u^{(3)}_5 + u^{(4)}_5\right), \\
\eta_4 &= \frac{1}{4}\left(-\left(u^{(3)}_1\right)^2 u^{(3)}_4 - u^{(3)}_1 u^{(3)}_5 - u^{(3)}_2 u^{(3)}_4 + u^{(4)}_1 u^{(3)}_1 u^{(3)}_4 + u^{(4)}_2 u^{(3)}_4 + u^{(4)}_1 u^{(3)}_5 - u^{(3)}_6 + u^{(4)}_6\right), \\
\eta_5 &= \frac{1}{4}\left(-\left(u^{(3)}_1\right)^2 u^{(3)}_5 - u^{(3)}_1 u^{(3)}_6 - u^{(3)}_2 u^{(3)}_5 + u^{(4)}_1 u^{(3)}_1 u^{(3)}_5 + u^{(4)}_2 u^{(3)}_5 + u^{(4)}_1 u^{(3)}_6 + u^{(4)}_7\right), \\
\eta_6 &= \frac{1}{4}\left(-\left(u^{(3)}_1\right)^2 u^{(3)}_6 - u^{(3)}_2 u^{(3)}_6 + u^{(4)}_1 u^{(3)}_1 u^{(3)}_6 + u^{(4)}_2 u^{(3)}_6 + u^{(4)}_8\right),
\end{aligned} \tag{6.36}$$

and the functions $u_\alpha^{(4)}(t,x)$ $(\alpha = 1,\dots,8)$ *satisfy the system*

$$\Delta_8 \equiv \begin{cases} u_{1,t}^{(4)} + u_1^{(4)} u_{1,x}^{(4)} - u_{1,xx}^{(4)} + u_{2,x}^{(4)} = 0, \\ u_{2,t}^{(4)} + u_2^{(4)} u_{1,x}^{(4)} - u_{2,xx}^{(4)} + u_{3,x}^{(4)} = 0, \\ u_{3,t}^{(4)} + u_3^{(4)} u_{1,x}^{(4)} - u_{3,xx}^{(4)} + u_{4,x}^{(4)} = 0, \\ u_{4,t}^{(4)} + u_4^{(4)} u_{1,x}^{(4)} - u_{4,xx}^{(4)} + u_{5,x}^{(4)} = 0, \\ u_{5,t}^{(4)} + u_5^{(4)} u_{1,x}^{(4)} - u_{5,xx}^{(4)} + u_{6,x}^{(4)} = 0, \\ u_{6,t}^{(4)} + u_6^{(4)} u_{1,x}^{(4)} - u_{6,xx}^{(4)} + u_{7,x}^{(4)} = 0, \\ u_{7,t}^{(4)} + u_7^{(4)} u_{1,x}^{(4)} - u_{7,xx}^{(4)} + u_{8,x}^{(4)} = 0, \\ u_{8,t}^{(4)} + u_8^{(4)} u_{1,x}^{(4)} - u_{8,xx}^{(4)} = 0. \end{cases} \tag{6.37}$$

Proof. Straightforward, by direct computation. □

The results heretofore obtained can be summarized as follows:

- there are Q-conditional symmetries of the system Δ_2 made by two coupled Burgers-like equations expressed in terms of four functions representing arbitrary solutions of the system Δ_4 made of four coupled Burgers-like equations;
- there are Q-conditional symmetries of Δ_4 expressed in terms of six functions representing arbitrary solutions of the system Δ_6 made of six coupled Burgers-like equations;
- there are Q-conditional symmetries of Δ_6 expressed in terms of eight functions representing arbitrary solutions of the system Δ_8 made of eight coupled Burgers-like equations.

These results suggest to conjecture that repeatedly searching for Q-conditional symmetries, and starting from a pair of coupled Burgers-like equations, a hierarchy of systems made of an even number of coupled Burgers-like equations arises.

Indeed, the latter conjecture and the one made in the previous Subsection, can be unified and proved to be true, as will be shown in the following Section.

6.4 The general hierarchy of Burgers-like equations

In this Section, we show that the existence of both hierarchies of Burgers-like equations, arising by searching at each step non trivial Q-conditional symmetries, can be proved in general. In fact, we have an infinite hierarchy of systems made of an odd number of coupled Burgers-like equations or made of an even number of coupled Burgers-like equations depending on the starting point.

Theorem 14. *Let* m *be a positive integer, and let* $k = \lceil m/2 \rceil$. *The system of Burgers-like equations (6.1) admits the* Q*-conditional symmetries associated to the vector field*

$$\Xi = \frac{\partial}{\partial t} + \xi(t,x,u_\beta^{(k)})\frac{\partial}{\partial x} + \sum_{\alpha=1}^{m} \eta_\alpha(t,x,u_\beta^{(k)})\frac{\partial}{\partial u_\alpha^{(k)}}, \tag{6.38}$$

where

$$
\begin{aligned}
\xi &= \frac{1}{2}\left(-u_1^{(k)} + u_1^{(k+1)}\right),\\
\eta_\alpha &= \frac{1}{4}\left(-\left(u_1^{(k)}\right)^2 u_\alpha^{(k)} - u_1^{(k)} u_{\alpha+1}^{(k)} - u_2^{(k)} u_\alpha^{(k)} + u_1^{(k+1)} u_1^{(k)} u_\alpha^{(k)}\right.\\
&\qquad \left. + u_2^{(k+1)} u_\alpha^{(k)} + u_1^{(k+1)} u_{\alpha+1}^{(k)} - u_{\alpha+2}^{(k)} + u_{\alpha+2}^{(k+1)}\right),\\
\eta_{m-1} &= \frac{1}{4}\left(-\left(u_1^{(k)}\right)^2 u_{m-1}^{(k)} - u_1^{(k)} u_m^{(k)} - u_2^{(k)} u_{m-1}^{(k)} + u_1^{(k+1)} u_1^{(k)} u_{m-1}^{(k)}\right.\\
&\qquad \left. + u_2^{(k+1)} u_{m-1}^{(k)} + u_1^{(k+1)} u_m^{(k)} + u_{m+1}^{(k+1)}\right),\\
\eta_m &= \frac{1}{4}\left(-\left(u_1^{(k)}\right)^2 u_m^{(k)} - (1-\delta_{1m}) u_2^{(k)} u_m^{(k)} + u_1^{(k+1)} u_1^{(k)} u_m^{(k)}\right.\\
&\qquad \left. + u_2^{(k+1)} u_m^{(k)} + u_{m+2}^{(k+1)}\right),
\end{aligned}
\tag{6.39}
$$

δ_{1m} being the Kronecker symbol, with $\alpha = 1, \dots, m-2$, provided that the functions $u_\alpha^{(k+1)}(t,x)$ satisfy the system

$$
\mathbf{\Delta}_{m+2} \equiv \begin{cases} u_{\alpha,t}^{(k+1)} + u_\alpha^{(k+1)} u_{1,x}^{(k+1)} - u_{\alpha,xx}^{(k+1)} + u_{\alpha+1,x}^{(k+1)} = 0,\\ u_{m+2,t}^{(k+1)} + u_{m+2}^{(k+1)} u_{1,x}^{(k+1)} - u_{m+2,xx}^{(k+1)} = 0, \end{cases}
\tag{6.40}
$$

with $\alpha = 1, \dots, m+1$.

Proof. It must be verified that the vector field (6.38), along with (6.39), is admitted by the system (6.1) along with the constraints (6.40). In fact, requiring

$$
\Xi^{(2)}\left(\mathbf{\Delta}_m\right)\Big|_{\mathcal{M}_m} = \mathbf{0},
\tag{6.41}
$$

where the manifold $\mathcal{M}_m$ of the jet space is defined by

$$
\begin{cases} \mathbf{\Delta}_m = \mathbf{0},\\ Q_\alpha \equiv \Xi\left(u_\alpha^{(k)} - u_\alpha^{(k)}(t,x)\right) = 0, \qquad \alpha = 1, \dots, m,\\ \dfrac{DQ_\alpha}{Dt} = \dfrac{DQ_\alpha}{Dx} = 0, \end{cases}
\tag{6.42}
$$

we get the following polynomial system of m equations in the variables $u_\alpha^{(k)}$ and $u_{\alpha,x}^{(k)}$:

$$\begin{aligned}
&\left(u_{1,t}^{(k+1)} + u_1^{(k+1)}u_{1,x}^{(k+1)} - u_{1,xx}^{(k+1)} + u_{2,x}^{(k+1)}\right) u_1^{(k)} u_\alpha^{(k)} \\
&\quad + \left(u_{2,t}^{(k+1)} + u_2^{(k+1)}u_{1,x}^{(k+1)} - u_{2,xx}^{(k+1)} + u_{3,x}^{(k+1)}\right) u_\alpha^{(k)} \\
&\quad + \left(u_{1,t}^{(k+1)} + u_1^{(k+1)}u_{1,x}^{(k+1)} - u_{1,xx}^{(k+1)} + u_{2,x}^{(k+1)}\right) u_{\alpha+1}^{(k)} \\
&\quad - 2\left(u_{1,t}^{(k+1)} + u_1^{(k+1)}u_{1,x}^{(k+1)} - u_{1,xx}^{(k+1)} + u_{2,x}^{(k+1)}\right) u_{\alpha,x}^{(k)} \\
&\quad + u_{\alpha+2,t}^{(k+1)} + u_{\alpha+2}^{(k+1)}u_{1,x}^{(k+1)} - u_{\alpha+2,xx}^{(k+1)} + u_{\alpha+3,x}^{(k+1)} = 0, \\
&\left(u_{1,t}^{(k+1)} + u_1^{(k+1)}u_{1,x}^{(k+1)} - u_{1,xx}^{(k+1)} + u_{2,x}^{(k+1)}\right) u_1^{(k)} u_m^{(k)} \\
&\quad + \left(u_{2,t}^{(k+1)} + u_2^{(k+1)}u_{1,x}^{(k+1)} - u_{2,xx}^{(k+1)} + u_{3,x}^{(k+1)}\right) u_m^{(k)} \\
&\quad - 2\left(u_{1,t}^{(k+1)} + u_1^{(k+1)}u_{1,x}^{(k+1)} - u_{1,xx}^{(k+1)} + u_{2,x}^{(k+1)}\right) u_{m,x}^{(k)} \\
&\quad + u_{m+2,t}^{(k+1)} + u_{m+2}^{(k+1)}u_{1,x}^{(k+1)} - u_{m+2,xx}^{(k+1)} = 0,
\end{aligned} \tag{6.43}$$

where $\alpha = 1, \ldots, m-1$.

Due to the arbitrariness of $u_\alpha^{(k)}$ and $u_{\alpha,x}^{(k)}$, the system (6.43) is satisfied if and only if

$$\begin{cases}
u_{1,t}^{(k+1)} + u_1^{(k+1)}u_{1,x}^{(k+1)} - u_{1,xx}^{(k+1)} + u_{2,x}^{(k+1)} = 0, \\
u_{2,t}^{(k+1)} + u_2^{(k+1)}u_{1,x}^{(k+1)} - u_{2,xx}^{(k+1)} + u_{3,x}^{(k+1)} = 0, \\
u_{\alpha+2,t}^{(k+1)} + u_{\alpha+2}^{(k+1)}u_{1,x}^{(k+1)} - u_{\alpha+2,xx}^{(k+1)} + u_{\alpha+3,x}^{(k+1)} = 0, \quad \alpha = 1, \ldots, m-1, \\
u_{m+2,t}^{(k+1)} + u_{m+2}^{(k+1)}u_{1,x}^{(k+1)} - u_{m+2,xx}^{(k+1)} = 0,
\end{cases} \tag{6.44}$$

i.e., the system $\mathbf{\Delta}_{m+2} = \mathbf{0}$ has to be satisfied. □

Remark 20. *Note that if m is odd (even, respectively), a hierarchy of systems with an odd (even, respectively) number of equations is generated.*

Each element of the infinite hierarchy of systems of Burgers-like equations can be written in the form of a matrix Burgers' equation [74, 75] that can be linearized by means of the matrix Hopf-Cole transformation [76].

In fact, defining the $m \times m$ matrix Ω as

$$\Omega = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ u_m^{(k)} & u_{m-1}^{(k)} & \cdots & u_2^{(k)} & u_1^{(k)} \end{bmatrix}, \tag{6.45}$$

the system (6.1) writes in the form of a matrix Burgers' equation, say

$$\Omega_{,t} + \Omega_{,x}\Omega - \Omega_{,xx} = 0. \tag{6.46}$$

The matrix Hopf-Cole transformation [74, 76],

$$\Omega = -2\Phi_{,x}\Phi^{-1}, \tag{6.47}$$

maps (6.46) to a matrix heat equation,

$$\Phi_{,t} - \Phi_{,xx} = 0; \tag{6.48}$$

moreover, from (6.47), it results $\Phi_{,x} = -\frac{1}{2}\Omega\Phi$, whereupon, computing the entries of $\Omega\Phi$, the solution of the system (6.1) is achieved from the linear algebraic system

$$\begin{cases} u_m^{(k)} v_1 + \displaystyle\sum_{j=1}^{m-1} (-2)^j u_{m-j}^{(k)} \frac{\partial^j v_1}{\partial x^j} = (-2)^m \frac{\partial^m v_1}{\partial x^m}, \\ u_m^{(k)} v_2 + \displaystyle\sum_{j=1}^{m-1} (-2)^j u_{m-j}^{(k)} \frac{\partial^j v_2}{\partial x^j} = (-2)^m \frac{\partial^m v_2}{\partial x^m}, \\ \dots, \\ u_m^{(k)} v_m + \displaystyle\sum_{j=1}^{m-1} (-2)^j u_{m-j}^{(k)} \frac{\partial^j v_m}{\partial x^j} = (-2)^m \frac{\partial^m v_m}{\partial x^m}, \end{cases}$$

where $v_\alpha(t, x)$ $(\alpha = 1, \dots, m)$ are m solutions of linear heat equations, *i.e.*,

$$v_{\alpha,t} - v_{\alpha,xx} = 0, \qquad \alpha = 1, \dots, m.$$

Chapter 7

Conclusions . . . not yet conclusive!

In this Chapter, we will discuss some open issues related to the SymboLie package; in fact, current investigation is devoted to add new functions, as well as to speed up the computation. Moreover, we present some preliminar results concerned with the optimal systems of all real five–dimensional Lie algebras characterized in [44] and of the Noether symmetries of geodesic equations. The complete analysis of these results, besides its own intrinsic interest, may also provide useful hints to detect possible bugs of the program. Actually, this Chapter is not really conclusive!

7.1 Open issues on the SymboLie Package

In this thesis, we focused on the problem of finding optimal systems of families of Lie subalgebras of finite dimensional Lie algebras almost automatically by means of a program written in the CAS *Wolfram Mathematica*™ [39]. To achieve the result, we introduced the definition of a p–family of Lie subalgebras: this allows us to prove the closure of a vector subspace with respect to the Lie bracket without the need of solving quadratic equations.

We have seen that, in some cases, this approach leaves out certain families of subalgebras (for instance, see Remark 14).

Throughout this doctoral dissertation we have seen two other such examples: the 3D optimal system discussed in [19] includes the Lie subalgebra $\{\Xi_1, \Xi_3 - \frac{1}{2}\Xi_4, \Xi_6\}$, and its corresponding `SymboLie` *object* $\{\Xi_1, \Xi_3 + a_1\Xi_4, \Xi_6\}$ is not a 4–family because the condition 4 of the definition of p–family is not satisfied. Another similar case occurs with the two-dimensional Lie subalgebras in the $2A_2$ algebra from the Patera and Winternitz classification.

Furthermore, in Section 5.7 regarding the optimal system of symmetries of the heat equation, we have seen that `SymboLie` return the 2–families $\{\Xi_3, \Xi_5\}$ and $\{\Xi_3, \Xi_6\}$ in addition to the two-dimensional optimal system found in [19]. It can be observed that they can be mapped into subfamilies of $\{\Xi_2, \Xi_3 + a_1\Xi_4\}$ and $\{\Xi_1, \Xi_3 + a_1\Xi_4\}$, respectively, so they do not satisfy point (3) of Definition 26.

Although this may happen, in most of the cases, the strategy adopted in SymboLie is effective and provides complete results. However, we plan to develop some new functions allowing for including in the analysis Lie subalgebras not included in our definition of p–family. This suggests the possibility of generalizing the definition of p–family to that of (p, q)–family, where instead of point (3), we consider

(3)′ the rank of the Jacobian matrix of $\{f_k^\alpha s_{k,\alpha},\ k = 1, \dots, d,\ \alpha = 1, \dots, r\}$ with respect to the elements of $\mathcal{P}$ is equal to q, with $q \leq p$.

For instance, there is an automorphism that maps the family $\{\Xi_3, \Xi_6\}$ to the family $\{-2t\Xi_1 - \Xi_3 + \Xi_4, 4t^2\Xi_1\}$, whose row echelon form is $\{\Xi_1, \Xi_3 - \Xi_4\}$.

Currently, we can observe these relations using the debug feature of `SymboLie` by setting

```
PrintDebug=1;
```

This line of code causes a debug file to be created, in which all the computations performed by the algorithm are recorded. In the case examined above, the debug file reveals an *error code* of 4. This indicates that the link between families $\{\Xi_3, \Xi_6\}$ and $\{\Xi_1, \Xi_3 + a_1\Xi_4\}$ exists, but the transformed family does not respect the maximum rank of the Jacobian matrix, namely, it is not a p–family.

In later developments of the algorithm, these types of ties can be highlighted automatically.

Furthermore, introducing a relation between families of subalgebras induced by the inner automorphisms (this relation is in general a *preorder*), we partition the families of Lie subalgebras and represent them by a graph: the simplest representatives of the connected components of this graph give the list of optimal systems of families of Lie subalgebras. The main algorithms of the package have been detailed, and some case studies presented.

We plan to extend the program by introducing additional features. In particular, it would be desirable to allow the program to suitably use special properties of the Lie algebra to be analyzed (for instance, especially for high dimensional Lie algebras, its decomposition as direct sum of an ideal and a subalgebra), as well as to implement some routines for constructing the submodels [77] once the optimal system of Lie subalgebras of a Lie algebra of symmetries of a differential equation is obtained. Last but not least, work is in progress to speed up the algorithms.

7.2 Optimal systems of subalgebras of all real 5D Lie algebras

Since the classification of real Lie algebras of dimensions 5 [44] and 6 [18] is complete, it is possible to compute their optimal systems of subalgebras, as we have done for the three- and four-dimensional algebras [26, 42]. In this section, the optimal systems of families of subalgebras for 5-dimensional algebras are computed using `SymboLie`. However, since such a classification does not exist in the literature, the reliability of these results must be verified. This aspect will be addressed in a future work, and could also help us to identify and correct possible bugs in the algorithm. Below are the results obtained with the `SymboLie` package.

We denote the Lie algebras using the notation from [44], maintaining the same order: the first six Lie algebras are nilpotent, while the remaining ones are solvable. Unlike the cases of 3D and 4D Lie algebras, here we do not consider 5-dimensional Lie algebras that can be decomposed as direct sums of lower-dimensional Lie algebras.

Algebra ($A_{5,1}$). Let $A_{5,1}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$[\Xi_3, \Xi_5] = \Xi_1, \quad [\Xi_4, \Xi_5] = \Xi_2.$$

The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\Theta^1 \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_5\}, \{\Xi_1 + a_1\Xi_2\}, \{\Xi_2 + a_1\Xi_3\}, \{\Xi_1 + a_1\Xi_4\},$$

$$
\begin{aligned}
&\{\Xi_3+a_1\Xi_4\},\{\Xi_3+a_1\Xi_5\},\{\Xi_4+a_1\Xi_5\},\{\Xi_1+a_1\Xi_3+a_2\Xi_4\},\\
&\{\Xi_3+a_1\Xi_4+a_2\Xi_5\}\},\\
\Theta^2\equiv\{&\{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_3\},\{\Xi_1,\Xi_4\},\{\Xi_1,\Xi_5\},\{\Xi_2,\Xi_3\},\{\Xi_2,\Xi_4\},\{\Xi_2,\Xi_5\},\\
&\{\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2+a_1\Xi_3\},\{\Xi_1,\Xi_3+a_1\Xi_4\},\{\Xi_1,\Xi_3+a_1\Xi_5\},\\
&\{\Xi_1,\Xi_4+a_1\Xi_5\},\{\Xi_2,\Xi_3+a_1\Xi_4\},\{\Xi_2,\Xi_3+a_1\Xi_5\},\{\Xi_2,\Xi_4+a_1\Xi_5\},\\
&\{\Xi_1+a_1\Xi_2,\Xi_3\},\{\Xi_1+a_1\Xi_2,\Xi_4\},\{\Xi_1+a_1\Xi_2,\Xi_5\},\{\Xi_1+a_1\Xi_3,\Xi_4\},\\
&\{\Xi_2+a_1\Xi_3,\Xi_4\},\{\Xi_1+a_1\Xi_4,\Xi_2\},\{\Xi_1+a_1\Xi_4,\Xi_3\},\\
&\{\Xi_1,\Xi_3+a_1\Xi_4+a_2\Xi_5\},\{\Xi_2,\Xi_3+a_1\Xi_4+a_2\Xi_5\},\{\Xi_1+a_1\Xi_2,\Xi_3+a_2\Xi_4\},\\
&\{\Xi_1+a_1\Xi_2,\Xi_3+a_2\Xi_5\},\{\Xi_1+a_1\Xi_2,\Xi_4+a_2\Xi_5\},\{\Xi_1+a_1\Xi_2+a_2\Xi_3,\Xi_4\},\\
&\{\Xi_1+a_1\Xi_4,\Xi_2+a_2\Xi_3\},\{\Xi_1+\alpha_1\Xi_3,\Xi_2+a_1\Xi_3+a_2\Xi_4\},\\
&\{\Xi_1+a_1\Xi_2,\Xi_3+a_2\Xi_4+a_3\Xi_5\},\{\Xi_1+a_1\Xi_4,\Xi_2+a_2\Xi_3+a_3\Xi_4\}\},\\
\Theta^3\equiv\{&\{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_3,\Xi_5\},\\
&\{\Xi_2,\Xi_3,\Xi_4\},\{\Xi_2,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_5\},\\
&\{\Xi_1,\Xi_2,\Xi_4+a_1\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4+a_1\Xi_5\},\{\Xi_1,\Xi_2+a_1\Xi_3,\Xi_4\},\\
&\{\Xi_1,\Xi_2+a_1\Xi_3,\Xi_5\},\{\Xi_2,\Xi_3+a_1\Xi_5,\Xi_4\},\{\Xi_1+a_1\Xi_2,\Xi_3,\Xi_4\},\\
&\{\Xi_1+a_1\Xi_4,\Xi_2,\Xi_3\},\{\Xi_1+a_1\Xi_4,\Xi_2,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_4+a_2\Xi_5\},\\
&\{\Xi_1,\Xi_2+a_1\Xi_3,\Xi_4+a_2\Xi_5\},\{\Xi_1+a_1\Xi_3,\Xi_2+a_2\Xi_3,\Xi_4\},\\
&\{\Xi_1+a_1\Xi_4,\Xi_2,\Xi_3+a_2\Xi_5\}\},\\
\Theta^4\equiv\{&\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_4,\Xi_5\},\\
&\{\Xi_1,\Xi_2,\Xi_3,\Xi_4+a_1\Xi_5\},\{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_4,\Xi_5\},\\
&\{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_5,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_5,\Xi_4+a_2\Xi_5\}\}.
\end{aligned}
$$

Algebra ($A_{5,2}$). Let $A_{5,2}$ be the 5D Lie algebra spanned by $\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5\}$ with the non–zero commutators:

$$[\Xi_2,\Xi_5]=\Xi_1,\quad [\Xi_3,\Xi_5]=\Xi_2,\quad [\Xi_4,\Xi_5]=\Xi_3.$$

The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$
\begin{aligned}
\Theta^1\equiv\{&\{\Xi_1\},\{\Xi_2\},\{\Xi_3\},\{\Xi_4\},\{\Xi_5\},\{\Xi_1+a_1\Xi_3\},\{\Xi_1+a_1\Xi_4\},\{\Xi_2+a_1\Xi_4\},\\
&\{\Xi_3+a_1\Xi_4\},\{\Xi_4+a_1\Xi_5\},\{\Xi_1+a_1\Xi_2+a_2\Xi_4\},\},\\
\Theta^2\equiv\{&\{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_3\},\{\Xi_1,\Xi_4\},\{\Xi_1,\Xi_5\},\{\Xi_2,\Xi_3\},\{\Xi_2,\Xi_4\},\\
&\{\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2+a_1\Xi_4\},\{\Xi_1,\Xi_4+a_1\Xi_5\},\{\Xi_2,\Xi_3+a_1\Xi_4\},\{\Xi_1+a_1\Xi_3,\Xi_2\},\\
&\{\Xi_1+a_1\Xi_2,\Xi_4\},\{\Xi_1+a_1\Xi_3,\Xi_4\},\{\Xi_2+a_1\Xi_3,\Xi_4\},\{\Xi_1+a_1\Xi_4,\Xi_2\},\\
&\{\Xi_1+a_1\Xi_4,\Xi_3\},\{\Xi_2+a_1\Xi_4,\Xi_3\},\{\Xi_1+a_1\Xi_4,\Xi_2+a_2\Xi_4\},\\
&\{\Xi_1+a_1\Xi_3,\Xi_2+a_2\Xi_4\},\{\Xi_1+a_1\Xi_2+a_2\Xi_3,\Xi_4\},\\
&\{\Xi_1+a_1\Xi_4,\Xi_2+a_2\Xi_3\},\{\Xi_1+a_1\Xi_4,\Xi_3+a_2\Xi_4\},\\
&\{\Xi_2+a_1\Xi_4,\Xi_3+a_2\Xi_4\},\{\Xi_1+a_1\Xi_2+a_2\Xi_4,\Xi_3\},\\
&\{\Xi_1+a_1\Xi_3,\Xi_2+a_2\Xi_3+a_3\Xi_4\}\},\\
\Theta^3\equiv\{&\{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4\},\{\Xi_2,\Xi_3,\Xi_4\},\\
&\{\Xi_1,\Xi_2,\Xi_4+a_1\Xi_5\},\{\Xi_1,\Xi_2+a_1\Xi_4,\Xi_3\},\{\Xi_1+a_1\Xi_2,\Xi_3,\Xi_4\},\\
&\{\Xi_1+a_1\Xi_3,\Xi_2,\Xi_4\},\{\Xi_1+a_1\Xi_4,\Xi_2,\Xi_3\},\{\Xi_1+a_1\Xi_4,\Xi_2,\Xi_3+a_2\Xi_4\}\},\\
\Theta^4\equiv\{&\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_4+a_1\Xi_5\}\}.
\end{aligned}
$$

Algebra ($A_{5,3}$)**.** Let $A_{5,3}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$[\Xi_3, \Xi_4] = \Xi_2, \quad [\Xi_3, \Xi_5] = \Xi_1, \quad [\Xi_4, \Xi_5] = \Xi_3.$$

The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\begin{aligned}
\Theta^1 \equiv \{ & \{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_5\}, \{\Xi_1 + a_1\Xi_2\}, \{\Xi_1 + a_1\Xi_4\}, \{\Xi_2 + a_1\Xi_5\}, \\
& \{\Xi_4 + a_1\Xi_5\}, \{\Xi_1 + a_1\Xi_4 + a_2\Xi_5\}\}, \\
\Theta^2 \equiv \{ & \{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_5\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_2, \Xi_5\}, \\
& \{\Xi_1, \Xi_2 + a_1\Xi_5\}, \{\Xi_1, \Xi_4 + a_1\Xi_5\}, \{\Xi_1 + a_1\Xi_4, \Xi_2\}, \{\Xi_2, \Xi_4 + a_1\Xi_5\}, \\
& \{\Xi_1 + a_1\Xi_2, \Xi_3\}, \{\Xi_1 + a_1\Xi_2, \Xi_4\}, \{\Xi_1 + a_1\Xi_2, \Xi_5\}, \{\Xi_1 + a_1\Xi_2, \Xi_4 + a_2\Xi_5\}, \\
& \{\Xi_1 + a_1\Xi_2, \Xi_3 + a_2\Xi_4 + a_3\Xi_5\}\}, \\
\Theta^3 \equiv \{ & \{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_5\}, \{\Xi_2, \Xi_3, \Xi_4\}, \\
& \{\Xi_1, \Xi_2, \Xi_4 + a_1\Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_5, \Xi_3\}, \{\Xi_1 + a_1\Xi_4, \Xi_2, \Xi_3\}\}, \\
\Theta^4 \equiv \{ & \{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_4 + a_1\Xi_5\}\}.
\end{aligned}$$

Algebra ($A_{5,4}$)**.** Let $A_{5,4}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$[\Xi_2, \Xi_4] = \Xi_1, \quad [\Xi_3, \Xi_5] = \Xi_1.$$

The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\begin{aligned}
\Theta^1 \equiv \{ & \{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_5\}, \{\Xi_2 + a_1\Xi_3\}, \{\Xi_2 + a_1\Xi_4\}, \{\Xi_3 + a_1\Xi_4\}, \\
& \{\Xi_2 + a_1\Xi_5\}, \{\Xi_3 + a_1\Xi_5\}, \{\Xi_4 + a_1\Xi_5\}, \{\Xi_2 + a_1\Xi_3 + a_2\Xi_4\}, \\
& \{\Xi_2 + a_1\Xi_3 + a_2\Xi_5\}, \{\Xi_2 + a_1\Xi_4 + a_2\Xi_5\}, \{\Xi_3 + a_1\Xi_4 + a_2\Xi_5\}, \\
& \{\Xi_2 + a_1\Xi_3 + a_2\Xi_4 + a_3\Xi_5\}, \}, \\
\Theta^2 \equiv \{ & \{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_5\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_5\}, \{\Xi_3, \Xi_4\}, \\
& \{\Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_3\}, \{\Xi_1, \Xi_2 + a_1\Xi_4\}, \{\Xi_1, \Xi_3 + a_1\Xi_4\}, \\
& \{\Xi_1, \Xi_2 + a_1\Xi_5\}, \{\Xi_1, \Xi_3 + a_1\Xi_5\}, \{\Xi_1, \Xi_4 + a_1\Xi_5\}, \{\Xi_2, \Xi_3 + a_1\Xi_5\}, \\
& \{\Xi_1 + \alpha_1\Xi_3, \Xi_2 + a_1\Xi_4\}, \{\Xi_2 + a_1\Xi_4, \Xi_5\}, \{\Xi_1 + \alpha_1\Xi_4, \Xi_3 + a_1\Xi_5\}, \\
& \{\Xi_1, \Xi_2 + a_1\Xi_3 + a_2\Xi_4\}, \{\Xi_1, \Xi_2 + a_1\Xi_3 + a_2\Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_4 + a_2\Xi_5\}, \\
& \{\Xi_1, \Xi_3 + a_1\Xi_4 + a_2\Xi_5\}, \{\Xi_2 + a_1\Xi_4, \Xi_3 + a_2\Xi_5\}, \\
& \{\Xi_1, \Xi_2 + a_1\Xi_3 + a_2\Xi_4 + a_3\Xi_5\}\}, \\
\Theta^3 \equiv \{ & \{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_5\}, \\
& \{\Xi_1, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_5\}, \\
& \{\Xi_1, \Xi_2, \Xi_4 + a_1\Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4 + a_1\Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_3, \Xi_4\}, \\
& \{\Xi_1, \Xi_2 + a_1\Xi_3, \Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_4, \Xi_3\}, \{\Xi_1, \Xi_2 + a_1\Xi_4, \Xi_5\}, \\
& \{\Xi_1, \Xi_3 + a_1\Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_5, \Xi_3\}, \{\Xi_1, \Xi_2 + a_1\Xi_5, \Xi_4\}, \\
& \{\Xi_1, \Xi_3 + a_1\Xi_5, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4 + a_2\Xi_5\}, \\
& \{\Xi_1, \Xi_2 + a_1\Xi_3, \Xi_4 + a_2\Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_4, \Xi_3 + a_2\Xi_4\}, \\
& \{\Xi_1, \Xi_2 + a_1\Xi_4, \Xi_3 + a_2\Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_3 + a_2\Xi_4, \Xi_5\}, \\
& \{\Xi_1, \Xi_2 + a_1\Xi_5, \Xi_3 + a_2\Xi_4\}, \{\Xi_1, \Xi_2 + a_1\Xi_5, \Xi_3 + a_2\Xi_5\},
\end{aligned}$$

$$
\begin{aligned}
&\{\Xi_1, \Xi_2 + a_1\Xi_5, \Xi_4 + a_2\Xi_5\}, \{\Xi_1, \Xi_3 + a_1\Xi_5, \Xi_4 + a_2\Xi_5\},\\
&\{\Xi_1, \Xi_2 + a_1\Xi_3 + a_2\Xi_5, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1\Xi_4 + a_2\Xi_5, \Xi_3\},\\
&\{\Xi_1, \Xi_2 + a_1\Xi_4, \Xi_3 + a_2\Xi_4 + a_3\Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_5, \Xi_3 + a_2\Xi_4 + a_3\Xi_5\},\\
&\{\Xi_1, \Xi_2 + a_1\Xi_3 + a_2\Xi_5, \Xi_4 + a_3\Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_4 + a_2\Xi_5, \Xi_3 + a_3\Xi_4\},\\
&\{\Xi_1, \Xi_2 + a_1\Xi_4 + a_2\Xi_5, \Xi_3 + a_3\Xi_5\},\\
&\{\Xi_1, \Xi_2 + a_1\Xi_4 + a_2\Xi_5, \Xi_3 + a_3\Xi_4 + a_4\Xi_5\}\},\\
\Theta^4 \equiv \{&\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4, \Xi_5\},\\
&\{\Xi_1, \Xi_2, \Xi_3, \Xi_4 + a_1\Xi_5\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_5, \Xi_4\},\\
&\{\Xi_1, \Xi_2 + a_1\Xi_3, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_4, \Xi_3, \Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_5, \Xi_3, \Xi_4\},\\
&\{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_5, \Xi_4 + a_2\Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_4, \Xi_3 + a_2\Xi_4, \Xi_5\},\\
&\{\Xi_1, \Xi_2 + a_1\Xi_5, \Xi_3, \Xi_4 + a_2\Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_5, \Xi_3 + a_2\Xi_5, \Xi_4\},\\
&\{\Xi_1, \Xi_2 + a_1\Xi_5, \Xi_3 + a_2\Xi_5, \Xi_4 + a_3\Xi_5\}, \}.
\end{aligned}
$$

Algebra ($A_{5,5}$). Let $A_{5,5}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$[\Xi_2, \Xi_5] = \Xi_1, \quad [\Xi_3, \Xi_4] = \Xi_1, \quad [\Xi_3, \Xi_5] = \Xi_2.$$

The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$
\begin{aligned}
\Theta^1 \equiv \{&\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_5\}, \{\Xi_2 + a_1\Xi_4\}, \{\Xi_3 + a_1\Xi_4\}, \{\Xi_3 + a_1\Xi_5\},\\
&\{\Xi_4 + a_1\Xi_5\}, \{\Xi_3 + a_1\Xi_4 + a_2\Xi_5\}\},\\
\Theta^2 \equiv \{&\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_5\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_4, \Xi_5\},\\
&\{\Xi_1, \Xi_2 + a_1\Xi_4\}, \{\Xi_1, \Xi_3 + a_1\Xi_4\}, \{\Xi_1, \Xi_3 + a_1\Xi_5\}, \{\Xi_1, \Xi_4 + a_1\Xi_5\},\\
&\{\Xi_2, \Xi_3 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_4, \Xi_5\}, \{\Xi_1, \Xi_3 + a_1\Xi_4 + a_2\Xi_5\}\},\\
\Theta^3 \equiv \{&\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_4, \Xi_5\},\\
&\{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_5\}, \{\Xi_1, \Xi_2, \Xi_4 + a_1\Xi_5\},\\
&\{\Xi_1, \Xi_2 + a_1\Xi_4, \Xi_3\}, \{\Xi_1, \Xi_2 + a_1\Xi_4, \Xi_5\}, \{\Xi_1, \Xi_3 + a_1\Xi_5, \Xi_4\},\\
&\{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4 + a_2\Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_4, \Xi_3 + a_2\Xi_5\}\},\\
\Theta^4 \equiv \{&\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_4 + a_1\Xi_5\},\\
&\{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_5, \Xi_4\},\\
&\{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_5, \Xi_4 + a_2\Xi_5\}\}.
\end{aligned}
$$

Algebra ($A_{5,6}$). Let $A_{5,6}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$[\Xi_2, \Xi_5] = \Xi_1, \quad [\Xi_3, \Xi_4] = \Xi_1, \quad [\Xi_3, \Xi_5] = \Xi_2, \quad [\Xi_4, \Xi_5] = \Xi_3.$$

The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$
\begin{aligned}
\Theta^1 \equiv \{&\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_5\}, \{\Xi_2 + a_1\Xi_4\}, \{\Xi_4 + a_1\Xi_5\}\},\\
\Theta^2 \equiv \{&\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_5\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\},\\
&\{\Xi_1, \Xi_2 + a_1\Xi_4\}, \{\Xi_1, \Xi_4 + a_1\Xi_5\}, \{\Xi_2, \Xi_3 + a_1\Xi_4\}\},\\
\Theta^3 \equiv \{&\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4\},
\end{aligned}
$$

$$\{\Xi_1,\Xi_2,\Xi_4+a_1\Xi_5\},\{\Xi_1,\Xi_2+a_1\Xi_4,\Xi_3\}\},$$
$$\Theta^4\equiv\{\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_4+a_1\Xi_5\}\}.$$

Algebra ($A^{abc}_{5,7}$). Let $A^{abc}_{5,7}$ be the 5D Lie algebra spanned by $\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5\}$ with the non–zero commutators:

$$[\Xi_1,\Xi_5]=\Xi_1,\quad [\Xi_2,\Xi_5]=a\Xi_2,\quad [\Xi_3,\Xi_5]=b\Xi_3,\quad [\Xi_4,\Xi_5]=c\Xi_4,$$

with $abc\neq 0$, $-1\leq c\leq b\leq a\leq 1$. The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\begin{aligned}
\Theta^1\equiv\{&\{\Xi_1\},\{\Xi_2\},\{\Xi_3\},\{\Xi_4\},\{\Xi_5\},\{\Xi_1+\alpha_1\Xi_2\},\{\Xi_1+\alpha_1\Xi_3\},\{\Xi_2+a_1\Xi_3\},\\
&\{\Xi_1+\alpha_1\Xi_4\},\{\Xi_2+a_1\Xi_4\},\{\Xi_3+a_1\Xi_4\},\{\Xi_1+\alpha_1\Xi_2+a_1\Xi_3\},\\
&\{\Xi_1+\alpha_1\Xi_2+a_1\Xi_4\},\{\Xi_1+\alpha_1\Xi_3+a_1\Xi_4\},\{\Xi_2+a_1\Xi_3+a_2\Xi_4\},\\
&\{\Xi_1+\alpha_1\Xi_2+a_1\Xi_3+a_2\Xi_4\}\},\\
\Theta^2\equiv\{&\{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_3\},\{\Xi_1,\Xi_4\},\{\Xi_1,\Xi_5\},\{\Xi_2,\Xi_3\},\{\Xi_2,\Xi_4\},\{\Xi_2,\Xi_5\},\\
&\{\Xi_3,\Xi_4\},\{\Xi_3,\Xi_5\},\{\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2+a_1\Xi_3\},\{\Xi_1,\Xi_2+a_1\Xi_4\},\\
&\{\Xi_1,\Xi_3+a_1\Xi_4\},\{\Xi_2,\Xi_3+a_1\Xi_4\},\{\Xi_1+\alpha_1\Xi_2,\Xi_3\},\{\Xi_1+\alpha_1\Xi_2,\Xi_4\},\\
&\{\Xi_1+\alpha_1\Xi_3,\Xi_2\},\{\Xi_1+\alpha_1\Xi_3,\Xi_4\},\{\Xi_2+a_1\Xi_3,\Xi_4\},\{\Xi_2+a_1\Xi_3,\Xi_5\},\\
&\{\Xi_1+\alpha_1\Xi_4,\Xi_2\},\{\Xi_1+\alpha_1\Xi_4,\Xi_3\},\{\Xi_2+a_1\Xi_4,\Xi_3\},\{\Xi_2+a_1\Xi_4,\Xi_5\},\\
&\{\Xi_3+a_1\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2+a_1\Xi_3+a_2\Xi_4\},\{\Xi_1+\alpha_1\Xi_2,\Xi_3+a_1\Xi_4\},\\
&\{\Xi_1+\alpha_1\Xi_3,\Xi_2+a_1\Xi_3\},\{\Xi_1+\alpha_1\Xi_3,\Xi_2+a_1\Xi_4\},\{\Xi_1+\alpha_1\Xi_2+a_1\Xi_3,\Xi_4\},\\
&\{\Xi_1+\alpha_1\Xi_4,\Xi_2+a_1\Xi_3\},\{\Xi_1+\alpha_1\Xi_4,\Xi_2+a_1\Xi_4\},\{\Xi_1+\alpha_1\Xi_4,\Xi_3+a_1\Xi_4\},\\
&\{\Xi_2+a_1\Xi_4,\Xi_3+a_2\Xi_4\},\{\Xi_1+\alpha_1\Xi_2+a_1\Xi_4,\Xi_3\},\{\Xi_1+\alpha_1\Xi_3+a_1\Xi_4,\Xi_2\},\\
&\{\Xi_2+a_1\Xi_3+a_2\Xi_4,\Xi_5\},\{\Xi_1+\alpha_1\Xi_3,\Xi_2+a_1\Xi_3+a_2\Xi_4\},\\
&\{\Xi_1+\alpha_1\Xi_4,\Xi_2+a_1\Xi_3+a_2\Xi_4\},\{\Xi_1+\alpha_1\Xi_2+a_1\Xi_4,\Xi_3+a_2\Xi_4\},\\
&\{\Xi_1+\alpha_1\Xi_3+a_1\Xi_4,\Xi_2+a_2\Xi_3\},\{\Xi_1+\alpha_1\Xi_3+a_1\Xi_4,\Xi_2+a_2\Xi_4\},\\
&\{\Xi_1+\alpha_1\Xi_3+a_1\Xi_4,\Xi_2+a_2\Xi_3+a_3\Xi_4\}\},\\
\Theta^3\equiv\{&\{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_3,\Xi_5\},\\
&\{\Xi_1,\Xi_4,\Xi_5\},\{\Xi_2,\Xi_3,\Xi_4\},\{\Xi_2,\Xi_3,\Xi_5\},\{\Xi_2,\Xi_4,\Xi_5\},\{\Xi_3,\Xi_4,\Xi_5\},\\
&\{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_4\},\{\Xi_1,\Xi_2+a_1\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2+a_1\Xi_3,\Xi_5\},\\
&\{\Xi_1,\Xi_2+a_1\Xi_4,\Xi_3\},\{\Xi_1,\Xi_2+a_1\Xi_4,\Xi_5\},\{\Xi_1,\Xi_3+a_1\Xi_4,\Xi_5\},\\
&\{\Xi_2,\Xi_3+a_1\Xi_4,\Xi_5\},\{\Xi_1+\alpha_1\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1+\alpha_1\Xi_3,\Xi_2,\Xi_4\},\\
&\{\Xi_2+a_1\Xi_3,\Xi_4,\Xi_5\},\{\Xi_1+\alpha_1\Xi_4,\Xi_2,\Xi_3\},\{\Xi_2+a_1\Xi_4,\Xi_3,\Xi_5\},\\
&\{\Xi_1,\Xi_2+a_1\Xi_4,\Xi_3+a_2\Xi_4\},\{\Xi_1,\Xi_2+a_1\Xi_3+a_2\Xi_4,\Xi_5\},\\
&\{\Xi_1+\alpha_1\Xi_3,\Xi_2+a_1\Xi_3,\Xi_4\},\{\Xi_1+\alpha_1\Xi_4,\Xi_2,\Xi_3+a_1\Xi_4\},\\
&\{\Xi_1+\alpha_1\Xi_4,\Xi_2+a_1\Xi_4,\Xi_3\},\{\Xi_2+a_1\Xi_4,\Xi_3+a_2\Xi_4,\Xi_5\},\\
&\{\Xi_1+\alpha_1\Xi_4,\Xi_2+a_1\Xi_4,\Xi_3+a_2\Xi_4\}\},\\
\Theta^4\equiv\{&\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4,\Xi_5\},\\
&\{\Xi_2,\Xi_3,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2+a_1\Xi_3,\Xi_4,\Xi_5\},\\
&\{\Xi_1,\Xi_2+a_1\Xi_4,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_2+a_1\Xi_4,\Xi_3+a_2\Xi_4,\Xi_5\}\}.
\end{aligned}$$

Algebra ($A^c_{5,8}$). Let $A^c_{5,8}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$[\Xi_2, \Xi_5] = \Xi_1, \quad [\Xi_3, \Xi_5] = \Xi_3, \quad [\Xi_4, \Xi_5] = c\Xi_4,$$

with $0 < |c| \leq 1$. The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\begin{aligned}
\Theta^1 \equiv \{&\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_5\}, \{\Xi_1 + \alpha_1\Xi_3\}, \{\Xi_2 + a_1\Xi_3\}, \{\Xi_1 + \alpha_1\Xi_4\},\\
&\{\Xi_2 + a_1\Xi_4\}, \{\Xi_3 + \alpha_1\Xi_4\}, \{\Xi_2 + a_1\Xi_5\}, \{\Xi_1 + \alpha_1\Xi_3 + a_1\Xi_4\},\\
&\{\Xi_2 + a_1\Xi_3 + a_2\Xi_4\}\},\\
\Theta^2 \equiv \{&\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_5\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_3, \Xi_5\},\\
&\{\Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_3\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_4\}, \{\Xi_1, \Xi_3 + \alpha_1\Xi_4\}, \{\Xi_1, \Xi_2 + a_1\Xi_5\},\\
&\{\Xi_2, \Xi_3 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_3, \Xi_2\}, \{\Xi_1 + \alpha_1\Xi_3, \Xi_4\}, \{\Xi_2 + a_1\Xi_3, \Xi_4\},\\
&\{\Xi_1 + a_1\Xi_4, \Xi_2\}, \{\Xi_1 + \alpha_1\Xi_4, \Xi_3\}, \{\Xi_2 + a_1\Xi_4, \Xi_3\}, \{\Xi_2 + a_1\Xi_5, \Xi_3\},\\
&\{\Xi_2 + a_1\Xi_5, \Xi_4\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_3 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_4\},\\
&\{\Xi_1 + a_1\Xi_4, \Xi_2 + a_2\Xi_3\}, \{\Xi_1 + \alpha_1\Xi_4, \Xi_3 + a_1\Xi_4\}, \{\Xi_2 + a_1\Xi_4, \Xi_3 + a_2\Xi_4\},\\
&\{\Xi_1 + a_1\Xi_3 + a_2\Xi_4, \Xi_2\}, \{\Xi_1 + a_1\Xi_3 + a_2\Xi_4, \Xi_2 + a_3\Xi_3\}\},\\
\Theta^3 \equiv \{&\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_5\}, \{\Xi_1, \Xi_4, \Xi_5\},\\
&\{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_3, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_3 + \alpha_1\Xi_4\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_3, \Xi_4\},\\
&\{\Xi_1, \Xi_2 + \alpha_1\Xi_4, \Xi_3\}, \{\Xi_1, \Xi_2 + a_1\Xi_5, \Xi_3\}, \{\Xi_1, \Xi_2 + a_1\Xi_5, \Xi_4\},\\
&\{\Xi_1 + a_1\Xi_3, \Xi_2, \Xi_4\}, \{\Xi_1 + a_1\Xi_4, \Xi_2, \Xi_3\}, \{\Xi_2 + a_1\Xi_5, \Xi_3, \Xi_4\},\\
&\{\Xi_1, \Xi_2 + \alpha_1\Xi_4, \Xi_3 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_4, \Xi_2, \Xi_3 + a_2\Xi_4\}\},\\
\Theta^4 \equiv \{&\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4, \Xi_5\},\\
&\{\Xi_1, \Xi_2 + a_1\Xi_5, \Xi_3, \Xi_4\}\}.
\end{aligned}$$

Algebra ($A^{bc}_{5,9}$). Let $A^{bc}_{5,9}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$[\Xi_1, \Xi_5] = \Xi_1, \quad [\Xi_2, \Xi_5] = \Xi_1 + \Xi_2, \quad [\Xi_3, \Xi_5] = b\Xi_3, \quad [\Xi_4, \Xi_5] = c\Xi_4,$$

with $0 \neq c \leq b$. The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\begin{aligned}
\Theta^1 \equiv \{&\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_5\}, \{\Xi_1 + \alpha_1\Xi_3\}, \{\Xi_2 + a_1\Xi_3\}, \{\Xi_1 + \alpha_1\Xi_4\},\\
&\{\Xi_2 + a_1\Xi_4\}, \{\Xi_3 + a_1\Xi_4\}, \{\Xi_1 + \alpha_1\Xi_3 + a_1\Xi_4\}, \{\Xi_2 + a_1\Xi_3 + a_2\Xi_4\}\},\\
\Theta^2 \equiv \{&\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_5\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_3, \Xi_5\},\\
&\{\Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_3\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_4\}, \{\Xi_1, \Xi_3 + a_1\Xi_4\}, \{\Xi_2, \Xi_3 + a_1\Xi_4\},\\
&\{\Xi_1 + a_1\Xi_3, \Xi_2\}, \{\Xi_1 + \alpha_1\Xi_3, \Xi_4\}, \{\Xi_2 + a_1\Xi_3, \Xi_4\}, \{\Xi_1 + a_1\Xi_4, \Xi_2\},\\
&\{\Xi_1 + \alpha_1\Xi_4, \Xi_3\}, \{\Xi_2 + a_1\Xi_4, \Xi_3\}, \{\Xi_3 + a_1\Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_3 + a_1\Xi_4\},\\
&\{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_4\}, \{\Xi_1 + a_1\Xi_4, \Xi_2 + a_2\Xi_3\}, \{\Xi_1 + \alpha_1\Xi_4, \Xi_3 + a_1\Xi_4\},\\
&\{\Xi_2 + a_1\Xi_4, \Xi_3 + a_2\Xi_4\}, \{\Xi_1 + a_1\Xi_3 + a_2\Xi_4, \Xi_2\},\\
&\{\Xi_1 + a_1\Xi_3 + a_2\Xi_4, \Xi_2 + a_3\Xi_3\}\},\\
\Theta^3 \equiv \{&\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_5\}, \{\Xi_1, \Xi_4, \Xi_5\},
\end{aligned}$$

$$\{\Xi_2,\Xi_3,\Xi_4\},\{\Xi_3,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_4\},\{\Xi_1,\Xi_2+\alpha_1\Xi_3,\Xi_4\},$$
$$\{\Xi_1,\Xi_2+\alpha_1\Xi_4,\Xi_3\},\{\Xi_1,\Xi_3+a_1\Xi_4,\Xi_5\},\{\Xi_1+a_1\Xi_3,\Xi_2,\Xi_4\},$$
$$\{\Xi_1+a_1\Xi_4,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2+\alpha_1\Xi_4,\Xi_3+a_1\Xi_4\},\{\Xi_1+a_1\Xi_4,\Xi_2,\Xi_3+a_2\Xi_4\}\},$$
$$\Theta^4\equiv\{\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4,\Xi_5\},$$
$$\{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_4,\Xi_5\}\}.$$

Algebra ($A_{5,10}$). Let $A_{5,10}$ be the 5D Lie algebra spanned by $\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5\}$ with the non–zero commutators:

$$[\Xi_2,\Xi_5]=\Xi_1,\quad [\Xi_3,\Xi_5]=\Xi_2,\quad [\Xi_4,\Xi_5]=\Xi_4.$$

The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\Theta^1\equiv\{\{\Xi_1\},\{\Xi_2\},\{\Xi_3\},\{\Xi_4\},\{\Xi_5\},\{\Xi_1+a_1\Xi_3\},\{\Xi_1+\alpha_1\Xi_4\},\{\Xi_2+a_1\Xi_4\},$$
$$\{\Xi_3+a_1\Xi_4\},\{\Xi_3+a_1\Xi_5\},\{\Xi_1+a_1\Xi_3+a_2\Xi_4\}\},$$
$$\Theta^2\equiv\{\{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_3\},\{\Xi_1,\Xi_4\},\{\Xi_1,\Xi_5\},\{\Xi_2,\Xi_3\},\{\Xi_2,\Xi_4\},\{\Xi_3,\Xi_4\},\{\Xi_4,\Xi_5\},$$
$$\{\Xi_1,\Xi_2+\alpha_1\Xi_4\},\{\Xi_1,\Xi_3+a_1\Xi_4\},\{\Xi_1,\Xi_3+a_1\Xi_5\},\{\Xi_2,\Xi_3+a_1\Xi_4\},$$
$$\{\Xi_1+a_1\Xi_3,\Xi_2\},\{\Xi_1+a_1\Xi_3,\Xi_4\},\{\Xi_1+a_1\Xi_4,\Xi_2\},\{\Xi_1+a_1\Xi_4,\Xi_3\},$$
$$\{\Xi_2+a_1\Xi_4,\Xi_3\},\{\Xi_3+a_1\Xi_5,\Xi_4\},\{\Xi_1+a_1\Xi_3+a_2\Xi_4,\Xi_2\},$$
$$\{\Xi_1+a_1\Xi_3,\Xi_2+a_2\Xi_4\},\{\Xi_1+a_1\Xi_4,\Xi_3+a_2\Xi_4\},$$
$$\{\Xi_2+a_1\Xi_4,\Xi_3+a_2\Xi_4\},\{\Xi_1+a_1\Xi_2+a_2\Xi_4,\Xi_3\},$$
$$\{\Xi_1+a_1\Xi_3,\Xi_2+a_2\Xi_3+a_3\Xi_4\}\},$$
$$\Theta^3\equiv\{\{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_4,\Xi_5\},$$
$$\{\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3+\alpha_1\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_5\},$$
$$\{\Xi_1,\Xi_2+a_1\Xi_4,\Xi_3\},\{\Xi_1,\Xi_3+a_1\Xi_5,\Xi_4\},\{\Xi_1+a_1\Xi_3,\Xi_2,\Xi_4\},$$
$$\{\Xi_1+a_1\Xi_4,\Xi_2,\Xi_3\},\{\Xi_1+a_1\Xi_4,\Xi_2,\Xi_3+a_2\Xi_4\}\},$$
$$\Theta^4\equiv\{\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_5,\Xi_4\}\}.$$

Algebra ($A^c_{5,11}$). Let $A_{5,11}$ be the 5D Lie algebra spanned by $\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5\}$ with the non–zero commutators:

$$[\Xi_1,\Xi_5]=\Xi_1,\quad [\Xi_2,\Xi_5]=\Xi_1+\Xi_2,\quad [\Xi_3,\Xi_5]=\Xi_2+\Xi_3,\quad [\Xi_4,\Xi_5]=c\Xi_4,$$

with $c\neq 0$. The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\Theta^1\equiv\{\{\Xi_1\},\{\Xi_2\},\{\Xi_3\},\{\Xi_4\},\{\Xi_5\},\{\Xi_1+a_1\Xi_3\},\{\Xi_1+\alpha_1\Xi_4\},\{\Xi_2+a_1\Xi_4\},$$
$$\{\Xi_3+a_1\Xi_4\},\{\Xi_1+a_1\Xi_3+a_2\Xi_4\}\},$$
$$\Theta^2\equiv\{\{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_3\},\{\Xi_1,\Xi_4\},\{\Xi_1,\Xi_5\},\{\Xi_2,\Xi_3\},\{\Xi_2,\Xi_4\},\{\Xi_3,\Xi_4\},\{\Xi_4,\Xi_5\},$$
$$\{\Xi_1,\Xi_2+\alpha_1\Xi_4\},\{\Xi_1,\Xi_3+a_1\Xi_4\},\{\Xi_2,\Xi_3+a_1\Xi_4\},\{\Xi_1+a_1\Xi_3,\Xi_2\},$$
$$\{\Xi_1+a_1\Xi_3,\Xi_4\},\{\Xi_1+a_1\Xi_4,\Xi_2\},\{\Xi_1+a_1\Xi_4,\Xi_3\},\{\Xi_2+a_1\Xi_4,\Xi_3\},$$
$$\{\Xi_1+a_1\Xi_3+a_2\Xi_4,\Xi_2\},\{\Xi_1+a_1\Xi_3,\Xi_2+a_2\Xi_4\},\{\Xi_1+a_1\Xi_4,\Xi_3+a_2\Xi_4\},$$
$$\{\Xi_2+a_1\Xi_4,\Xi_3+a_2\Xi_4\},\{\Xi_1+a_1\Xi_2+a_2\Xi_4,\Xi_3\},$$
$$\{\Xi_1+a_1\Xi_3,\Xi_2+a_2\Xi_3+a_3\Xi_4\}\},$$

$$\Theta^3 \equiv \{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_4, \Xi_5\}, \{\Xi_2, \Xi_3, \Xi_4\},$$
$$\{\Xi_1, \Xi_2, \Xi_3 + \alpha_1\Xi_4\}, \{\Xi_1, \Xi_2 + a_1\Xi_4, \Xi_3\}, \{\Xi_1 + a_1\Xi_3, \Xi_2, \Xi_4\},$$
$$\{\Xi_1 + a_1\Xi_4, \Xi_2, \Xi_3\}, \{\Xi_1 + a_1\Xi_4, \Xi_2, \Xi_3 + a_2\Xi_4\}\},$$
$$\Theta^4 \equiv \{\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_4, \Xi_5\}\}.$$

Algebra ($A_{5,12}$). Let $A_{5,12}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$[\Xi_1, \Xi_5] = \Xi_1, \quad [\Xi_2, \Xi_5] = \Xi_1 + \Xi_2, \quad [\Xi_3, \Xi_5] = \Xi_2 + \Xi_3, \quad [\Xi_4, \Xi_5] = \Xi_3 + \Xi_4.$$

The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\Theta^1 \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_5\}, \{\Xi_1 + a_1\Xi_3\}, \{\Xi_1 + a_1\Xi_4\}, \{\Xi_2 + a_1\Xi_4\},$$
$$\{\Xi_3 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_2 + a_2\Xi_4\}\},$$
$$\Theta^2 \equiv \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_5\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\},$$
$$\{\Xi_1, \Xi_2 + a_1\Xi_4\}, \{\Xi_2, \Xi_3 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_3, \Xi_2\}, \{\Xi_1 + a_1\Xi_2, \Xi_4\},$$
$$\{\Xi_1 + a_1\Xi_3, \Xi_4\}, \{\Xi_2 + a_1\Xi_3, \Xi_4\}, \{\Xi_1 + a_1\Xi_4, \Xi_2\}, \{\Xi_1 + a_1\Xi_4, \Xi_3\},$$
$$\{\Xi_2 + a_1\Xi_4, \Xi_3\}, \{\Xi_1 + a_1\Xi_4, \Xi_2 + a_2\Xi_4\}, \{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_4\},$$
$$\{\Xi_1 + a_1\Xi_2 + a_2\Xi_3, \Xi_4\}, \{\Xi_1 + a_1\Xi_4, \Xi_2 + a_2\Xi_3\}, \{\Xi_1 + a_1\Xi_4, \Xi_3 + a_2\Xi_4\},$$
$$\{\Xi_2 + a_1\Xi_4, \Xi_3 + a_2\Xi_4\}, \{\Xi_1 + a_1\Xi_2 + a_2\Xi_4, \Xi_3\},$$
$$\{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_3 + a_3\Xi_4\}, \},$$
$$\Theta^3 \equiv \{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\},$$
$$\{\Xi_1, \Xi_2 + a_1\Xi_4, \Xi_3\}, \{\Xi_1 + a_1\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1 + a_1\Xi_3, \Xi_2, \Xi_4\},$$
$$\{\Xi_1 + a_1\Xi_4, \Xi_2, \Xi_3\}, \{\Xi_1 + a_1\Xi_4, \Xi_2, \Xi_3 + a_2\Xi_4\}\},$$
$$\Theta^4 \equiv \{\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_5\}\}.$$

Algebra ($A^{apq}_{5,13}$). Let $A^{apq}_{5,13}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$[\Xi_1, \Xi_5] = \Xi_1, \quad [\Xi_2, \Xi_5] = a\Xi_2, \quad [\Xi_3, \Xi_5] = p\Xi_3 - q\Xi_4, \quad [\Xi_4, \Xi_5] = p\Xi_4 + q\Xi_3,$$

with $aq \neq 0$, $|a| \leq 1$. The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\Theta^1 \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_5\}, \{\Xi_1 + \alpha_1\Xi_2\}, \{\Xi_1 + a_1\Xi_3\}, \{\Xi_2 + a_1\Xi_3\},$$
$$\{\Xi_1 + a_1\Xi_2 + a_2\Xi_3\}\},$$
$$\Theta^2 \equiv \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_5\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_5\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1\Xi_3\},$$
$$\{\Xi_1 + a_1\Xi_2, \Xi_3\}, \{\Xi_1 + a_1\Xi_3, \Xi_2\}, \{\Xi_1 + a_1\Xi_3, \Xi_4\}, \{\Xi_2 + a_1\Xi_3, \Xi_4\},$$
$$\{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_3\}, \{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_4\}, \{\Xi_1 + a_1\Xi_2 + a_2\Xi_3, \Xi_4\},$$
$$\{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_3 + a_3\Xi_4\}\},$$
$$\Theta^3 \equiv \{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_3, \Xi_4, \Xi_5\},$$
$$\{\Xi_1, \Xi_2 + a_1\Xi_3, \Xi_4\}, \{\Xi_1 + \alpha_1\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1 + a_1\Xi_3, \Xi_2, \Xi_4\},$$
$$\{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_3, \Xi_4\}\},$$
$$\Theta^4 \equiv \{\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4, \Xi_5\}, \{\Xi_2, \Xi_3, \Xi_4, \Xi_5\}\}.$$

Algebra ($A^p_{5,14}$). Let $A^p_{5,14}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$[\Xi_2, \Xi_5] = \Xi_1, \quad [\Xi_3, \Xi_5] = p\Xi_3 - \Xi_4, \quad [\Xi_4, \Xi_5] = \Xi_3 + p\Xi_4,$$

with $p \in \mathbb{R}$. The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\begin{aligned}
\Theta^1 \equiv \{&\{\Xi_1\}, \{\Xi_1 + \alpha_1\Xi_2\}, \{\Xi_3\}, \{\Xi_5\}, \{\Xi_1 + a_1\Xi_3\}, \{\Xi_2 + a_1\Xi_3\}, \{\Xi_2 + a_1\Xi_4\},\\
&\{\Xi_2 + a_1\Xi_5\}, \{\Xi_1 + a_1\Xi_2 + a_2\Xi_3\}\},\\
\Theta^2 \equiv \{&\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_5\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1\Xi_3\},\\
&\{\Xi_1, \Xi_2 + a_1\Xi_5\}, \{\Xi_1 + a_1\Xi_2, \Xi_3\}, \{\Xi_1 + a_1\Xi_3, \Xi_2\}, \{\Xi_1 + a_1\Xi_3, \Xi_4\},\\
&\{\Xi_2 + a_1\Xi_3, \Xi_4\}, \{\Xi_1 + a_1\Xi_4, \Xi_2\}, \{\Xi_2 + a_1\Xi_4, \Xi_3\}, \{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_3\},\\
&\{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_4\}, \{\Xi_1 + a_1\Xi_2 + a_2\Xi_3, \Xi_4\}, \{\Xi_1 + a_1\Xi_4, \Xi_2 + a_2\Xi_3\},\\
&\{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_3 + a_3\Xi_4\}\},\\
\Theta^3 \equiv \{&\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1 + \alpha_1\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_3, \Xi_4, \Xi_5\},\\
&\{\Xi_1, \Xi_2 + a_1\Xi_3, \Xi_4\}, \{\Xi_1 + a_1\Xi_3, \Xi_2, \Xi_4\}, \{\Xi_1 + a_1\Xi_4, \Xi_2, \Xi_3\},\\
&\{\Xi_2 + a_1\Xi_5, \Xi_3, \Xi_4\}, \{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_3, \Xi_4\}\},\\
\Theta^4 \equiv \{&\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_5, \Xi_3, \Xi_4\}\}.
\end{aligned}$$

Algebra ($A^a_{5,15}$). Let $A^a_{5,15}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$[\Xi_1, \Xi_5] = \Xi_1, \quad [\Xi_2, \Xi_5] = \Xi_1 + \Xi_2, \quad [\Xi_3, \Xi_5] = a\Xi_3, \quad [\Xi_4, \Xi_5] = a\Xi_4 + \Xi_3,$$

with $|a| \leq 1$. The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\begin{aligned}
\Theta^1 \equiv \{&\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_5\}, \{\Xi_1 + \alpha_1\Xi_3\}, \{\Xi_2 + a_1\Xi_3\}, \{\Xi_1 + a_1\Xi_4\},\\
&\{\Xi_2 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_2 + a_2\Xi_4\}\},\\
\Theta^2 \equiv \{&\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_5\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_3, \Xi_5\},\\
&\{\Xi_1, \Xi_2 + \alpha_1\Xi_3\}, \{\Xi_1, \Xi_2 + a_1\Xi_4\}, \{\Xi_2, \Xi_3 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_3, \Xi_2\},\\
&\{\Xi_1 + a_1\Xi_3, \Xi_4\}, \{\Xi_2 + a_1\Xi_3, \Xi_4\}, \{\Xi_1 + a_1\Xi_4, \Xi_2\}, \{\Xi_1 + \alpha_1\Xi_4, \Xi_3\},\\
&\{\Xi_2 + a_1\Xi_4, \Xi_3\}, \{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_4\}, \{\Xi_1 + a_1\Xi_2 + a_2\Xi_3, \Xi_4\},\\
&\{\Xi_1 + a_1\Xi_4, \Xi_2 + a_2\Xi_3\}, \{\Xi_1 + a_1\Xi_4, \Xi_2 + a_2\Xi_4\},\\
&\{\Xi_1 + a_1\Xi_4, \Xi_2 + a_2\Xi_3 + a_3\Xi_4\}\},\\
\Theta^3 \equiv \{&\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_5\}, \{\Xi_2, \Xi_3, \Xi_4\},\\
&\{\Xi_3, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_4, \Xi_3\}, \{\Xi_1 + a_1\Xi_3, \Xi_2, \Xi_4\},\\
&\{\Xi_1 + a_1\Xi_4, \Xi_2, \Xi_3\}, \{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_3, \Xi_4\}\},\\
\Theta^4 \equiv \{&\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4, \Xi_5\}\}.
\end{aligned}$$

Algebra ($A^{pq}_{5,16}$). Let $A^{pq}_{5,16}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$[\Xi_1, \Xi_5] = \Xi_1, \quad [\Xi_2, \Xi_5] = \Xi_1 + \Xi_2, \quad [\Xi_3, \Xi_5] = p\Xi_3 - q\Xi_4, \quad [\Xi_4, \Xi_5] = p\Xi_4 + q\Xi_3,$$

$p \in \mathbb{R}$, $q \neq 0$. The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$
\begin{aligned}
\Theta^1 \equiv \{ & \{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_5\}, \{\Xi_1 + a_1\Xi_3\}, \{\Xi_2 + a_1\Xi_3\}, \{\Xi_2 + a_1\Xi_4\}, \\
& \{\Xi_1 + a_1\Xi_2 + a_2\Xi_3\}\}, \\
\Theta^2 \equiv \{ & \{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_5\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1\Xi_3\}, \\
& \{\Xi_1 + a_1\Xi_2, \Xi_3\}, \{\Xi_1 + a_1\Xi_3, \Xi_2\}, \{\Xi_1 + a_1\Xi_3, \Xi_4\}, \{\Xi_2 + a_1\Xi_3, \Xi_4\}, \\
& \{\Xi_1 + a_1\Xi_4, \Xi_2\}, \{\Xi_2 + a_1\Xi_4, \Xi_3\}, \{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_3\}, \\
& \{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_4\}, \{\Xi_1 + a_1\Xi_2 + a_2\Xi_3, \Xi_4\}, \{\Xi_1 + a_1\Xi_4, \Xi_2 + a_2\Xi_3\}, \\
& \{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_3 + a_3\Xi_4\}\}, \\
\Theta^3 \equiv \{ & \{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1 + \alpha_1\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_3, \Xi_4, \Xi_5\}, \\
& \{\Xi_1, \Xi_2 + a_1\Xi_3, \Xi_4\}, \{\Xi_1 + a_1\Xi_3, \Xi_2, \Xi_4\}, \{\Xi_1 + a_1\Xi_4, \Xi_2, \Xi_3\}, \\
& \{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_3, \Xi_4\}\}, \\
\Theta^4 \equiv \{ & \{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4, \Xi_5\}\}.
\end{aligned}
$$

Algebra ($A^{spq}_{5,17}$). Let $A^{spq}_{5,17}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$
\begin{aligned}
& [\Xi_1, \Xi_5] = p\Xi_1 - \Xi_2, \quad [\Xi_2, \Xi_5] = \Xi_1 + p\Xi_2, \quad [\Xi_3, \Xi_5] = q\Xi_3 - s\Xi_4, \\
& [\Xi_4, \Xi_5] = s\Xi_3 + q\Xi_4,
\end{aligned}
$$

with $p, q \in \mathbb{R}$, $s \neq 0$. The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$
\begin{aligned}
\Theta^1 \equiv \{ & \{\Xi_1\}, \{\Xi_3\}, \{\Xi_5\}, \{\Xi_1 + a_1\Xi_3\}, \{\Xi_2 + a_1\Xi_3\}, \{\Xi_1 + a_1\Xi_4\}, \{\Xi_2 + a_1\Xi_4\}, \\
& \{\Xi_1 + a_1\Xi_2 + a_2\Xi_3\}\}, \\
\Theta^2 \equiv \{ & \{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1\Xi_3\}, \{\Xi_1, \Xi_2 + a_1\Xi_4\}, \\
& \{\Xi_1, \Xi_3 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_3, \Xi_4\}, \{\Xi_2 + a_1\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1\Xi_3 + a_2\Xi_4\}, \\
& \{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_4\}, \{\Xi_1 + a_1\Xi_2 + a_2\Xi_3, \Xi_4\}, \{\Xi_1 + a_1\Xi_4, \Xi_2 + a_2\Xi_3\}, \\
& \{\Xi_1 + \alpha_1\Xi_3 + a_1\Xi_4, \Xi_2 + a_2\Xi_3 + a_3\Xi_4\}\}, \\
\Theta^3 \equiv \{ & \{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_3, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_3, \Xi_4\}, \\
& \{\Xi_1, \Xi_2 + a_1\Xi_4, \Xi_3\}, \{\Xi_1, \Xi_2 + a_1\Xi_4, \Xi_3 + a_2\Xi_4\}\}, \\
\Theta^4 \equiv \{ & \{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}\}.
\end{aligned}
$$

Algebra ($A^{p}_{5,18}$). Let $A^{p}_{5,18}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$
\begin{aligned}
& [\Xi_1, \Xi_5] = p\Xi_1 - \Xi_2, \quad [\Xi_2, \Xi_5] = \Xi_1 + p\Xi_2, \quad [\Xi_3, \Xi_5] = \Xi_1 - \Xi_4 + p\Xi_3, \\
& [\Xi_4, \Xi_5] = \Xi_2 + \Xi_3 + p\Xi_4,
\end{aligned}
$$

with $p \geq 0$. The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$
\begin{aligned}
\Theta^1 \equiv \{ & \{\Xi_1\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_5\}, \{\Xi_1 + a_1\Xi_3\}, \{\Xi_2 + a_1\Xi_3\}, \{\Xi_1 + a_1\Xi_4\}, \\
& \{\Xi_2 + a_1\Xi_4\}, \{\Xi_3 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_2 + a_2\Xi_3\}\}, \\
\Theta^2 \equiv \{ & \{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_2 + a_1\Xi_3 + a_2\Xi_4\}, \{\Xi_1 + a_1\Xi_3, \Xi_2 + a_2\Xi_4\},
\end{aligned}
$$

$$\{\Xi_1, \Xi_{+}a_1\Xi_3 2\}, \{\Xi_1, \Xi_3 + a_1\Xi_4\}, \{\Xi_2, \Xi_3 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_2, \Xi_3\},$$
$$\{\Xi_1 + a_1\Xi_2, \Xi_4\}, \{\Xi_1 + a_1\Xi_3, \Xi_4\}, \{\Xi_2 + a_1\Xi_3, \Xi_4\}, \{\Xi_1 + a_1\Xi_4, \Xi_3\},$$
$$\{\Xi_2 + a_1\Xi_4, \Xi_3\}, \{\Xi_1 + a_1\Xi_2 + a_2\Xi_3, \Xi_4\}, \{\Xi_1 + a_1\Xi_4, \Xi_2 + a_2\Xi_3\},$$
$$\{\Xi_1 + a_1\Xi_4, \Xi_3 + a_2\Xi_4\}, \{\Xi_2 + a_1\Xi_4, \Xi_3 + a_2\Xi_4\},$$
$$\{\Xi_1 + a_1\Xi_2 + a_2\Xi_4, \Xi_3\}, \{\Xi_1 + \alpha_1\Xi_3 + a_1\Xi_4, \Xi_2 + a_2\Xi_3 + a_3\Xi_4\}\},$$
$$\Theta^3 \equiv \{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_4, \Xi_3\}, \{\Xi_1, \Xi_2 + a_1\Xi_3, \Xi_4\},$$
$$\{\Xi_1 + a_1\Xi_4, \Xi_2, \Xi_3\}, \{\Xi_1 + \alpha_1\Xi_4, \Xi_2 + a_1\Xi_4, \Xi_3 + a_2\Xi_4\}\},$$
$$\Theta^4 \equiv \{\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}\}.$$

Algebra ($A^{ab}_{5,19}$). Let $A^{ab}_{5,19}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$[\Xi_1, \Xi_5] = a\Xi_1, \quad [\Xi_2, \Xi_3] = \Xi_1, \quad [\Xi_2, \Xi_5] = \Xi_2, \quad [\Xi_3, \Xi_5] = (a-1)\Xi_3,$$
$$[\Xi_4, \Xi_5] = b\Xi_4,$$

with $a \in \mathbb{R}, b \neq 0$. The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\Theta^1 \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_5\}, \{\Xi_2 + \alpha_1\Xi_3\}, \{\Xi_1 + a_1\Xi_4\}, \{\Xi_2 + \alpha_1\Xi_4\},$$
$$\{\Xi_3 + \alpha_1\Xi_4\}, \{\Xi_2 + \alpha_1\Xi_3 + a_1\Xi_4\}\},$$
$$\Theta^2 \equiv \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_5\}, \{\Xi_2, \Xi_4\}, \{\Xi_2, \Xi_5\}, \{\Xi_3, \Xi_4\}, \{\Xi_3, \Xi_5\},$$
$$\{\Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_3\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_4\}, \{\Xi_1, \Xi_3 + \alpha_1\Xi_4\},$$
$$\{\Xi_2 + \alpha_1\Xi_3, \Xi_4\}, \{\Xi_1 + a_1\Xi_4, \Xi_2\}, \{\Xi_1 + a_1\Xi_4, \Xi_3\}, \{\Xi_1 + a_1\Xi_4, \Xi_5\},$$
$$\{\Xi_1, \Xi_2 + \alpha_1\Xi_3 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_4, \Xi_2 + \alpha_1\Xi_3\}\},$$
$$\Theta^3 \equiv \{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_5\}, \{\Xi_1, \Xi_4, \Xi_5\},$$
$$\{\Xi_2, \Xi_4, \Xi_5\}, \{\Xi_3, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_3 + \alpha_1\Xi_4\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_3, \Xi_4\},$$
$$\{\Xi_1, \Xi_2 + \alpha_1\Xi_4, \Xi_3\}, \{\Xi_1 + a_1\Xi_4, \Xi_2, \Xi_5\}, \{\Xi_1 + a_1\Xi_4, \Xi_3, \Xi_5\},$$
$$\{\Xi_1, \Xi_2 + \alpha_1\Xi_4, \Xi_3 + a_1\Xi_4\}\},$$
$$\Theta^4 \equiv \{\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4, \Xi_5\}\}.$$

Algebra ($A^{a}_{5,20}$). Let $A^{a}_{5,20}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$[\Xi_1, \Xi_5] = a\Xi_1, \quad [\Xi_2, \Xi_3] = \Xi_1, \quad [\Xi_2, \Xi_5] = \Xi_2, \quad [\Xi_3, \Xi_5] = (a-1)\Xi_3,$$
$$[\Xi_4, \Xi_5] = a\Xi_4 + \Xi_1,$$

with $a \in \mathbb{R}$. The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\Theta^1 \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_5\}, \{\Xi_2 + \alpha_1\Xi_3\}, \{\Xi_2 + a_1\Xi_4\}, \{\Xi_3 + a_1\Xi_4\},$$
$$\{\Xi_2 + a_1\Xi_3 + a_2\Xi_4\}\},$$
$$\Theta^2 \equiv \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_5\}, \{\Xi_2, \Xi_4\}, \{\Xi_2, \Xi_5\}, \{\Xi_3, \Xi_4\}, \{\Xi_3, \Xi_5\},$$
$$\{\Xi_1, \Xi_2 + \alpha_1\Xi_3\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_4\}, \{\Xi_1, \Xi_3 + \alpha_1\Xi_4\}, \{\Xi_2 + a_1\Xi_3, \Xi_4\},$$
$$\{\Xi_1, \Xi_2 + \alpha_1\Xi_3 + a_1\Xi_4\}\},$$
$$\Theta^3 \equiv \{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_5\}, \{\Xi_1, \Xi_4, \Xi_5\},$$

$$\{\Xi_1,\Xi_2,\Xi_3+\alpha_1\Xi_4\},\{\Xi_1,\Xi_2+\alpha_1\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2+\alpha_1\Xi_4,\Xi_3\},$$
$$\{\Xi_1,\Xi_2+\alpha_1\Xi_4,\Xi_3+a_1\Xi_4\}\},$$
$$\Theta^4 \equiv \{\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4,\Xi_5\}\}.$$

Algebra ($A_{5,21}$). Let $A_{5,21}$ be the 5D Lie algebra spanned by $\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5\}$ with the non–zero commutators:

$$[\Xi_1,\Xi_5]=2\Xi_1,\quad [\Xi_2,\Xi_3]=\Xi_1,\quad [\Xi_2,\Xi_5]=\Xi_2+\Xi_3,\quad [\Xi_3,\Xi_5]=\Xi_3+\Xi_4,$$
$$[\Xi_4,\Xi_5]=\Xi_4.$$

The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\Theta^1 \equiv \{\{\Xi_1\},\{\Xi_2\},\{\Xi_3\},\{\Xi_4\},\{\Xi_5\},\{\Xi_2+a_1\Xi_4\},\{\Xi_1+\alpha_1\Xi_4\}\},$$
$$\Theta^2 \equiv \{\{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_3\},\{\Xi_1,\Xi_4\},\{\Xi_1,\Xi_5\},\{\Xi_2,\Xi_4\},\{\Xi_3,\Xi_4\},\{\Xi_4,\Xi_5\},$$
$$\{\Xi_1,\Xi_2+a_1\Xi_4\},\{\Xi_1+a_1\Xi_4,\Xi_2\},\{\Xi_1+a_1\Xi_4,\Xi_3\}\},$$
$$\Theta^3 \equiv \{\{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\{\Xi_1,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_4,\Xi_5\},\{\Xi_3,\Xi_4,\Xi_5\},$$
$$\{\Xi_1,\Xi_2+a_1\Xi_4,\Xi_3\}\},$$
$$\Theta^4 \equiv \{\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_3,\Xi_4,\Xi_5\}\}.$$

Algebra ($A_{5,22}$). Let $A_{5,22}$ be the 5D Lie algebra spanned by $\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5\}$ with the non–zero commutators:

$$[\Xi_2,\Xi_3]=\Xi_1,\quad [\Xi_2,\Xi_5]=\Xi_3,\quad [\Xi_4,\Xi_5]=\Xi_4.$$

The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\Theta^1 \equiv \{\{\Xi_1\},\{\Xi_2\},\{\Xi_3\},\{\Xi_4\},\{\Xi_5\},\{\Xi_1+\alpha_1\Xi_4\},\{\Xi_2+a_1\Xi_4\},\{\Xi_3+\alpha_1\Xi_4\},$$
$$\{\Xi_1+a_1\Xi_5\},\{\Xi_2+a_1\Xi_5\}\},$$
$$\Theta^2 \equiv \{\{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_3\},\{\Xi_1,\Xi_4\},\{\Xi_1,\Xi_5\},\{\Xi_2,\Xi_4\},\{\Xi_3,\Xi_4\},\{\Xi_3,\Xi_5\},\{\Xi_4,\Xi_5\},$$
$$\{\Xi_1,\Xi_2+a_1\Xi_4\},\{\Xi_1,\Xi_3+\alpha_1\Xi_4\},\{\Xi_1,\Xi_2+a_1\Xi_5\},\{\Xi_1+a_1\Xi_5,\Xi_3\},$$
$$\{\Xi_1+a_1\Xi_4,\Xi_2\},\{\Xi_1+\alpha_1\Xi_4,\Xi_3\},\{\Xi_1+a_1\Xi_5,\Xi_4\},\{\Xi_2+a_1\Xi_5,\Xi_4\}\},$$
$$\Theta^3 \equiv \{\{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\{\Xi_1,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_4,\Xi_5\},\{\Xi_3,\Xi_4,\Xi_5\},$$
$$\{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_4\},\{\Xi_1,\Xi_2+\alpha_1\Xi_4,\Xi_3\},\{\Xi_1,\Xi_2+a_1\Xi_5,\Xi_3\},$$
$$\{\Xi_1,\Xi_2+a_1\Xi_5,\Xi_4\},\{\Xi_1+a_1\Xi_5,\Xi_3,\Xi_4\}\},$$
$$\Theta^4 \equiv \{\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2+a_1\Xi_5,\Xi_3,\Xi_4\}\}.$$

Algebra ($A^b_{5,23}$). Let $A^b_{5,23}$ be the 5D Lie algebra spanned by $\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5\}$ with the non–zero commutators:

$$[\Xi_1,\Xi_5]=2\Xi_1,\quad [\Xi_2,\Xi_3]=\Xi_1,\quad [\Xi_2,\Xi_5]=\Xi_2+\Xi_3,\quad [\Xi_3,\Xi_5]=\Xi_3,$$
$$[\Xi_4,\Xi_5]=b\Xi_4,$$

with $b \neq 0$. The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\Theta^1 \equiv \{\{\Xi_1\},\{\Xi_2\},\{\Xi_3\},\{\Xi_4\},\{\Xi_5\},\{\Xi_1+\alpha_1\Xi_4\},\{\Xi_2+a_1\Xi_4\},\{\Xi_3+\alpha_1\Xi_4\}\},$$

$$\begin{aligned}
\Theta^2 \equiv \{ & \{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_3\},\{\Xi_1,\Xi_4\},\{\Xi_1,\Xi_5\},\{\Xi_2,\Xi_4\},\{\Xi_3,\Xi_4\},\{\Xi_3,\Xi_5\},\{\Xi_4,\Xi_5\},\\
& \{\Xi_1,\Xi_2+a_1\Xi_4\},\{\Xi_1,\Xi_3+\alpha_1\Xi_4\},\{\Xi_1+a_1\Xi_4,\Xi_2\},\{\Xi_1+\alpha_1\Xi_4,\Xi_3\}\},\\
\Theta^3 \equiv \{ & \{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\{\Xi_1,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_4,\Xi_5\},\{\Xi_3,\Xi_4,\Xi_5\},\\
& \{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_4\},\{\Xi_1,\Xi_2+\alpha_1\Xi_4,\Xi_3\}\},\\
\Theta^4 \equiv \{ & \{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4,\Xi_5\}\}.
\end{aligned}$$

Algebra ($A^{\epsilon}_{5,24}$). Let $A^{\epsilon}_{5,24}$ be the 5D Lie algebra spanned by $\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5\}$ with the non–zero commutators:

$$\begin{aligned}
&[\Xi_1,\Xi_5]=2\Xi_1, \quad [\Xi_2,\Xi_3]=\Xi_1, \quad [\Xi_2,\Xi_5]=\Xi_2+\Xi_3,\\
&[\Xi_3,\Xi_5]=\Xi_3, \quad [\Xi_4,\Xi_5]=2\Xi_4+\epsilon\Xi_1,
\end{aligned}$$

with $\epsilon=\pm1$. The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\begin{aligned}
\Theta^1 \equiv \{ & \{\Xi_1\},\{\Xi_2\},\{\Xi_3\},\{\Xi_4\},\{\Xi_5\},\{\Xi_2+a_1\Xi_4\},\{\Xi_3+a_1\Xi_4\}\},\\
\Theta^2 \equiv \{ & \{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_3\},\{\Xi_1,\Xi_4\},\{\Xi_1,\Xi_5\},\{\Xi_2,\Xi_4\},\{\Xi_3,\Xi_4\},\{\Xi_3,\Xi_5\},\\
& \{\Xi_1,\Xi_2+a_1\Xi_4\},\{\Xi_1,\Xi_3+\alpha_1\Xi_4\},\{\Xi_2+a_1\Xi_3,\Xi_4\}\},\\
\Theta^3 \equiv \{ & \{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\{\Xi_1,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_4,\Xi_5\},\\
& \{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_4\},\{\Xi_1,\Xi_2+\alpha_1\Xi_4,\Xi_3\}\},\\
\Theta^4 \equiv \{ & \{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4,\Xi_5\}\}.
\end{aligned}$$

Algebra ($A^{bp}_{5,25}$). Let $A^{bp}_{5,25}$ be the 5D Lie algebra spanned by $\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5\}$ with the non–zero commutators:

$$\begin{aligned}
&[\Xi_1,\Xi_5]=2p\Xi_1, \quad [\Xi_2,\Xi_3]=\Xi_1, \quad [\Xi_2,\Xi_5]=\Xi_3+p\Xi_2, \quad [\Xi_3,\Xi_5]=p\Xi_3-\Xi_2,\\
&[\Xi_4,\Xi_5]=b\Xi_4,
\end{aligned}$$

with $p\in\mathbb{R}$, $b\neq0$. The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\begin{aligned}
\Theta^1 \equiv \{ & \{\Xi_1\},\{\Xi_2\},\{\Xi_4\},\{\Xi_5\},\{\Xi_1+\alpha_1\Xi_4\},\{\Xi_2+a_1\Xi_4\}\},\\
\Theta^2 \equiv \{ & \{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_4\},\{\Xi_1,\Xi_5\},\{\Xi_2,\Xi_4\},\{\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2+a_1\Xi_4\},\\
& \{\Xi_1+a_1\Xi_4,\Xi_2\}\},\\
\Theta^3 \equiv \{ & \{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\{\Xi_1,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_4\}\},\\
\Theta^4 \equiv \{ & \{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_5\}\}.
\end{aligned}$$

Algebra ($A^{p\epsilon}_{5,26}$). Let $A^{p\epsilon}_{5,26}$ be the 5D Lie algebra spanned by $\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5\}$ with the non–zero commutators:

$$\begin{aligned}
&[\Xi_1,\Xi_5]=2p\Xi_1, \quad [\Xi_2,\Xi_3]=\Xi_1, \quad [\Xi_2,\Xi_5]=\Xi_3+p\Xi_2, \quad [\Xi_3,\Xi_5]=p\Xi_3-\Xi_2,\\
&[\Xi_4,\Xi_5]=2p\Xi_4+\epsilon\Xi_1
\end{aligned}$$

with $p\in\mathbb{R}$, $\epsilon=\pm1$. The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\Theta^1 \equiv \{\{\Xi_1\},\{\Xi_2\},\{\Xi_4\},\{\Xi_5\},\{\Xi_2+a_1\Xi_4\}\},$$

$$\Theta^2 \equiv \{\{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_4\},\{\Xi_1,\Xi_5\},\{\Xi_1+a_1\Xi_4,\Xi_2\},\{\Xi_1,\Xi_2+a_1\Xi_4\}\},$$
$$\Theta^3 \equiv \{\{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\{\Xi_1,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_4\}\},$$
$$\Theta^4 \equiv \{\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_5\}\}.$$

Algebra ($A_{5,27}$). Let $A_{5,27}$ be the 5D Lie algebra spanned by $\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5\}$ with the non–zero commutators:

$$[\Xi_1,\Xi_5]=\Xi_1,\quad [\Xi_2,\Xi_3]=\Xi_1,\quad [\Xi_3,\Xi_5]=\Xi_3+\Xi_4,\quad [\Xi_4,\Xi_5]=\Xi_1+\Xi_4.$$

The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\Theta^1 \equiv \{\{\Xi_1\},\{\Xi_2\},\{\Xi_3\},\{\Xi_4\},\{\Xi_5\},\{\Xi_2+a_1\Xi_3\},\{\Xi_2+a_1\Xi_4\},\{\Xi_2+a_1\Xi_5\}\},$$
$$\Theta^2 \equiv \{\{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_3\},\{\Xi_1,\Xi_4\},\{\Xi_1,\Xi_5\},\{\Xi_2,\Xi_4\},\{\Xi_2,\Xi_5\},\{\Xi_3,\Xi_4\},$$
$$\{\Xi_1,\Xi_2+a_1\Xi_3\},\{\Xi_1,\Xi_2+\alpha_1\Xi_4\},\{\Xi_1,\Xi_2+a_1\Xi_5\},\{\Xi_2+a_1\Xi_3,\Xi_4\}\},$$
$$\Theta^3 \equiv \{\{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_4,\Xi_5\},$$
$$\{\Xi_1,\Xi_2+\alpha_1\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2+a_1\Xi_4,\Xi_3\},\{\Xi_1,\Xi_2+a_1\Xi_5,\Xi_4\}\},$$
$$\Theta^4 \equiv \{\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4,\Xi_5\},$$
$$\{\Xi_1,\Xi_2+a_1\Xi_5,\Xi_3,\Xi_4\}\}.$$

Algebra ($A^a_{5,28}$). Let $A^a_{5,28}$ be the 5D Lie algebra spanned by $\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5\}$ with the non–zero commutators:

$$[\Xi_1,\Xi_5]=a\Xi_1,\quad [\Xi_2,\Xi_3]=\Xi_1,\quad [\Xi_2,\Xi_5]=(a-1)\Xi_2,\quad [\Xi_3,\Xi_5]=\Xi_3+\Xi_4,$$
$$[\Xi_4,\Xi_5]=\Xi_4,$$

with $a\in\mathbb{R}$. The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\Theta^1 \equiv \{\{\Xi_1\},\{\Xi_2\},\{\Xi_3\},\{\Xi_4\},\{\Xi_5\},\{\Xi_2+a_1\Xi_3\},\{\Xi_1+\alpha_1\Xi_4\},\{\Xi_2+\alpha_1\Xi_4\}\},$$
$$\Theta^2 \equiv \{\{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_3\},\{\Xi_1,\Xi_4\},\{\Xi_1,\Xi_5\},\{\Xi_2,\Xi_4\},\{\Xi_2,\Xi_5\},\{\Xi_3,\Xi_4\},\{\Xi_4,\Xi_5\},$$
$$\{\Xi_1,\Xi_2+a_1\Xi_3\},\{\Xi_1,\Xi_2+\alpha_1\Xi_4\},\{\Xi_2+\alpha_1\Xi_3,\Xi_4\},\{\Xi_1+\alpha_1\Xi_4,\Xi_2\},$$
$$\{\Xi_1+a_1\Xi_4,\Xi_3\},\{\Xi_1+a_1\Xi_4,\Xi_2+a_2\Xi_3\}\},$$
$$\Theta^3 \equiv \{\{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_4,\Xi_5\},\{\Xi_2,\Xi_4,\Xi_5\},$$
$$\{\Xi_3,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2+\alpha_1\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2+a_1\Xi_4,\Xi_3\}\},$$
$$\Theta^4 \equiv \{\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4,\Xi_5\}\}.$$

Algebra ($A_{5,29}$). Let $A_{5,29}$ be the 5D Lie algebra spanned by $\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5\}$ with the non–zero commutators:

$$[\Xi_1,\Xi_5]=\Xi_1,\quad [\Xi_2,\Xi_4]=\Xi_1,\quad [\Xi_2,\Xi_5]=\Xi_2,\quad [\Xi_4,\Xi_5]=\Xi_3.$$

The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\Theta^1 \equiv \{\{\Xi_1\},\{\Xi_2\},\{\Xi_3\},\{\Xi_4\},\{\Xi_5\},\{\Xi_1+\alpha_1\Xi_3\},\{\Xi_2+\alpha_1\Xi_3\},\{\Xi_2+a_1\Xi_4\},$$
$$\{\Xi_4+a_1\Xi_5\}\},$$

$$
\begin{aligned}
\Theta^2 \equiv \{ & \{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_5\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_5\}, \{\Xi_3, \Xi_4\}, \{\Xi_3, \Xi_5\}, \\
& \{\Xi_1, \Xi_2 + \alpha_1 \Xi_3\}, \{\Xi_1, \Xi_2 + a_1 \Xi_4\}, \{\Xi_1, \Xi_4 + a_1 \Xi_5\}, \{\Xi_2, \Xi_3 + a_1 \Xi_5\}, \\
& \{\Xi_3, \Xi_4 + a_1 \Xi_5\}, \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2\}, \{\Xi_1 + a_1 \Xi_3, \Xi_4\}, \{\Xi_2 + \alpha_1 \Xi_4, \Xi_3\}, \\
& \{\Xi_1 + a_1 \Xi_3, \Xi_2 + a_2 \Xi_4\}\}, \\
\Theta^3 \equiv \{ & \{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_5\}, \{\Xi_2, \Xi_3, \Xi_5\}, \\
& \{\Xi_3, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_4 + a_1 \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4 + a_1 \Xi_5\}, \{\Xi_1, \Xi_2 + a_1 \Xi_3, \Xi_4\}, \\
& \{\Xi_1, \Xi_2 + \alpha_1 \Xi_4, \Xi_3\}\}, \\
\Theta^4 \equiv \{ & \{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_4 + a_1 \Xi_5\}\}.
\end{aligned}
$$

Algebra ($A_{5,30}$). Let $A_{5,30}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$
\begin{aligned}
& [\Xi_1, \Xi_5] = (a+1)\Xi_1, \quad [\Xi_2, \Xi_4] = \Xi_1, \quad [\Xi_2, \Xi_5] = a\Xi_2, \quad [\Xi_3, \Xi_4] = \Xi_2, \\
& [\Xi_3, \Xi_5] = (a-1)\Xi_3, \quad [\Xi_4, \Xi_5] = \Xi_4.
\end{aligned}
$$

The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$
\begin{aligned}
\Theta^1 \equiv \{ & \{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_5\}, \{\Xi_1 + \alpha_1 \Xi_3\}, \{\Xi_3 + \alpha_1 \Xi_4\}, \{\Xi_1 + a_1 \Xi_5\}\}, \\
\Theta^2 \equiv \{ & \{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_5\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_5\}, \{\Xi_3, \Xi_5\}, \{\Xi_4, \Xi_5\}, \\
& \{\Xi_1, \Xi_3 + \alpha_1 \Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2\}, \{\Xi_1 + a_1 \Xi_5, \Xi_2\}, \{\Xi_1 + a_1 \Xi_5, \Xi_3\}, \\
& \{\Xi_1 + a_1 \Xi_5, \Xi_4\}\}, \\
\Theta^3 \equiv \{ & \{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_5\}, \{\Xi_1, \Xi_4, \Xi_5\}, \{\Xi_2, \Xi_3, \Xi_5\}, \\
& \{\Xi_1, \Xi_2, \Xi_3 + \alpha_1 \Xi_4\}, \{\Xi_1 + a_1 \Xi_5, \Xi_2, \Xi_3\}\}, \\
\Theta^4 \equiv \{ & \{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_4, \Xi_5\}\}.
\end{aligned}
$$

Algebra ($A_{5,31}$). Let $A_{5,31}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$
\begin{aligned}
& [\Xi_1, \Xi_5] = 3\Xi_1, \quad [\Xi_2, \Xi_4] = \Xi_1, \quad [\Xi_2, \Xi_5] = 2\Xi_2, \quad [\Xi_3, \Xi_4] = \Xi_2, \quad [\Xi_3, \Xi_5] = \Xi_3, \\
& [\Xi_4, \Xi_5] = \Xi_3 + \Xi_4.
\end{aligned}
$$

The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$
\begin{aligned}
\Theta^1 \equiv \{ & \{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_5\}, \{\Xi_1 + \alpha_1 \Xi_3\}\}, \\
\Theta^2 \equiv \{ & \{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_5\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_5\}, \{\Xi_3, \Xi_5\}, \\
& \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2\}\}, \\
\Theta^3 \equiv \{ & \{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_5\}, \{\Xi_2, \Xi_3, \Xi_5\}\}, \\
\Theta^4 \equiv \{ & \{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_5\}\}.
\end{aligned}
$$

Algebra ($A^a_{5,32}$). Let $A^a_{5,32}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$
\begin{aligned}
& [\Xi_1, \Xi_5] = \Xi_1, \quad [\Xi_2, \Xi_4] = \Xi_1, \quad [\Xi_2, \Xi_5] = \Xi_2, \quad [\Xi_3, \Xi_4] = \Xi_2, \\
& [\Xi_3, \Xi_5] = a\Xi_1 + \Xi_3,
\end{aligned}
$$

with $a \in \mathbb{R}$. The complete optimal system of families of Lie subalgebras computed by SymboLie is as follows:

$$\begin{aligned}
\Theta^1 \equiv \{&\{\Xi_1\},\{\Xi_2\},\{\Xi_3\},\{\Xi_4\},\{\Xi_5\},\{\Xi_3+a_1\Xi_4\},\{\Xi_4+a_1\Xi_5\}\},\\
\Theta^2 \equiv \{&\{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_3\},\{\Xi_1,\Xi_4\},\{\Xi_1,\Xi_5\},\{\Xi_2,\Xi_3\},\{\Xi_2,\Xi_5\},\{\Xi_4,\Xi_5\},\\
&\{\Xi_1,\Xi_3+\alpha_1\Xi_4\},\{\Xi_1,\Xi_4+a_1\Xi_5\}\},\\
\Theta^3 \equiv \{&\{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_4,\Xi_5\},\\
&\{\Xi_1,\Xi_2,\Xi_3+\alpha_1\Xi_4\},\{\Xi_1,\Xi_2,\Xi_4+a_1\Xi_5\}\},\\
\Theta^4 \equiv \{&\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_4,\Xi_5\},\\
&\{\Xi_1,\Xi_2,\Xi_3,\Xi_4+a_1\Xi_5\}\}
\end{aligned}$$

Algebra ($A^{ab}_{5,33}$). Let $A^{ab}_{5,33}$ be the 5D Lie algebra spanned by $\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5\}$ with the non–zero commutators:

$$[\Xi_1,\Xi_4]=\Xi_1,\quad [\Xi_2,\Xi_5]=\Xi_2,\quad [\Xi_3,\Xi_4]=b\Xi_3,\quad [\Xi_3,\Xi_5]=a\Xi_3,$$

with $a^2+b^2\neq 0$. The complete optimal system of families of Lie subalgebras computed by SymboLie is as follows:

$$\begin{aligned}
\Theta^1 \equiv \{&\{\Xi_1\},\{\Xi_2\},\{\Xi_3\},\{\Xi_4\},\{\Xi_5\},\{\Xi_1+\alpha_1\Xi_2\},\{\Xi_1+\alpha_1\Xi_3\},\\
&\{\Xi_2+\alpha_1\Xi_3\},\{\Xi_2+\alpha_1\Xi_4\},\{\Xi_1+\alpha_1\Xi_5\},\{\Xi_4+a_1\Xi_5\},\\
&\{\Xi_1+\alpha_1\Xi_2+\alpha_2\Xi_3\}\},\\
\Theta^2 \equiv \{&\{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_3\},\{\Xi_1,\Xi_4\},\{\Xi_1,\Xi_5\},\{\Xi_2,\Xi_3\},\{\Xi_2,\Xi_4\},\{\Xi_2,\Xi_5\},\\
&\{\Xi_3,\Xi_4\},\{\Xi_3,\Xi_5\},\{\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2+\alpha_1\Xi_3\},\{\Xi_1,\Xi_2+\alpha_1\Xi_4\},\\
&\{\Xi_1,\Xi_4+a_1\Xi_5\},\{\Xi_2,\Xi_4+a_1\Xi_5\},\{\Xi_1+\alpha_1\Xi_2,\Xi_3\},\{\Xi_3,\Xi_4+a_1\Xi_5\},\\
&\{\Xi_1+\alpha_1\Xi_3,\Xi_2\},\{\Xi_2+\alpha_1\Xi_4,\Xi_3\},\{\Xi_1+\alpha_1\Xi_5,\Xi_2\},\\
&\{\Xi_1+\alpha_1\Xi_5,\Xi_3\},\{\Xi_1+\alpha_1\Xi_3,\Xi_2+\alpha_2\Xi_3\}\},\\
\Theta^3 \equiv \{&\{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_3,\Xi_5\},\\
&\{\Xi_1,\Xi_4,\Xi_5\},\{\Xi_2,\Xi_3,\Xi_4\},\{\Xi_2,\Xi_3,\Xi_5\},\{\Xi_2,\Xi_4,\Xi_5\},\{\Xi_3,\Xi_4,\Xi_5\},\\
&\{\Xi_1,\Xi_2,\Xi_4+a_1\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4+a_1\Xi_5\},\{\Xi_1,\Xi_2+\alpha_1\Xi_4,\Xi_3\},\\
&\{\Xi_2,\Xi_3,\Xi_4+a_1\Xi_5\},\{\Xi_1+\alpha_1\Xi_5,\Xi_2,\Xi_3\}\},\\
\Theta^4 \equiv \{&\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4,\Xi_5\},\\
&\{\Xi_2,\Xi_3,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_4+a_1\Xi_5\}\}.
\end{aligned}$$

Algebra ($A^{a}_{5,34}$). Let $A^{a}_{5,34}$ be the 5D Lie algebra spanned by $\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5\}$ with the non–zero commutators:

$$[\Xi_1,\Xi_4]=a\Xi_1,\quad [\Xi_1,\Xi_5]=\Xi_1,\quad [\Xi_2,\Xi_4]=\Xi_2,\quad [\Xi_3,\Xi_4]=\Xi_3,\quad [\Xi_3,\Xi_5]=\Xi_2,$$

with $a \in \mathbb{R}$. The complete optimal system of families of Lie subalgebras computed by SymboLie is as follows:

$$\begin{aligned}
\Theta^1 \equiv \{&\{\Xi_1\},\{\Xi_2\},\{\Xi_3\},\{\Xi_4\},\{\Xi_5\},\{\Xi_1+\alpha_1\Xi_2\},\{\Xi_1+\alpha_1\Xi_3\},\{\Xi_3+\alpha_1\Xi_5\},\\
&\{\Xi_4+a_1\Xi_5\}\},\\
\Theta^2 \equiv \{&\{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_3\},\{\Xi_1,\Xi_4\},\{\Xi_1,\Xi_5\},\{\Xi_2,\Xi_3\},\{\Xi_2,\Xi_4\},\{\Xi_2,\Xi_5\},\{\Xi_3,\Xi_4\},\\
&\{\Xi_4,\Xi_5\},\{\Xi_1,\Xi_3+\alpha_1\Xi_5\},\{\Xi_1,\Xi_4+a_1\Xi_5\},\{\Xi_2,\Xi_3+\alpha_1\Xi_5\},\{\Xi_2,\Xi_4+a_1\Xi_5\},
\end{aligned}$$

$\{\Xi_1+\alpha_1\Xi_2,\Xi_3\},\{\Xi_1+\alpha_1\Xi_3,\Xi_2\}\},$

$\Theta^3 \equiv \{\{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_4,\Xi_5\},\{\Xi_2,\Xi_3,\Xi_4\},$
$\{\Xi_2,\Xi_3,\Xi_5\},\{\Xi_2,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_3+\alpha_1\Xi_5\},\{\Xi_1,\Xi_2,\Xi_4+a_1\Xi_5\},$
$\{\Xi_2,\Xi_3,\Xi_4+a_1\Xi_5\}\},$

$\Theta^4 \equiv \{\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_4,\Xi_5\},\{\Xi_2,\Xi_3,\Xi_4,\Xi_5\},$
$\{\Xi_1,\Xi_2,\Xi_3,\Xi_4+a_1\Xi_5\}\}.$

Algebra ($A^{ab}_{5,35}$). Let $A^{ab}_{5,35}$ be the 5D Lie algebra spanned by $\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5\}$ with the non–zero commutators:

$$[\Xi_1,\Xi_4]=b\Xi_1,\quad [\Xi_1,\Xi_5]=a\Xi_1,\quad [\Xi_2,\Xi_4]=\Xi_2,\quad [\Xi_2,\Xi_5]=-\Xi_3,\quad [\Xi_3,\Xi_4]=\Xi_3,$$
$$[\Xi_3,\Xi_5]=\Xi_2,$$

with $a^2+b^2\neq 0$. The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$\Theta^1 \equiv \{\{\Xi_1\},\{\Xi_2\},\{\Xi_4\},\{\Xi_5\},\{\Xi_1+\alpha_1\Xi_2\},\{\Xi_4+a_1\Xi_5\}\},$

$\Theta^2 \equiv \{\{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_4\},\{\Xi_1,\Xi_5\},\{\Xi_2,\Xi_3\},\{\Xi_2,\Xi_4\},\{\Xi_4,\Xi_5\},\{\Xi_1,\Xi_4+a_1\Xi_5\},$
$\{\Xi_1+\alpha_1\Xi_2,\Xi_3\}\},$

$\Theta^3 \equiv \{\{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\{\Xi_1,\Xi_4,\Xi_5\},\{\Xi_2,\Xi_3,\Xi_4\},\{\Xi_2,\Xi_3,\Xi_5\},$
$\{\Xi_2,\Xi_3,\Xi_4+a_1\Xi_5\}\},$

$\Theta^4 \equiv \{\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_5\},\{\Xi_2,\Xi_3,\Xi_4,\Xi_5\},$
$\{\Xi_1,\Xi_2,\Xi_3,\Xi_4+a_1\Xi_5\}\}.$

Algebra ($A_{5,36}$). Let $A_{5,36}$ be the 5D Lie algebra spanned by $\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5\}$ with the non–zero commutators:

$$[\Xi_1,\Xi_4]=\Xi_1,\quad [\Xi_2,\Xi_3]=\Xi_1,\quad [\Xi_2,\Xi_4]=\Xi_2,\quad [\Xi_2,\Xi_5]=-\Xi_2,\quad [\Xi_3,\Xi_5]=\Xi_3.$$

The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$\Theta^1 \equiv \{\{\Xi_1\},\{\Xi_2\},\{\Xi_3\},\{\Xi_4\},\{\Xi_5\},\{\Xi_2+\alpha_1\Xi_3\},\{\Xi_3+\alpha_1\Xi_4\},\{\Xi_1+\alpha_1\Xi_5\},$
$\{\Xi_4+a_1\Xi_5\}\},$

$\Theta^2 \equiv \{\{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_3\},\{\Xi_1,\Xi_4\},\{\Xi_1,\Xi_5\},\{\Xi_2,\Xi_4\},\{\Xi_2,\Xi_5\},\{\Xi_3,\Xi_4\},\{\Xi_3,\Xi_5\},$
$\{\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2+\alpha_1\Xi_3\},\{\Xi_1,\Xi_3+\alpha_1\Xi_4\},\{\Xi_1,\Xi_4+a_1\Xi_5\},\{\Xi_2,\Xi_4+a_1\Xi_5\},$
$\{\Xi_3,\Xi_4+a_1\Xi_5\},\{\Xi_1+\alpha_1\Xi_5,\Xi_2\},\{\Xi_1+\alpha_1\Xi_5,\Xi_3\}\},$

$\Theta^3 \equiv \{\{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_4,\Xi_5\},$
$\{\Xi_2,\Xi_4,\Xi_5\},\{\Xi_3,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_3+\alpha_1\Xi_4\},\{\Xi_1,\Xi_2,\Xi_4+a_1\Xi_5\},$
$\{\Xi_1,\Xi_3,\Xi_4+a_1\Xi_5\}\},$

$\Theta^4 \equiv \{\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4,\Xi_5\},$
$\{\Xi_1,\Xi_2,\Xi_3,\Xi_4+a_1\Xi_5\}\}.$

Algebra ($A_{5,37}$). Let $A_{5,37}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$[\Xi_1, \Xi_4] = 2\Xi_1, \quad [\Xi_2, \Xi_3] = \Xi_1, \quad [\Xi_2, \Xi_4] = \Xi_2, \quad [\Xi_2, \Xi_5] = -\Xi_3,$$
$$[\Xi_3, \Xi_4] = \Xi_3, \quad [\Xi_3, \Xi_5] = \Xi_2.$$

The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\Theta^1 \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_4\}, \{\Xi_5\}, \{\Xi_1 + \alpha_1\Xi_5\}, \{\Xi_4 + a_1\Xi_5\}\},$$
$$\Theta^2 \equiv \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_5\}, \{\Xi_2, \Xi_4\}, \{\Xi_4, \Xi_5\}, \{\Xi_1, \Xi_4 + a_1\Xi_5\}\},$$
$$\Theta^3 \equiv \{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_4, \Xi_5\}\},$$
$$\Theta^4 \equiv \{\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_4 + a_1\Xi_5\}\}.$$

Algebra ($A_{5,38}$). Let $A_{5,38}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$[\Xi_1, \Xi_4] = \Xi_1, \quad [\Xi_2, \Xi_5] = \Xi_2, \quad [\Xi_4, \Xi_5] = \Xi_3.$$

The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\begin{aligned}
\Theta^1 \equiv \{&\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_5\}, \{\Xi_1 + \alpha_1\Xi_2\}, \{\Xi_1 + \alpha_1\Xi_3\}, \{\Xi_2 + \alpha_1\Xi_3\},\\
&\{\Xi_2 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_5\}, \{\Xi_4 + a_1\Xi_5\}, \{\Xi_1 + \alpha_1\Xi_2 + a_1\Xi_3\}\},\\
\Theta^2 \equiv \{&\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_5\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_2, \Xi_5\}, \{\Xi_3, \Xi_4\},\\
&\{\Xi_3, \Xi_5\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_3\}, \{\Xi_1, \Xi_2 + a_1\Xi_4\}, \{\Xi_1, \Xi_4 + a_1\Xi_5\}, \{\Xi_2, \Xi_4 + a_1\Xi_5\},\\
&\{\Xi_1 + \alpha_1\Xi_2, \Xi_3\}, \{\Xi_3, \Xi_4 + a_1\Xi_5\}, \{\Xi_1 + \alpha_1\Xi_3, \Xi_2\}, \{\Xi_1 + a_1\Xi_3, \Xi_5\},\\
&\{\Xi_2 + a_1\Xi_3, \Xi_4\}, \{\Xi_2 + \alpha_1\Xi_4, \Xi_3\}, \{\Xi_1 + a_1\Xi_5, \Xi_2\}, \{\Xi_1 + \alpha_1\Xi_5, \Xi_3\},\\
&\{\Xi_1 + \alpha_1\Xi_3, \Xi_2 + \alpha_2\Xi_3\}\},\\
\Theta^3 \equiv \{&\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_5\}, \{\Xi_2, \Xi_3, \Xi_4\},\\
&\{\Xi_2, \Xi_3, \Xi_5\}, \{\Xi_3, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_4 + a_1\Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4 + a_1\Xi_5\},\\
&\{\Xi_1, \Xi_2 + a_1\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_4, \Xi_3\},\\
&\{\Xi_2, \Xi_3, \Xi_4 + a_1\Xi_5\}, \{\Xi_1 + a_1\Xi_3, \Xi_2, \Xi_5\}, \{\Xi_1 + \alpha_1\Xi_5, \Xi_2, \Xi_3\}\},\\
\Theta^4 \equiv \{&\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4, \Xi_5\}, \{\Xi_2, \Xi_3, \Xi_4, \Xi_5\},\\
&\{\Xi_1, \Xi_2, \Xi_3, \Xi_4 + a_1\Xi_5\}\}.
\end{aligned}$$

Algebra ($A_{5,39}$). Let $A_{5,39}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$[\Xi_1, \Xi_4] = \Xi_1, \quad [\Xi_1, \Xi_5] = -\Xi_2, \quad [\Xi_2, \Xi_4] = \Xi_2, \quad [\Xi_2, \Xi_5] = \Xi_1, \quad [\Xi_4, \Xi_5] = \Xi_3.$$

The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\begin{aligned}
\Theta^1 \equiv \{&\{\Xi_1\}, \{\Xi_3\}, \{\Xi_3 + \alpha_1\Xi_4\}, \{\Xi_5\}, \{\Xi_1 + \alpha_1\Xi_3\}, \{\Xi_4 + a_1\Xi_5\}\},\\
\Theta^2 \equiv \{&\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_3 + a_1\Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_3, \Xi_5\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_3\},\\
&\{\Xi_3, \Xi_4 + a_1\Xi_5\}\},\\
\Theta^3 \equiv \{&\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_3 + \alpha_1\Xi_4\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_3, \Xi_4, \Xi_5\},
\end{aligned}$$

$$\{\Xi_1, \Xi_2, \Xi_4 + a_1\Xi_5\}\},$$
$$\Theta^4 \equiv \{\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_4 + a_1\Xi_5\}\}.$$

Algebra ($A_{5,40}$). Let $A_{5,40}$ be the 5D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}$ with the non–zero commutators:

$$[\Xi_1, \Xi_2] = 2\Xi_1, \quad [\Xi_1, \Xi_3] = -\Xi_2, \quad [\Xi_1, \Xi_4] = \Xi_5, \quad [\Xi_2, \Xi_3] = 2\Xi_3,$$
$$[\Xi_2, \Xi_4] = \Xi_4, \quad [\Xi_2, \Xi_5] = -\Xi_5, \quad [\Xi_3, \Xi_5] = \Xi_4.$$

The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\Theta^1 \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_4\}, \{\Xi_1 + \alpha_1\Xi_3\}, \{\Xi_1 + \alpha_1\Xi_4\}\},$$
$$\Theta^2 \equiv \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_5\}, \{\Xi_2, \Xi_4\}, \{\Xi_4, \Xi_5\}, \{\Xi_1 + \alpha_1\Xi_4, \Xi_5\}\},$$
$$\Theta^3 \equiv \{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_4, \Xi_5\}, \{\Xi_2, \Xi_4, \Xi_5\}, \{\Xi_1 + \alpha_1\Xi_3, \Xi_4, \Xi_5\}\},$$
$$\Theta^4 \equiv \{\{\Xi_1, \Xi_2, \Xi_4, \Xi_5\}\}.$$

7.3 Optimal systems of Noether symmetries of geodesic equations

A very important example of nonlinear ordinary differential equations is the set of geodesic equations in an n–dimensional manifold. Let

$$ds^2 = \sum_{i,k=1}^{n} g_{ik} dq^i dq^k$$

be the metric in an n–dimensional manifold. The geodesic equations read [78]

$$\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = 0 \quad (i, j, k = 1, \ldots, n),$$

where Γ^i_{jk} stands for the *Christoffel symbol* defined by

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\ell}(g_{j\ell,k} + g_{k\ell,j} - g_{jk,\ell}),$$

with g^{ik} denoting the inverse of the metric tensor, such that $g_{ij}g^{ik} = \delta^k_j$, and $g_{ij,k} = \partial g_{ij}/\partial q^k$.

This system is composed of second-order, homogeneous, and coupled ODEs. Here, q^i denotes the generalized coordinates, while $\dot{q}^i$ represents their derivatives with respect to the parameter s. These equations describe the paths taken by objects under the influence of a gravitational field. Studying their symmetries provides insights into the geometric and physical properties of the space-time in question.

The underlying concept of geodesics extends the notion of straight lines in Euclidean space to Riemannian manifolds.

In [21], the authors focus on the Lie algebra associated with the Noether symmetries of the geodesic equations in some different metrics.

In particular, Jamil et al. [21] limit themselves to find the optimal systems of one-dimensional subalgebras of Noether symmetries associated with systems of geodesic equations. Using `SymboLie`, we determine the complete optimal systems from dimension one to $r-1$ (where r is the dimension of the corresponding Lie algebra) for the cases they considered.

Let us now give the definition of variational symmetries. Consider a vector field

$$\Xi = \xi(s, x^j)\frac{\partial}{\partial s} + \eta_i(s, x^j)\frac{\partial}{\partial x^i},$$

where s, the arc length parameter, is the independent variable and $x^i = x^i(s)$ ($i = 1, \ldots, n$) are the dependent variables. The corresponding Lie group of transformations

$$s^\star = s^\star(s, \mathbf{x}), \qquad \mathbf{x}^\star = \mathbf{x}^\star(s, \mathbf{x}) \tag{7.1}$$

leaving the action integral of the Lagrangian $\mathcal{L}(s, x^i, \dot{x}^i)$ invariant, *i.e.*,

$$\int_{\Omega^\star} \mathcal{L}(s^\star, \mathbf{x}^\star, \dot{\mathbf{x}}^\star)\, ds^\star = \int_\Omega \mathcal{L}(s, \mathbf{x}, \dot{\mathbf{x}})\, ds$$

where $\Omega^\star$ is the image of Ω under the point transformation. Then, we have

$$\begin{aligned} ds^\star &= \left(\frac{\partial s^\star}{\partial s} + \dot{x}^i \frac{\partial s^\star}{\partial x^i}\right) ds \\ &= D_s(s^\star(s, \mathbf{x}(s)))ds \\ &= D_s(s + a\xi(s, x^i) + O(a^2))ds \\ &= \left(1 + a\left(\frac{\partial \xi}{\partial s} + \dot{x}^i \frac{\partial}{\partial x^i}\right) + O(a^2)\right) ds \\ &= \left(1 + a(D_s\xi) + O(a^2)\right) ds, \end{aligned}$$

where $D_s = \dfrac{\partial}{\partial s} + \dot{x}^i \dfrac{\partial}{\partial x^i}$, and the dot denotes differentiation with respect to s.

Since (7.1) is a Lie group of point transformations, by using the exponential map of the first–order infinitesimal generator, it is

$$\mathcal{L}(s^\star, \mathbf{x}^\star, \dot{\mathbf{x}}^\star) = \exp\left(a\Xi^{(1)}\right) \mathcal{L}(s, \mathbf{x}, \dot{\mathbf{x}}).$$

Therefore, (7.1) is a Lie point symmetry of the action integral of the Lagrangian $\mathcal{L}(s, \mathbf{x}, \dot{\mathbf{x}})$ if and only if

$$\begin{aligned} 0 \equiv& \int_\Omega \left(\left(1 + a(D_s\xi) + o(a^2)\right) \exp\left(a\Xi^{(1)}\right) - 1\right) \mathcal{L}(s, \mathbf{x}, \dot{\mathbf{x}})ds = \\ &= a \int_\Omega \mathcal{L}(s, \mathbf{x}, \dot{\mathbf{x}}) \left(D_s\xi(s, \mathbf{x}) + \Xi^{(1)}\mathcal{L}(s, \mathbf{x}, \dot{\mathbf{x}})\right) ds + O(a^2), \end{aligned}$$

and this implies that the $O(a)$ terms have to vanish, *i.e.*,

$$\mathcal{L}(s, \mathbf{x}, \dot{\mathbf{x}})\left(D_s\xi(s, \mathbf{x})\right) + \Xi^{(1)}\mathcal{L}(s, \mathbf{x}, \dot{\mathbf{x}}) = 0.$$

A Lie symmetry of the action integral of the Lagrangian is called a *variational symmetry* or *Noether symmetry*.

Whereas, Ξ is a *divergence symmetry* of the Lagrangian $\mathcal{L}(s, \mathbf{x}, \dot{\mathbf{x}})$ if there exists a gauge function, $A(s, x^i)$, such that

$$\mathcal{L}(s, \mathbf{x}, \dot{\mathbf{x}})\left(D_s\xi(s, \mathbf{x})\right) + \Xi^{(1)}\mathcal{L}(s, \mathbf{x}, \dot{\mathbf{x}}) = D_s A.$$

In out framework, the Lagrangian has the form

$$\mathcal{L}(s, \mathbf{x}, \dot{\mathbf{x}}) = g_{ij}(x^k)\dot{x}^i\dot{x}^j.$$

By deriving the Euler–Lagrange equations

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^h}\right) - \frac{\partial \mathcal{L}}{\partial x^h} = 0,$$

with respect to the given metric tensor, we obtain the corresponding geodesic equations.

7.3.1 System admitting five Noether symmetries

Consider the geodetic Lagrangian of the metric

$$ds^2 = e^{2\nu(x)}dt^2 - dx^2 - e^{2\mu(x)}\left(dy^2 + dz^2\right)$$

with $\mu(x) = \frac{x}{a}$ and $\nu(x)$ an arbitrary function. The Lie group of Noether symmetries has a Lie algebra $\mathcal{L}_5$ which is generated by

$$\Xi_1 = \frac{\partial}{\partial t}, \quad \Xi_2 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad \Xi_3 = \frac{\partial}{\partial z}, \quad \Xi_4 = \frac{\partial}{\partial y}, \quad \Xi_5 = \frac{\partial}{\partial s}.$$

In the aforementioned paper, their notation corresponds to ours as follows: X_{i-1} is equivalent to Ξ_i for $i = 1, \ldots, 4$, and Y_0 is equivalent to Ξ_5. The system of geodesic equations is given as

$$\begin{aligned} &\ddot{t} + 2\nu'(x)\dot{t}\dot{x} = 0, && \ddot{y} + \frac{2}{a}\dot{x}\dot{y} = 0,\\ &\ddot{x} + \nu'(x)e^{2\nu(x)}\dot{t}^2 - \frac{1}{a}e^{\frac{2x}{a}}\left(\dot{y}^2 + \dot{z}^2\right) = 0, && \ddot{z} + \frac{2}{a}\dot{x}\dot{z} = 0. \end{aligned}$$

Using `SymboLie`, we obtain the complete optimal system of families of Lie subalgebras:

$$\begin{aligned}
\Theta^1 \equiv \{&\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_5\}, \{\Xi_1 + a_1\Xi_2\}, \{\Xi_1 + a_1\Xi_3\}, \{\Xi_1 + a_1\Xi_5\},\\
&\{\Xi_2 + a_1\Xi_5\}, \{\Xi_3 + a_1\Xi_5\}, \{\Xi_1 + a_1\Xi_2 + a_2\Xi_5\}, \{\Xi_1 + a_1\Xi_3 + a_2\Xi_5\}\},\\
\Theta^2 \equiv \{&\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_5\}, \{\Xi_2, \Xi_5\}, \{\Xi_3, \Xi_4\}, \{\Xi_3, \Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_5\},\\
&\{\Xi_1, \Xi_3 + a_1\Xi_5\}, \{\Xi_1 + a_1\Xi_2, \Xi_5\}, \{\Xi_3, \Xi_4 + a_1\Xi_5\}, \{\Xi_1 + a_1\Xi_3, \Xi_4\},\\
&\{\Xi_1 + a_1\Xi_3, \Xi_5\}, \{\Xi_1 + a_1\Xi_5, \Xi_2\}, \{\Xi_1 + a_1\Xi_5, \Xi_3\}, \{\Xi_1 + a_1\Xi_3, \Xi_4 + a_2\Xi_5\},\\
&\{\Xi_1 + a_1\Xi_5, \Xi_2 + a_2\Xi_5\}, \{\Xi_1 + a_1\Xi_5, \Xi_3 + a_2\Xi_5\}, \{\Xi_1 + a_1\Xi_3 + a_2\Xi_5, \Xi_4\},\\
&\{\Xi_1 + a_1\Xi_4, \Xi_3 + a_2\Xi_4 + a_3\Xi_5\}\},\\
\Theta^3 \equiv \{&\{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_5\}, \{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_3, \Xi_4, \Xi_5\},\\
&\{\Xi_1, \Xi_3, \Xi_4 + a_1\Xi_5\}, \{\Xi_1 + a_1\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1 + a_1\Xi_3, \Xi_4, \Xi_5\},\\
&\{\Xi_1 + a_1\Xi_5, \Xi_3, \Xi_4\}, \{\Xi_2 + a_1\Xi_5, \Xi_3, \Xi_4\}, \{\Xi_1 + a_1\Xi_5, \Xi_3, \Xi_4 + a_2\Xi_5\},\\
&\{\Xi_1 + a_1\Xi_2 + a_2\Xi_5, \Xi_3, \Xi_4\}\},\\
\Theta^4 \equiv \{&\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4, \Xi_5\}, \{\Xi_2, \Xi_3, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_5, \Xi_3, \Xi_4\},\\
&\{\Xi_1 + a_1\Xi_2, \Xi_3, \Xi_4, \Xi_5\}, \{\Xi_1 + a_1\Xi_5, \Xi_2, \Xi_3, \Xi_4\},\\
&\{\Xi_1 + a_1\Xi_5, \Xi_2 + a_2\Xi_5, \Xi_3, \Xi_4\}\},
\end{aligned}$$

with $a_1, a_2, a_3 \neq 0$. In particular, in the one-dimensional case, `SymboLie` returns 11 representative families, whereas [21] reports 15 subalgebras. This means that four additional families are provided in the one-dimensional optimal system, specifically:

$$\{\Xi_4\}, \quad \{\Xi_1 + a_1\Xi_4\}, \quad \{\Xi_4 + a_1\Xi_5\}, \quad \{\Xi_1 + a_1\Xi_4 + a_2\Xi_5\}.$$

By considering the inner automorphism

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{t^2+1} - 1 & -\frac{2t}{t^2+1} & 0 \\ 0 & 0 & \frac{2t}{t^2+1} & \frac{2}{t^2+1} - 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

it becomes evident that the families $\{\Xi_4\}$, $\{\Xi_1 + a_1\Xi_4\}$, $\{\Xi_4 + a_1\Xi_5\}$, and $\{\Xi_1 + a_1\Xi_4 + a_2\Xi_5\}$ are mapped to $\{\Xi_3\}$, $\{\Xi_1 + \widetilde{a_1}\Xi_3\}$, $\{\Xi_3 + \widetilde{a_1}\Xi_5\}$, and $\{\Xi_1 + \widetilde{a_1}\Xi_3 + a_2\Xi_5\}$, respectively, thus confirming the correctness of the `SymboLie` result.

7.3.2 System admitting six Noether symmetries

The geodetic Lagrangian of the metric

$$ds^2 = e^{\frac{2x}{a}} dt^2 - dx^2 - e^{\frac{2x}{b}} \left(dy^2 + dz^2\right) \qquad (a, b \neq 0)$$

admits six Noether symmetries $\mathcal{L}_6$ which are listed as follows:

$$\Xi_1 = \frac{\partial}{\partial t}, \quad \Xi_2 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad \Xi_3 = \frac{\partial}{\partial z}, \quad \Xi_4 = \frac{\partial}{\partial y},$$
$$\Xi_5 = \frac{\partial}{\partial x} - \frac{t}{a}\frac{\partial}{\partial t} - \frac{1}{b}\left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right), \quad \Xi_6 = \frac{\partial}{\partial s}.$$

The corresponding system of geodesic equations is

$$\ddot{t} + \frac{2}{a}\dot{t}\dot{x} = 0, \qquad \ddot{x} + \frac{1}{a}e^{\frac{2x}{a}}\dot{t}^2 - \frac{1}{b}e^{\frac{2x}{b}}\left(\dot{y}^2 + \dot{z}^2\right) = 0,$$
$$\ddot{y} + \frac{2}{b}\dot{x}\dot{y} = 0, \qquad \ddot{z} + \frac{2}{b}\dot{x}\dot{z} = 0.$$

The complete optimal system of families of Lie subalgebras computed by `SymboLie` is as follows:

$$\begin{aligned}
\Theta^1 \equiv \{ & \{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_5\}, \{\Xi_6\}, \{\Xi_1 + \alpha_1\Xi_2\}, \{\Xi_1 + a_1\Xi_3\}, \{\Xi_2 + a_1\Xi_5\}, \\
& \{\Xi_1 + \alpha_1\Xi_6\}, \{\Xi_2 + a_1\Xi_6\}, \{\Xi_3 + \alpha_1\Xi_6\}, \{\Xi_5 + a_1\Xi_6\}, \{\Xi_1 + \alpha_1\Xi_2 + a_1\Xi_6\}, \\
& \{\Xi_1 + \alpha_1\Xi_6 + a_1\Xi_3\}, \{\Xi_2 + a_1\Xi_5 + a_2\Xi_6\}\}, \\
\Theta^2 \equiv \{ & \{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_5\}, \{\Xi_1, \Xi_6\}, \{\Xi_2, \Xi_5\}, \{\Xi_2, \Xi_6\}, \{\Xi_3, \Xi_4\}, \{\Xi_3, \Xi_5\}, \\
& \{\Xi_3, \Xi_6\}, \{\Xi_5, \Xi_6\}, \{\Xi_1, \Xi_2 + a_1\Xi_5\}, \{\Xi_1, \Xi_2 + a_1\Xi_6\}, \{\Xi_1, \Xi_3 + \alpha_1\Xi_6\}, \\
& \{\Xi_1, \Xi_5 + a_1\Xi_6\}, \{\Xi_2, \Xi_5 + a_1\Xi_6\}, \{\Xi_1 + \alpha_1\Xi_2, \Xi_6\}, \{\Xi_3, \Xi_4 + \alpha_1\Xi_6\}, \\
& \{\Xi_3, \Xi_5 + a_1\Xi_6\}, \{\Xi_1 + a_1\Xi_3, \Xi_4\}, \{\Xi_1 + a_1\Xi_3, \Xi_5\}, \{\Xi_1 + a_1\Xi_3, \Xi_6\}, \\
& \{\Xi_2 + a_1\Xi_5, \Xi_6\}, \{\Xi_1 + \alpha_1\Xi_6, \Xi_2\}, \{\Xi_1 + \alpha_1\Xi_6, \Xi_3\}, \{\Xi_1 + \alpha_1\Xi_5, \Xi_2 + a_1\Xi_6\}, \\
& \{\Xi_1, \Xi_2 + a_1\Xi_5 + a_2\Xi_6\}, \{\Xi_1 + a_1\Xi_3, \Xi_4 + \alpha_1\Xi_6\}, \{\Xi_1 + a_1\Xi_3, \Xi_5 + a_2\Xi_6\}, \\
& \{\Xi_1 + \alpha_1\Xi_6, \Xi_2 + a_1\Xi_6\}, \{\Xi_1 + \alpha_1\Xi_6, \Xi_3 + a_1\Xi_6\}, \{\Xi_2 + a_1\Xi_6, \Xi_5 + a_2\Xi_6\},
\end{aligned}$$

$$\begin{aligned}
&\{\Xi_1+\alpha_1\Xi_6+a_1\Xi_3,\Xi_4\},\{\Xi_1+a_1\Xi_4,\Xi_3+\alpha_1\Xi_6+a_2\Xi_4\}\},\\
\Theta^3 \equiv\ &\{\{\Xi_1,\Xi_2,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_6\},\{\Xi_1,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_3,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_6\},\\
&\{\Xi_1,\Xi_5,\Xi_6\},\{\Xi_2,\Xi_3,\Xi_4\},\{\Xi_2,\Xi_5,\Xi_6\},\{\Xi_3,\Xi_4,\Xi_5\},\{\Xi_3,\Xi_4,\Xi_6\},\\
&\{\Xi_3,\Xi_5,\Xi_6\},\{\Xi_1,\Xi_2,\Xi_5+a_1\Xi_6\},\{\Xi_1,\Xi_3,\Xi_4+\alpha_1\Xi_6\},\\
&\{\Xi_1,\Xi_3,\Xi_5+a_1\Xi_6\},\{\Xi_1,\Xi_2+a_1\Xi_5,\Xi_6\},\{\Xi_1,\Xi_2+a_1\Xi_6,\Xi_5\},\\
&\{\Xi_1+\alpha_1\Xi_2,\Xi_3,\Xi_4\},\{\Xi_3,\Xi_4,\Xi_5+a_1\Xi_6\},\{\Xi_1+a_1\Xi_3,\Xi_4,\Xi_5\},\\
&\{\Xi_1+a_1\Xi_3,\Xi_4,\Xi_6\},\{\Xi_1+a_1\Xi_3,\Xi_5,\Xi_6\},\{\Xi_2+a_1\Xi_5,\Xi_3,\Xi_4\},\\
&\{\Xi_1+\alpha_1\Xi_6,\Xi_3,\Xi_4\},\{\Xi_2+a_1\Xi_6,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2+a_1\Xi_6,\Xi_5+a_2\Xi_6\},\\
&\{\Xi_1+a_1\Xi_3,\Xi_4,\Xi_5+a_2\Xi_6\},\{\Xi_1+\alpha_1\Xi_6,\Xi_3,\Xi_4+a_1\Xi_6\},\\
&\{\Xi_1+\alpha_1\Xi_2+a_1\Xi_6,\Xi_3,\Xi_4\},\{\Xi_2+a_1\Xi_5+a_2\Xi_6,\Xi_3,\Xi_4\}\},\\
\Theta^4 \equiv\ &\{\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_5,\Xi_6\},\{\Xi_1,\Xi_3,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_3,\Xi_4,\Xi_6\},\\
&\{\Xi_1,\Xi_3,\Xi_5,\Xi_6\},\{\Xi_2,\Xi_3,\Xi_4,\Xi_5\},\{\Xi_2,\Xi_3,\Xi_4,\Xi_6\},\{\Xi_3,\Xi_4,\Xi_5,\Xi_6\},\\
&\{\Xi_1,\Xi_3,\Xi_4,\Xi_5+a_1\Xi_6\},\{\Xi_1,\Xi_2+a_1\Xi_5,\Xi_3,\Xi_4\},\\
&\{\Xi_1,\Xi_2+a_1\Xi_6,\Xi_3,\Xi_4\},\{\Xi_2,\Xi_3,\Xi_4,\Xi_5+a_1\Xi_6\},\\
&\{\Xi_1+\alpha_1\Xi_2,\Xi_3,\Xi_4,\Xi_6\},\{\Xi_1+a_1\Xi_3,\Xi_4,\Xi_5,\Xi_6\},\\
&\{\Xi_2+a_1\Xi_5,\Xi_3,\Xi_4,\Xi_6\},\{\Xi_1+\alpha_1\Xi_6,\Xi_2,\Xi_3,\Xi_4\},\\
&\{\Xi_1+\alpha_1\Xi_5,\Xi_2+a_1\Xi_6,\Xi_3,\Xi_4\},\{\Xi_1,\Xi_2+a_1\Xi_5+a_2\Xi_6,\Xi_3,\Xi_4\},\\
&\{\Xi_1+\alpha_1\Xi_6,\Xi_2+a_1\Xi_6,\Xi_3,\Xi_4\},\{\Xi_2+a_1\Xi_6,\Xi_3,\Xi_4,\Xi_5+a_2\Xi_6\}\},\\
\Theta^5 \equiv\ &\{\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_6\},\{\Xi_1,\Xi_3,\Xi_4,\Xi_5,\Xi_6\},\\
&\{\Xi_2,\Xi_3,\Xi_4,\Xi_5,\Xi_6\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5+a_1\Xi_6\},\{\Xi_1,\Xi_2+a_1\Xi_5,\Xi_3,\Xi_4,\Xi_6\},\\
&\{\Xi_1,\Xi_2+a_1\Xi_6,\Xi_3,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2+a_1\Xi_6,\Xi_3,\Xi_4,\Xi_5+a_2\Xi_6\}\},
\end{aligned}$$

with $\alpha_1 = \pm 1$ and $a_1, a_2 \neq 0$.

Again, there are differences between the one-dimensional optimal system calculated by `SymboLie` and the one reported in [21]. However, verifying the correctness of the results of our algorithm will be the subject of future work.

7.3.3 System admitting seven Noether symmetries

Consider the geodetic Lagrangian of the metric

$$ds^2 = dt^2 - \left(\frac{x}{a}\right)^{2b} dx^2 - \left(\frac{x}{K}\right)^2 \left(dy^2 + dz^2\right)$$

where constants $a, b, K \neq 0$. The Lie group of Noether symmetries has a Lie algebra $\mathcal{L}_7$ which is generated by

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial t}, \quad \Xi_2 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad \Xi_3 = \frac{\partial}{\partial z}, \quad \Xi_4 = \frac{\partial}{\partial y}, \quad \Xi_5 = \frac{\partial}{\partial s},\\
&\Xi_6 = s\frac{\partial}{\partial t}, \quad \Xi_7 = s\frac{\partial}{\partial s} + \frac{t}{2}\frac{\partial}{\partial t} + \frac{1}{2b+2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right),
\end{aligned}$$

with $A = 2t$. The corresponding geodesic equations are given by

$$\begin{aligned}
&\ddot{t} = 0, &&\ddot{x} + \frac{b}{x}\dot{x}^2 - \frac{x}{K^2\left(\frac{x}{a}\right)^{2b}}\left(\dot{y}^2 + \dot{z}^2\right) = 0,\\
&\ddot{y} + \frac{2}{x}\dot{x}\dot{y} = 0, &&\ddot{z} + \frac{2}{x}\dot{x}\dot{z} = 0.
\end{aligned}$$

The complete optimal system of $\mathcal{L}_7$ computed by `SymboLie` is the following:

$$
\begin{aligned}
\Theta^1 \equiv \{ & \{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_5\}, \{\Xi_6\}, \{\Xi_7\}, \{\Xi_1 + \alpha_1\Xi_2\}, \{\Xi_1 + \alpha_1\Xi_3\}, \\
& \{\Xi_2 + \alpha_1\Xi_5\}, \{\Xi_3 + \alpha_1\Xi_5\}, \{\Xi_2 + \alpha_1\Xi_6\}, \{\Xi_3 + a_1\Xi_6\}, \{\Xi_5 + \alpha_1\Xi_6\}, \\
& \{\Xi_2 + a_1\Xi_7\}, \{\Xi_2 + \alpha_1\Xi_5 + a_1\Xi_6\}, \{\Xi_3 + \alpha_1\Xi_5 + a_1\Xi_6\}\},
\end{aligned}
$$

$$
\begin{aligned}
\Theta^2 \equiv \{ & \{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_5\}, \{\Xi_1, \Xi_6\}, \{\Xi_1, \Xi_7\}, \{\Xi_2, \Xi_5\}, \{\Xi_2, \Xi_6\}, \\
& \{\Xi_2, \Xi_7\}, \{\Xi_3, \Xi_4\}, \{\Xi_3, \Xi_5\}, \{\Xi_3, \Xi_6\}, \{\Xi_3, \Xi_7\}, \{\Xi_5, \Xi_7\}, \{\Xi_6, \Xi_7\}, \\
& \{\Xi_1, \Xi_2 + \alpha_1\Xi_5\}, \{\Xi_1, \Xi_3 + \alpha_1\Xi_5\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_6\}, \{\Xi_1, \Xi_3 + a_1\Xi_6\}, \\
& \{\Xi_1, \Xi_5 + \alpha_1\Xi_6\}, \{\Xi_1, \Xi_2 + a_1\Xi_7\}, \{\Xi_2, \Xi_5 + \alpha_1\Xi_6\}, \{\Xi_1 + \alpha_1\Xi_2, \Xi_5\}, \\
& \{\Xi_1 + \alpha_1\Xi_2, \Xi_6\}, \{\Xi_3, \Xi_4 + \alpha_1\Xi_5\}, \{\Xi_3, \Xi_4 + a_1\Xi_6\}, \{\Xi_3, \Xi_5 + \alpha_1\Xi_6\}, \\
& \{\Xi_1 + \alpha_1\Xi_3, \Xi_4\}, \{\Xi_1 + \alpha_1\Xi_3, \Xi_5\}, \{\Xi_1 + \alpha_1\Xi_3, \Xi_6\}, \{\Xi_3 + a_1\Xi_6, \Xi_7\}, \\
& \{\Xi_2 + a_1\Xi_7, \Xi_5\}, \{\Xi_2 + a_1\Xi_7, \Xi_6\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_5 + a_1\Xi_6\}, \\
& \{\Xi_1, \Xi_3 + \alpha_1\Xi_5 + a_1\Xi_6\}, \{\Xi_1 + \alpha_1\Xi_2, \Xi_5 + a_1\Xi_6\}, \{\Xi_3, \Xi_4 + \alpha_1\Xi_5 + a_1\Xi_6\}, \\
& \{\Xi_1 + \alpha_1\Xi_3, \Xi_4 + a_1\Xi_5\}, \{\Xi_1 + \alpha_1\Xi_3, \Xi_4 + a_1\Xi_6\}, \{\Xi_1 + \alpha_1\Xi_3, \Xi_5 + a_1\Xi_6\}, \\
& \{\Xi_1 + \alpha_1\Xi_3, \Xi_4 + a_1\Xi_5 + a_2\Xi_6\}\},
\end{aligned}
$$

$$
\begin{aligned}
\Theta^3 \equiv \{ & \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_6\}, \{\Xi_1, \Xi_2, \Xi_7\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_5\}, \\
& \{\Xi_1, \Xi_3, \Xi_6\}, \{\Xi_1, \Xi_3, \Xi_7\}, \{\Xi_1, \Xi_5, \Xi_6\}, \{\Xi_1, \Xi_5, \Xi_7\}, \{\Xi_1, \Xi_6, \Xi_7\}, \\
& \{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_2, \Xi_5, \Xi_7\}, \{\Xi_2, \Xi_6, \Xi_7\}, \{\Xi_3, \Xi_4, \Xi_5\}, \{\Xi_3, \Xi_4, \Xi_6\}, \\
& \{\Xi_3, \Xi_4, \Xi_7\}, \{\Xi_3, \Xi_5, \Xi_7\}, \{\Xi_3, \Xi_6, \Xi_7\}, \{\Xi_1, \Xi_2, \Xi_5 + \alpha_1\Xi_6\}, \\
& \{\Xi_1, \Xi_3, \alpha_1\Xi_5 + \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4 + a_1\Xi_6\}, \{\Xi_1, \Xi_3, \Xi_5 + \alpha_1\Xi_6\}, \\
& \{\Xi_1, \Xi_2 + \alpha_1\Xi_5, \Xi_6\}, \{\Xi_1, \Xi_3 + \alpha_1\Xi_5, \Xi_6\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_6, \Xi_5\}, \\
& \{\Xi_1, \Xi_3 + a_1\Xi_6, \Xi_5\}, \{\Xi_1, \Xi_3 + a_1\Xi_6, \Xi_7\}, \{\Xi_1, \Xi_2 + a_1\Xi_7, \Xi_5\}, \\
& \{\Xi_1, \Xi_2 + a_1\Xi_7, \Xi_6\}, \{\Xi_1 + \alpha_1\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_3, \Xi_4, \Xi_5 + \alpha_1\Xi_6\}, \\
& \{\Xi_3, \Xi_4 + a_1\Xi_6, \Xi_7\}, \{\Xi_1 + \alpha_1\Xi_3, \Xi_4, \Xi_5\}, \{\Xi_1 + \alpha_1\Xi_3, \Xi_4, \Xi_6\}, \\
& \{\Xi_2 + \alpha_1\Xi_5, \Xi_3, \Xi_4\}, \{\Xi_2 + \alpha_1\Xi_6, \Xi_3, \Xi_4\}, \{\Xi_2 + a_1\Xi_7, \Xi_3, \Xi_4\}, \\
& \{\Xi_1, \Xi_3, \Xi_4 + \alpha_1\Xi_5 + a_1\Xi_6\}, \{\Xi_1, \Xi_3 + \alpha_1\Xi_5, \Xi_4 + a_1\Xi_6\}, \\
& \{\Xi_1, \Xi_2 + \alpha_1\Xi_6, \Xi_5 + a_1\Xi_6\}, \{\Xi_1, \Xi_3 + a_1\Xi_6, \Xi_5 + \alpha_1\Xi_6\}, \\
& \{\Xi_1 + \alpha_1\Xi_3, \Xi_4, \Xi_5 + a_1\Xi_6\}, \{\Xi_2 + \alpha_1\Xi_5 + a_1\Xi_6, \Xi_3, \Xi_4\}, \\
& \{\Xi_1, \Xi_3 + \alpha_1\Xi_5, \Xi_4 + a_1\Xi_5 + a_2\Xi_6\}\},
\end{aligned}
$$

$$
\begin{aligned}
\Theta^4 \equiv \{ & \{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_5, \Xi_6\}, \{\Xi_1, \Xi_2, \Xi_5, \Xi_7\}, \{\Xi_1, \Xi_2, \Xi_6, \Xi_7\}, \\
& \{\Xi_1, \Xi_3, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4, \Xi_6\}, \{\Xi_1, \Xi_3, \Xi_4, \Xi_7\}, \{\Xi_1, \Xi_3, \Xi_5, \Xi_6\}, \\
& \{\Xi_1, \Xi_3, \Xi_5, \Xi_7\}, \{\Xi_1, \Xi_3, \Xi_6, \Xi_7\}, \{\Xi_1, \Xi_5, \Xi_6, \Xi_7\}, \{\Xi_2, \Xi_3, \Xi_4, \Xi_5\}, \\
& \{\Xi_2, \Xi_3, \Xi_4, \Xi_6\}, \{\Xi_2, \Xi_3, \Xi_4, \Xi_7\}, \{\Xi_3, \Xi_4, \Xi_5, \Xi_7\}, \{\Xi_3, \Xi_4, \Xi_6, \Xi_7\}, \\
& \{\Xi_1, \Xi_3, \Xi_4, \Xi_5 + \alpha_1\Xi_6\}, \{\Xi_1, \Xi_3, \Xi_4 + \alpha_1\Xi_5, \Xi_6\}, \{\Xi_1, \Xi_3, \Xi_4 + a_1\Xi_6, \Xi_5\}, \\
& \{\Xi_1, \Xi_3, \Xi_4 + a_1\Xi_6, \Xi_7\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_5, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + \alpha_1\Xi_6, \Xi_3, \Xi_4\}, \\
& \{\Xi_1, \Xi_3 + a_1\Xi_6, \Xi_5, \Xi_7\}, \{\Xi_1, \Xi_2 + a_1\Xi_7, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1\Xi_7, \Xi_5, \Xi_6\}, \\
& \{\Xi_2, \Xi_3, \Xi_4, \Xi_5 + \alpha_1\Xi_6\}, \{\Xi_1 + \alpha_1\Xi_2, \Xi_3, \Xi_4, \Xi_5\}, \\
& \{\Xi_1 + \alpha_1\Xi_2, \Xi_3, \Xi_4, \Xi_6\}, \{\Xi_2 + a_1\Xi_7, \Xi_3, \Xi_4, \Xi_5\}, \\
& \{\Xi_2 + a_1\Xi_7, \Xi_3, \Xi_4, \Xi_6\}, \{\Xi_1, \Xi_3, \Xi_4 + a_1\Xi_6, \Xi_5 + \alpha_1\Xi_6\}, \\
& \{\Xi_1, \Xi_2 + \alpha_1\Xi_5 + a_1\Xi_6, \Xi_3, \Xi_4\}, \{\Xi_1 + \alpha_1\Xi_2, \Xi_3, \Xi_4, \Xi_5 + a_1\Xi_6\}\},
\end{aligned}
$$

$$
\begin{aligned}
\Theta^5 \equiv \{ & \{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_6\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_7\}, \\
& \{\Xi_1, \Xi_2, \Xi_5, \Xi_6, \Xi_7\}, \{\Xi_1, \Xi_3, \Xi_4, \Xi_5, \Xi_6\}, \{\Xi_1, \Xi_3, \Xi_4, \Xi_5, \Xi_7\},
\end{aligned}
$$

$$
\begin{aligned}
&\{\Xi_1,\Xi_3,\Xi_4,\Xi_6,\Xi_7\},\{\Xi_1,\Xi_3,\Xi_5,\Xi_6,\Xi_7\},\{\Xi_2,\Xi_3,\Xi_4,\Xi_5,\Xi_7\},\\
&\{\Xi_2,\Xi_3,\Xi_4,\Xi_6,\Xi_7\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5+\alpha_1\Xi_6\},\\
&\{\Xi_1,\Xi_3,\Xi_4+a_1\Xi_6,\Xi_5,\Xi_7\},\{\Xi_1,\Xi_2+\alpha_1\Xi_5,\Xi_3,\Xi_4,\Xi_6\},\\
&\{\Xi_1,\Xi_2+\alpha_1\Xi_6,\Xi_3,\Xi_4,\Xi_5\},\{\Xi_1,\Xi_2+a_1\Xi_7,\Xi_3,\Xi_4,\Xi_5\},\\
&\{\Xi_1,\Xi_2+a_1\Xi_7,\Xi_3,\Xi_4,\Xi_6\},\{\Xi_1,\Xi_2+\alpha_1\Xi_6,\Xi_3,\Xi_4,\Xi_5+a_1\Xi_6\}\},\\
\Theta^6 \equiv \{&\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5,\Xi_6\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_5,\Xi_7\},\{\Xi_1,\Xi_2,\Xi_3,\Xi_4,\Xi_6,\Xi_7\},\\
&\{\Xi_1,\Xi_3,\Xi_4,\Xi_5,\Xi_6,\Xi_7\},\{\Xi_1,\Xi_2+a_1\Xi_7,\Xi_3,\Xi_4,\Xi_5,\Xi_6\}\}.
\end{aligned}
$$

with $\alpha_1 = \pm 1$ and $a_1, a_2 \neq 0$.

In this case, however, `SymboLie` returns two additional representative families in the one-dimensional optimal system. It will be a task for future work to determine whether these two representatives are indeed unmappable to the remainders.

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