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Regular measures of noncompactness and Ascoli–Arzelà type compactness criteria in spaces of vector-valued functions

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Abstract

In this paper we estimate the Kuratowski and the Hausdorff measures of noncompactness of bounded subsets of spaces of vector-valued bounded functions and of vector-valued bounded differentiable functions. To this end, we use a quantitative characteristic modeled on a new equicontinuity-type concept and classical quantitative characteristics related to pointwise relative compactness. We obtain new regular measures of noncompactness in the spaces taken into consideration. The established inequalities reduce to precise formulas in some classes of subsets. We derive Ascoli–Arzelà type compactness criteria.

Keywords Banach space · Bounded function · Differentiable function · Measure of noncompactness · Ascoli–Arzelà theorem

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1 Introduction

Let Y be a real Banach space, possibly infinite-dimensional. Throughout the paper we will deal with Y-valued functions. In [1] Ambrosetti has extended the Ascoli–Arzelà theorem to the space of Y-valued functions defined and continuous on a compact metric space and equipped with the supremum norm, obtaining that a bounded subset of the space is relatively compact if and only if it is equicontinuous and pointwise relatively compact, and establishing a precise formula for the Kuratowski measure of noncompactness of bounded and equicontinuous subsets of the space. Later, Nussbaum [24] has estimated both the Kuratowski and the Hausdorff measures of noncompactness of bounded subsets of that space, finding, as a special case, the result of Ambrosetti. On the other hand, Bartle [11] extended the Ascoli–Arzelà theorem to the space $\mathcal{BC}(\Omega, \mathbb{R})$ of all real-valued functions defined, continuous and bounded on a topological space Ω. Precisely, a bounded subset M of $\mathcal{BC}(\Omega, \mathbb{R})$ is relatively compact if and only if for any positive ε there is a finite partition $\{A_1, \ldots, A_n\}$ of Ω such that if x, y belong to the same A_i, then $|f(x) - f(y)| \leq \varepsilon$, for all $f \in M$. In [18] the estimates of Nussbaum have been extended to the space of Y-valued functions defined and bounded on a general set Ω, by means of quantitative characteristics which unfortunately do not allow to obtain a compactness criterion for all bounded subsets of the whole space. Similar results have been obtained in [5], where the Hausdorff measure of noncompactness has been estimated in the space of totally bounded Y-valued functions defined on a set Ω, obtaining implicitly, when Ω is a topological space, a generalization of the Bartle criterion. We also mention that in [7–9] measures of noncompactness have been investigated in the space of Y-valued functions defined, continuous and bounded on unbounded intervals. Actually, the compactness criterion of Ascoli–Arzelà has been extended to several more general cases which find applications in many fields of mathematical analysis. Of interest, in addition to the already mentioned cases, are the results that generalize the criterion to spaces of differentiable functions (among others, we recall [3, 6, 15, 23, 27]). In particular, in [15] the relative compactness has been characterized for subsets of the space of Y-valued functions defined, k-times continuously differentiable and bounded with all differentials up to the order k on unbounded intervals. While in [3] the case of real-valued functions defined either on a compact subset of $\mathbb{R}^n$, or on $\mathbb{R}^n$ itself when functions vanish at infinity, has been considered. Finally, we have to recall that a general version of the Ascoli–Arzelà theorem concerns the characterization of relative compactness in the space $\mathcal{D}(\Omega, T)$, that is, the space $\mathcal{C}(\Omega, T)$ of continuous functions between two topological spaces Ω and T endowed with the topology of compact convergence (see [19, Theorem 18]).

The results of this paper will cover and extend the mentioned classical and more recent results on the subject. Our first aim is to construct regular measures of noncompactness equivalent to the Kuratowski and the Hausdorff ones in the space $\mathcal{B}(\Omega, Y)$ of Y-valued functions defined and bounded on a nonempty set Ω, made into a Banach space by the supremum norm. The main condition is a new equicontinuity-type concept which we will refer to as extended equicontinuity. Then a quantitative characteristic measuring the degree of extended equicontinuity, together with the classical quantitative characteristics μ_α, σ_α, μ_γ, σ_γ (where α and γ stand for the Kuratowski and the Hausdorff measures, respectively) measuring the degree of pointwise relative compactness (see [1, 5, 18, 24]), will allow us to estimate the Kuratowski and the Hausdorff measures of noncompactness of bounded subsets of the space. When Ω is an open subset of a Banach space, of independent interest are the results we are able to obtain in the space $\mathcal{BC}^k(\Omega, Y)$ of Y-valued functions defined, k-times continuously differentiable and bounded with all differentials up to the order k on Ω and endowed with the norm $\|f\|_{\mathcal{BC}^k} = \max\{\|f\|_\infty, \|\mathrm{d}f\|_\infty, \dots, \|\mathrm{d}^k f\|_\infty\}$, and also in the complete locally convex space $\mathcal{D}^k(\Omega, Y)$, that is, the space $\mathcal{C}^k(\Omega, Y)$ of Y-valued functions defined and k-times continuously differentiable on Ω, endowed with the topology of compact convergence for all differentials. It is worth mentioning that in $\mathcal{BC}^k(\Omega, Y)$ and $\mathcal{D}^k(\Omega, Y)$ the formulation of our equicontinuity-type concept, as well as of all quantitative characteristics there considered, will depend on each space in a natural way. From our results we obtain Ascoli–Arzelà type compactness criteria in spaces of Y-valued functions in very general settings.

The paper is organized as follows. In Sect. 2, we introduce some definitions and preliminary facts on measures of noncompactness. Then we consider in $\mathcal{B}(\Omega, Y)$ the generalized measure of non-equicontinuity ω (see [5, 18]) and the quantitative characteristics $\mu_\alpha, \sigma_\alpha, \mu_\gamma, \sigma_\gamma$. In particular, we put in evidence that ω associated with any of these quantities (which actually are equivalent), differently from what happens in the space of totally bounded functions, does not allow in general to characterize compactness in $\mathcal{B}(\Omega, Y)$. The main results of the paper are presented in the following two sections. In Sect. 3, we introduce our new equicontinuity-type concept, then we obtain inequalities and compactness criteria in any Banach subspace of $\mathcal{B}(\Omega, Y)$. It is worthwhile to notice that, as a particular case when Ω is a topological space, the results in $\mathcal{BC}(\Omega, Y)$ hold when Ω is not necessarily compact and the functions not necessarily totally bounded, so we generalize at the same time the result of Nussbaum and the Bartle criterion. In Sect. 4, we obtain inequalities and compactness criteria in Banach subspaces of $\mathcal{BC}^k(\Omega, Y)$ and in the space $\mathcal{D}^k(\Omega, Y)$. In all the spaces, we always construct regular measures of noncompactness equivalent to the Kuratowski and the Hausdorff measures. A precise formula for the Kuratowski measure of noncompactness is obtained for bounded and extendedly equicontinuous subsets of Banach subspaces of $\mathcal{B}(\Omega, Y)$. Analogous results are obtained in Banach subspaces of $\mathcal{BC}^k(\Omega, Y)$ and in the space $\mathcal{D}^k(\Omega, Y)$. Further, precise formulas for the Hausdorff measure of noncompactness are given for bounded and equicontinuous subsets of the spaces $\mathcal{TB}(\Omega, Y)$ and $\mathcal{TBC}^k(\Omega, Y)$, consisting of totally bounded functions and functions of $\mathcal{BC}^k(\Omega, Y)$ which are compact with all differentials, respectively. An analogous formula is obtained in $\mathcal{D}^k(\Omega, Y)$. In the last section, we obtain some results for pointwise relatively compact subsets of $\mathcal{B}(\Omega, Y)$ under the hypothesis that Y is a Lindenstrauss space.

In the literature a different approach is sometimes used to obtain measures of noncompactness in some Banach spaces of Y-valued functions (see, for example, [3, 7, 8]), but not always such measures enjoy the property of regularity.

2 Preliminaries

In the following we will consider real linear spaces. Given a Banach space E with zero element θ, we denote by $B(x, r)$ the closed ball with center x and radius $r > 0$, and $B(E)$ will stand for $B(\theta, 1)$. If M is a subset of E we denote by $\overline{M}$, co$M$ and $\overline{\text{co}}M$ the closure, the convex hull and the closed convex hull of M, respectively. We use the symbol $\text{diam}_E(M)$ for the diameter of M in E, or simply $\text{diam}(M)$ if no confusion can arise. If M and N are subsets of E and $\lambda \in \mathbb{R}$, then $M + N$ and λM will denote the algebraic operations on sets. Next, let $\mathfrak{M}_E$ be the family of all nonempty bounded subsets of E and let $\mathfrak{N}_E$ be its subfamily consisting of all relatively compact sets. Given a set function $\mu : \mathfrak{M}_E \to [0, +\infty)$, the family $\ker \mu = \{M \in \mathfrak{M}_E : \ \mu(M) = 0\}$ is called kernel of μ. Following [10], we introduce the concept of measure of noncompactness.

Definition 2.1 A set function $\mu : \mathfrak{M}_E \to [0, +\infty)$ is said to be a measure of noncompactness in E if the following conditions hold for $M, N \in \mathfrak{M}_E$:

(i) $\ker \mu$ is nonempty and $\ker \mu \subseteq \mathfrak{N}_E$;
(ii) $M \subseteq N$ implies $\mu(M) \leq \mu(N)$;
(iii) $\mu(\overline{M}) = \mu(M)$;
(iv) $\mu(\text{co}M) = \mu(M)$;
(v) $\mu(\lambda M + (1 - \lambda)N) \leq \lambda\mu(M) + (1 - \lambda)\mu(N)$, for $\lambda \in [0, 1]$;
(vi) if $(M_n)_n$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $M_{n+1} \subseteq M_n$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty} \mu(M_n) = 0$, then the intersection set $M_\infty = \bigcap_{n=1}^{\infty} M_n$ is nonempty.

We will say that μ is a full measure if $\ker \mu = \mathfrak{N}_E$. A measure μ is called sublinear if it is homogeneous and subadditive, i.e.

(vii) $\mu(\lambda M) = |\lambda|\mu(M)$ for $\lambda \in \mathbb{R}$, and $\mu(M + N) \leq \mu(M) + \mu(N)$,

moreover, μ is said to have the maximum property if

(viii) $\mu(M \cup N) = \max\{\mu(M), \mu(N)\}$.

An important class of measures of noncompactness is that constituted by regular measures, which are full, sublinear measures with the maximum property. We recall that given a set M in $\mathfrak{M}_E$ the Kuratowski measure of noncompactness of M, denoted by $\alpha(M)$, is the infimum of all $\varepsilon > 0$ such that M can be covered by finitely many sets of diameters not greater than ε and the Hausdorff measure of noncompactness of M, denoted by $\gamma_E(M)$, is the infimum of all $\varepsilon > 0$ such that M has a finite ε-net in E. These measures of noncompactness are regular, besides they are equivalent, since $\gamma_E(M) \leq \alpha(M) \leq 2\gamma_E(M)$. For our purposes, it is also useful to recall that the Istratescu measure of noncompactness $\beta(M)$ of M is the infimum of all $\varepsilon > 0$ such that M does not have an infinite ε-separation, i.e there is no infinite set in M such that $\|x - y\| \geq \varepsilon$ for all x, y in this set, with $x \neq y$. Moreover, the inequalities $\beta(M) \leq \alpha(M) \leq 2\beta(M)$ hold true. For more details on measures of noncompactness the reader is referred to [4, 10].

Throughout, Ω will be a nonempty set and $(Y, \|\cdot\|)$ a Banach space, γ will always stand for γ_Y. We denote by $\mathcal{F}(\Omega, Y)$ the linear space of all functions $f : \Omega \to Y$.

Given a set of functions M in $\mathcal{F}(\Omega, Y)$, $x \in \Omega$ and $A \subseteq \Omega$ we define the subsets $M(x)$ and $M(A)$ of the Banach space Y by letting

$$M(x) = \{f(x) : f \in M\}, \quad M(A) = \{f(x) : x \in A,\ f \in M\}. \tag{2.1}$$

The symbol $\mathcal{B}(\Omega, Y)$ will stand for the Banach space of all bounded functions in $\mathcal{F}(\Omega, Y)$, endowed with the supremum norm

$$\|f\|_\infty = \sup\{\|f(x)\|,\ x \in \Omega\}.$$

We denote by $\mathcal{TB}(\Omega, Y)$ the space of all Y-valued functions defined and totally bounded on Ω, i.e. such that $f(\Omega)$ is relatively compact, and whenever Ω is a topological space, we denote by $\mathcal{BC}(\Omega, Y)$ the spaces of all Y-valued functions defined, bounded and continuous on Ω. Both $\mathcal{TB}(\Omega, Y)$ and $\mathcal{BC}(\Omega, Y)$ are Banach subspaces of $\mathcal{B}(\Omega, Y)$. A function $f \in \mathcal{BC}(\Omega, Y) \cap \mathcal{TB}(\Omega, Y)$ is called compact. Throughout the paper we will use the symbols $\mathfrak{M}_\mathcal{B}$ and $\mathfrak{N}_\mathcal{B}$ instead of $\mathfrak{M}_{\mathcal{B}(\Omega,Y)}$ and $\mathfrak{N}_{\mathcal{B}(\Omega,Y)}$ if no misunderstanding is possible, and analogously for all the spaces we will consider. The same shortcuts will be used in the notations of the quantitative characteristics.

We devote the remaining part of this section to introduce and discuss in $\mathcal{B}(\Omega, Y)$ the quantitative characteristics, based on the classical results on compactness given in [11, 17, 28], which have been useful tools for the study of compactness, for example, in spaces of totally bounded or compact functions. Given $M \in \mathfrak{M}_\mathcal{B}$, we consider (see [2, 5, 13, 18, 24, 29]) the quantitative characteristic

$$\begin{aligned}\omega(M) = \inf\{\varepsilon > 0 :\ &\text{there is a finite partition } \{A_1, \dots, A_n\} \text{ of } \Omega \\ &\text{such that, for all } f \in M,\ \operatorname{diam}(f(A_i)) \le \varepsilon \text{ for } i = 1, \dots, n\},\end{aligned} \tag{2.2}$$

which, according to [18], generalizes the "measure of non-equicontinuity" of Nussbaum [24], and the quantitative characteristics (see [1, 18, 24])

$$\mu_\alpha(M) = \sup_{x\in\Omega} \alpha(M(x)), \quad \mu_\gamma(M) = \sup_{x\in\Omega} \gamma(M(x)).$$

It is easy to verify that $\mu_\gamma(M) \le \mu_\alpha(M) \le 2\mu_\gamma(M)$. A set $M \in \mathfrak{M}_\mathcal{B}$ is called pointwise relatively compact if $\mu_\alpha(M) = 0$ (or $\mu_\gamma(M) = 0$). We also consider (see [5, 18]) the quantitative characteristics

$$\sigma_\alpha(M) = \alpha(M(\Omega)), \quad \sigma_\gamma(M) = \gamma(M(\Omega)).$$

We have $\sigma_\gamma(M) \le \sigma_\alpha(M) \le 2\sigma_\gamma(M)$. Moreover, $\mu_\alpha(M) \le \sigma_\alpha(M)$ and $\mu_\gamma(M) \le \sigma_\gamma(M)$.

For the sake of completeness we recall the result of Nussbaum on the estimates of the Kuratowski measure of noncompactness.

Theorem 2.2 [24, Theorem 1] *Let (Ω, d) be a compact metric space, (T, s) a metric space and $\mathcal{C}(\Omega, T)$ the space of functions defined and continuous on Ω with values* *in* T, *made into a metric space by* $d_\infty(f, g) = \sup_{x\in\Omega} s(f(x), g(x))$. *Let* M *be a bounded set in* $\mathcal{C}(\Omega, T)$, *set*

$$a = \inf\{\omega_N(\delta, M) : \delta \geq 0\},$$

where $\omega_N(\delta, M) = \sup\{s(f(x), f(y)) : x, y \in \Omega;\ d(x, y) \leq \delta,\ f \in M\}$ *is the modulus of continuity of* M, *then*

$$\max\{\mu_\alpha(M), \frac{1}{2}a\} \leq \alpha(M) \leq \mu_\alpha(M) + 2a.$$

The above theorem contains the classical result of Ambrosetti [1, Theorem 2.3]. Precisely, if M is a bounded and equicontinuous subset of $\mathcal{C}(\Omega, T)$, then $\alpha(M) = \mu_\alpha(M)$. Let us observe that in [18] Heinz has extended the result of Nussbaum to bounded subsets of $\mathcal{B}(\Omega, Y)$. In particular, in his paper (see [18, Theorem 2 and Proposition 2 (i)]) the following inequalities, which we state using the notations of this paper, are proved:

$$\max\left\{\mu_\alpha(M), \frac{1}{2}\left(\omega(M) - \sup_{f\in M} \sigma_\alpha(\{f\})\right)\right\} \leq \alpha(M) \leq \mu_\alpha(M) + 2\omega(M). \tag{2.3}$$

Therefore, if $M \in \mathfrak{M}_{\mathcal{TB}}$ then $\sup_{f\in M} \sigma_\alpha(\{f\})) = 0$, so that $\ker(\mu_\alpha + 2\omega) = \mathfrak{N}_{\mathcal{TB}}$ and consequently inequalities (2.3) furnish a criterion of compactness in the space $\mathcal{TB}(\Omega, Y)$ of totally bounded functions (see also [5, Theorem 2.1]). But the same is not true in the space $\mathcal{B}(\Omega, Y)$. In fact, the following Example 1 shows that the left-hand side of (2.3) can be equal to zero for a given set $M \in \mathfrak{M}_{\mathcal{B}}$ without being $\alpha(M) = 0$, which means that the left-hand side of (2.3) vanishes on sets $M \notin \mathfrak{N}_{\mathcal{B}}$. While Example 2 shows that there are sets $M \in \mathfrak{N}_{\mathcal{B}}$ such that the right-hand side of (2.3) does not vanish on M. In other words, $\ker(\mu_\alpha + 2\omega) = \mathfrak{N}_{\mathcal{TB}}$ and it is a proper subset of $\mathfrak{N}_{\mathcal{B}}$, so that the relations given in (2.3) are not adequate to characterize compactness in $\mathcal{B}(\Omega, Y)$. Before giving the examples, let us observe that given a function $f \in \mathcal{B}$, we have $\alpha(\{f\}) = \gamma_{\mathcal{B}}(\{f\}) = 0$ and also $\mu_\alpha(\{f\}) = \mu_\gamma(\{f\}) = 0$, due to the fact that $\alpha(\{f(x)\}) = \gamma(\{f(x)\}) = 0$ for all $x \in \Omega$. Moreover, for $f \in \mathcal{B}$ we have

$$\omega(\{f\}) = \sigma_\alpha(\{f\}). \tag{2.4}$$

Indeed, given $a > \sigma_\alpha(\{f\})$, that is $a > \alpha(f(\Omega))$, and $\{Y_1, \ldots, Y_n\}$ a partition of $f(\Omega)$ with $\operatorname{diam}(Y_i) \leq a$ for $i = 1, \ldots, n$, considering the finite partition $\{A_1, \ldots, A_n\}$ of Ω with $A_i = f^{-1}(Y_i)$, we find $\omega(\{f\}) \leq a$. Vice versa, given $b > \omega(\{f\})$ and $\{A_1, \ldots, A_n\}$ a finite partition of Ω with $\operatorname{diam}(f(A_i)) \leq b$ for $i = 1, \ldots, n$, we have $f(\Omega) = \bigcup_{i=1}^n f(A_i)$, which implies $\alpha(f(\Omega)) \leq b$. Thus (2.4) follows.

Example 1 Let $\Omega = [0, +\infty)$ and $Y = \ell_\infty$. Consider the sequence $(f_k)_k$ in $\mathcal{B}([0, +\infty), \ell_\infty)$ defined by

$$f_k(x) = \begin{cases} e_n & \text{for } x = k - \frac{1}{n}, \text{ for } n = 1, 2, \dots \\ \theta & \text{otherwise.} \end{cases}$$

Set $M = \{f_k : k = 1, 2, \dots\}$. Then given $x \in [0, +\infty)$, $M(x) = \{\theta, e_n\}$ if $x \in \bigcup_{k=1}^{\infty} \left\{n - \frac{1}{k} : n = 1, 2, \dots\right\}$ and $M(x) = \{\theta\}$ if x does not belong to that set. Thus M is pointwise relatively compact. Using (2.4) and taking into account that diam $(\{e_n : n = 1, 2, \dots\}) = 1$ we deduce, for each $k \in \mathbb{N}$,

$$\omega(\{f_k\}) = \alpha\left(f_k([0, +\infty))\right) = \alpha\left(\{e_n : n = 1, 2, \dots\}\right) \le 1.$$

Since, looking at the Istratescu measure, we have $\beta\left(\{e_n : n = 1, 2, \dots\}\right) = 1$, we find $\alpha\left(\{e_n : n = 1, 2, \dots\}\right) = 1$, so $\omega(\{f_k\}) = 1$ which implies $\omega(M) \ge 1$. On the other hand, $\text{diam}(f_k([0, +\infty)) = \text{diam}\left(\{e_n : n = 1, 2, \dots\}\right) = 1$ for all k, thus taking the partition $\{[0, +\infty)\}$ we obtain $\omega(M) \le 1$, thus $\omega(M) = 1$. Now we prove $\alpha(M) = 1$. First we notice that for $k \ne s$ we have

$$\|f_k - f_s\|_\infty = \sup_{x \in \Omega} \|f_k(x) - f_s(x)\| = 1,$$

so that $\beta(M) \ge 1$, which implies $\alpha(M) \ge 1$. At the same time we have $\text{diam}(M) = \sup_{k,s \in \mathbb{N}} \|f_k - f_s\|_\infty = 1$, so that $\alpha(M) \le 1$, and our assert follows. Then, on the one hand $\alpha(M) = 1$, on the other hand

$$\mu_\alpha(M) = 0, \quad \omega(M) = 1 = \sup_{k \in \mathbb{N}} \sigma_\alpha(\{f_k\}),$$

which in turn give $M \notin \mathfrak{N}_{\mathcal{B}([0,+\infty),\ell_\infty)}$ and

$$\max\left\{\mu_\alpha(M), \frac{1}{2}\left(\omega(M) - \sup_{k \in \mathbb{N}} \sigma_\alpha(\{f_k\})\right)\right\} = 0.$$

Example 2 Assume the Banach space Y to be infinite-dimensional and $\Omega = B(Y)$. Denote by I the identity function on $B(Y)$ and let $M = \{I\}$ in $\mathcal{B}(B(Y), Y)$. Then $\alpha(M) = \gamma_{\mathcal{B}(B(Y),Y)}(M) = \mu_\alpha(M) = \mu_\gamma(M) = 0, \sigma_\alpha(M) = \omega(M) = 2$, and $\sigma_\gamma(M) = 1$. In particular, $M \in \mathfrak{N}_{\mathcal{B}(B(Y),Y)}$, but the right-hand side of (2.3) does not vanish on M.

Further, in view of the above example, the quantitative characteristics ω, σ_α and σ_γ cannot be used in general to characterize compactness in $\mathcal{B}(\Omega, Y)$. The goal of the next section is to use a new equicontinuity-type concept, and a quantitative characteristic modeled on it, to fill in this gap.

To close this section, given $\psi : \mathfrak{M}_\mathcal{X} \to [0, \infty)$ any set function in $\{\mu_\alpha, \sigma_\alpha, \mu_\gamma, \sigma_\gamma\}$, from the properties of α and γ we derive that ψ satisfies axioms (ii)–(v) of Definition 2.1.

Proposition 2.3 *Assume $\psi \in \{\mu_\alpha, \sigma_\alpha, \mu_\gamma, \sigma_\gamma\}$. Let $M, N \in \mathfrak{M}_{\mathcal{X}}$ and $\lambda \in [0,1]$. Then*

(ii) *$M \subseteq N$ implies $\psi(M) \le \psi(N)$;*
(iii) *$\psi(\overline{M}) = \psi(M)$;*
(iv) *$\psi(coM) = \psi(M)$;*
(v) *$\psi(\lambda M + (1-\lambda)N) \le \lambda\psi(M) + (1-\lambda)\psi(N)$.*

Proof As the proof works in the same way for α and γ, we assume $\psi \in \{\mu_\alpha, \sigma_\alpha\}$. Let $M, N \in \mathfrak{M}_{\mathcal{X}}$ and $\lambda \in [0,1]$. If $M \subseteq N$, property (ii) follows immediately from the definition of ψ.

(iii) Using (ii) we have $\psi(M) \le \psi(\overline{M})$. We prove the converse inequality.

Let $\psi = \mu_\alpha$. Let $x \in \Omega$, $y \in \overline{M}(x)$ and $f \in \overline{M}$ such that $y = f(x)$. Let $(f_n)_n$ be a sequence of functions in M such that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$, then $\|f_n(x) - f(x)\| \to 0$. Hence $f(x) = y \in \overline{M(x)}$, so that $\overline{M}(x) \subseteq \overline{M(x)}$ and $\gamma(\overline{M}(x)) \le \gamma(\overline{M(x)})$. Since $\alpha(\overline{M(x)}) = \alpha(M(x))$, we obtain $\alpha(\overline{M}(x)) \le \alpha(M(x))$, for all $x \in \Omega$, which implies $\mu_\alpha(\overline{M}) \le \mu_\alpha(M)$.

Let $\psi = \sigma_\alpha$. Let $x \in \Omega$, $y \in \overline{M}(\Omega)$ and $f \in \overline{M}$ such that $y = f(x)$. Repeating the same argument as before, we find $\overline{M}(\Omega) \subseteq \overline{M(\Omega)}$ and then $\sigma_\alpha(\overline{M}) \le \sigma_\alpha(M)$.

(iv) Using (ii) we have $\psi(M) \le \psi(\text{co}M)$. We prove the converse inequality. Let $\psi = \mu_\alpha$. Let $x \in \Omega$, $y \in (\text{co}M)(x)$ and $f \in \text{co}M$ such that $y = f(x)$. Fix $f_1, \dots, f_n \in M$ and $\lambda_1, \dots, \lambda_n \in [0,1]$ with $\sum_{i=1}^n \lambda_i = 1$ such that $f = \sum_{i=1}^n \lambda_i f_i$. Then $y = f(x) \in \text{co}M(x)$. Therefore $(\text{co}M)(x) \subseteq \text{co}M(x)$ for all $x \in \Omega$. Hence $\alpha((\text{co}M)(x)) \le \alpha(\text{co}M(x)) = \alpha(M(x))$, and from this follows $\mu_\alpha(\text{co}M) \le \mu_\alpha(M)$, as desired.

If $\psi = \sigma_\alpha$, the result follows from the fact that one proves $(\text{co}M)(\Omega) \subseteq \text{co}M(\Omega)$.

(v) It is immediate, in one case from $(\lambda M + (1-\lambda)N)(x) \subseteq (\lambda M)(x) + ((1-\lambda)N)(x)$ for all $x \in \Omega$, and in the other, from $(\lambda M + (1-\lambda)N)(\Omega) \subseteq (\lambda M)(\Omega) + ((1-\lambda)N)(\Omega)$. □

Remark 2.4 From [24, Example 1] we see that the set functions μ_α, σ_α, μ_γ and σ_γ are not in general measures of noncompactness. Let $E = \mathcal{C}([0,1], \mathbb{R})$ and $M = \{f_n : n = 3, 4, \dots\} \in \mathfrak{M}_E$, where

$$f_n(t) = \begin{cases} 0 & \text{for } 0 \le t < \frac{1}{2} - \frac{1}{n}, \\ \left(t - \frac{1}{2} + \frac{1}{n}\right)\frac{n}{2} & \text{for } \frac{1}{2} - \frac{1}{n} \le t < \frac{1}{2} + \frac{1}{n}, \\ 1 & \text{for } \frac{1}{2} + \frac{1}{n} \le t \le 1. \end{cases}$$

Then $M \notin \mathfrak{N}_E$ and $\mu_\gamma(M) = \mu_\alpha(M) = \sigma_\gamma(M) = \sigma_\alpha(M) = 0$, so that, none of the given set functions satisfies condition (i) of Definition 2.1.

3 Compactness in Banach subspaces of $\mathcal{B}(\Omega, Y)$

Throughout this section, $\mathcal{X}$ will stand for a Banach subspace of $\mathcal{B}(\Omega, Y)$, possibly $\mathcal{B}(\Omega, Y)$ itself, endowed with $\|\cdot\|_\infty$. Let us introduce the following equicontinuity-type concept in our general setting.

Definition 3.1 We say that a set $M \in \mathfrak{M}_{\mathcal{X}}$ is extendedly equicontinuous if for any $\varepsilon > 0$ there are a finite partition $\{A_1, \dots, A_n\}$ of Ω and a finite set $\{\varphi_1, \dots, \varphi_m\}$ of functions in $\mathcal{X}$ such that, for all $f \in M$, there is $j \in \{1, \dots, m\}$ with $\operatorname{diam}((f - \varphi_j)(A_i)) \leq \varepsilon$ for $i = 1, \dots, n$.

Next, for a set $M \in \mathfrak{M}_{\mathcal{X}}$ we introduce the new quantitative characteristic $\omega_{\mathcal{X}}(M)$ that will measure the degree of extended equicontinuity of M.

Definition 3.2 We define the set function $\omega_{\mathcal{X}} : \mathfrak{M}_{\mathcal{X}} \to [0, +\infty)$ by setting

$$\begin{aligned}\omega_{\mathcal{X}}(M) = \inf\{\varepsilon > 0 :\ & \text{there are a finite partition } \{A_1, \dots, A_n\} \text{ of } \Omega \\ & \text{and a finite set } \{\varphi_1, \dots, \varphi_m\} \text{ in } \mathcal{X} \text{ such that, for all } f \in M, \\ & \text{there is } j \in \{1, \dots, m\} \text{ with } \operatorname{diam}((f - \varphi_j)(A_i)) \leq \varepsilon \\ & \text{for } i = 1, \dots, n\}.\end{aligned}$$

Clearly a set M is extendedly equicontinuous if and only if $\omega_{\mathcal{X}}(M) = 0$. Let us notice that, for $M \in \mathfrak{M}_{\mathcal{X}}$, we have $\omega_{\mathcal{B}}(M) \leq \omega_{\mathcal{X}}(M) \leq 2\omega_{\mathcal{B}}(M)$. The left inequality is immediate. To show the right one, let $a > \omega_{\mathcal{B}}(M)$, $\{A_1, \dots, A_n\}$ a finite partition of Ω and $\{\varphi_1, \dots, \varphi_m\}$ a finite set in $\mathcal{B}(\Omega, Y)$ such that, for all $f \in M$ there is $j \in \{1, \dots, m\}$ with $\operatorname{diam}((f - \varphi_j)(A_i)) \leq a$ for $i = 1, \dots, n$. Next for each j, set $M_j = \{f \in M : \operatorname{diam}((f - \varphi_j)(A_i)) \leq a\}$ and choose ψ_j arbitrarily in M_j. Then, $\{\psi_1, \dots, \psi_m\}$ is a finite set in $\mathcal{X}$ such that, for all $f \in M$ there is $j \in \{1, \dots, m\}$ with $\operatorname{diam}((f - \psi_j)(A_i)) \leq \operatorname{diam}((f - \varphi_j)(A_i)) + \operatorname{diam}((\psi_j - \varphi_j)(A_i)) \leq 2a$ for $i = 1, \dots, n$, as expected.

Proposition 3.3 *If $\mathcal{X} \subseteq \mathcal{TB}(\Omega, Y)$ and $M \in \mathfrak{M}_{\mathcal{X}}$, then*

$$\omega(M) = \omega_{\mathcal{X}}(M).$$

Proof Take $\{\varphi_0\}$, where φ_0 denotes the null function in $\mathcal{X}$, as a finite set of functions in the definition of $\omega_{\mathcal{X}}$, then $\omega_{\mathcal{X}}(M) \leq \omega(M)$. Now, we prove the reverse inequality. Let $a > \omega_{\mathcal{X}}(M)$, let $\{A_1, \dots, A_n\}$ be a finite partition of Ω and $\{\varphi_1, \dots, \varphi_m\}$ a finite set in $\mathcal{B}(\Omega, Y)$ such that, for all $f \in M$ there is $j \in \{1, \dots, m\}$ such that $\operatorname{diam}((f - \varphi_j)(A_i)) \leq a$, all for $i = 1, \dots, n$. Set $T = \bigcup_{j=1}^{m} \varphi_j(\Omega)$. Since $\varphi_j \in \mathcal{TB}(\Omega, Y)$ for $j = 1, \dots, m$ we have $\gamma(T) = 0$. Therefore, for $\delta > 0$ arbitrarily fixed, there are $y_1, \dots, y_k \in Y$ such that $T \subseteq \bigcup_{l=1}^{k} B(y_l, \delta)$. Hence $T = \bigcup_{l=1}^{k} K_l$, where $K_l = T \cap B(y_l, \delta)$, for $l = 1, \dots, k$. Set $B_l = K_l \backslash (\bigcup_{s=0}^{\ell-1} K_s)$, for $l = 1, \dots, k$, where $K_0 = \emptyset$. Then $\{B_1, \dots, B_k\}$ is a finite partition of T, and for all $j \in \{1, \dots, m\}$ the family $\{\varphi_j^{-1}(B_1), \dots, \varphi_j^{-1}(B_k)\}$ is a finite partition of Ω. Let $\{S_1, \dots, S_q\}$ be the partition of Ω generated by the partitions $\{\varphi_j^{-1}(B_1), \dots, \varphi_j^{-1}(B_k)\}$ $(j = 1, \dots, m)$ and by the partition $\{A_1, \dots, A_n\}$. Then, for all $f \in M$ and for all $r \in \{1, \dots, q\}$ we can choose $j \in \{1, \dots, m\}$ such that

$$\operatorname{diam}(f(S_r)) = \sup_{x, y \in S_r} \|f(x) - f(y)\|$$

$$\leq \sup_{x,y\in S_r} \|(f-\varphi_j)(x)-(f-\varphi_j)(y)\| + \sup_{x,y\in S_r} \|\varphi_j(x)-\varphi_j(y)\|$$
$$\leq a + 2\delta,$$

so that $\omega(M) \leq \omega_{\mathcal{X}}(M)$. □

The set function $\omega_{\mathcal{X}} : \mathfrak{M}_{\mathcal{X}} \to [0, +\infty)$ satisfies axioms (ii)–(v) of Definition 2.1. Clearly, it is not in general a measure of noncompactness in $\mathcal{X}$.

Proposition 3.4 *Let* $M, N \in \mathfrak{M}_{\mathcal{X}}$ *and* $\lambda \in [0, 1]$. *Then*

(ii) $M \subseteq N$ *implies* $\omega_{\mathcal{X}}(M) \leq \omega_{\mathcal{X}}(N)$;
(iii) $\omega_{\mathcal{X}}(\overline{M}) = \omega_{\mathcal{X}}(M)$;
(iv) $\omega_{\mathcal{X}}(\mathrm{coM}) = !_{\mathcal{X}}(\mathrm{M})$;
(v) $\omega_{\mathcal{X}}(\lambda M + (1-\lambda)N) \leq \lambda\omega_{\mathcal{X}}(M) + (1-\lambda)\omega_{\mathcal{X}}(N)$.

Proof Throughout this proof, given $M \in \mathfrak{M}_{\mathcal{X}}$, $a > \omega_{\mathcal{X}}(M)$ we denote by $\{A_1, \ldots, A_n\}$ a finite partition of Ω and by $\{\varphi_1, \ldots, \varphi_m\}$ a finite set in $\mathcal{X}$ such that for all $f \in M$ there is $j \in \{1, \ldots, m\}$ such that $\mathrm{diam}((f-\varphi_j)(A_i)) \leq a$ for $i = 1, \ldots, n$.

Property (ii) follows immediately from the definition of $\omega_{\mathcal{X}}$.

Next, observe that the inequality $\omega_{\mathcal{X}}(M) \leq \omega_{\mathcal{X}}(\overline{M})$ follows from (ii). To prove the converse inequality, let $\delta > 0$ be arbitrarily fixed and let $g \in \overline{M}$. Choose a function $f \in M$ and $j \in \{1, \ldots, m\}$ such that $\|f - g\|_\infty \leq \delta$ and $\mathrm{diam}((f-\varphi_j)(A_i)) \leq a$ for $i = 1, \ldots, n$. Thus, for all i, we have

$$\begin{aligned}\mathrm{diam}((g-\varphi_j)(A_i)) &= \sup_{x,y\in A_i} \|(g-\varphi_j)(x)-(g-\varphi_j)(y)\| \\ &\leq \sup_{x,y\in A_i} (\|(f-\varphi_j)(x)-(f-\varphi_j)(y)\| + \|g(x)-f(x)\| \\ &\quad + \|g(y)-f(y)\|) \leq a + 2\delta.\end{aligned}$$

Hence, by the arbitrariness of δ, it follows $\mathrm{diam}((g-\varphi_j)(A_i)) \leq a$, and so, by the arbitrariness of a, we obtain $\omega_{\mathcal{X}}(\overline{M}) \leq \omega_{\mathcal{X}}(M)$. We have proved (iii).

To prove (iv) enough to show $\omega_{\mathcal{X}}(\mathrm{coM}) \leq !_{\mathcal{X}}(\mathrm{M})$. Let $g \in \mathrm{co}M$ be arbitrarily fixed. Let $f_1, \ldots, f_k \in M$ and $\lambda_1, \ldots, \lambda_k \in [0, 1]$ with $\sum_{s=1}^k \lambda_s = 1$ such that $g = \sum_{s=1}^k \lambda_s f_s$. Denote by H the set of all functions $i \to h(i)$ of $\{1, \ldots k\}$ into $\{1, \ldots, m\}$. Fix $h \in H$ such that, for all $s \in \{1, \ldots, k\}$, we have

$$\mathrm{diam}((f_s - \varphi_{h(s)})(A_i)) \leq a \quad \text{for } i = 1, \ldots, n.$$

We observe that $\mathrm{co}\{\varphi_1, \ldots, \varphi_m\}$ is a compact subset of $\mathcal{X}$. Hence, given $\delta > 0$, we can choose a finite $\|\cdot\|_\infty$-δ-net $\{\psi_1, \ldots, \psi_l\}$ for $\mathrm{co}\{\varphi_1, \ldots, \varphi_m\}$ in $\mathcal{X}$. Then the function $\sum_{s=1}^k \lambda_s \varphi_{h(s)}$ belongs to $\mathrm{co}\{\varphi_1, \ldots, \varphi_m\}$, so that there is $v \in \{1, \ldots, l\}$ such that

$$\left\| \sum_{s=1}^k \lambda_s \varphi_{h(s)} - \psi_v \right\|_\infty \leq \delta.$$

Therefore, for all $i \in \{1, \ldots, n\}$, we obtain

$$\operatorname{diam}((g-\psi_v)(A_i)) = \sup_{x,y\in A_i} \left\| \left(\sum_{s=1}^{k} \lambda_s f_s - \psi_v\right)(x) - \left(\sum_{s=1}^{k} \lambda_s f_s - \psi_v\right)(y) \right\|$$
$$\leq \sup_{x,y\in A_i} \left(\left\| \sum_{s=1}^{k} \lambda_s \left(f_s - \varphi_{h(s)}\right)(x) - \sum_{s=1}^{k} \lambda_s \left(f_s - \varphi_{h(s)}\right)(y) \right\| \right.$$
$$\left. + \left\| \sum_{s=1}^{k} \lambda_s \varphi_{h(s)}(x) - \psi_v(x) \right\| + \left\| \sum_{s=1}^{k} \lambda_s \varphi_{h(s)}(y) - \psi_v(y) \right\| \right) \leq a + 2\delta.$$

By the arbitrariness of δ and a we have the desired result. So the proof of (iv) is complete.

Finally, we prove (v). Given $N \in \mathfrak{M}_{\mathcal{X}}$, $b > \omega_{\mathcal{X}}(N)$, without loss of generality we may still assume that $\{A_1, \ldots, A_n\}$ is a finite partition of Ω and $\{\varphi_1, \ldots, \varphi_m\}$ is a finite set in $\mathcal{X}$ such that for all $g \in N$ there is $k \in \{1, \ldots, m\}$ such that $\operatorname{diam}((g-\varphi_k)(A_i)) \leq b$, for $i = 1, \ldots, n$.

Let $\lambda \in [0, 1]$ and $w \in \lambda M + (1-\lambda)N$, then $w = \lambda f + (1-\lambda)g$ with $f \in M$ and $g \in N$. Choose $j, k \in \{1, \ldots, m\}$ such that $\operatorname{diam}((f-\varphi_j)(A_i)) \leq a$ and $\operatorname{diam}((g-\varphi_k)(A_i)) \leq b$ for all $i \in \{1, \ldots, n\}$. Note that $\operatorname{co}\{\varphi_1, \ldots, \varphi_m\}$ is a compact set in $\mathcal{X}$. Thus, given $\delta > 0$, we can choose a finite $\|\cdot\|_\infty$-δ-net $\{\psi_1, \ldots, \psi_l\}$ for $\operatorname{co}\{\varphi_1, \ldots, \varphi_m\}$ in $\mathcal{X}$.

Next, let $s \in \{1, \ldots, l\}$ such that $\|\psi_s - (\lambda\varphi_j + (1-\lambda)\varphi_k)\|_\infty \leq \delta$. Then, for $i \in \{1, \ldots, n\}$, we have

$$\operatorname{diam}((w-\psi_s)(A_i)) = \sup_{x,y\in A_i} \|(w-\psi_s)(x) - (w-\psi_s)(y)\|$$
$$= \sup_{x,y\in A_i} \|(\lambda f + (1-\lambda)g - \psi_s)(x) - (\lambda f + (1-\lambda)g - \psi_s)(y)\|$$
$$\leq \sup_{x,y\in A_i} \Big(\|\psi_s(x) - (\lambda\varphi_j + (1-\lambda)\varphi_k)(x)\| + \|\psi_s(y) - (\lambda\varphi_j + (1-\lambda)\varphi_k)(y)\|$$
$$+ \|\lambda(f-\varphi_j)(x) - \lambda(f-\varphi_j)(y)\|$$
$$+ \|(1-\lambda)(g-\varphi_k)(x) - (1-\lambda)(g-\varphi_k)(y)\| \Big)$$
$$\leq 2\delta + \lambda a + (1-\lambda)b.$$

In virtue of the arbitrariness of δ, a and b we obtain (v). □

3.1 MNC equivalent to α

In this subsection, we prove equivalent relations in Banach subspaces $\mathcal{X}$ of $\mathcal{B}(\Omega, Y)$ for the Kuratowski measure of noncompactness. We establish a criterion of compactness and a precise formula for the Kuratowski measure of bounded extendedly equicontinuous subsets of the space. The results are new when $\mathcal{X}$ properly contains $\mathcal{TB}(\Omega, Y)$.

Theorem 3.5 *Let $\mathcal{X} \subseteq \mathcal{B}(\Omega, Y)$ and let $M \in \mathfrak{M}_{\mathcal{X}}$. Then*

$$\max\{\mu_\alpha(M),\ \frac{1}{2}\omega_{\mathcal{X}}(M)\} \le \alpha(M) \le \mu_\alpha(M) + 2\omega_{\mathcal{X}}(M). \tag{3.1}$$

Proof We prove the left inequality. Let $a > \alpha(M)$ and let $\{M_1, \ldots, M_n\}$ be a finite cover of M such that $\operatorname{diam}(M_i) \le a$ for $i = 1, \ldots, n$. For all $x \in \Omega$, we have $M(x) \subseteq \bigcup_{i=1}^n M_i(x)$. Moreover, for all $i \in \{1, \ldots, n\}$, we have

$$\operatorname{diam}(M_i(x)) = \sup_{f,g \in M_i} \|f(x) - g(x)\| \le \sup_{f,g \in M_i} \|f - g\|_\infty = \operatorname{diam}(M_i) \le a.$$

Hence $\alpha(M(x)) \le a$ for all $x \in \Omega$, i.e. $\mu_\alpha(M) \le a$. By the arbitrariness of a, we have

$$\mu_\alpha(M) \le \alpha(M). \tag{3.2}$$

Now, choose $\{\Omega\}$ as a partition of Ω and $\{\varphi_1, \ldots, \varphi_n\}$ as a finite set in $\mathcal{X}$, taking $\varphi_i \in M_i$ for $i = 1, \ldots, n$. Let $f \in M$, fix $i \in \{1, \ldots, n\}$ such that $\|f - \varphi_i\|_\infty \le a$, then

$$\operatorname{diam}((f - \varphi_i)(\Omega)) = \sup_{x,y \in \Omega} \|(f - \varphi_i)(x) - (f - \varphi_i)(y)\| \le 2\|f - \varphi_i\|_\infty \le 2a.$$

Thus we derive that $\omega_{\mathcal{X}}(M) \le 2a$, and, by the arbitrariness of a,

$$\omega_{\mathcal{X}}(M) \le 2\alpha(M). \tag{3.3}$$

Then (3.2) and (3.3) give $\max\{\mu_\alpha(M),\ \frac{1}{2}\omega_{\mathcal{X}}(M)\} \le \alpha(M)$.

Next, we prove the right inequality. First let $a > \omega_{\mathcal{X}}(M)$. Choose a finite partition $\{A_1, \ldots, A_n\}$ of Ω and a finite set $\{\varphi_1, \ldots, \varphi_m\}$ in $\mathcal{X}$ such that, for all $f \in M$, there is $j \in \{1, \ldots, m\}$ such that $\operatorname{diam}((f - \varphi_j)(A_i)) \le a$, for all $i = 1, \ldots, n$. Set $M_j = \{f \in M : \operatorname{diam}((f - \varphi_j)(A_i)) \le a \text{ for } i = 1, \ldots, n\}$, for each j. Fix $x_i \in A_i$, for each i, and observe that for each j we have

$$\alpha\left(\bigcup_{i=1}^n M_j(x_i)\right) = \max_{i=1}^n \alpha(M_j(x_i)) \le \max_{i=1}^n \alpha(M(x_i)) \le \mu_\alpha(M).$$

Then, let $b > \mu_\alpha(M)$ and, for any fixed j, let $\{B_1^j, \ldots, B_{l_j}^j\}$ be a finite cover of $\bigcup_{i=1}^n M_j(x_i)$ such that $\operatorname{diam}(B_s^j) \le b$ for $s = 1, \ldots, l_j$. Let j be fixed. Denote by H^j the set of all functions $h^j : \{1, \ldots n\} \to \{1, \ldots, l_j\}$, and for $h^j \in H^j$ consider the set

$$M_{j,h^j} = \{f \in M_j : f(x_i) \in B_{h^j(i)}^j, \text{ for } i = 1, \ldots, n\}.$$

Then, $\{M_{j,h^j} : h^j \in H^j\}$ is a finite cover of M_j, and $\mathrm{diam}(M_{j,h^j}) \le b + 2a$ for all j. In fact, for $f, g \in M_{j,h^j}$, we have

$$\|f - g\|_\infty = \max_{i=1}^{n} \sup_{x\in A_i} \|f(x) - g(x)\|$$
$$\le \max_{i=1}^{n} \sup_{x\in A_i} (\|f(x_i) - g(x_i)\| + \|(f - \varphi_j)(x) - (f - \varphi_j)(x_i)\|$$
$$+ \|(g - \varphi_j)(x) - (g - \varphi_j)(x_i)\|) \le b + 2a.$$

Since $\{M_j : j = 1, \ldots, m\}$ is a finite cover of M, we infer $\alpha(M) \le b + 2a$, and by the arbitrariness of a and b we get $\alpha(M) \le \mu_\alpha(M) + 2\omega_{\mathcal{X}}(M)$. □

Now, from the inequalities we have proved, we obtain the following compactness criterion in $\mathcal{X}$.

Corollary 3.6 *Let $\mathcal{X} \subseteq \mathcal{B}(\Omega, Y)$. A subset M of $\mathcal{X}$ is relatively compact if and only if it is bounded, extendedly equicontinuous and pointwise relatively compact.*

The inequalities (3.1) reduce, for a class of subsets, to a precise formula of Ambrosetti-type.

Corollary 3.7 *Let $\mathcal{X} \subseteq \mathcal{B}(\Omega, Y)$ and assume that $M \in \mathfrak{M}_{\mathcal{X}}$ is extendedly equicontinuous. Then*

$$\alpha(M) = \mu_\alpha(M).$$

In the general case, we get a regular measure of noncompactness equivalent to that of Kuratowski.

Corollary 3.8 *Given $\mathcal{X} \subseteq \mathcal{B}(\Omega, Y)$, the set function $\mu_\alpha + 2\omega_{\mathcal{X}} : \mathfrak{M}_{\mathcal{X}} \to [0, +\infty)$ is a regular measure of noncompactness in $\mathcal{X}$ equivalent to the Kuratowski measure α.*

Proof From Propositions 2.3 and 3.4 it follows that $\mu_\alpha + 2\omega_{\mathcal{X}}$ satisfies conditions (ii)–(v) of Definition 2.1. Further conditions (i) and (vi) are consequences of Corollary 3.6. Finally, it can be easily verified that both μ_α and $\omega_{\mathcal{X}}$ are sublinear and enjoy the maximum property, so the same is true for $\mu_\alpha + 2\omega_{\mathcal{X}}$. The equivalence follows from Theorem 3.5. □

We end this subsection looking at Banach subspaces of $\mathcal{TB}(\Omega, Y)$. In such a case, the quantitative characteristic σ_α can be used to improve the left-hand side of (3.1). We need the following lemma.

Lemma 3.9 *Let $\mathcal{X} \subseteq \mathcal{TB}(\Omega, Y)$ and let $M \in \mathfrak{M}_{\mathcal{X}}$. Then*

$$\sigma_\alpha(M) \le \alpha(M).$$

Proof Let $M \in \mathfrak{M}_{\mathcal{X}}$, $a > \alpha(M)$ and let $\{M_1, \ldots, M_n\}$ be a finite cover of M with diam $(M_i) \leq a$. Take $f_i \in M_i$, for $i = 1, \ldots n$. Since each f_i is totally bounded we have that $\alpha(\bigcup_{i=1}^n f_i(\Omega)) = 0$. Thus, given $\varepsilon > 0$ arbitrarily fixed, we choose a finite cover $\{B_1, \ldots, B_m\}$ of $\bigcup_{i=1}^n f_i(\Omega)$ such that $\operatorname{diam}(B_j) \leq \varepsilon$, for $j = 1, \ldots, m$. Next, for each j, fix $y_j \in B_j$. Now let $f \in M$ and $x \in \Omega$ be arbitrarily fixed. First choose $i \in \{1, \ldots, n\}$ such that $\|f - f_i\|_\infty \leq a$, then $j \in \{1, \ldots, m\}$ such that $f_i(x) \in B_j$, that is $\|f_i(x) - y_j\|_\infty \leq \varepsilon$. Then $\|f(x) - y_j\|_\infty \leq a + \varepsilon$, and by the arbitrariness of ε we have $\|f(x) - y_j\|_\infty \leq a$, so that $\{B(y_1, a), \ldots B(y_m, a)\}$ is a finite cover of $M(\Omega)$. As $\sigma_\alpha(M) = \alpha(M(\Omega))$, the proof is complete. □

Theorem 3.10 *If* $\mathcal{X} \subseteq \mathcal{TB}(\Omega, Y)$ *and* $M \in \mathfrak{M}_{\mathcal{X}}$, *then*

$$\max\{\sigma_\alpha(M), \frac{1}{2}\omega(M)\} \leq \alpha(M) \leq \mu_\alpha(M) + 2\omega(M).$$

Proof From Lemma 3.9 and Proposition 3.3 we have $\sigma_\alpha(M) \leq \alpha(M)$ and $\omega_{\mathcal{X}}(M) = \omega(M)$. Hence, Theorem 3.5 gives the thesis. □

From the above result we get that $\omega(M) = 0$ implies $\alpha(M) = \mu_\alpha(M) = \sigma_\alpha(M)$. This extends [1, Lemma 2.2] from the case of sets of Y-valued functions defined and continuous on a compact metric space to the case of sets of Y-valued functions defined and totally bounded on a general set Ω.

3.2 MNC equivalent to γ

We now provide estimates for the Hausdorff measure of noncompactness, more accurate than those one can derive from the previous results using the known equivalence between measures of noncompactness and the involved quantitative characteristics. At first, the following lemma gives the upper estimate of the Hausdorff measure in $\mathcal{B}(\Omega, Y)$.

Lemma 3.11 *Let* $M \in \mathfrak{M}_{\mathcal{B}}$, *then*

$$\gamma_{\mathcal{B}}(M) \leq \mu_\gamma(M) + \omega_{\mathcal{B}}(M).$$

Proof Let $a > \omega_{\mathcal{B}}(M)$. Let $\{A_1, \ldots, A_n\}$ be a finite partition of Ω and $\{\varphi_1, \ldots, \varphi_m\}$ a finite set in $\mathcal{B}$ such that, for all $f \in M$, there is $j \in \{1, \ldots, m\}$ such that $\operatorname{diam}((f - \varphi_j)(A_i)) \leq a$, for all $i = 1, \ldots, n$. Set

$$M_j = \{f \in M : \operatorname{diam}((f - \varphi_j)(A_i)) \leq a \text{ for } i = 1, \ldots, n\}, \quad \text{for } j = 1, \ldots, m.$$

Now, for each $i = 1, \ldots, n$, let $x_i \in A_i$ be fixed. Then

$$\gamma\left(\bigcup_{i=1}^n M_j(x_i)\right) = \max_{i=1}^n \gamma(M_j(x_i)) \leq \max_{i=1}^n \gamma(M(x_i)) \leq \mu_\gamma(M).$$

Next, let $b > \mu_\gamma(M)$ and let $\{y_1^j, \dots, y_{k_j}^j\}$ be a $\|\cdot\|$-b-net for $\bigcup_{i=1}^n M_j(x_i)$ in Y. Finally, let H^j be the set of all functions $h^j : \{1, \dots n\} \to \{1, \dots, k_j\}$, and for $h^j \in H^j$, define $\psi_{j,h^j} : \Omega \to Y$ by setting, for $x \in \Omega$,

$$\psi_{j,h^j}(x) = \sum_{i=1}^n \chi_{A_i}(x)\left(\varphi_j(x) - \varphi_j(x_i) + y^j_{h^j(i)}\right).$$

For each fixed j, we show that $\{\psi_{j,h^j} : h^j \in H^j\}$ is a $\|\cdot\|_\infty$-$(a+b)$-net for M_j in $\mathcal{B}(\Omega, Y)$. To this end, given $f \in M_j$ choose $h^j \in H^j$ such that $\|f(x_i) - y^j_{h^j(i)}\| \le b$, for each $i \in \{1, \dots, n\}$. Then for $x \in \Omega$, by letting $i \in \{1, \dots, n\}$ such that $x \in A_i$, we have

$$\begin{aligned}\|f(x) - \psi_{j,h^j}(x)\| &= \|f(x) - \varphi_j(x) + \varphi_j(x_i) - y^j_{h^j(i)}\| \\ &\le \|(f - \varphi_j)(x) - (f - \varphi_j)(x_i)\| + \|f(x_i) - y^j_{h^j(i)}\| \\ &\le a + b,\end{aligned}$$

which implies

$$\|f - \psi_{j,h^j}\|_\infty = \max_{i=1}^n \sup_{x \in A_i} \|f(x) - \psi_{j,h^j}(x)\| \le a + b.$$

Consequently the set $\bigcup_{j=1}^m \{\psi_{j,h^j} : h^j \in H^j\}$ is a finite $\|\cdot\|_\infty$-$(a+b)$-net for M in $\mathcal{B}(\Omega, Y)$. The arbitrariness of a and b implies $\gamma_B(M) \le \mu_\gamma(M) + \omega_{\mathcal{B}}(M)$, and the proof is complete. □

Theorem 3.12 *Let $\mathcal{X} \subseteq \mathcal{B}(\Omega, Y)$ and let $M \in \mathfrak{M}_{\mathcal{X}}$. Then*

$$\max\{\mu_\gamma(M),\ \frac{1}{2}\omega_{\mathcal{X}}(M)\} \le \gamma_{\mathcal{X}}(M) \le 2(\mu_\gamma(M) + \omega_{\mathcal{X}}(M)). \tag{3.4}$$

Proof We prove the left inequality. Let $a > \gamma_{\mathcal{X}}(M)$ and let $\{\varphi_1, \dots, \varphi_n\}$ be a $\|\cdot\|_\infty$-a-net for M in $\mathcal{X}$. Then it is easy to check that $\{\varphi_1(x), \dots, \varphi_n(x)\}$ is a $\|\cdot\|$-a-net for $M(x)$ in Y. Hence $\sup_{x\in\Omega} \gamma(M(x)) \le a$, and, by the arbitrariness of a, we have

$$\mu_\gamma(M) \le \gamma_{\mathcal{X}}(M). \tag{3.5}$$

Next let $f \in M$ be arbitrarily fixed and choose $j \in \{1, \dots, n\}$ such that $\|f - \varphi_j\|_\infty \le a$, then

$$\begin{aligned}\operatorname{diam}((f - \varphi_j)(\Omega)) &= \sup_{x,y\in\Omega} \|(f - \varphi_j)(x) - (f - \varphi_j)(y)\| \\ &\le 2\|f - \varphi_j\|_\infty \le 2a.\end{aligned}$$

By the arbitrariness of a, in view of the definition of $\omega_{\mathcal{X}}(M)$, choosing $\{\Omega\}$ as a partition of Ω and $\{\varphi_1, \ldots, \varphi_n\}$ as a finite set in $\mathcal{X}$ we have

$$\omega_{\mathcal{X}}(M) \leq 2\gamma_{\mathcal{X}}(M). \tag{3.6}$$

Hence by (3.5) and (3.6), the desired result follows. The right inequality follows from Lemma 3.11, taking into account that $\gamma_{\mathcal{X}}(M) \leq 2\gamma_{\mathcal{B}}(M)$ and $\omega_{\mathcal{B}}(M) \leq \omega_{\mathcal{X}}(M)$. □

The compactness criterion given in Corollary 3.6 can be deduced as well from Theorem 3.12. Moreover, we notice that the set function $\mu_\gamma + \omega_{\mathcal{X}}$ is a regular measure of noncompactness in $\mathcal{X}$ equivalent to $\gamma_{\mathcal{X}}$. From Lemma 3.11 and the left hand side of (3.4) we obtain the following theorem.

Theorem 3.13 *Let $M \in \mathfrak{M}_{\mathcal{B}}$. Then*

$$\max\{\mu_\gamma(M),\ \frac{1}{2}\omega_{\mathcal{B}}(M)\} \leq \gamma_{\mathcal{B}}(M) \leq \mu_\gamma(M) + \omega_{\mathcal{B}}(M).$$

Corollary 3.14 *Let $M \in \mathfrak{M}_{\mathcal{B}}$ and assume $\omega_{\mathcal{B}}(M) = 0$. Then*

$$\gamma_{\mathcal{B}}(M) = \mu_\gamma(M).$$

Next, we look at the inequalities in $\mathcal{TB}(\Omega, Y)$. The following lemma will be useful.

Lemma 3.15 *If $\mathcal{X} \subseteq \mathcal{TB}(\Omega, Y)$ and $M \in \mathfrak{M}_{\mathcal{X}}$, then*

$$\sigma_\gamma(M) \leq \gamma_{\mathcal{X}}(M).$$

Proof Let $M \in \mathfrak{M}_{\mathcal{X}}$, $a > \gamma_{\mathcal{X}}(M)$ and let $\{\varphi_1, \ldots, \varphi_n\}$ be a $\|\cdot\|_\infty$-a-net for M in $\mathcal{X}$. By hypothesis we have $\gamma(\bigcup_{i=1}^n \varphi_i(\Omega)) = 0$, thus given $\varepsilon > 0$ we find a $\|\cdot\|$-b-net $\{\xi_1, \ldots, \xi_m\}$ for $\bigcup_{i=1}^n \varphi_i(\Omega)$ in Y. Then $\{\xi_1, \ldots, \xi_m\}$ is a $\|\cdot\|$-$(a+\varepsilon)$-net for $f(\Omega)$ in Y. Indeed, given $f \in M$ and $x \in \Omega$ arbitrarily fixed, choosing $i \in \{1, \ldots, n\}$ such that $\|f(x) - \varphi_i(x)\| \leq a$ and then $j \in \{1, \ldots, m\}$ such that $\|\varphi_i(x) - \xi_j\| \leq \varepsilon$, we have $\|f(x) - \xi_j\| \leq a + \varepsilon$. Then $\sigma_\gamma(M) = \gamma(M(\Omega)) \leq a + \varepsilon$ and, by the arbitrariness of ε, the result follows. □

Theorem 3.16 *If $M \in \mathfrak{M}_{\mathcal{TB}}$, then*

$$\max\left\{\sigma_\gamma(M), \frac{1}{2}\omega(M)\right\} \leq \gamma_{\mathcal{TB}}(M) \leq \mu_\gamma(M) + \omega(M).$$

Proof By Lemma 3.15 we have $\sigma_\gamma(M) \leq \gamma_{\mathcal{TB}}(M)$. Proposition 3.3 gives $\omega_{\mathcal{TB}}(M) = \omega(M)$, hence the left inequality follows from Theorem 3.12. The right inequality can be proved in the same way as in Lemma 3.11, with the simplifications due to the use of the quantitative characteristic ω instead of that of extended equicontinuity. □

We point out that Theorem 3.16 recovers the result proved in [5, Theorem 2.1 and Proposition 3.1]. Moreover, if $M \in \mathfrak{M}_{\mathcal{TB}}$ and $\omega(M) = 0$ we have $\gamma_{\mathcal{TB}}(M) = \mu_\gamma(M) = \sigma_\gamma(M)$.

Finally, the following example shows that given a set M in $\mathfrak{M}_{\mathcal{TB}}$, in general $\gamma_{\mathcal{TB}}(M) \neq \gamma_{\mathcal{B}}(M)$.

Example 3 Let $\Omega = \mathbb{N}$, $Y = \ell_1$ (so $\|\cdot\|$ and γ denote the norm and the Hausdorff measure of noncompactness in ℓ_1, respectively), and let $\{e_n\}_{n=1}^\infty$ be the standard basis in ℓ_1. We consider the bounded set $M = \{f_k : k = 1, 2, \dots\}$ in $\mathcal{TB}(\mathbb{N}, \ell_1)$, where

$$f_k(n) = \begin{cases} e_k & \text{for } n = k \\ 0 & \text{for } n \neq k. \end{cases}$$

Then we define $g : \mathbb{N} \to \ell_1$ by setting $g(n) = \frac{1}{2}e_n$ for $n \in \mathbb{N}$; so to have $g \in \mathcal{B}(\mathbb{N}, \ell_1)$, but $g \notin \mathcal{TB}(\mathbb{N}, \ell_1)$. Given $k \in \mathbb{N}$, we have

$$\begin{aligned} \|f_k - g\|_\infty &= \sup_{n\in\mathbb{N}} \|f_k(n) - g(n)\| \\ &= \max\left\{ \|f_k(k) - g(k)\|,\ \sup_{\substack{n\in\mathbb{N}\\ n\neq k}} \|f_k(n) - g(n)\| \right\} \\ &= \max\left\{ \left\|e_k - \frac{1}{2}e_k\right\|,\ \sup_{\substack{n\in\mathbb{N}\\ n\neq k}} \left\|\frac{1}{2}e_n\right\| \right\} = \frac{1}{2}. \end{aligned}$$

This shows that $\{g\}$ is a $\|\cdot\|_\infty$-$\frac{1}{2}$-net for M in $\mathcal{B}(\mathbb{N}, \ell_1)$, thus $\gamma_{\mathcal{B}(\mathbb{N},\ell_1)}(M) \leq \frac{1}{2}$. Next, on the one hand

$$\sigma_\gamma(M) = \gamma(M(\mathbb{N})) = \gamma\left(\cup_{n\in\mathbb{N}} M(n)\right) = \gamma\left(\cup_{n\in\mathbb{N}}\{e_n, 0\}\right) = 1,$$

hence from Corollary 3.16 it follows $\gamma_{\mathcal{TB}(\mathbb{N},\ell_1)}(M) \geq 1$. On the other hand, taking into account that $\gamma_{\mathcal{B}(\mathbb{N},\ell_1)}(M) \leq \gamma_{\mathcal{TB}(\mathbb{N},\ell_1)}(M) \leq 2\gamma_{\mathcal{B}(\mathbb{N},\ell_1)}(M)$ we infer $\gamma_{\mathcal{TB}(\mathbb{N},\ell_1)}(M) = 1$ and $\gamma_{\mathcal{B}(\mathbb{N},\ell_1)}(M) = \frac{1}{2}$.

4 Compactness in the spaces $\mathcal{BC}^k(\Omega, Y)$ and $\mathcal{D}^k(\Omega, Y)$

Throughout this section, the differentiability of functions is considered in the Fréchet sense. Let us start by introducing the spaces of interest. As before, Y denotes a Banach space with norm $\|\cdot\|$. Moreover, we assume that Ω is an open set in a Banach space Z and that $k \in \mathbb{N}$ is fixed. The symbol $\mathcal{C}^k(\Omega, Y)$ will stand for the space of all Y-valued functions defined and k-times continuously differentiable on Ω. For $f \in \mathcal{C}^k(\Omega, Y)$ we denote by $\mathrm{d}^p f : \Omega \to \mathcal{L}(Z^p, Y)$, for $p = 0, \dots k$, the differential of f of order p, where $\mathcal{L}(Z^p, Y)$ denotes the Banach space of all bounded p-linear operators endowed with the standard operator norm $\|\cdot\|_{\text{on}}$. We have $\mathcal{L}(Z^0, Y) = Y$, $\mathrm{d}^0 f = f$ and

$\mathcal{C}^0(\Omega, Y) = \mathcal{C}(\Omega, Y)$, the space of Y-valued functions defined and continuous on Ω. We denote, for each $p \in \{0, \dots, k\}$, by W_p the space $\mathcal{L}(Z^p, Y)$, where $W_0 = Y$, thus each $\mathrm{d}^p f$ is an element of the space $\mathcal{C}(\Omega, W_p)$. Moreover, we will denote by α_p and γ_p $(p = 0, \dots, k)$, respectively, the Kuratowski and the Hausdorff measure of noncompactness in W_p, where $\alpha_0 = \alpha$ and $\gamma_0 = \gamma$ in Y. Finally, given a set M in $\mathcal{C}^k(\Omega, Y)$ and $p \in \{0, \dots, k\}$, we define

$$M^p = \{\mathrm{d}^p f : f \in M\} \subseteq \mathcal{C}(\Omega, W_p), \tag{4.1}$$

where $M^0 = M$. Then, for $x \in \Omega$ and $A \subseteq \Omega$, the sets $M^p(x)$ and $M^p(A)$ will be the subsets of W_p described according to (2.1) by

$$M^p(x) = \{\mathrm{d}^p f(x) : f \in M\}, \quad M^p(A) = \{\mathrm{d}^p f(A) : x \in A, \ f \in M\}.$$

Now we denote by $\mathcal{BC}^k(\Omega, Y)$ the space consisting of all functions $f \in \mathcal{C}^k(\Omega, Y)$ which are bounded with all differentials up to the order k, made into a Banach space by the norm

$$\|f\|_{\mathcal{BC}^k} = \max\{\|f\|_\infty, \|\mathrm{d}f\|_\infty, \dots, \|\mathrm{d}^k f\|_\infty\},$$

where $\|\mathrm{d}^p f\|_\infty = \sup_{x \in \Omega} \|\mathrm{d}^p f(x)\|_{\mathrm{on}}$.

4.1 Results in $\mathcal{BC}^k(\Omega, Y)$

For $M \in \mathfrak{M}_{\mathcal{BC}^k}$ we define the following quantitative characteristics based on those considered in $\mathcal{B}(\Omega, Y)$, precisely

$$\mu_{\overline{\alpha}}(M) = \max_{p=0}^{k} \mu_{\alpha_p}(M^p), \quad \mu_{\overline{\gamma}}(M) = \max_{p=0}^{k} \mu_{\gamma_p}(M^p)$$

and also

$$\sigma_{\overline{\alpha}}(M) = \max_{p=0}^{k} \sigma_{\alpha_p}(M^p), \quad \sigma_{\overline{\gamma}}(M) = \max_{p=0}^{k} \sigma_{\gamma_p}(M^p).$$

A subset M is said to be pointwise k-relatively compact if $\mu_{\overline{\alpha}}(M) = 0$ (or $\mu_{\overline{\gamma}}(M) = 0$). Further, we introduce

$$\omega_{\mathcal{BC}^k}(M) = \inf \Big\{ \varepsilon > 0 : \text{ there are a finite partition } \{A_1, \dots, A_n\} \text{ of } \Omega \text{ and a finite set } \{\varphi_1, \dots, \varphi_m\} \text{ in } \mathcal{BC}^k(\Omega, Y) \text{ such that, for all } f \in M, \text{ there is } j \in \{1, \dots, m\} \text{ with } \max_{p=0}^{k} \operatorname{diam}_{W_p}(\mathrm{d}^p(f - \varphi_j))(A_i)) \le \varepsilon \text{ for } i = 1, \dots, n \Big\}.$$

We call M extendedly k-equicontinuous if $\omega_{\mathcal{BC}^k}(M) = 0$. If $\mathcal{V}$ denotes a Banach subspace of $\mathcal{BC}^k(\Omega, Y)$ equipped with the same norm, then $\omega_{\mathcal{V}}$ is defined as above by taking $\{\varphi_1, \dots, \varphi_m\}$ in $\mathcal{V}$. Note that for $k = 0$ all these quantities evidently reduce to the corresponding quantities introduced in Sect. 2 and considered in the space $\mathcal{BC}(\Omega, Y)$. We are ready to prove the following equivalent relations in the space $\mathcal{BC}^k(\Omega, Y)$. The proof is substantially different from that of Theorem 3.5, although it follows the same steps of it.

Theorem 4.1 *Let $M \in \mathfrak{M}_{\mathcal{BC}^k}$, then*

$$\max\{\mu_{\overline{\alpha}}(M),\ \frac{1}{2}\omega_{\mathcal{BC}^k}(M)\} \le \alpha(M) \le \mu_{\overline{\alpha}}(M) + 2\omega_{\mathcal{BC}^k}(M). \tag{4.2}$$

Proof We prove the left inequality. Let $a > \alpha(M)$ and let $\{M_1, \dots, M_n\}$ be a finite cover of M such that $\operatorname{diam}(M_i) \le a$ for $i = 1, \dots, n$. For all $x \in \Omega$, we have $M^p(x) \subseteq \bigcup_{i=1}^n M_i^p(x)$, for $p = 0, \dots, k$. Then,

$$\begin{aligned} \operatorname{diam}_{W_p}(M_i^p(x)) &= \sup_{f,g\in M_i} \|\mathrm{d}^p f(x) - \mathrm{d}^p g(x)\|_{\mathrm{on}} \le \sup_{f,g\in M_i} \|\mathrm{d}^p f - \mathrm{d}^p g\|_\infty \\ &\le \sup_{f,g\in M_i} \|f - g\|_{\mathcal{BC}^k} = \operatorname{diam}_{\mathcal{BC}^k}(M_i) \le a. \end{aligned}$$

Hence $\alpha_p(M^p(x)) \le a$ for all $x \in \Omega$, which implies $\mu_{\overline{\alpha}}(M) \le a$. By the arbitrariness of a, we have

$$\mu_{\overline{\alpha}}(M) \le \alpha(M). \tag{4.3}$$

Next, choose $\{\Omega\}$ as a finite partition of Ω and $\{\varphi_1, \dots, \varphi_n\}$ as a finite set in $\mathcal{BC}^k(\Omega, Y)$, where $\varphi_i \in M_i$ for $i = 1, \dots, n$. Let $f \in M$, fix $i \in \{1, \dots, n\}$ such that $\|f - \varphi_i\|_{\mathcal{BC}^k} \le a$. Then

$$\begin{aligned} \max_{p=0}^{k} \operatorname{diam}_{W_p}((\mathrm{d}^p f - \mathrm{d}^p\varphi_i)(\Omega)) &= \max_{p=0}^{k} \sup_{x,y\in\Omega} \|\mathrm{d}^p(f - \varphi_i)(x) - \mathrm{d}^p(f - \varphi_i)(y)\|_{\mathrm{on}} \\ &\le 2 \max_{p=0}^{k} \|f - \varphi_i\|_{\mathcal{BC}^k} \le 2a. \end{aligned}$$

Hence $\omega_{\mathcal{BC}^k}(M) \le 2a$, and, by the arbitrariness of a,

$$\omega_{\mathcal{BC}^k}(M) \le 2\alpha(M). \tag{4.4}$$

Combining (4.3) and (4.4), we obtain the left inequality of (4.2).

Now, let us prove the right inequality. Let $a > \omega_{\mathcal{BC}^k}(M)$, $\{A_1, \dots, A_n\}$ a finite partition of Ω and $\{\varphi_1, \dots, \varphi_m\}$ a finite set in $\mathcal{BC}^k(\Omega, Y)$ such that, for all $f \in M$, there is $j \in \{1, \dots, m\}$ such that

$$\max_{p=0}^{k} \operatorname{diam}_{W_p}(\mathrm{d}^p(f - \varphi_j)(A_i)) \le a, \quad \text{for } i = 1, \dots, n.$$

For each $j = 1, \dots, m$ set

$$M_j = \{f \in M : \max_{p=0}^{k} \operatorname{diam}_{W_p}(\mathrm{d}^p(f - \varphi_j)(A_i)) \le a, \text{ for } i = 1, \dots, n\}.$$

Fix $x_i \in A_i$ for $i = 1, \dots, n$. Then for $p = 0, \dots, k$

$$\alpha\left(\bigcup_{i=1}^{n} M_j^p(x_i)\right) \le \alpha\left(\bigcup_{i=1}^{n} M^p(x_i)\right) = \max_{i=1}^{n} \alpha_p(M^p(x_i)) \le \mu_{\alpha_p}(M^p) \le \mu_{\overline{\alpha}}(M).$$

Moreover, let $b > \mu_{\overline{\alpha}}(M)$ and, for each $p = 0, \dots, k$, let $\{B_1^j, \dots, B_{l_{p(j)}}^j\}$ be a finite cover of $\bigcup_{i=1}^{n} M_j^p(x_i)$ such that diam $(B_s^j) \le b$.

Let $j \in \{1, \dots, m\}$ be arbitrarily fixed, and let H^j be the set of all functions $h^j = (h_0^j, \dots, h_k^j)$ where, for each $p \in \{0, \dots k\}$, h_p^j maps $\{1, \dots n\}$ into $\{1, \dots, l_{p(j)}\}$. Set

$$M_{j,h^j} = \{f \in M_j : \text{for } p = 0, \dots, k,\ \mathrm{d}^p f(x_i) \in B^j_{h_p^j(i)}, \text{ for } i = 1, \dots, n\}.$$

Then $\{M_{j,h^j} : h^j \in H^j\}$ is a finite cover of M_j in $\mathcal{BC}^k(\Omega, Y)$. Moreover, for all $h^j \in H^j$, we have

$$\begin{aligned}
\operatorname{diam}_{\mathcal{BC}^k}(M_{j,h^j}) &= \sup_{f,g \in M_{j,h^j}} \|f - g\|_{\mathcal{BC}^k} = \sup_{f,g \in M_{j,h^j}} \max_{p=0}^{k} \|\mathrm{d}^p(f - g)\|_\infty \\
&= \max_{p=0}^{k} \sup_{f,g \in M_{j,h^j}} \|\mathrm{d}^p(f - g)\|_\infty.
\end{aligned}$$

Now, for all $p \in \{0, \dots k\}$ and for all $f, g \in M_{j,h^j}$, we have

$$\begin{aligned}
\|\mathrm{d}^p f - \mathrm{d}^p g\|_\infty &= \max_{i=1}^{n} \sup_{x \in A_i} \|\mathrm{d}^p f(x) - \mathrm{d}^p g(x)\|_{\mathrm{on}} \\
&\le \max_{i=1}^{n} \sup_{x \in A_i} (\|\mathrm{d}^p(f - \varphi_j)(x) - \mathrm{d}^p(f - \varphi_j)(x_i)\| \\
&\quad + \|\mathrm{d}^p(g - \varphi_j)(x) - \mathrm{d}^p(g - \varphi_j)(x_i)\| + \|\mathrm{d}^p f(x_i) - \mathrm{d}^p g(x_i)\|) \\
&\le 2 \max_{p=0}^{k} \operatorname{diam}_{W_p} \mathrm{d}^p(f - \varphi_j)(A_i)) \\
&\quad + \max_{i=1}^{n} \sup_{x \in A_i} \|\mathrm{d}^p f(x_i) - \mathrm{d}^p g(x_i)\| \le b + 2a.
\end{aligned}$$

Therefore, $\operatorname{diam}_{\mathcal{BC}^k}(M_{j,h^j}) \le b + 2a$ so that $\alpha(M_j) \le b + 2a$. Since $\{M_j : j = 1, \dots, m\}$ is a finite cover of M, we get $\alpha(M) \le b + 2a$ and by the arbitrariness of a and b we find $\alpha(M) \le \mu_{\overline{\alpha}}(M) + 2\omega_{\mathcal{BC}^k}(M)$. □

We obtain the following new criterion of compactness in $\mathcal{BC}^k(\Omega, Y)$.

Corollary 4.2 *A subset M of $\mathcal{BC}^k(\Omega, Y)$ is relatively compact if and only if it is bounded, extendedly k-equicontinuous and pointwise k-relatively compact.*

In the literature (see, for example, [3, 7, 8, 15]) there are results that characterize compactness in $\mathcal{BC}^k(\Omega, Y)$, or in proper subspaces of it, for particular Ω or Y. Let us now consider, for example, the case $Z = \mathbb{R}$. Then each space $W_p = \mathcal{L}(Z^p, Y)$, for $p = 0, \dots, k$, can be identified with Y itself. Therefore, whenever Ω is an open subset of $\mathbb{R}$ and M a bounded subset of $\mathcal{BC}^k(\overline{\Omega}, Y)$, the estimates (4.2) hold with $\mu_{\overline{\alpha}}(M) = \max_{p=0}^k \mu_\alpha(M^p)$. We also notice that in the definition of $\omega_{\mathcal{BC}^k}(M)$, in such a case, all diameters will actually be calculated in Y. Hence we obtain the following new criterion of compactness in $\mathcal{BC}^k(\overline{\Omega}, Y)$, which in particular recovers the case $\mathcal{BC}^k([0, +\infty), Y)$ (cf. [15]).

Corollary 4.3 *Let Ω be an open subset of $\mathbb{R}$ and let M be a subset of $\mathcal{BC}^k(\overline{\Omega}, Y)$. Then M is relatively compact if and only if each M^p for $p \in \{0, \dots, k\}$ is bounded and pointwise relatively compact and M is extendedly k-equicontinuous.*

Going back to the general case, from (4.2) we obtain an Ambrosetti-type formula also in the space $\mathcal{BC}^k(\Omega, Y)$, and a regular measure of noncompactness equivalent to that of Kuratowski.

Corollary 4.4 *Let $M \in \mathfrak{M}_{\mathcal{BC}^k}$ be extendedly k-equicontinuous. Then*

$$\alpha(M) = \mu_{\overline{\alpha}}(M).$$

Corollary 4.5 *The set function $\mu_{\overline{\alpha}} + 2\omega_{\mathcal{BC}^k} : \mathfrak{M}_{\mathcal{BC}^k} \to [0, +\infty)$ is a regular measure of noncompactness in $\mathcal{BC}^k(\Omega, Y)$ equivalent to the Kuratowski measure α.*

Remark 4.6 If $\mathcal{V}$ is a Banach subspace of $\mathcal{BC}^k(\Omega, Y)$, Theorem 4.1 and the subsequent corollaries hold true in $\mathcal{V}$.

Now we focus our attention on the Banach space $\mathcal{TBC}^k(\Omega, Y)$ consisting of all functions $f \in \mathcal{BC}^k(\Omega, Y)$ which are compact with all differentials up to the order k. The following remark shows that the hypothesis that each $\mathrm{d}^p f$ $(p = 1, \dots, k)$ is compact is not redundant.

Remark 4.7 It is well known (see, for instance, [16]) that if $f \in \mathcal{BC}^k(\Omega, Y)$ is a compact function, then for each $x \in \Omega$ the differentials $\mathrm{d}^p f(x)$ of f at x, for $p \in \{1, \dots k\}$, are compact linear operators. On the other hand, there are compact functions $f \in \mathcal{BC}^k(\Omega, Y)$ such that the functions $\mathrm{d}^p f : \Omega \to \mathcal{L}(Z^p, Y)$ are not compact. For example, let us consider $Z = \mathbb{R}$, $\Omega = \bigcup_{n=1}^\infty I_n$ with $I_n = \left(n - \frac{1}{2n}, n + \frac{1}{2n}\right)$, $Y = \ell_1$ and $\{e_n\}_{n=1}^\infty$ the standard basis in ℓ_1. Then we define $f \in \mathcal{BC}^1(\Omega, \ell_1)$ by setting

$$f(x) = (x - n)e_n \quad \text{for } x \in I_n \ (n = 1, 2 \dots).$$

Clearly f is a compact function. On the other hand, since $\mathrm{d}f(x) = e_n$ if $x \in I_n$, we deduce $\mathrm{d}f(\Omega) = \{e_1, \dots, e_n, \dots\}$ and this shows that $\mathrm{d}f$ is not compact.

Now given $M \in \mathfrak{M}_{\mathcal{TBC}^k}$, we define $\overline{\omega}(M)$ extending the definition of ω given in (2.2). We set

$$\overline{\omega}(M) = \inf \Big\{ \varepsilon > 0 : \text{ there are a finite partition } \{A_1, \ldots, A_n\} \text{ of } \Omega \\ \text{such that, for all } f \in M,\ \max_{p=0}^{k} \operatorname{diam}_{W_p}(\mathrm{d}^p(f(A_i)) \leq \varepsilon \\ \text{for } i = 1, \ldots, n \Big\},$$

By the definition, it is immediate to see that

$$\overline{\omega}(M) = \max_{p=0}^{k} \omega(M^p). \tag{4.5}$$

Next, repeating the arguments of Proposition 3.3, given $M \in \mathfrak{M}_{\mathcal{TBC}^k}$ we find $\omega_{\mathcal{TBC}^k}(M) = \overline{\omega}(M)$. Arguing similarly as in Lemma 3.9, given $M \in \mathfrak{M}_{\mathcal{TBC}^k}$ we can prove $\sigma_{\overline{\alpha}}(M^p) \leq \alpha(M)$. Therefore, we obtain the following result as consequence of Theorem 4.1.

Theorem 4.8 *Let $M \in \mathfrak{M}_{\mathcal{TBC}^k}$. Then*

$$\max\{\sigma_{\overline{\alpha}}(M),\ \frac{1}{2}\overline{\omega}(M)\} \leq \alpha_{\mathcal{TBC}^k}(M) \leq \mu_{\overline{\alpha}}(M) + 2\overline{\omega}(M).$$

Now observe that, if Ω is an open bounded subset of $\mathbb{R}^n$, $\mathcal{TBC}^k(\overline{\Omega}, Y) = \mathcal{BC}^k(\overline{\Omega}, Y) = \mathcal{C}^k(\overline{\Omega}, Y)$. Therefore, we can state the following result.

Corollary 4.9 *If Ω be an open bounded subset of $\mathbb{R}^n$ and let $M \in \mathfrak{M}_{\mathcal{C}^k(\overline{\Omega},Y)}$. Then*

$$\max\{\sigma_{\overline{\alpha}}(M),\ \frac{1}{2}\overline{\omega}(M)\} \leq \alpha_{\mathcal{C}^k(\overline{\Omega},Y)}(M) \leq \mu_{\overline{\alpha}}(M) + 2\overline{\omega}(M).$$

If Ω is an open subset of R^n and, in addition, $Y = \mathbb{R}$, among others, some results of [3] are recovered. To this end, let us mention that if $f \in \mathcal{BC}^k(\Omega, \mathbb{R})$ then

$$\|f\|_{\mathcal{BC}^k} = \max_{0 \leq |\alpha| \leq k} \|D^\alpha f\|_\infty,$$

where $\|D^\alpha f\|_\infty = \sup\{|D^\alpha(x)| : x \in \Omega\}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $D^\alpha f = \frac{\delta^{\alpha_1}}{\delta x_1^{\alpha_1}} \cdots \frac{\delta^{\alpha_n}}{\delta x_n^{\alpha_n}} f$. Keeping in mind this and taking into account (4.5) we deduce the following compactness criteria.

Corollary 4.10 *Let Ω be an open bounded subset of $\mathbb{R}^n$ and let M be a subset of $\mathcal{C}^k(\overline{\Omega}, \mathbb{R})$. Then the following are equivalent*:

(i) *M is $\|\cdot\|_{\mathcal{C}^k(\overline{\Omega},\mathbb{R})}$-relatively compact,*

(ii) *M is $\|\cdot\|_{\mathcal{C}^k(\overline{\Omega},\mathbb{R})}$-bounded and $\overline{\omega}(M)=0$,*
(iii) *M^p is $\|\cdot\|_\infty$-bounded and equicontinuous for all $p=0,\dots,k$,*
(iv) *$M^\alpha=\{D^\alpha f : f\in M\}$ are $\|\cdot\|_\infty$-bounded and equicontinuous for all $0\leq|\alpha|\leq k$.*

In particular, the above condition (iv) recovers Theorem 2.1 of [3]. Moreover, denoting by $|\cdot|_n$ the Euclidean norm in $\mathbb{R}^n$ we define the subspace $\mathcal{C}_0^k(\mathbb{R}^n,\mathbb{R})$ of $\mathcal{BC}^k(\mathbb{R}^n,\mathbb{R})$ as follows:

$$\mathcal{C}_0^k(\mathbb{R}^n,\mathbb{R})=\{f\in\mathcal{C}^k(\mathbb{R}^n,\mathbb{R}) : D^\alpha f\in\mathcal{C}_0 \text{ for } 0\leq|\alpha|\leq k\},$$

where $\mathcal{C}_0=\{f\in\mathcal{BC}^k(\mathbb{R}^n,\mathbb{R}) : \lim_{|x|_n\to\infty}f(x)=0\}$, endowed with the norm $\|f\|_{\mathcal{C}_0^k(\mathbb{R}^n,\mathbb{R})}=\max_{0\leq|\alpha|\leq k}\|D^\alpha f\|_\infty$. Then, from Corollary 4.2 and in view of Remark 4.6, we recover the compactness criterion in the space $\mathcal{C}_0^k(\mathbb{R}^n,\mathbb{R})$ given in [3, Theorem 3.1].

We complete this section by stating, without proofs, estimates and precise formulas for the Hausdorff measure of noncompactness, involving the quantitative characteristics $\mu_{\overline{Y}}$ and $\sigma_{\overline{Y}}$, in the general case of the spaces $\mathcal{BC}^k(\Omega,Y)$ and $\mathcal{TBC}^k(\Omega,Y)$.

Theorem 4.11 *Let $M\in\mathfrak{M}_{\mathcal{BC}^k}$, then*

$$\max\left\{\mu_{\overline{Y}}(M),\ \frac{1}{2}\omega_{\mathcal{BC}^k}(M)\right\}\leq\gamma_{\mathcal{BC}^k}(M)\leq 2(\mu_{\overline{Y}}(M)+\omega_{\mathcal{BC}^k}(M)).$$

Theorem 4.12 *Let $M\in\mathfrak{M}_{\mathcal{TBC}^k}$, then*

$$\max\left\{\sigma_{\overline{Y}}(M),\ \frac{1}{2}\overline{\omega}(M)\right\}\leq\gamma_{\mathcal{TBC}^k}(M)\leq\mu_{\overline{Y}}(M)+\overline{\omega}(M).$$

Hence given a subset M of $\mathfrak{M}_{\mathcal{TBC}^k}$ which satisfies $\overline{\omega}(M)=0$, the formula $\gamma_{\mathcal{TBC}^k}(M)=\sigma_{\overline{Y}}(M)$ (or $\gamma_{\mathcal{TBC}^k}(M)=\mu_{\overline{Y}}(M)$) holds.

4.2 Results in $\mathcal{D}^k(\Omega,Y)$

Finally, we apply results of Sect. 3 to derive compactness results in $\mathcal{C}^k(\Omega,Y)$ made into a complete locally convex space by the topology τ of compact convergence for all differentials, i.e. the topology generated by the family of seminorms

$$\|f\|_{\mathcal{C}^k,K}=\max\left\{\sup_{x\in K}\|f(x)\|,\ \sup_{x\in K}\|df(x)\|,\dots,\sup_{x\in K}\|d^kf(x)\|\right\}\quad K\in\mathcal{K},$$

where the symbol $\mathcal{K}$ denotes the family of all compact subsets of Ω. We set $\mathcal{D}^k(\Omega,Y)=(\mathcal{C}^k(\Omega,Y),\tau)$. In particular, $\mathcal{D}^0(\Omega,Y)$ reduces to the space, simply denoted by $\mathcal{D}(\Omega,Y)$, of all continuous functions from Ω to Y endowed with the usual topology of uniform convergence on compacta. Further, for a fixed K in $\mathcal{K}$, we denote by $\mathcal{D}^k_K(\Omega, Y)$ the complete seminormed space of all k-times continuously differentiable functions endowed with the seminorm $\|\cdot\|_{\mathcal{C}^k,K}$. We use the notation $\mathfrak{M}_{\mathcal{D}^k}$ for the family of all τ-bounded subsets of $\mathcal{D}^k(\Omega, Y)$. Let us now equip the linear space of all functions from $\mathcal{K}$ to $[0, +\infty)$ with the usual order and with the topology of pointwise convergence. Then, according to [26, Definition 1.2.1], for a subset M of $\mathfrak{M}_{\mathcal{D}^k}$, the Kuratowski and the Hausdorff measures of noncompactness generated by the family of seminorms $\{\|\cdot\|_{\mathcal{C}^k,K}\}_{K\in\mathcal{K}}$ are functions $\alpha_{\mathcal{D}^k}(M),\ \ \gamma_{\mathcal{D}^k}(M) : \ \mathcal{K} \to [0, +\infty)$ where $\alpha_{\mathcal{D}^k}(M)(K) = \alpha_{\mathcal{D}^k_K}(M)$, that is, $\alpha_{\mathcal{D}^k}(M)(K)$ is the Kuratowski measure of noncompactness of M with respect to the seminorm $\|\cdot\|_{\mathcal{C}^k,K}$, and analogously $\gamma_{\mathcal{D}^k}(M)(K) = \gamma_{\mathcal{D}^k_K}(M)$. We refer to [26, Theorem 1.2.3] for the properties of these generalized measures of noncompactness. In a similar way, we will introduce the quantitative characteristics useful to prove our estimates as functions from $\mathcal{K}$ to $[0, +\infty)$. To this end, for $M \in \mathfrak{M}_{\mathcal{D}^k}$, and $p = 0, \ldots, k$, we define M^p as in (4.1), and, given $x \in \Omega$ and $K \in \mathcal{K}$, we define consequently also $M^p(x)$ and $M^p(K)$. Moreover, for $M \in \mathfrak{M}_{\mathcal{D}^k}$ and $K \in \mathcal{K}$ we use the following notations

$$\mu_{\overline{\alpha},K}(M) = \max_{p=0}^{k} \mu_{\alpha_p,K}(M^p) \quad \text{with } \mu_{\alpha_p,K}(M^p) = \sup_{x\in K} \alpha_p(M^p(x)),$$
$$\mu_{\overline{\gamma},K}(M) = \max_{p=0}^{k} \mu_{\gamma_p,K}(M^p) \quad \text{with } \mu_{\gamma_p,K}(M^p) = \sup_{x\in K} \gamma_p(M^p(x)),$$
$$\sigma_{\overline{\alpha},K}(M) = \max_{p=0}^{k} \sigma_{\alpha_p,K}(M^p) \quad \text{with } \sigma_{\alpha_p,K}(M^p) = \sup_{x\in K} \alpha_p(M^p(x)),$$
$$\sigma_{\overline{\gamma},K}(M) = \max_{p=0}^{k} \sigma_{\gamma_p,K}(M^p) \quad \text{with } \sigma_{\gamma_p,K}(M^p) = \sup_{x\in K} \gamma_p(M^p(x)),$$

and

$$\overline{\omega}_K(M) = \max_{p=0}^{k} \Big\{ \inf\{\varepsilon > 0 :\ \text{there is a finite partition } \{A_1, \ldots, A_n\} \text{ of } K$$
$$\text{such that, for all } f \in M,\ \operatorname{diam}_{W_p} \mathrm{d}^p f(A_i) \le \varepsilon,\ i = 1, \ldots, n\}\Big\}.$$

Now, for a given $M \in \mathfrak{M}_{\mathcal{D}^k}$ we define the set functions

$$\mu_{\alpha_\tau}(M),\ \mu_{\gamma_\tau}(M),\ \sigma_{\alpha_\tau}(M),\ \sigma_{\gamma_\tau}(M),\ \omega_{\mathcal{D}^k}(M) :\ \mathcal{K} \to [0, +\infty)$$

by setting for $K \in \mathcal{K}$

$$\mu_{\alpha_\tau}(M)(K) = \mu_{\overline{\alpha},K}(M)$$
$$\mu_{\gamma_\tau}(M)(K) = \mu_{\overline{\gamma},K}(M)$$
$$\sigma_{\alpha_\tau}(M)(K) = \sigma_{\overline{\alpha},K}(M)$$
$$\sigma_{\gamma_\tau}(M)(K) = \sigma_{\overline{\gamma},K}(M)$$

and

$$\omega_{\mathcal{D}^k}(M)(K) = \overline{\omega}_K(M).$$

A set M in $\mathfrak{M}_{\mathcal{D}^k}$ will be called pointwise τ-relatively compact if $\mu_{\alpha_\tau}(M) = 0$ or $\mu_{\gamma_\tau}(M) = 0$, and τ-equicontinuous if $\omega_{\mathcal{D}^k}(M) = 0$.

Theorem 4.13 *Let $M \in \mathfrak{M}_{\mathcal{D}^k}$, then*

$$\max\{\mu_{\alpha_\tau}(M),\ \frac{1}{2}\omega_{\mathcal{D}^k}(M)\} \le \alpha_{\mathcal{D}^k}(M) \le \mu_{\alpha_\tau}(M) + 2\omega_{\mathcal{D}^k}(M).$$

Proof Let $M \in \mathfrak{M}_{\mathcal{D}^k}$. We have to prove, for each $K \in \mathcal{K}$

$$\max\{\mu_{\overline{\alpha},K}(M),\ \frac{1}{2}\overline{\omega}_K(M)\} \le \alpha_{\mathcal{D}^k_K}(M) \le \mu_{\overline{\alpha},K}(M) + 2\overline{\omega}_K(M). \tag{4.6}$$

Let us make an intermediate step. Let $(W, \|\cdot\|_W)$ be a given Banach space. Coherently with our previous notations, we denote by $\mathcal{D}_K(\Omega, W)$ the complete seminormed space $(\mathcal{C}(\Omega, W), \|\cdot\|_{\mathcal{C},K})$, where

$$\|f\|_{\mathcal{C},K} = \sup_{x\in K} \|f(x)\|_W,$$

and we focus our attention on this space. Set $N = \{f \in \mathcal{D}_K(\Omega, W) : \|f\|_{\mathcal{C},K} = 0\}$ and let us still denote by $\mathcal{D}_K(\Omega, W)$ the Banach quotient space $\mathcal{D}_K(\Omega, W)/N$ of equivalence classes, by f the class $f + N$ of $\mathcal{D}_K(\Omega, W)/N$ and the same for the norm $\|\cdot\|_{\mathcal{C},K}$. Then, let us observe that the Banach space $\mathcal{D}_K(\Omega, W)$ is isometric to the Banach space $(\mathcal{C}(K, W), \|\cdot\|_\infty)$. Therefore, for $p = 0, \ldots, k$, the quotient Banach spaces $\mathcal{D}_K(\Omega, W_p)$, endowed with the norm $\|\cdot\|_{\mathcal{C},K}$, are isometric to the Banach spaces $(\mathcal{C}(K, W_p), \|\cdot\|_\infty)$, so that $\alpha_{\mathcal{C}(K,W_p)}(M^p) = \alpha_{\mathcal{D}_K(\Omega,W_p)}(M^p)$. Hence in view of Corollary 3.10, for each p, we have

$$\max\left\{\sigma_{\alpha_p,K}(M^p),\ \frac{1}{2}\omega_K(M^p)\right\} \le \alpha_{\mathcal{D}_K(\Omega,W_p)}(M^p) \le \mu_{\alpha_p,K}(M^p) + 2\omega_K(M^p).$$

Taking the maximum for $p = 0, \ldots, k$ we obtain (4.6) as desired. □

Corollary 4.14 *A subset M of $\mathcal{D}^k(\Omega, Y)$ is relatively compact if and only if it is bounded, τ-equicontinuous and pointwise τ-relatively compact.*

Let us observe that in each space $\mathcal{D}^k(\Omega, Y)$ the function $\mu_{\alpha_\tau} + 2\omega_{\mathcal{D}^k}$ is a regular generalized measure of noncompactness equivalent to the Kuratowski measure $\alpha_{\mathcal{D}^k}$. We also underline that, for τ-equicontinuous sets M of $\mathcal{D}^k(\Omega, Y)$, we obtain the formula $\alpha_{\mathcal{D}^k}(M) = \mu_{\alpha_\tau}(M)$. Moreover, as a particular case of Theorem 4.13, we obtain estimates for the Kuratowski measure in the space $\mathcal{D}^k(\mathbb{R}^n, \mathbb{R})$ (defined for instance in [20]) and the consequent compactness criterion. For $k = 0$, Corollary 4.14 is a special case of the well known general Ascoli–Arzelà theorem [19, Theorem 18]. Finally, the same reasoning of Theorem 4.13, using Theorem 3.16, leads to the following inequalities, which estimate the Hausdorff measure of noncompactness $\gamma_{\mathcal{D}^k}(M)$ of sets $M \in \mathfrak{M}_{\mathcal{D}^k}$,

$$\max\left\{\sigma_{\gamma_\tau}(M), \frac{1}{2}\omega_{\mathcal{D}^k}(M)\right\} \leq \gamma_{\mathcal{D}^k}(M) \leq \mu_{\gamma_\tau}(M) + \omega_{\mathcal{D}^k}(M).$$

When $\omega_{\mathcal{D}^k}(M) = 0$, we obtain the formula $\gamma_{\mathcal{D}^k}(M) = \sigma_{\gamma_\tau}(M)$ or $\gamma_{\mathcal{D}^k}(M) = \mu_{\gamma_\tau}(M)$.

5 A remark in $\mathcal{B}(\Omega, Y)$ when Y is a Lindenstrauss space

A real Banach space Y is said to be an L_1-predual provided its dual Y^* is isometric to $L^1(\mu)$ for some measure μ. Such spaces are often referred to as Lindenstrauss spaces and play a central role in the Banach space theory. The Banach space $\mathcal{C}(K, \mathbb{R})$ of real-valued functions defined and continuous on the compact Hausdorff space K, under the supremum norm, is the most natural example of a Lindenstrauss space. We mainly refer to [12, 14, 22, 25], and to [21] for a survey of results on such spaces. Let us recall that given a bounded subset H of Y, the Chebyshev radius $r_C(H)$ is defined as the infimum of all numbers $c > 0$ such that H can be covered with a ball of a radius c. Thus we have $r_C(H) = \inf\{c > 0 : \; y \in Y,\; H \subseteq B(y, c)\}$ with $\frac{1}{2}\text{diam}(H) \leq r(H) \leq \text{diam}(H)$. A point $\bar{z} \in Y$ is said to be a Chebyshev centre of H if $H \subseteq B(\bar{z}, r_C(H))$. The set H is said to be centrable if $r_C(H) = \frac{1}{2}\text{diam}(H)$. In [25, Theorem 1] Lindenstrauss spaces are characterized as those Banach spaces in which every finite set is centrable. Moreover, if Y is a Lindenstrauss space then every finite set has a Chebyshev centre and every compact set is centrable (cf. [25, Corollary 1 and Remark 1]). Whenever Y is a Lindenstrauss space we find a better lower estimate for the Kuratowski measure of noncompactness of bounded and pointwise relatively compact subsets of the space $\mathcal{B}(\Omega, Y)$.

Proposition 5.1 *Assume that Y is a Lindenstrauss space and that $M \in \mathfrak{M}_{\mathcal{B}}$ is pointwise relatively compact. Then $\omega_{\mathcal{B}}(M) \leq \alpha(M)$.*

Proof Let $a > \alpha(M)$ and choose $M_1, \dots, M_n$ such that $M = \cup_{i=1}^n M_i$ and $\text{diam} M_i \leq a$ for $i = 1, \dots n$. Let $\delta > 0$ be arbitrarily fixed. Let $i \in \{1, \dots, n\}$. Fix $x \in \Omega$ and let $F_{i,x} \subseteq Y$ be a finite inner $\|\cdot\|$-δ-net for $M_i(x)$. Let $z_{i,x}$ be a Chebyshev center of $F_{i,x}$ in Y, so that $F_{i,x} \subseteq B(z_{i,x},\; r_C(F_{i,x}))$, where, $r_C(F_{i,x}) = \frac{1}{2}\text{diam}(F_{i,x})$. By the hypothesis $\overline{M_i(x)}$ is a compact set, so that it is centrable, that is, $r_C(\overline{M_i(x)}) = \frac{1}{2}\text{diam}(\overline{M_i(x)})$. Now, we define the mapping $\varphi_i : \Omega \to Y$ by setting $\varphi_i(x) = z_{i,x}$, for $x \in \Omega$. Then, for $f \in M_i$ arbitrarily fixed, we have

$$\|f - \varphi_i\|_\infty \leq \frac{1}{2}\text{diam}(M_i) + \delta. \tag{5.1}$$

Indeed, for each $x \in \Omega$ choosing $y_{i,x} \in F_{i,x}$ with $\|f(x) - y_{i,x}\| \le \delta$ we have

$$\begin{aligned}\|f - \varphi_i\|_\infty &= \sup_{x\in\Omega} \|f(x) - \varphi_i(x)\| = \sup_{x\in\Omega} \|f(x) - z_{i,x}\| \\ &\le \sup_{x\in\Omega} \big(\|f(x) - y_{i,x}\| + \|z_{i,x} - y_{i,x}\|\big) \le \sup_{x\in\Omega} \|z_{i,x} - y_{i,x}\| + \delta \\ &\le \frac{1}{2} \sup_{x\in\Omega} \operatorname{diam}(F_{i,x}) + \delta \le \frac{1}{2} \sup_{x\in\Omega} \operatorname{diam}(M_i(x)) + \delta \\ &\le \frac{1}{2} \operatorname{diam}(M_i) + \delta.\end{aligned}$$

We also get $\varphi_i \in \mathcal{B}(\Omega, Y)$. Finally, let $f \in M$ and choose i such that $f \in M_i$. Then using (5.1) we find

$$\begin{aligned}\operatorname{diam}((f - \varphi_i)(\Omega)) &= \sup_{x,y\in\Omega} \|(f - \varphi_i)(x) - (f - \varphi_i)(y)\| \\ &\le \sup_{x,y\in\Omega} (\|(f - \varphi_i)(x)\| + \|(f - \varphi_i)(y)\|) \le 2\|f - \varphi_i\|_\infty \\ &\le 2\left(\frac{1}{2}\operatorname{diam}(M_i) + \delta\right) = \operatorname{diam}(M_i) + 2\delta \le a + 2\delta.\end{aligned}$$

Taking $\{\Omega\}$ as a partition of Ω and $\{\varphi_1, \ldots, \varphi_n\}$ as a finite set in $\mathcal{B}(\Omega, Y)$, from the arbitrariness of a and δ, it follows $\omega_{\mathcal{B}}(M) \le \alpha(M)$, which is the thesis. ◻

Then combining the previous result with Theorem 3.5 we derive the following estimates.

Theorem 5.2 *Assume that Y is a Lindenstrauss space and that $M \in \mathfrak{M}_{\mathcal{B}}$ is pointwise relatively compact. Then*

$$\omega_{\mathcal{B}}(M) \le \alpha(M) \le 2\omega_{\mathcal{B}}(M).$$

The following two examples show that the inequalities given in Theorem 5.2 are the best possible.

Example 4 Let Y be an infinite-dimensional Lindenstrauss space with origin θ. Let $(y_n)_n$ be a sequence in Y which satisfies $\|y_n - y_m\| \ge 2$ when $n \ne m$. We now consider the closed balls $B(Y)$ and $B(y_n, 1) = y_n + B(Y)$ for all $n = 1, 2, \ldots$, which for short we will denote by B and B_n, respectively. Clearly, the sets of the sequence $(B_n)_n$ are pairwise disjoint. Let us define $f_n : Y \to Y$ for $n = 1, 2, \ldots$, by setting

$$f_n(y) = \begin{cases} \theta & \text{for } y \notin B_n \\ y - y_n & \text{for } y \in B_n. \end{cases}$$

Let us consider the subset $M = \{f_n : n = 1, 2, \ldots\}$ of $\mathcal{B}(Y, Y)$. Given $y \in Y$, $M(y) = \theta$ if $y \notin \cup_{k=1}^\infty B_k$ and $M(y) = \{\theta, y - y_k\}$ if $y \in B_k$, so that M is pointwise relatively compact. Now, let us observe that $f_n(Y) = f_n(B_n) = \{y - y_n : y \in B_n\} = B$, for all n, hence by (2.4) we get

$$\omega(\{f_n\}) = \alpha(f_n(Y)) = \alpha(B) = 2,$$

so that $\omega(M) \geq 2$. On the other hand, since $\operatorname{diam}(f_n(Y)) = \operatorname{diam}(B) = 2$ for all n, taking $\{Y\}$ as partition of Y we obtain $\omega(M) \leq 2$, thus $\omega(M) = 2$.

Next, we prove $\alpha(M) = 1$. To this end, having in mind the definition of f_n, first we notice that for all $n, m \in \mathbb{N}$ with $n \neq m$ we have

$$\|f_n - f_m\|_\infty = \sup_{y \in Y} \|f_n(y) - f_m(y)\| = 1$$

so that, considering the Istratescu measure of noncompactness of M, we have $\beta(M) \geq 1$. Since $\beta(M) \leq \alpha(M)$, we have $\alpha(M) \geq 1$. On the other hand

$$\operatorname{diam}(M) = \sup_{n,m \in \mathbb{N}} \|f_n - f_m\|_\infty = 1,$$

so that $\alpha(M) \leq 1$, and our assert follows.

Now we show $\omega_{\mathcal{B}(Y,Y)}(M) = 1$. Set $\varphi(y) = \frac{1}{2}\sum_{n=1}^{\infty} f_n(y)$, for all $y \in Y$. Then to evaluate $\omega_{\mathcal{B}(Y,Y)}(M)$ we consider $\{Y\}$ as a partition of Y and $\{\varphi\}$ as a finite set in $\mathcal{B}(Y, Y)$. Then given $y, z \in Y$ and $n \in \mathbb{N}$, we have

$$\|f_n(y) - \varphi(y) - f_n(z) + \varphi(z)\| = \frac{1}{2}\|f_n(y) - f_n(z)\| = \frac{1}{2}\|y - z\| \leq 1 \quad \text{if } y, z \in B_n$$

and $\|f_n(y) - \varphi(y) - f_n(z) + \varphi(z)\| \leq \frac{1}{2}$ if y, z are not simultaneously both in B_n. Consequently $\operatorname{diam}(f_n - \varphi)(Y)) \leq 1$, which implies $\omega_{\mathcal{B}(Y,Y)}(M) \leq 1$. Assume by contradiction that $\omega_{\mathcal{B}(Y,Y)}(M) < 1$ and let $\omega_{\mathcal{B}(Y,Y)}(M) = 1 - \delta$. Then there are a finite partition $\{A_1, \ldots, A_n\}$ of Y, and a finite set $\{\varphi_1, \ldots, \varphi_m\}$ in Y such that for all $f \in M$, there is $j \in \{1, \ldots, m\}$ with $\operatorname{diam}((f - \varphi_j)(A_i)) \leq 1 - \delta$ for $i = 1, \ldots, n$. We set, for $j \in \{1, \ldots, m\}$,

$$M_j = \{f \in M : \operatorname{diam}((f - \varphi_j)(A_i)) \leq 1 - \delta \text{ for } i = 1, \ldots, n\},$$

without loss of generality, we may assume that each M_j is an infinite set. Moreover, since $\omega(M) = 2$, there is $f_s \in M$ and $i \in \{1, \ldots, n\}$ such that $\operatorname{diam}(f_s(A_i)) \geq 2 - \delta$. Fix $y, z \in A_i$ such that $\|f_s(y) - f_s(z)\| \geq 2 - \delta$, and let $j \in \{1, \ldots, m\}$ such that $f_s \in M_j$. Then

$$1 - \delta \geq \|(f_s - \varphi_j)(y) - (f_s - \varphi_j)(z))\| \geq \Big| \|f_s(y) - f_s(z)\| - \|\varphi_j(y) - \varphi_j(z)\| \Big|,$$

so it follows $\|\varphi_j(y) - \varphi_j(z)\| \geq 1$. On the other hand, taking $f_l \in M_j$ with $l \neq s$ we have $f_l(y) = f_l(z) = 0$, thus

$$\|f_l(y) - \varphi_j(y) - (f_l(z) - \varphi_j(z))\| = \|\varphi_j(y) - \varphi_j(z))\| \leq 1 - \delta,$$

which is a contradiction, consequently $\omega_{\mathcal{B}(Y,Y)}(M) = 1$ and thus $\alpha(M) = \omega_{\mathcal{B}(Y,Y)}(M)$.

Example 5 Let $\Omega = [0, +\infty)$ and $Y = (\mathcal{C}([0,1]), \|\cdot\|_\infty)$, hence Y a Lindenstrauss space. Now we define $f_k : \Omega \to Y$ for $k = 1, 2, \ldots,$ by putting

$$f_k(x) = \begin{cases} \psi_n & \text{for } x = k - \frac{1}{n} \text{ for } n = 1, 2, \ldots \\ \psi_0 & \text{otherwise,} \end{cases}$$

with $\|\psi_n\|_\infty = 1$ for $n = 1, 2, \ldots$ and $\psi_0(t) = 0$ for all $t \in [0, 1]$, so that $f_k \in \mathcal{B}(\Omega, Y)$. Setting $M = \{f_k,\ \ k = 1, 2, \ldots\}$ we have that M is pointwise relatively compact. Moreover, $\|f_k - f_s\|_\infty = \sup_{x\in\Omega} \|f_k(x) - f_s(x)\|_\infty = 1$ (for $k \neq s$) and $\operatorname{diam}(M) = \sup_{k,s\in\mathbb{N}} \|f_k - f_s\|_\infty = 1$, hence similarly as in the case of Example 4, we deduce $\alpha(M) = 1$.

On the other hand, let us consider, for $n = 1, 2, \ldots$ and $t \in [0, 1]$,

$$\psi_n(t) = \begin{cases} 0 & \text{for } t \le 1 - \frac{1}{n} \\ n\left(t - \left(1 - \frac{1}{n}\right)\right) & \text{for } 1 - \frac{1}{n} < t \le 1, \end{cases}$$

and put $\varphi(x) = \frac{1}{2}\sum_{k=1}^{\infty} f_k(x)$ for $x \in \Omega$. To evaluate $\omega_{\mathcal{B}}(M)$ we consider $\{\Omega\}$ as a partition of Ω and $\{\varphi\}$ as a finite set in $\mathcal{B}(\Omega, Y)$. Then it is easy to check that

$$\|f_k - \varphi\|_\infty \le \frac{1}{2},$$

whence $\operatorname{diam}((f - \varphi)(Y)) \le \frac{1}{2}$, so that $\omega_{\mathcal{B}}(M) \le \frac{1}{2}$. Since $\alpha(M) = 1$ and Theorem 5.2 implies $\frac{1}{2}\alpha(M) \le \omega_{\mathcal{B}}(M)$, we have $\alpha(M) = 2\omega_{\mathcal{B}}(M)$.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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