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CQ *-algebras of measurable operators

SALVATORE TRIOLO[1]

ABSTRACT. We study, from a quite general point of view, a CQ*-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ possessing a sufficient family of bounded positive tracial sesquilinear forms. Non-commutative L^2-spaces are shown to constitute examples of a class of CQ*-algebras and any abstract CQ*-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ possessing a sufficient family of bounded positive tracial sesquilinear forms can be represented as a direct sum of non-commutative L^2-spaces.

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1. Introduction and preliminaries

In this paper we will continue the analysis undertaken in [4]-[23]. Since any C*-algebra $\mathfrak{A}_0$ has a faithful *-representation π, in [4] using the Segal-Nelson theory [18, 16] of non-commutative integration the authors have proved that any CQ*-algebra can be realized as a quasi *-algebra of operators affiliated to $\pi(\mathfrak{A}_0)''$.

The aim of this new work instead is of proved that some CQ*-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ can be represented as a direct sum of non-commutative L^2-spaces.

A *quasi *-algebra* is a couple $(\mathfrak{X}, \mathfrak{A}_0)$, where $\mathfrak{X}$ is a vector space with involution *, $\mathfrak{A}_0$ is a *-algebra and a vector subspace of $\mathfrak{X}$ and $\mathfrak{X}$ is an $\mathfrak{A}_0$-bimodule whose module operations and involution extend those of $\mathfrak{A}_0$. Quasi *-algebras were introduced by Lassner [14, 15, 17] to provide an appropriate mathematical framework where discussing certain quantum physical systems for which the usual algebraic approach made in terms of C*-algebras revealed to be insufficient. In these applications they usually arise by taking the completion of the C*-algebra of observables in a weaker topology satisfying certain physical requirements [2]. The case where

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[1] *Dipartimento di Ingegneria, Università di Palermo, I-90123 Palermo (Italy).*
e-mail: salvatore.triolo@unipa.it .

this weaker topology is a norm topology has been considered in a series of previous papers [4]-[3], where CQ*-algebras were introduced: a CQ*-algebra is, indeed, a quasi *-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ where $\mathfrak{X}$ is a Banach space with respect to a norm $\| \ \|$ possessing an isometric involution and $\mathfrak{A}_0$ is a C*-algebra with respect to a norm $\| \ \|_0$, which is dense in $\mathfrak{X}[\| \ \|]$.

Since any C*-algebra $\mathfrak{A}_0$ has a faithful *-representation π, it is natural to pose the question if this completion also can be realized as a quasi *-algebra of operators "of type $L^2(\varphi)$".

The paper is organized as follows. In Section 2 we show that non-commutative L^2-spaces constructed starting from a von Neumann algebra $\mathfrak{M}$ and a normal, semifinite, faithful trace τ are CQ*-algebras. In particular if φ is finite, then $(L^p(\varphi), \mathfrak{M})$ is a CQ*-algebra. If $p \geq 2$, they even possess a *sufficient* family of positive sesquilinear forms enjoying certain *invariance* properties.

In Section 3, we prove that any CQ*-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ possessing a sufficient family of bounded positive tracial sesquilinear forms can be continuously embedded into the direct sum of non-commutative L^2-spaces.

In order to keep the paper sufficiently self-contained, we collect below some preliminary definitions and propositions that will be used in what follows.

Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a quasi *-algebra. The *unit* of $(\mathfrak{X}, \mathfrak{A}_0)$ is an element $e \in \mathfrak{A}_0$ such that $xe = ex = x$, for every $x \in \mathfrak{X}$. A quasi *-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ is said to be *locally convex* if $\mathfrak{X}$ is endowed with a topology τ which makes of $\mathfrak{X}$ a locally convex space and such that the involution $a \mapsto a^*$ and the multiplications $a \mapsto ab$, $a \mapsto ba$, $b \in \mathfrak{A}_0$, are continuous. If τ is a norm topology and the involution is isometric with respect to the norm, we say that $(\mathfrak{X}, \mathfrak{A}_0)$ is a *normed quasi *-algebra* and, if it is complete, we say it is a *Banach quasi*-algebra*.

Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra with norm $\| \ \|$ and involution *. Assume that on $\mathfrak{A}_0$ a second norm $\| \ \|_0$ is defined and that the following conditions are satisfied:

(a.1) $\|a^*a\|_0 = \|a\|_0^2, \quad \forall a \in \mathfrak{A}_0$

(a.2) $\|a\| \leq \|a\|_0, \quad \forall a \in \mathfrak{A}_0$

(a.3) $\|ax\| \leq \|a\|_0 \|x\|, \quad \forall a \in \mathfrak{A}_0, x \in \mathfrak{X}$

Then by- (a.2), the identity map $i : \mathfrak{A}_0[\| \ \|_0] \to \mathfrak{A}_0[\| \ \|]$ has a continuous extension $\hat{i}$ from the completion $\mathfrak{A}$ of $\mathfrak{A}_0[\| \ \|_0]$ ($\mathfrak{A}$ is, of course, a C*-algebra) into $\mathfrak{X}[\| \ \|]$. If $\hat{i}$ is injective, then $\mathfrak{A}$ is (identified with) a subspace of $\mathfrak{X}$.

Definition 1.1. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *algebra with norm $\| \ \|$ and involution *.*

Assume that on $\mathfrak{A}_0$ a second norm $\| \ \|_0$, satisfying the conditions (a.1)-(a.3) above, is defined and with $\mathfrak{A} \subset \mathfrak{X}$ (up to an identification), then we say that $(\mathfrak{X}, \mathfrak{A})$ is a CQ-algebra.*

Remark 1.2. In previous papers the name CQ*-algebra was given to a more complicated structure where two different involutions were considered on $\mathfrak{A}_0$. When these involutions coincide, we spoke of a *proper* CQ *-algebra. In this paper only this case will be considered and so we systematically omit the term *proper*.

The following basic definitions and results on non-commutative measure theory are also needed in what follows.

Let $\mathfrak{M}$ be a von Neumann algebra and φ a normal faithful semifinite trace defined on $\mathfrak{M}_+$.

Put

$$\mathcal{J} = \{X \in \mathfrak{M} : \varphi(|X|) < \infty\}.$$

$\mathcal{J}$ is a *-ideal of $\mathfrak{M}$.

Definition 1.3. *We say that an operator T is affiliated with a von Neumann algebra $\mathfrak{M}$, written $T\,\eta\,\mathfrak{M}$, whean T is closed, densely defined and $TU \supseteq UT$ for every unitary operator $U \in \mathfrak{M}'$.*

Definition 1.4. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a CQ*-algebra. We denote with $\mathcal{T}(\mathfrak{X})$ the set of all sesquilinear forms Ω on $\mathfrak{X} \times \mathfrak{X}$ with the following properties*

(i) $\Omega(x, x) \geq 0 \quad \forall x \in \mathfrak{X}$.
(ii) $\Omega(xa, b) = \Omega(a, x^*b) \quad \forall x \in \mathfrak{X}, \;\; \forall a, b \in \mathfrak{A}_0$,
(iii) $|\Omega(x, y)| \leq \| x \| \| y \| \quad \forall x, y \in \mathfrak{X}$,
(iv) $\Omega(x, x) = \Omega(x^*, x^*), \;\; \forall x \in \mathfrak{X}$,

A subfamily $\mathcal{A}$ of $\mathcal{T}(\mathfrak{X})$ is called sufficient *if $x \in \mathfrak{X}$, $\Omega(x, x) = 0$, for every $\Omega \in \mathcal{A}$, implies $x = 0$.*

Definition 1.5. *A CQ*-algebra $(\mathfrak{X}[\| \cdot \|], \mathfrak{A}_0[\| \cdot \|_0])$ is said to be strongly regular if $\mathcal{T}(\mathfrak{X})$ is sufficient and*

$$\|x\| = \sup_{\Omega \in \mathcal{T}(\mathfrak{X})} \Omega(x, x)^{1/2}, \quad \forall x \in \mathfrak{X}.$$

2. Direct sum of non-commutative L^2-spaces

In this Section we will discuss the structure of the non-commutative L^2-spaces as quasi *-algebras. We begin with recalling the basic definitions.

Let $\mathfrak{M}$ be a von Neumann algebra and φ a normal faithful finite trace defined on $\mathfrak{M}_+$. We denote with $L^2(\varphi)$ the Banach space completion of $\mathfrak{M}$ with respect to the norm

$$\|X\|_{2,\varphi} := \varphi(|X|^2)^{1/2}, \quad X \in \mathfrak{M}.$$

As shown in [16], if $T \in L^2(\varphi)$, then T is a an affiliated operator with a von Neumann algebra $\mathfrak{M}$, i.e. $T\,\eta\,\mathfrak{M}$. By Proposition 2.1 of [8] if $\mathfrak{M}$ is a von Neumann algebra and φ a normal faithful finite trace on $\mathfrak{M}_+$. Then $(L^2(\varphi), \mathfrak{M})$ is a CQ*-algebra.

Let $\mathfrak{M}$ be a von Neumann algebra and $\mathfrak{F} = \{\varphi_\alpha;\, \alpha \in \mathcal{I}\}$ be a family of normal, *semi finite* traces on $\mathfrak{M}$. As usual, we say that the family $\mathfrak{F}$ is *sufficient* if for $X \in \mathfrak{M}$, $X \geq 0$ and $\varphi_\alpha(X) = 0$ for every $\alpha \in \mathcal{I}$, then $X = 0$ (clearly, if $\mathfrak{F} = \{\varphi\}$, then $\mathfrak{F}$ is sufficient if, and only if, φ is faithful). Put $\mathfrak{M}_\alpha = \mathfrak{M}P_\alpha$, where, as before, P_α denotes the support of φ_α. Each $\mathfrak{M}_\alpha$ is a von Neumann algebra and φ_α is faithful in $\mathfrak{M}P_\alpha$ [20, Proposition V. 2.10].

More precisely,

$$\mathfrak{M}_\alpha := \mathfrak{M}P_\alpha = \{Z = XP_\alpha, \text{ for some } X \in \mathfrak{M}\}.$$

In this case, putting $\mathcal{H}_\alpha = P_\alpha\mathcal{H}$, we have

$$\mathcal{H} = \bigoplus_{\alpha \in \mathcal{I}} \mathcal{H}_\alpha = \{(f_\alpha) : \; f_\alpha \in \mathcal{H}_\alpha, \; \sum_{\alpha \in I} \|f_\alpha\|^2 < \infty\}.$$

Each vector $X = \{f_\alpha\}_{\alpha \in \mathcal{I}} \in \mathcal{H}$ is denoted by $X = \sum_{\alpha \in I}^{\oplus} f_\alpha$ (Definition 3.4, [20]). For each bounded sequencese $\{A_\alpha\}_{\alpha \in \mathcal{I}}$ in $\prod_{\alpha \in \mathcal{I}} \mathfrak{M}_\alpha$, we define an operator A (following [20]) on $\mathcal{H}$ by

$$AX := A\sum_{\alpha \in I}^{\oplus} f_\alpha = \sum_{\alpha \in I}^{\oplus} A_\alpha f_\alpha.$$

Clearly A is a bounded operator on $\mathcal{H}$ we denote it by $A = \sum_{\alpha\in I}^{\oplus} A_\alpha$.
Let $\sum_{\alpha\in I}^{\oplus} M_\alpha$ the set of all such A, by Proposition 3.3 [20], $\sum_{\alpha\in I}^{\oplus} M_\alpha$ is a von Neumann algebra on $\mathcal{H}$. The algebra $\sum_{\alpha\in I}^{\oplus} M_\alpha$ is called the diret sum of $\{\mathfrak{M}_\alpha\}$.

If $T_\alpha \in Ł^2(\widetilde{\varphi_\Omega})$ as shown in [16], then T_α densely defined on $D_\alpha := D(T_\alpha)$. For each bounded sequencese $\{T_\alpha\}_{\alpha\in\mathcal{I}}$ in $\prod_{\alpha\in\mathcal{I}} L^2(\widetilde{\varphi_\Omega})$, we define an operator T on $D = \prod_{\alpha\in\mathcal{I}} D(T_\alpha)$in this way:

$$TX := T\sum_{\alpha\in I}^{\oplus} f_\alpha = \sum_{\alpha\in I}^{\oplus} T_\alpha f_\alpha.$$

Clearly T is closed, densely defined on the von Neumann algebra $\sum_{\alpha\in I}^{\oplus} M_\alpha$. We define

$$\bigoplus_{\Omega\in\mathcal{T}(\mathfrak{X})} L^2(\widetilde{\varphi_\Omega}),$$

the set of all such (D, T).

3. A representation theorem

Once we have constructed in the previous section some CQ*-algebras of operators affiliated to a given von Neumann algebra, it is natural to pose the question under which conditions can an abstract CQ*-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ be realized as a CQ*-algebra of this type.

Let $(\mathfrak{X}[\|\cdot\|], \mathfrak{A}_0[\|\cdot\|_0])$ be a proper CQ*-algebra with unit e.

By the Gelfand - Naimark theorem each C^*-algebra is isometrically *-isomorphic to a C^*-algebra of bounded operators in Hilbert space. This isometric *-isomorphism is called the *universal *-representation*.

Thus, let π be the universal *-representation of $\mathfrak{A}_0$ and $\pi(\mathfrak{A}_0)''$ the von Neumann algebra generated by $\pi(\mathfrak{A}_0)$.

For every $\Omega \in \mathcal{T}(\mathfrak{X})$ and $a \in \mathfrak{A}_0$, we put

$$\varphi_\Omega(\pi(a)) = \omega_\Omega(a).$$

Then, for each $\Omega \in \mathcal{T}(\mathfrak{X})$, φ_Ω is a positive bounded linear functional on the operator algebra $\pi(\mathfrak{A}_0)$.

Clearly,

$$\varphi_\Omega(\pi(a)) = \omega_\Omega(a) = \Omega(a, e),$$

$$| \varphi_\Omega(\pi(a)) |=| \omega_\Omega(a) || \Omega(a, e) | \leq \|a\|\|e\| \leq \|a\|_0\|e\|^2 = \|\pi(a)\| \, \|e\|^2.$$

Thus φ_Ω is continuous on $\pi(\mathfrak{A}_0)$.

By [12, Theorem 10.1.2], φ_Ω is weakly continuous and so it extends uniquely to $\pi(\mathfrak{A}_0)''$. Moreover, since φ_Ω is a trace on $\pi(\mathfrak{A}_0)$, the extension $\widetilde{\varphi_\Omega}$ is a trace on $\mathfrak{M} := \pi(\mathfrak{A}_0)''$ too.

Let now $(\mathfrak{X}[\|\cdot\|], \mathfrak{A}_0[\|\cdot\|_0])$ be a CQ*-algebra with unit e and sufficient $\mathcal{T}(\mathfrak{X})$. Let $\pi : \mathfrak{A}_0 \hookrightarrow \mathcal{B}(\mathcal{H})$ be the universal representation of $\mathfrak{A}_0$. Assume that the C^*algebra $\pi(\mathfrak{A}_0) := \mathfrak{M}$ is a von Neumann algebra. Clearly, $\mathfrak{A}_0$ can be identified with $\mathfrak{M}$.

Theorem 3.1. *Let $(\mathfrak{X}[\|\cdot\|], \mathfrak{A}_0[\|\cdot\|_0])$ be a CQ*-algebra with unit e and and sufficient $\mathcal{T}(\mathfrak{X})$.*

Then there exist a von Neumann algebra $\mathfrak{M}$ and a monomorphism

$$\Phi : x \in \mathfrak{X} \to \Phi(x) := \widetilde{X} \in \bigoplus_{\Omega \in \mathcal{T}(\mathfrak{X})} L^2(\widetilde{\varphi_\Omega})$$

with the following properties:

*(i) Φ extends the universal *-representation π of $\mathfrak{A}_0$;*

(ii) $\Phi(x^*) = \Phi(x)^*, \quad \forall x \in \mathfrak{X}$;

(iii) $\Phi(xy) = \Phi(x)\Phi(y)$ for every $x, y \in \mathfrak{X}$ such that $x \in \mathfrak{A}_0$ or $y \in \mathfrak{A}_0$.

Then $\mathfrak{X}$ can be identified with a space of $\bigoplus_{\Omega \in \mathcal{T}(\mathfrak{X})} L^2(\widetilde{\varphi_\Omega})$.

If, in addition, $(\mathfrak{X}, \mathfrak{A}_0)$ is strongly regular, then

(iv) Φ is an isometry of $\mathfrak{X}$ into $\bigoplus_{\Omega \in \mathcal{T}(\mathfrak{X})} L^2(\widetilde{\varphi_\Omega})$;

(v) If $\mathfrak{A}_0$ is a W-algebra, then Φ is an isometric *-isomorphism of $\mathfrak{X}$ onto $\bigoplus_{\Omega \in \mathcal{T}(\mathfrak{X})} L^2(\widetilde{\varphi_\Omega})$.*

Proof. By Proof of Theorem 4.6 of [21] a given CQ*-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ can be embedded in a CQ*-algebra $(\mathfrak{K}, \mathfrak{B}_0)$ where $\mathfrak{B}_0$ is a W*-algebra. Thus let π be the universal representation of $\mathfrak{A}_0$ and assume that $\pi(\mathfrak{A}_0) =: \mathfrak{M}$.

For every element $x \in \mathfrak{X}$, there exists a sequence $\{a_n\}$ of elements of $\mathfrak{A}_0$ converging to x with respect to the norm of $\mathfrak{X}(\|\cdot\|)$. Put $X_n = \pi(a_n)$, $n \in \mathbb{N}$. Then, for every Ω

$$\begin{aligned} \|X_n - X_m\|_{2,\widetilde{\varphi_\Omega}} &= [\Omega((a_n - a_m)^*(a_n - a_m), e)]^{1/2} \\ &= [\Omega(a_n - a_m, a_n - a_m)]^{1/2} \le \|a_n - a_m\| \to 0. \end{aligned}$$

Let $\widetilde{X^\Omega}$ be the $\|\cdot\|_{2,\widetilde{\varphi_\Omega}}$-limit of the sequence (X_n) in $L^2(\widetilde{\varphi_\Omega})$ and putting

$$\widetilde{X} := (\widetilde{X^\Omega}) \in \bigoplus_{\Omega \in \mathcal{T}(\mathfrak{X})} L^2(\widetilde{\varphi_\Omega}).$$

We define $\Phi(x) := \widetilde{X}$.

For each $x \in \mathfrak{X}$, we put

$$p_{\mathcal{T}(\mathfrak{X})}(x) = \sup_{\Omega \in \mathcal{T}(\mathfrak{X})} \Omega(x, x)^{1/2}.$$

Then, owed to the sufficiency of $\mathcal{T}(\mathfrak{X})$, $p_{\mathcal{T}(\mathfrak{X})}$ is a norm on $\mathfrak{X}$ weaker than $\| \, \|$. This implies that

$$\lim_{n\to\infty} \sup_{\Omega \in \mathcal{T}(\mathfrak{X})} \Omega(a_n, a_n) = \lim_{n\to\infty} p_{\mathcal{T}(\mathfrak{X})}(a_n)^2 p_{\mathcal{T}(\mathfrak{X})}(x)^2.$$

From this equality it follows easily that the linear map Φ is well defined and injective. The condition (iii) can be easily proved. If $(\mathfrak{X}, \mathfrak{A}_0)$ is strongly regular, then, for every $x \in \mathfrak{X}$, $p_{\mathcal{T}(\mathfrak{X})}(x) = \|x\|$. Thus Φ is isometric. Moreover, in this case, Φ is surjective; indeed, if $T \in \bigoplus_{\Omega \in \mathcal{T}(\mathfrak{X})} L^2(\widetilde{\varphi_\Omega})$, then there exists a sequence T_n of bounded operators of $\pi(\mathfrak{A}_0)$ which converges to T with respect to the norm $\|\cdot\|_{2,\mathfrak{n}_{\mathcal{T}(\mathfrak{A}_0)}}$. The corresponding sequence $\{t_n\} \subset \mathfrak{A}_0$, $T_n = \Phi(t_n)$, converges to t with respect to the norm of $\mathfrak{X}$ and $\Phi(t) = T$ by definition. Therefore Φ is an isometric *-isomorphism.

If $(\mathfrak{X}, \mathfrak{A}_0)$ is regular, but $\pi(\mathfrak{A}_0) \subset \pi(\mathfrak{A}_0)''$, then Φ is an isometry of $\mathfrak{X}$ into $\bigoplus_{\Omega \in \mathcal{T}(\mathfrak{X})} L^2(\widetilde{\varphi_\Omega})$, but needs not be surjective. □

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