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Research Article

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Doubling measure and regularity to *K*-quasiminimizers of double-phase energy

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Abstract: We consider an integral functional involving the double-phase operator in a metric measure space equipped with a doubling measure. On the basis of suitable hypotheses on the function governing the phase changes, we design a unifying approach to establish Sobolev-Poincaré inequalities. By using such inequalities together with a Caccioppoli type estimate, we obtain the local boundedness and the local Hölder continuity of K-quasiminimizers of the double-phase energy. Here, we provide a direct approach for establishing the Sobolev-Poincaré inequalities and local Hölder continuity, that is, we do not use neither the method of separation of phases nor auxiliary frozen functional. Finally, we establish the local gradient higher integrability for K-quasiminimizers of the integral functional. We prove our results via the De Giorgi's method, imposing that the involved measure is just doubling.

Keywords: double-phase energy, doubling measure, K-quasiminimizer, metric measure space, upper gradient

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1 Introduction

The starting point of this work is the celebrated De Giorgi's method to study nonlinear elliptic equations. It is well known that De Giorgi [7] developed this technique in 1957 to solve the ninenteenth Hilbert problem, hence establishing regularity properties of variational solutions to some elliptic problems. Roughly speaking, De Giorgi designed a geometric approach to ensure boundedness and regularity of solutions, in the case of discontinuous coefficients in the equations too (basically, the coefficients are required to be measurable and bounded). The developed idea has been successfully applied in various contexts, based on local energy estimate for elliptic equations and suitable Sobolev inequality between two related balls of $\mathbb{R}^N$. Differently from some already existing approaches, using now nonperturbation arguments, several a priori estimates related to equations in divergence form and regularity properties of solutions were obtained. A relevant feature of the De Giorgi's method is that it does not depend on the linearity of the elliptic equation. So, it was extended, for example, to the setting of quasilinear and nonlinear elliptic equations by Ladyzenskaja and Ural'ceva [16] who proved Hölder continuity of bounded functions satisfying certain Sobolev inequalities in the sense of De Giorgi (namely, functions in the De Giorgi's class). Another salient point of the literature is the contribution of DiBenedetto and Trudinger [9] who established that every function in the De Giorgi's class fulfils Harnack type inequality. Furthermore, this class of functions is useful for a wide variety of applications

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to the study of minimizers to variational integrals whose integrand satisfies nonstandard growth conditions of (p, q)-type (hence, for the standard growth condition of $m(x)$-type). Briefly, let us consider the variational integral

$$u \mapsto \int_\Omega f(x, Du)\mathrm{d}x,$$

where $\Omega \subset \mathbb{R}^N$ and $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ admits the following growth condition

$$|t|^p \leq |t|^{m(x)} \leq f(x, t) \leq c_0[1 + |t|^{m(x)}] \leq c[1 + |t|^q], \tag{1.1}$$

$q > m(x) > p > 1$, for a.a. $x \in \Omega$, all $t \in \mathbb{R}^N$, some $c_0, c > 0$.

Such integrals were studied by Marcellini [21] and Zhikov [24] in the context of problems of the calculus of variations and of nonlinear elasticity theory (e.g., to model both homogenization of composite and strongly anisotropic materials). Precisely, they investigated regularity of local minimizers under the aforementioned p & q-growth condition (1.1). A milestone of our story here is the special variational integral

$$u \mapsto \int_\Omega (|Du|^p + a(x)|Du|^q)\mathrm{d}x, \quad a(\cdot) \geq 0,$$

where the integrand is the so-called double-phase functional, so that the problem changes its ellipticity depending on the positivity of the coefficient $a \in L^\infty(\Omega)$. This functional attracted the attention of many researchers in the last decades; in particular, it is relevant in the analysis of the so-called Lavrentiev's phenomenon [25]. Clearly, in the case $a(x) = 0$, we retrieve the classical p-Laplacian (hence we see a p-growth related to p-phase) with transition to a q-growth (hence, q-phase) as $a(x) > 0$. Following the works of Colombo and Mingione [3,4], we know that suitable conditions about regularity of minimizers are given by

$$a \in C^{0,\alpha}(\Omega), \quad \alpha \in (0, 1] \quad \text{and} \quad \frac{q}{p} \leq 1 + \frac{\alpha}{N}.$$

After these contributions, functionals with double-phase type have become a topic of intense study, see the works of Byun, Harjulehto, Papageorgiou, and colleagues [2,5,6,17], and the references cited therein. This is a well-established theory in dealing with the single p-Laplacian, both in the scalar and in the vectorial case, and the reader can refer to previous studies [20,22] for some specific comments and further discussion.

Next, we point out that the similar theories as mentioned earlier, can be posed in the setting of metric measure spaces, equipped with (doubling) measure and supporting a Poincaré type inequality (see Definitions 2.1 and 2.8 of Section 2). A feature of this setting is that the Newtonian spaces can be defined on a metric space without providing a concept of partial derivative [15]. Let (X, d) be a metric space endowed with a (doubling) Borel regular measure μ. For a bounded open set $\Omega \subset X$, we refer to a general variational integral of the form

$$u \mapsto \int_\Omega f(x, g_u)\mathrm{d}\mu,$$

where this time we consider $f : \Omega \times \mathbb{R} \to \mathbb{R}$, and g_u is the minimal p-weak upper gradient of u. Precisely, the triple (X, d, μ) will mean a complete doubling metric measure space that supports a weak $(1, p)$-Poincaré inequality for some $p > 1$. We recall that Euclidean spaces and compact Riemannian manifolds support such inequality; these are classical results. Hence, a crucial notion to reformulate the Poincaré inequality in the metric measure space setting is that of upper gradient of u. Namely, it is a substitute for the derivative (weak gradient of u) in the case of metric measure spaces. The salient point in developing the theory in metric measure spaces is the notion of K-quasiminimizer ($K \geq 1$), as introduced and discussed by Giaquinta and Giusti [10,11], based on the De Giorgi's results. In fact, one of the advantages of De Giorgi's method is that it is applicable to K-quasiminimizers as well. Recall that De Giorgi's method is based on two ingredients: Sobolev and Caccioppoli type estimates. Now, we remark that K-quasiminimizers fit naturally into the analysis on metric measure spaces, as these use variational integrals, but not partial differential equations. Here, as already mentioned earlier, we are interested to regularity properties as boundedness and local Hölder continuity. The developed approach follows the similar arguments and general presentation of Giaquinta and Giusti [10,11] and of DiBenedetto [8]. In particular, Colombo and Mingione [3] obtained local regularity of the gradient for minimizers of certain double-phase functionals. So turning to the double-phase functional, we can consider the variational integral:

$$N^{1,1}(\Omega) \ni u \mapsto \mathcal{H}(u, \Omega) := \int_{\Omega} H(x, g_u)\mathrm{d}\mu, \tag{1.2}$$

where g_u is the minimal p-weak upper gradient of u and

$$H(x, t) := |t|^p + a(x)|t|^q, \quad 1 < p < q, \tag{1.3}$$

where the function $a(\cdot)$ satisfies the following assumptions:

$$a \in C^{0,\alpha}(\Omega), \quad a(\cdot) \geq 0, \quad \alpha \in (0, 1], \quad \frac{q}{p} \leq 1 + \frac{\alpha}{N_\mu}, \tag{1.4}$$

and N_μ is given in Lemma 2.2 [1]; the precise definition of the Newtonian space $N^{1,p}(\Omega)$ is given in Section 2. A classical result that has proved important in establishing higher integrability results, is known as Gehring's lemma, see Theorem 6.1 of Section 6 and also [18,23]. This lemma says that when a weight function fits a reverse Hölder inequality for some exponent, then the same function fulfils the analogous reverse Hölder inequality with slightly larger exponent. In a recent work (on metric space), Kinnunen et al. [14] considered the double-phase function in (1.3) in the case the doubling measure μ is Ahlfors-David regular (see Remark 2.6 in [14]) and in the case the function $a(\cdot)$ governing the change of phases is in $C^{0,\alpha}(\Omega)$ with respect to a quasi-distance associated to μ. They establish local and global higher integrability properties for the K-quasiminimizers.

Here, we establish the Sobolev-Poincaré inequalities and local Hölder continuity following a direct approach, in the sense that we do not use neither the method of separation of phases nor any auxiliary frozen functional. Further, we obtain the local gradient higher integrability for K-quasiminimizers of the integral functional. More precisely, we involve the De Giorgi's method, imposing that the measure μ is just doubling.

We briefly summarize the content of each section of present work. In Section 2, we collect the mathematical background, from the notion of metric measure space, with its additional features, to the notion of Newtonian functional space. In Section 3, we establish our versions of Sobolev-Poincaré and Caccioppoli inequalities. In Section 4, we prove local boundedness of local K-quasiminimizers to the energy functional (1.2) (namely, Theorem 4.2), by combining again our Sobolev-Poincaré inequality with Hölder inequality and Caccioppoli inequalities. In Section 5, we obtain useful estimates and limit property for the measure of certain sets (namely, Lemmas 5.1, 5.2, and 5.4). Finally, we conclude by speculating about local Hölder continuity of K-quasiminimizers in Theorem 5.6. In Section 6, we focus on the inner local gradient higher integrability of K-quasiminimizers, and we establish Theorem 6.2 by using the aforementioned Sobolev-Poincaré inequality together with Gehring's lemma.

2 Mathematical background

In this section, we collect the basic notions and main features of the functional framework required to develop our study. A comprehensive presentation of the theory of metric measure spaces can be found in the book of Björn and Björn [1].

By (X, d, μ), we denote a complete metric measure space, endowed with a Borel regular measure μ such that $\mu(E) > 0$ for every nonempty open set $E \subset X$, and $\mu(A) < +\infty$ for every bounded set $A \subset X$. By $B_R(y) \subset X$, we mean a ball with center $y \in X$ and radius $R > 0$; we will write simply B_R instead of $B_R(y)$ when there is no confusion from the context, and also use the notation $B_{\tau R}$ for a ball with the same center as B_R but τ times its radius (namely, $\tau > 0$ is a “dilation factor”). If $u : A \to \mathbb{R}$ is a measurable function, where $A \subset X$ is measurable with finite and positive μ-measure $\mu(A) > 0$, then the integral average of u over A is given by

$$u_A = \fint_A u \, d\mu = \frac{1}{\mu(A)} \int_A u d\mu.$$

From now on, we will use the letter C to denote any positive constant that may be different at each occurrence; here, C_κ means the constant depends on κ and $C \equiv C(\kappa_1, \kappa_2, \kappa_3)$ says that the constant depends on the data $\kappa_1, \kappa_2, \kappa_3$. For our purpose, we need to impose the following property of measure, see [1, Section 3.1] for more information and details.

Definition 2.1. If (X, d, μ) is a metric measure space, then we say that μ is a doubling measure on X if we can find a constant $C \geq 1$ satisfying the following condition:

$$0 < \mu(B_{2R}) \leq C\mu(B_R) < +\infty, \tag{2.1}$$

for all $B_R \subset X$. The doubling constant of μ is defined to be $C_\mu \coloneqq \inf\{C \in (1, +\infty) : (2.1) \text{ holds true}\}$.

We note that every complete metric measure space (X, d, μ) endowed with a doubling measure is proper (i.e., every closed and bounded subset of X is compact). The doubling property of μ is crucial to obtain the following result, where the involved constant N_μ brings us to a counterpart of space dimension related to μ, in the sense of an upper bound (see [1, Lemma 3.3]).

Lemma 2.2. *If (X, d, μ) is a metric measure space and μ is a doubling measure, then we can find $N_\mu > 0$ satisfying the following inequality*:

$$\frac{\mu(B_r(y))}{\mu(B_R(x))} \geq 4^{-N_\mu} \left(\frac{r}{R}\right)^{N_\mu} \tag{2.2}$$

for all $0 < r \leq R$, $x \in X$, $y \in B_R(x)$, *where* $N_\mu \coloneqq \log_2 C_\mu$.

Remark 2.3. The doubling condition implies also that for balls B_R and B_r with the same center and $0 < r \leq R$, we have

$$\frac{\mu(B_R)}{\mu(B_r)} \leq C_\mu \left(\frac{R}{r}\right)^{N_\mu} = 2^{N_\mu} \left(\frac{R}{r}\right)^{N_\mu}.$$

Since μ is not necessarily invariant to translation, the following result is crucial.

Proposition 2.4. *If (X, d, μ) is a complete metric measure space with μ doubling and $\Omega \Subset X$, then for each $R > 0$, we can find $C_R \equiv C(N_\mu, R, \Omega) > 0$ satisfying the condition*:

$$\mu(B_R(x)) \geq C_R \quad \text{for all } x \in \Omega.$$

Proof. Since $\overline{\Omega}$ is a compact subset of X, then we can find $B_R(x_i) \subset X$, $i = 1, \ldots, n$, such that $\Omega \subset \bigcup_{i=1}^n B_R(x_i)$. Now, for some $1 \leq i_0 \leq n$, we have

$$\mu(B_R(x_{i_0})) = \min_{1 \leq i \leq n} \mu(B_R(x_i)).$$

Fix $B_R(x)$ with $x \in \Omega$, then $x \in B_R(x_i)$ for some $1 \leq i \leq n$. By using the inequality (2.2), we obtain that

$$\frac{\mu(B_R(x))}{\mu(B_R(x_i))} \geq 4^{-N_\mu},$$

and hence, we conclude that

$$\mu(B_R(x)) \geq 4^{-N_\mu}\mu(B_R(x_{i_0})) \coloneqq C_R$$

for all $B_R(x)$ with $x \in \Omega$. □

The second key notion to introduce is that of upper gradient. This concept is aimed to overcome the lack of differentiable structure in the metric setting. Roughly speaking, the upper gradient is a way to generalize the modulus of the gradient in the Euclidean setting. More information and details can be found in Björn and Björn [1], here we recall some well-known facts. Let $\mathring{R} \coloneqq [-\infty, +\infty]$ and denote by Γ_{rect} the family of all rectifiable arc length parameterized paths $\gamma : [a, b] \to X$.

Definition 2.5. [1, Definition 1.13] Given a function $u : X \to \mathring{R}$, we say that a non-negative Borel measurable function g is an upper gradient of u if the following inequality holds

$$|u(x) - u(y)| \leq \int_\gamma g \, \mathrm{d}s, \tag{2.3}$$

for each $\gamma \in \Gamma_{\text{rect}}$ connecting x and y, whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g \, \mathrm{d}s = +\infty$ otherwise.

If g is an upper gradient of a function u and ψ is a non-negative Borel measurable function, then $g + \psi$ is still an upper gradient of u. Hence, we have a lack of uniqueness, and to overcome this situation, a notion of p-weak upper gradient of u is given in the following sense.

Definition 2.6. [1, Definition 1.32] If (2.3) is satisfied for all paths $\gamma \in \Gamma_{\text{rect}} \backslash \Gamma$ with $\Gamma \subset \Gamma_{\text{rect}}$ that has p-modulus zero, then g is said a p-weak upper gradient of u.

Clearly, every upper gradient is a p-weak upper gradient. In fact, differently from Definition 2.6, the notion of upper gradient is independent from p (as it does not involve any p-modulus or measure). The following result establishes the existence of a minimal element to the set of p-weak upper gradients (see also [1, Theorem 2.5]). This minimal element is μ-a.e. uniquely determined by u.

Theorem 2.7. *If $u \in L^p(X)$ admits a p-weak upper gradient in $L^p(X)$, $1 < p < +\infty$, then we can find a p-weak upper gradient $g_u \in L^p(X)$ satisfying the condition*

$$g_u \leq g \ \ \mu\text{-a.e. in } X,$$

for every p-weak upper gradient $g \in L^p(X)$ of u. Such g_u is called the minimal p-weak upper gradient of u.

In a Euclidean setting, the minimal p-weak upper gradient of u reduces to the modulus of the gradient of u. However, we point out that the upper gradient of u does not necessarily bring us to any control of the function itself. This additional aspect requires to impose a Poincaré inequality to the space, in the metric setting (see [1, Definition 4.1]).

Definition 2.8. If (X, d, μ) is a metric measure space and $1 \leq p < +\infty$, then X supports a weak $(1, p)$-Poincaré inequality provided that we can find $C_P > 0$ and $\sigma \geq 1$ satisfying the following inequality:

$$⨍_{B_R} \frac{|u - u_{B_R}|}{R} \mathrm{d}\mu \leq C_P \left(⨍_{B_{\sigma R}} g_u^p \mathrm{d}\mu \right)^{\frac{1}{p}} \tag{2.4}$$

for all balls $B_R \subset X$ and for all $u \in L^1_{\text{loc}}(X)$.

We note that the exponent $p \geq 1$ involved into (2.4) links such inequality to the first-order calculus on the metric measure space setting. Furthermore, we remark that lower values of p correspond to stronger (i.e., more restrictive) *a priori* bounds. If a metric measure space supports a weak $(1, p)$-Poincaré inequality, then by

Hölder inequality, it also supports a weak $(1, q)$-Poincaré inequality for all $q > p$. The following result of Keith and Zhong [13] reflects a self-improving property of weak $(1, p)$-Poincaré inequalities.

Theorem 2.9. [13, Theorem 1.0.1] *If (X, d, μ) is a complete metric measure space endowed with a Borel and doubling measure, and supporting a weak $(1, p)$-Poincaré inequality for $p > 1$, then we can find $\varepsilon > 0$ such that X supports a weak $(1, p_0)$-Poincaré inequality for every $p_0 > p - \varepsilon$.*

In Theorem 2.9, $\varepsilon > 0$ and the constants associated with the weak $(1, p_0)$-Poincaré inequality depend only on p, C_μ and C_P (we refer to Definition 2.1 and (2.4)). Then, we also have the following result that can be found in [1] and gives us a self-improving property of the weak $(1, p)$-Poincaré inequality, too (see [1], Theorem 4.21 and Corollary 4.26). Here, by p^*, we denote the critical Sobolev exponent defined by

$$p^* = \begin{cases} \dfrac{N_\mu p}{N_\mu - p} & \text{if } p < N_\mu, \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 2.10. *Assume that X supports a weak $(1, p)$-Poincaré inequality with dilation factor σ, and N_μ is given in Lemma 2.2. We distinguish two cases:*

(*a*) *If $N_\mu > p$, then X supports a weak (p^*, p)-Poincaré inequality with dilation constant 2σ. In fact, we have*

$$\left(\fint_{B_R} \left| \frac{u - u_{B_R}}{R} \right|^{p^*} \mathrm{d}\mu \right)^{\frac{1}{p^*}} \le C \left(\fint_{B_{2\sigma R}} g_u^p \mathrm{d}\mu \right)^{\frac{1}{p}},$$

for all balls $B_R \subset X$. The constant $C > 0$ depends on C_μ and C_P of Definitions 2.1 and 2.8.

(*b*) *If $N_\mu \le p$, then X supports a weak (t, p)-Poincaré inequality for all $1 \le t < +\infty$.*

Remark 2.11. By Hölder inequality, we see that a weak (p^*, p)-Poincaré inequality implies the same inequality for smaller values of p^*. Hence, X supports a weak (t, p)-Poincaré inequality for every $1 \le t \le p^*$.

The study of variational integral (1.2) leads to a functional framework that requires the use of certain Newtonian spaces. We first consider the space of all p-integrable functions u on X that have a p-integrable (p-weak) upper gradient g on X, namely, the space $\widetilde{N}^{1,p}(X)$. Such space can be endowed with the seminorm given as follows:

$$\|u\|_{\widetilde{N}^{1,p}(X)} = \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},$$

where the infimum is taken over all (p-weak) upper gradients of u. We say that $u \sim v$ (namely, u is equivalent to v) in $\widetilde{N}^{1,p}(X)$ if $\|u - v\|_{\widetilde{N}^{1,p}(X)} = 0$. Consequently, we can define the Newtonian space $N^{1,p}(X)$ as $\widetilde{N}^{1,p}(X)/\sim$, using the norm $\|u\|_{N^{1,p}(X)} = \|u\|_{\widetilde{N}^{1,p}(X)} = \|u\|_{L^p(X)} + \|g_u\|_{L^p(X)}$, too.

If Ω is an open subset of X, then $N^{1,p}(\Omega)$ is defined by considering Ω as a metric measure space with the metric d and the measure μ restricted to Ω. The corresponding local Newtonian space $N^{1,p}_{\mathrm{loc}}(\Omega)$ is defined as usual.

For an open and bounded set $\Omega \subset X$, by $N_0^{1,p}(\Omega)$, we mean the space of functions $u \in N^{1,p}(X)$ that are zero on $X \backslash \Omega$ μ-a.e.. This space can be also endowed with the norm

$$\|u\|_{N_0^{1,p}(\Omega)} = \|u\|_{N^{1,p}(\Omega)}.$$

Now, we give the definition of K-quasiminimizer of the functional (1.2).

Definition 2.12. We say that $u \in N^{1,1}_{\mathrm{loc}}(\Omega)$ is a K-quasiminimizer of $\mathcal{H}(\cdot,\Omega)$ if $H(\cdot,g_u) \in L^1_{\mathrm{loc}}(\Omega)$, and we can find $K \geq 1$ such that, for every bounded open subset $\Omega' \Subset \Omega$ and all $w \in N^{1,1}(\Omega')$ with $u - w \in N^{1,1}_0(\Omega')$, the following inequality holds

$$\int_{\Omega' \cap \{u \neq w\}} H(x, g_u)\mathrm{d}\mu \leq K \int_{\Omega' \cap \{u \neq w\}} H(x, g_w)\mathrm{d}\mu. \tag{2.5}$$

We also say that $u \in N^{1,1}(\Omega)$ is a global K-quasiminimizer on Ω if inequality (2.5) holds for Ω in place of Ω', hence for all $w \in N^{1,1}(\Omega)$ with $u - w \in N^{1,1}_0(\Omega)$.

From the previous definition, it follows that $g_u \in L^p_{\mathrm{loc}}(\Omega)$, whenever u is a K-quasiminimizer. So, if X supports a weak $(1,p)$-Poincaré inequality, then $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$. If u is a global K-quasiminimizer, then $u \in N^{1,p}(\Omega)$.

3 Sobolev-Poincaré and Caccioppoli inequalities

We first recall the terminology and notation, then note some properties of the function H defined in (1.3). In fact, we consider a complete metric measure space (X, d, μ), and we assume that X is endowed with a doubling Borel regular measure μ. We also suppose that X supports a weak $(1,p)$-Poincaré inequality for some $p > 1$. If $p \leq N_\mu$ we choose $p_0^* \in (q, p^*)$ so that

$$p_0 = \frac{N_\mu p_0^*}{N_\mu + p_0^*} \in (\max\{1, p - \varepsilon\}, p), \tag{3.1}$$

and if $p > N_\mu$, we choose

$$p_0^* \in (q, +\infty) \quad \text{such that} \quad p_0 \in (\max\{N_\mu, p - \varepsilon\}, p), \tag{3.2}$$

where ε is given in Theorem 2.9. This ensures that X supports a weak (t, p_0)-Poincaré inequality for all $1 \leq t \leq p_0^*$ (recall Theorems 2.9 and 2.10, as well as Remark 2.11). Roughly speaking, this framework gives us constant C and dilation factor $\sigma \geq 1$ satisfying the inequality

$$\left(\fint_{B_R} \left| \frac{u - u_{B_R}}{R} \right|^t \mathrm{d}\mu \right)^{\frac{1}{t}} \leq C \left(\fint_{B_{\sigma R}} g_u^{p_0} \mathrm{d}\mu \right)^{\frac{1}{p_0}}, \quad 1 \leq t \leq p_0^*, \tag{3.3}$$

for all balls $B_R \subset X$ and $u \in L^1_{\mathrm{loc}}(X)$. We recall that C depends on p, C_μ, C_P, and σ depends on the dilation factor of Definition 2.8 (see Remark 2.11).

Remark 3.1. Let X be a metric measure space endowed with a doubling measure supporting a weak $(1,p)$-Poincaré inequality. Then Lemma 2.1 of [15] brings us to a Sobolev type inequality if $u \in N^{1,p}_0(B_R)$. That is, we can find $C \equiv C(C_\mu, C_P) > 0$ so that, for every ball $B_R \subset X$ and every $u \in N^{1,p}_0(B_R)$, the following inequality holds

$$\left(\fint_{B_R} \left(\frac{|u|}{R} \right)^t \mathrm{d}\mu \right)^{\frac{1}{t}} \leq C \left(\fint_{B_R} g_u^p \mathrm{d}\mu \right)^{\frac{1}{p}} \quad 1 \leq t \leq p^*.$$

We also assume that Ω is a bounded open subset of X such that $X \setminus \Omega \neq \emptyset$ and the non-negative function $a \in C^{0,\alpha}(\Omega)$ ($\alpha \in (0,1]$). Denote by $L_{0,\alpha}$ a suitable constant satisfying the inequality

$$|a(x) - a(y)| \leq L_{0,\alpha}(d(x,y))^\alpha \quad \text{for all } \; x, y \in \Omega. \tag{3.4}$$

Given $R > 0$, we will always deal with concentric balls $B_{\tau R} \subset \Omega$ (hence, we do not need to precise the center of balls). Later on, we will involve in our analysis the following set:

$$S_{k,R} = \{x \in B_R \cap \Omega : u(x) > k\},$$

where $k \in \mathbb{R}$ and $R > 0$.

In our proofs, we will use the following properties of the function H, namely, we have

$$H(\cdot,\beta t) \leq \begin{cases} \beta^p H(\cdot,t) & \text{if } \beta \leq 1, \\ \beta^q H(\cdot,t) & \text{if } \beta > 1, \end{cases} \tag{3.5}$$

and

$$H(\cdot,t_1 + t_2) \leq C(H(\cdot,t_1) + H(\cdot,t_2)) \tag{3.6}$$

for some $C \equiv C(q) > 0$. Further, as $a(\cdot)$ is bounded in each ball $B \subset \Omega$, we deduce

$$\int_B H(x,1)\mathrm{d}\mu < +\infty. \tag{3.7}$$

The constants involved in the estimates will depend essentially on the starting quantities assigned in the problem, that is on: N_μ, p, q, $L_{0,\alpha}$, C_μ, C_P. So, in the following, we denote

$$\text{data} \equiv (N_\mu, p, q, L_{0,\alpha}, C_\mu, C_P).$$

As a first step of the De Giorgi's method, we give now a Sobolev-Poincaré inequality in the setting of complete metric measure spaces. Before that, we briefly discuss the condition

$$N_\mu\left(\frac{q}{p} - 1\right) \leq \alpha, \tag{3.8}$$

given in hypothesis (1.4). Let $\Omega \subset X$ be a bounded open set and $B_R \subset \Omega$. In the proof of Sobolev-Poincaré inequality, we use an estimate of the quotient

$$\frac{R^\alpha}{\mu(B_R)^{\frac{q}{p}-1}}, \tag{3.9}$$

that is, we show that it is bounded for $R \in (0, \mathrm{diam}(\Omega)/2]$. We know that (3.9) is bounded provided that (3.8) holds and the measure μ is Ahlfors-David regular, that is,

$$c_1 R^{N_\mu} \leq \mu(B_R) \leq c_2 R^{N_\mu}, \tag{3.10}$$

for every $B_R \subset X$ and $R \in (0, \mathrm{diam}(X)]$, where $0 < c_1 \leq c_2$ are suitable constants. We note that (3.10) is satisfied by Lebesgue measure in the Euclidean setting. When μ is doubling and (3.8) is satisfied, then (3.9) is bounded by Proposition 2.4. Precisely, from $B_R \subset \Omega$, we obtain $R \leq \frac{\mathrm{diam}(\Omega)}{2}$. So, by using Proposition 2.4 and Remark 2.3, we deduce

$$\begin{aligned} R^\alpha\left(\frac{1}{\mu(B_R)}\right)^{\frac{q}{p}-1} &= \frac{R^\alpha}{(\mu(B_{\mathrm{diam}(\Omega)/2}))^{\frac{q}{p}-1}}\left(\frac{\mu(B_{\mathrm{diam}(\Omega)/2})}{\mu(B_R)}\right)^{\frac{q}{p}-1} \\ &\leq \frac{R^\alpha}{(C_{\mathrm{diam}(\Omega)/2})^{\frac{q}{p}-1}}\left(\frac{\mathrm{diam}(\Omega)}{R}\right)^{N_\mu(\frac{q}{p}-1)} \\ &\leq \frac{(\mathrm{diam}(\Omega))^{N_\mu(\frac{q}{p}-1)}}{(C_{\mathrm{diam}(\Omega)/2})^{\frac{q}{p}-1}} R^{\alpha - N_\mu\left(\frac{q}{p}-1\right)} \\ &\leq \frac{\max\{1, \mathrm{diam}(\Omega)\}}{(C_{\mathrm{diam}(\Omega)/2})^{\frac{q}{p}-1}} < +\infty. \end{aligned} \tag{3.11}$$

We are ready to establish our Sobolev-Poincaré inequality, in the following theorem, where p_0 is given by (3.1) or (3.2).

Theorem 3.2. *Let Ω be a bounded open subset of X and $u \in N^{1,1}_{\mathrm{loc}}(\Omega)$ such that $H(\cdot, g_u) \in L^1_{\mathrm{loc}}(\Omega)$. If (1.4) holds, then there exist a constant $C \equiv C(\mathrm{data}) > 0$ and an exponent $\theta = \frac{p_0}{p} \in (0,1)$ such that*

$$\begin{aligned} &\fint_{B_R} H\left(x, \frac{u - u_{B_R}}{R}\right) \mathrm{d}\mu \\ &\leq C\left(1 + \frac{\max\{1, \mathrm{diam}(\Omega)\}}{(C_{\mathrm{diam}(\Omega)/2})^{\frac{q}{p}-1}} \|H(\cdot, g_u)\|_{L^1(B_{\sigma R})}^{\frac{q}{p}-1}\right)\left(\fint_{B_{\sigma R}} H(x, g_u)^\theta \mathrm{d}\mu\right)^{\frac{1}{\theta}}, \end{aligned} \tag{3.12}$$

whenever $B_{\sigma R} \subset \Omega$, where $C_{\mathrm{diam}(\Omega)/2} \equiv C(N_\mu, \mathrm{diam}(\Omega)/2, \Omega)$ is given in Proposition 2.4.

Proof. From $B_{\sigma R} \subset \Omega$, hence, $\sigma R \leq \frac{\mathrm{diam}(\Omega)}{2}$, by (3.11), we obtain

$$(\sigma R)^\alpha \left(\frac{1}{\mu(B_{\sigma R})}\right)^{\frac{q}{p}-1} \leq \frac{\max\{1, \mathrm{diam}(\Omega)\}}{(C_{\mathrm{diam}(\Omega)/2})^{\frac{q}{p}-1}}. \tag{3.13}$$

Now, put $i(a, \sigma R) = \inf\{a(x) : x \in B_{\sigma R}\}$ and using the assumption (1.4) together with (3.13), we obtain

$$\begin{aligned} &\fint_{B_R} H\left(x, \frac{u - u_{B_R}}{R}\right) \mathrm{d}\mu \\ &= \fint_{B_R} H\left(x, \frac{u - u_{B_R}}{R}\right) \mathrm{d}\mu - \fint_{B_R} i(a, \sigma R)\left|\frac{u - u_{B_R}}{R}\right|^q \mathrm{d}\mu + \fint_{B_R} i(a, \sigma R)\left|\frac{u - u_{B_R}}{R}\right|^q \mathrm{d}\mu \\ &= \fint_{B_R} \left|\frac{u - u_{B_R}}{R}\right|^p \mathrm{d}\mu + \fint_{B_R} i(a, \sigma R)\left|\frac{u - u_{B_R}}{R}\right|^q \mathrm{d}\mu + \fint_{B_R} [a(x) - i(a, \sigma R)]\left|\frac{u - u_{B_R}}{R}\right|^q \mathrm{d}\mu \\ &\leq \fint_{B_R} \left(\left|\frac{u - u_{B_R}}{R}\right|^p + i(a, \sigma R)\left|\frac{u - u_{B_R}}{R}\right|^q\right) \mathrm{d}\mu + 3L_{0,\alpha}(\sigma R)^\alpha \fint_{B_R} \left|\frac{u - u_{B_R}}{R}\right|^q \mathrm{d}\mu \quad \text{(by (3.4))} \\ &\leq C\left(\fint_{B_{\sigma R}} g_u^{p_0} \mathrm{d}\mu\right)^{\frac{p}{p_0}} + i(a, \sigma R) C\left(\fint_{B_{\sigma R}} g_u^{p_0} \mathrm{d}\mu\right)^{\frac{q}{p_0}} + C(\sigma R)^\alpha \left(\fint_{B_{\sigma R}} g_u^{p_0} \mathrm{d}\mu\right)^{\frac{q}{p_0}} \quad \text{(by (3.3))} \\ &\leq C\left(\fint_{B_{\sigma R}} (g_u^p)^{\frac{p_0}{p}} \mathrm{d}\mu\right)^{\frac{p}{p_0}} + C\left(\fint_{B_{\sigma R}} (i(a, \sigma R) g_u^q)^{\frac{p_0}{q}} \mathrm{d}\mu\right)^{\frac{q}{p_0}} + C(\sigma R)^\alpha \left(\fint_{B_{\sigma R}} (g_u^p)^{\frac{p_0}{p}} \mathrm{d}\mu\right)^{\frac{q}{p_0}} \\ &\leq C\left(\fint_{B_{\sigma R}} H(x, g_u)^\theta \mathrm{d}\mu\right)^{\frac{1}{\theta}} + C(\sigma R)^\alpha \left(\fint_{B_{\sigma R}} H(x, g_u)^\theta \mathrm{d}\mu\right)^{\frac{1}{\theta}\frac{q}{p}} \quad \left(\text{with } \theta = \frac{p_0}{p}\right) \\ &= C\left\{1 + (\sigma R)^\alpha \left(\fint_{B_{\sigma R}} H(x, g_u)^\theta \mathrm{d}\mu\right)^{\frac{1}{\theta}\left(\frac{q}{p}-1\right)}\right\}\left(\fint_{B_{\sigma R}} H(x, g_u)^\theta \mathrm{d}\mu\right)^{\frac{1}{\theta}} \\ &\leq C\left(1 + \frac{\max\{1, \mathrm{diam}(\Omega)\}}{(C_{\mathrm{diam}(\Omega)/2})^{\frac{q}{p}-1}} \|H(\cdot, g_u)\|_{L^1(B_{\sigma R})}^{\frac{q}{p}-1}\right)\left(\fint_{B_{\sigma R}} H(x, g_u)^\theta \mathrm{d}\mu\right)^{\frac{1}{\theta}}. \end{aligned}$$

The result is now established. □

Remark 3.3. From the inequality (3.12), as $C_{\mathrm{diam}(\Omega)/2}$ depends on Ω, we obtain

$$\fint_{B_R} H\left(x, \frac{u - u_{B_R}}{R}\right) d\mu \leq C_S \left(\fint_{B_{\sigma R}} H(x, g_u)^\theta d\mu \right)^{\frac{1}{\theta}}, \tag{3.14}$$

where $C_S \equiv C_S(\text{data}, \Omega, \|H(\cdot, g_u)\|_{L^1(B_{\sigma R})})$.

As a consequence of the Sobolev-Poincaré inequality, we deduce a Sobolev-Poincaré inequality for functions that are zero on a set of positive measure.

Theorem 3.4. *Let Ω be a bounded open subset of X and $u \in N^{1,1}_{\text{loc}}(\Omega)$ with $H(\cdot, g_u) \in L^1_{\text{loc}}(\Omega)$. Suppose that* (1.4) *holds and that $u = 0$ on a measurable set $A \subset B_R$ such that $\mu(A) > 0$, then*

$$\fint_{B_R} H\left(x, \frac{u}{R}\right) d\mu \leq C_S \left(1 + \frac{\mu(B_R)}{\mu(A)}\right) \left(\fint_{B_{\sigma R}} H(x, g_u)^\theta d\mu \right)^{\frac{1}{\theta}},$$

whenever $B_{\sigma R} \subset \Omega$.

Proof. By using the hypothesis $u = 0$ on the set A, we deduce

$$\int_A \left| \frac{u - u_{B_R}}{R} \right|^s d\mu = \mu(A) \left| \frac{u_{B_R}}{R} \right|^s \leq \int_{B_R} \left| \frac{u - u_{B_R}}{R} \right|^s d\mu. \tag{3.15}$$

So, by using (3.15), we obtain

$$\int_{B_R} \left| \frac{u}{R} \right|^s d\mu \leq C \int_{B_R} \left| \frac{u - u_{B_R}}{R} \right|^s d\mu + C\mu(B_R) \left| \frac{u_{B_R}}{R} \right|^s \leq C \left(1 + \frac{\mu(B_R)}{\mu(A)}\right) \int_{B_R} \left| \frac{u - u_{B_R}}{R} \right|^s d\mu.$$

Next, from the previous inequality, we obtain the following:

$$\begin{aligned}
\fint_{B_R} H\left(x, \frac{u}{R}\right) d\mu &= \fint_{B_R} H\left(x, \frac{u}{R}\right) d\mu \pm i(a, \sigma R) \fint_{B_R} \left| \frac{u}{R} \right|^q d\mu \\
&= \fint_{B_R} \left| \frac{u}{R} \right|^p d\mu + \fint_{B_R} i(a, \sigma R) \left| \frac{u}{R} \right|^q d\mu + \fint_{B_R} [a(x) - i(a, \sigma R)] \left| \frac{u}{R} \right|^q d\mu \\
&\leq \fint_{B_R} \left(\left| \frac{u}{R} \right|^p + i(a, \sigma R) \left| \frac{u}{R} \right|^q \right) d\mu + 3L_{0,\alpha} (\sigma R)^\alpha \fint_{B_R} \left| \frac{u}{R} \right|^q d\mu \\
&\leq C \left(1 + \frac{\mu(B_R)}{\mu(A)}\right) \fint_{B_R} \left(\left| \frac{u - u_{B_R}}{R} \right|^p + i(a, \sigma R) \left| \frac{u - u_{B_R}}{R} \right|^q \right) d\mu + C \left(1 + \frac{\mu(B_R)}{\mu(A)}\right) (\sigma R)^\alpha \fint_{B_R} \left| \frac{u - u_{B_R}}{R} \right|^q d\mu.
\end{aligned}$$

Now, to complete the proof, one can simply follow the similar steps to the ones in the proof of Theorem 3.2. □

As a consequence of the Sobolev-Poincaré inequality for functions that are zero on a set of positive measure, we establish a Sobolev-Poincaré inequality for functions with zero boundary values.

Theorem 3.5. *Let Ω be a bounded open subset of X and $u \in N^{1,1}_{\text{loc}}(\Omega)$ with $H(\cdot, g_u) \in L^1_{\text{loc}}(\Omega)$. Suppose that* (1.4) *holds and that $u \in N^{1,1}_0(B_R)$, then*

$$\fint_{B_R} H\left(x, \frac{u}{R}\right) d\mu \leq C_S \left(\fint_{B_R} H(x, g_u)^\theta d\mu \right)^{\frac{1}{\theta}}, \tag{3.16}$$

whenever $B_{\lambda\sigma R} \Subset \Omega$ for some $\lambda \in (1, 2]$.

Proof. The assumption $B_{\lambda\sigma R} \Subset \Omega$ together with $X\backslash\Omega \neq \emptyset$, by Proposition 4.2 of [1], ensures that there exists $z \in \partial B_{\frac{\lambda+1}{2}R}$. So, $B_{\frac{\lambda-1}{2}R}(z) \subset B_{\lambda R} \subset \Omega$ and $B_{\frac{\lambda-1}{2}R}(z) \subset A = \{x \in B_{\lambda R} : u(x) = 0\}$. By using Theorem 3.4, we obtain

$$\fint_{B_R} H\left(x, \frac{u}{R}\right) \mathrm{d}\mu \le C \fint_{B_{\lambda R}} H\left(x, \frac{u}{R}\right) \mathrm{d}\mu \le C_S\left(1 + \frac{\mu(B_{\lambda R})}{\mu(A)}\right)\left(\fint_{B_{\lambda\sigma R}} H(x, g_u)^\theta \mathrm{d}\mu\right)^{\frac{1}{\theta}}.$$

From $u \in N_0^{1,1}(B_R)$, we deduce that $g_u = 0$ μ-a.e. in $B_{\lambda\sigma R}\backslash B_R$, and so we have

$$\fint_{B_R} H\left(x, \frac{u}{R}\right) \mathrm{d}\mu \le C_S\left(1 + \frac{\mu(B_{\lambda R})}{\mu(A)}\right)\left(\fint_{B_R} H(x, g_u)^\theta \mathrm{d}\mu\right)^{\frac{1}{\theta}},$$

where $C_S \equiv C_S(\text{data}, \Omega, \|H(\cdot, g_u)\|_{L^1(B_R)})$. We note that from (2.2), we can deduce the estimate

$$\frac{\mu(B_{\lambda R})}{\mu(A)} \le C_\mu^2\left(\frac{2\lambda}{\lambda - 1}\right)^{N_\mu} = 4^{N_\mu}\left(\frac{2\lambda}{\lambda - 1}\right)^{N_\mu}.$$

This ensures that (3.16) holds. □

As second ingredient of the De Giorgi's method described in Section 1, we look for suitable Caccioppoli type estimates for K-quasiminimizer of the functional (1.2). An easy adaptation of Lemma 6.1 in Giusti's book [12] is given in the form of Lemma 3.6.

Lemma 3.6. *If $P(\cdot)$ is a bounded non-negative function in the interval $[\rho, R] \subset \mathbb{R}$ and $\theta \in [0, 1)$ satisfies*

$$P(r) \le \int_{S_{k,R}} H\left(x, \frac{u-k}{s-r}\right) \mathrm{d}\mu + \theta P(s)$$

for all $\rho \le r < s \le R$ with $B_R \subset \Omega$, then we have

$$P(\rho) \le C \int_{S_{k,R}} H\left(x, \frac{u-k}{R-\rho}\right) \mathrm{d}\mu,$$

where $C \equiv C(\theta, q)$.

Then, we establish our Caccioppoli type estimate as follows.

Lemma 3.7. *Let Ω be an open subset of X and $0 < \rho < R$ so that $B_R \Subset \Omega$. Then, we can find $C \equiv C(K, q) > 0$ satisfying the following inequality*:

$$\int_{S_{k,\rho}} H(x, g_u) \mathrm{d}\mu \le C \int_{S_{k,R}} H\left(x, \frac{u-k}{R-\rho}\right) \mathrm{d}\mu, \tag{3.17}$$

namely, H-Caccioppoli type inequality, where $u \in N^{1,1}_{\mathrm{loc}}(\Omega)$ is a K-quasiminimizer of the $\mathcal{H}$-energy integral.

Proof. For a fixed ball $B_R \Subset \Omega$, consider the concentric balls $B_\rho \subset B_r \subset B_s \subset B_R$ with $\rho \le r < s \le R$. We introduce the $\frac{1}{s-r}$-Lipschitz cut-off function η such that $0 \le \eta \le 1$, $\eta \equiv 1$ on B_r and the support of η is contained in B_s, hence, we consider the function

$$w = u - \eta(u-k)_+ = \begin{cases} (1-\eta)(u-k) + k & \text{in } S_{k,R}, \\ u & \text{otherwise.} \end{cases}$$

By Leibniz rule (see Lemma 2.18 of [1]), we deduce that

$$g_w \le (u-k)g_\eta + (1-\eta)g_u \le g_u + \frac{u-k}{s-r} \quad \mu\text{-a.e. on } S_{k,R},$$

and using the "triangular" property of H in (3.6), we obtain

$$H(x, g_w) \le C\left(H(x, g_u) + H\left(x, \frac{u-k}{s-r}\right)\right) \quad \mu\text{-a.e. on } S_{k,R}. \tag{3.18}$$

Since u is a K-quasiminimizer of $\mathcal{H}$, using (3.18), $g_w = 0$ μ-a.e. on B_r, we obtain

$$\begin{aligned}
\int_{S_{k,r}} H(x, g_u)\mathrm{d}\mu &\le \int_{S_{k,s}} H(x, g_u)\mathrm{d}\mu \\
&\le K\int_{S_{k,s}} H(x, g_w)\mathrm{d}\mu \\
&= K\int_{S_{k,s}\setminus S_{k,r}} H(x, g_w)\mathrm{d}\mu \\
&\le C\int_{S_{k,s}\setminus S_{k,r}} H\left(x, \frac{u-k}{s-r}\right)\mathrm{d}\mu + C\int_{S_{k,s}\setminus S_{k,r}} H(x, g_u)\mathrm{d}\mu \\
&\le C\int_{S_{k,R}} H\left(x, \frac{u-k}{s-r}\right)\mathrm{d}\mu + C\int_{S_{k,s}\setminus S_{k,r}} H(x, g_u)\mathrm{d}\mu.
\end{aligned} \tag{3.19}$$

Next, we use the hole-filling method, by adding to both sides of (3.19) the quantity

$$C\int_{S_{k,r}} H(x, g_u)\mathrm{d}\mu,$$

then we obtain

$$(1+C)\int_{S_{k,r}} H(x, g_u)\mathrm{d}\mu \le C\int_{S_{k,R}} H\left(x, \frac{u-k}{s-r}\right)\mathrm{d}\mu + C\int_{S_{k,s}} H(x, g_u)\mathrm{d}\mu.$$

Immediately, we rewrite this inequality as follows:

$$\int_{S_{k,r}} H(x, g_u)\mathrm{d}\mu \le \int_{S_{k,R}} H\left(x, \frac{u-k}{s-r}\right)\mathrm{d}\mu + \theta\int_{S_{k,s}} H(x, g_u)\mathrm{d}\mu,$$

where $\theta = \frac{C}{1+C} < 1$. By appealing to Lemma 3.6 with $P(r) = \int_{S_{k,r}} H(x, g_u)\mathrm{d}\mu$, we conclude inequality (3.17), and hence, the result is established. □

Remark 3.8. If we set $(u-k)_+ = \max\{u-k, 0\}$ and $(u-k)_- = \max\{k-u, 0\}$, then we can first note that the H-Caccioppoli type estimate in (3.17) is equivalent to the following one:

$$\int_{B_\rho} H(x, g_{(u-k)_+})\mathrm{d}\mu \le C\int_{B_R} H\left(x, \frac{(u-k)_+}{R-\rho}\right)\mathrm{d}\mu. \tag{3.20}$$

Similarly from (3.20), we also deduce that

$$\int_{B_\rho} H(x, g_{(u-k)_-})\mathrm{d}\mu \le C\int_{B_R} H\left(x, \frac{(u-k)_-}{R-\rho}\right)\mathrm{d}\mu, \tag{3.21}$$

as $-u$ is a K-quasiminimizer too.

4 Locally boundedness

In this section, we aim to show that K-quasiminimizers of the $\mathcal{H}$-energy integral (1.2) are essentially locally bounded. Hence, the strategy designed by De Giorgi is applied to obtain an estimate of essential supremum of K-quasiminimizers. We obtain this result by the Sobolev-Poincaré inequality (3.16) and H-Caccioppoli type estimate. Our result is a counterpart to the well-known result of [15] for K-quasiminimizers of p-Dirichlet integrals.

The next lemma is also frequently used in the literature (in this context) and will be applied here.

Lemma 4.1. [12, Lemma 7.1] *Let $\gamma > 0$ and let $\{\Psi_j\}_{j\geq 0}$ be a sequence of real positive numbers satisfying*

$$\Psi_{j+1} \leq \kappa B^j \Psi_j^{1+\gamma}$$

for all $j \geq 0$ with $\kappa > 0$ and $B > 1$. If $\Psi_0 \leq \kappa^{-1/\gamma} B^{-1/\gamma^2}$, then $\Psi_j \to 0$ as $j \to +\infty$.

Theorem 4.2. *Let Ω be a bounded open subset of X. If $u \in N^{1,1}_{\mathrm{loc}}(\Omega)$ is a K-quasiminimizer of the functional $\mathcal{H}(\cdot,\Omega)$ defined in* (1.2) *and if* (1.4) *holds, then u is essentially locally bounded. Consequently, $u \in L^\infty_{\mathrm{loc}}(\Omega)$.*

Proof. For $0 \leq h < k$ and fixed ball B_R such that $B_{\sigma R} \Subset \Omega$ with $R/2 \leq \rho < R \leq 1$, consider the concentric balls $B_\rho \subset B_s \subset B_R$ with $s := (R+\rho)/2$. We introduce the $\frac{1}{R-\rho}$-Lipschitz cut-off function η such that

$$\eta \equiv 1 \text{ on } B_\rho \text{ and the support of } \eta \text{ is contained in } B_s.$$

Hence, we consider the test function $w = \eta(u-k)_+ \in N^{1,1}_0(B_s)$. By using the Sobolev-Poincaré inequality (3.16) together with Hölder inequality, then we obtain

$$\begin{aligned}\int_{B_s} H(x, \eta(u-k)_+)\mathrm{d}\mu &\leq \int_{B_s} H\left(x, \frac{\eta(u-k)_+}{s}\right)\mathrm{d}\mu \\ &\leq C_S(\mu(B_s))^{1-\frac{1}{\theta}}\left(\int_{B_s} H(x, g_{\eta(u-k)_+})^\theta \mathrm{d}\mu\right)^{\frac{1}{\theta}} \\ &\leq C_S\left(\frac{\mu(S_{k,s})}{\mu(B_s)}\right)^{\frac{1-\theta}{\theta}} \int_{B_s} H(x, g_{\eta(u-k)_+})\mathrm{d}\mu,\end{aligned} \tag{4.1}$$

where $\theta = \frac{p_0}{p} \in (0,1)$ and $C_S \equiv C_S(\text{data}, \Omega, \|H(\cdot, g_u)\|_{L^1(B_s)})$. On the other hand, the properties of H and η lead to the following estimate:

$$\int_{B_s} H(x, g_{\eta(u-k)_+})\mathrm{d}\mu \leq C\int_{B_s} H\left(x, \frac{(u-k)_+}{R-\rho}\right)\mathrm{d}\mu + C\int_{B_s} H(x, g_{(u-k)_+})\mathrm{d}\mu. \tag{4.2}$$

By using estimates (4.1) and (4.2) together with the H-Caccioppoli type estimate (3.20) in the case $\rho = s$, then since $h < k$, we conclude that

$$\int_{S_{k,\rho}} H(x, u-k)\mathrm{d}\mu \leq C_S\left(\frac{\mu(S_{k,R})}{\mu(B_\rho)}\right)^{\frac{1-\theta}{\theta}} \int_{S_{h,R}} H\left(x, \frac{u-h}{R-\rho}\right)\mathrm{d}\mu. \tag{4.3}$$

Next, we introduce a sequence of concentric balls $\{B_{\rho_j}\}$ to B_R, where $\rho_j := R(1+2^{-j})/2$ for $j \geq 0$, and for appropriate value $D > 0$ (whose expression will be precised below), we consider the levels $k_j := 2D(1-2^{-(j+1)})$. If we use the inequality (4.3) for $\rho \equiv \rho_{j+1}$, $R \equiv \rho_j$, $k \equiv k_{j+1}$ and $h \equiv k_j$, then for $j \geq 0$, we deduce that

$$\int_{S_{k_{j+1},\rho_{j+1}}} H(x, u-k_{j+1})\mathrm{d}\mu \leq \frac{C_S}{R^q} 2^{jq}\left(\frac{\mu(S_{k_{j+1},\rho_j})}{\mu(B_{R/2})}\right)^{\frac{1-\theta}{\theta}} \int_{S_{k_j,\rho_j}} H(x, u-k_j)\mathrm{d}\mu.$$

We also note that

$$(k_{j+1}-k_j)^p\mu(S_{k_{j+1},\rho_j}) = \int\limits_{S_{k_{j+1},\rho_j}} (k_{j+1}-k_j)^p \mathrm{d}\mu \le \int\limits_{S_{k_{j+1},\rho_j}} H(x,u-k_j)\mathrm{d}\mu \le \int\limits_{S_{k_j,\rho_j}} H(x,u-k_j)\mathrm{d}\mu,$$

and hence, we have

$$\mu(S_{k_{j+1},\rho_j}) \le 2^{(j+1)p}D^{-p}\int\limits_{S_{k_j,\rho_j}} H(x,u-k_j)\mathrm{d}\mu.$$

It follows that

$$D^{-p}\int\limits_{S_{k_{j+1},\rho_{j+1}}} H(x,u-k_{j+1})\mathrm{d}\mu \le \frac{C_S}{R^q}(2^{\frac{q}{\theta}})^j(\mu(B_{R/2}))^{\frac{\theta-1}{\theta}}\left(D^{-p}\int\limits_{S_{k_j,\rho_j}} H(x,u-k_j)\mathrm{d}\mu\right)^{\frac{1}{\theta}}.$$

Setting

$$\Psi_j \coloneqq D^{-p}\int\limits_{S_{k_j,\rho_j}} H(x,u-k_j)\mathrm{d}\mu,$$

the aforementioned inequality reduces to

$$\Psi_{j+1} \le \frac{C_S}{R^q}(\mu(B_{R/2}))^{\frac{\theta-1}{\theta}}(2^{\frac{q}{\theta}})^j\Psi_j^{\frac{1}{\theta}},$$

for every $j \ge 0$, and some C_S depending only on: data, Ω and $\|H(\cdot,g_u)\|_{L^1(B_R)}$. Now, involving the standard iteration Lemma 4.1, we deduce that $\Psi_j \to 0$ (as $j \to +\infty$) provided that

$$\Psi_0 \le D^{-p}\int\limits_{B_R} H(x,(u-D)_+)\mathrm{d}\mu \le c^{-p}\mu(B_{R/2}),$$

where $c \ge 1$, depending only on C_S, θ, q, R, is such that

$$\frac{1}{c^p} \le (R^{-q}C_S)^{-\frac{1}{\gamma}}\,(2^{\frac{q}{\theta}})^{-\frac{1}{\gamma^2}},$$

where $\frac{1}{\gamma} = \frac{\theta}{1-\theta}$. Now, from

$$\Psi_0 \le D^{-p}\int\limits_{B_R} H(x,(u-D)_+)\mathrm{d}\mu \le c^{-p}\mu(B_{R/2}),$$

we deduce that $\Psi_j \to 0$ as $j \to +\infty$ if for D we choose the value

$$D \coloneqq c\left(\mu(B_{R/2})^{-1}\int\limits_{B_R} H(x,(u-D)_+)\mathrm{d}\mu\right)^{\frac{1}{p}} \le c\;4^{\frac{N_\mu}{p}}\left(\fint\limits_{B_R} H(x,(u-D)_+)\mathrm{d}\mu\right)^{\frac{1}{p}}.$$

So, from

$$0 \le \int\limits_{S_{2D,R/2}} H(x,u-2D)\mathrm{d}\mu \le D^p\lim_{j\to+\infty}\Psi_j = 0,$$

we obtain

$$\operatorname*{ess\,sup}_{B_{R/2}} u \le 2D \le 2c\ 4^{\frac{N_\mu}{p}}\left(\fint_{B_R} H(x,(u-D)_+)\mathrm{d}\mu\right)^{\frac{1}{p}}$$
$$\le C\left(\fint_{B_R} H(x,u_+)\mathrm{d}\mu\right)^{\frac{1}{p}}.$$

The similar calculations as mentioned earlier can be developed with respect to $-u$; this is a K-quasiminimizer of the functional $\mathcal{H}(\cdot,\Omega)$ too. Consequently, the conclusion of the theorem is reached. As a byproduct, we deduce that $u \in L^\infty_{\mathrm{loc}}(\Omega)$. □

5 Hölder continuity

Here, we assume that X supports a weak $(1,p)$-Poincaré inequality, and so by Theorem 2.9, it supports a weak $(1,p_0)$-Poincaré inequality for some $1 < p_0 < p < p_0^*$ too. Given a ball $B_R \subset \Omega$ and a function $u \in N^{1,1}_{\mathrm{loc}}(\Omega) \cap N^\infty_{\mathrm{loc}}(\Omega)$, we adopt the following notation

$$s(u,R) = \operatorname{ess\,sup}_{B_R} u,\quad i(u,R) = \operatorname*{ess\,inf}_{B_R} u \quad \text{and} \quad w(u,R) = s(u,R) - i(u,R).$$

The first result of this section is the following auxiliary lemma, which provides an useful tool to estimate the measure of the set $\mu(S_{k,R})$, in the case when k is close to the supremum of u (see Lemma 5.2).

Lemma 5.1. *If the function $u \in N^{1,1}_{\mathrm{loc}}(\Omega)$ with $H(\cdot,g_u) \in L^1_{\mathrm{loc}}(\Omega)$ satisfies the H-Caccioppoli inequality* (3.17), $R > 0$ *is such that $B_{\lambda\sigma R} \Subset \Omega$ for some $\lambda \in (1,2]$, and $h > 0$ is such that*

$$\mu(S_{h,R}) \le (1-\gamma)\mu(B_R) \quad \text{for some } \gamma \in (0,1), \tag{5.1}$$

then for $k > h$, we have

$$(k-h)\mu(S_{k,R}) \le CR\mu(B_R)^{1-\frac{1}{p_0}}\Delta_{h,k}^{\frac{1}{p_0}-\frac{1}{p}}\left(\int_{S_{h,\lambda\sigma R}} H\left(x,\frac{u-h}{R}\right)\mathrm{d}\mu\right)^{\frac{1}{p}}, \tag{5.2}$$

where $p_0 < p$ is given by (3.1) *if $p \le N_\mu$ or* (3.2) *if $p > N_\mu$ and*

$$\Delta_{h,k} = \mu(S_{h,\sigma R}) - \mu(S_{k,\sigma R}).$$

Proof. Fixed $k > h$, we define the truncation function:

$$w(x) = \begin{cases} 0 & \text{if } u(x) \le h, \\ u(x)-h & \text{if } h < u(x) < k, \\ k-h & \text{if } u(x) \ge k. \end{cases} \tag{5.3}$$

Since $u \in N^{1,1}_{\mathrm{loc}}(\Omega)$, then we have $w \in N^{1,1}_{\mathrm{loc}}(\Omega)$ too. Now, inequality (5.1) and truncation (5.3) imply that

$$\mu(\{x \in B_R : w(x) = 0\}) \ge \gamma\mu(B_R) > 0.$$

By using Hölder inequality with respect to p and Theorem 3.4, we obtain

$$
\begin{aligned}
(k-h)\mu(S_{k,R}) &= \int_{S_{k,R}} w \,\mathrm{d}\mu \le \int_{B_R} w\,\mathrm{d}\mu \\
&\le \mu(B_R)^{1-\frac{1}{p}}\left(\int_{B_R} w^p \mathrm{d}\mu\right)^{\frac{1}{p}} \\
&= R\mu(B_R)^{1-\frac{1}{p}}\left(\int_{B_R}\left(\frac{w}{R}\right)^p \mathrm{d}\mu\right)^{\frac{1}{p}} \\
&\le R\mu(B_R)\left(\fint_{B_R} H\left(x,\frac{w}{R}\right)\mathrm{d}\mu\right)^{\frac{1}{p}} \\
&\le CR\mu(B_R)\left(\fint_{B_{\sigma R}} H(x,g_w)^{\theta}\mathrm{d}\mu\right)^{\frac{1}{\theta p}} \quad \left(\text{with } \theta = \frac{p_0}{p}\right) \\
&\le CR\mu(B_R)^{1-\frac{1}{p_0}}\left(\int_{B_{\sigma R}} H(x,g_w)^{\theta}\mathrm{d}\mu\right)^{\frac{1}{\theta p}} \\
&\le CR\mu(B_R)^{1-\frac{1}{p_0}}\left(\int_{S_{h,\sigma R}\setminus S_{k,\sigma R}} H(x,g_u)^{\theta}\mathrm{d}\mu\right)^{\frac{1}{\theta p}} \quad (\text{by } (5.3)) \\
&\le CR\mu(B_R)^{1-\frac{1}{p_0}}\Delta_{h,k}^{\frac{1}{p_0}-\frac{1}{p}}\left(\int_{S_{h,\sigma R}} H(x,g_u)\mathrm{d}\mu\right)^{\frac{1}{p}}.
\end{aligned}
$$

Now, using the H-Caccioppoli inequality (3.17) with $\rho = \sigma R$ and R replaced by $\lambda\sigma R$, we obtain the inequality (5.2). The constant C depends on C_S and γ. □

Now, we prove the following key lemma.

Lemma 5.2. *Let* $R > 0$ *and* $\lambda \in (1, 2]$ *such that* $B_{\lambda\sigma R} \Subset \Omega$, $u \in N^{1,1}_{\mathrm{loc}}(\Omega) \cap N^{\infty}_{\mathrm{loc}}(\Omega)$ *with* $H(\cdot, g_u) \in L^1_{\mathrm{loc}}(\Omega)$ *be a function satisfying* (3.17) *together with* $-u$ *for every* $k \in \mathbb{R}$, *and let* $2k_0 = s(u, \lambda\sigma R) + i(u, \lambda\sigma R)$. *If* $\mu(S_{k_0,R}) \le (1-\gamma)\mu(B_R)$ *for some* $0 < \gamma < 1$, *then*

$$\lim_{k\to s(u,\lambda\sigma R)^-} \mu(S_{k,R}) = 0.$$

Proof. For convenience, in the proof, we write $S = s(u, \lambda\sigma R)$ and $I = i(u, \lambda\sigma R)$, hence we set $k_j = S - 2^{-(j+1)}(S - I)$, $j \ge 0$. Consequently, we obtain that

$$k_0 = \frac{S+I}{2} \le k_j \uparrow S \quad \text{as } j \to +\infty.$$

Of course, we have $S - k_j = 2^{-(j+1)}(S-I)\downarrow 0$ as $j \to +\infty$ and $k_{j+1} - k_j = 2^{-(j+2)}(S - I)$, and we note that $u - k_j \le S - k_j$ μ-a.e. on $S_{k_j,\lambda\sigma R}$. So, there exists $n_0 > 0$ such that

$$R^{-1}(S - k_j) \le 1 \quad \text{for all } j \ge n_0. \tag{5.4}$$

The inequality $\mu(S_{k_j,R}) \le \mu(S_{k_0,R})$ for all $j \ge 0$ allows us to use the inequality (5.2) with $k = k_{j+1}$ and $h = k_j$, then we obtain

$$(k_{j+1}-k_j)\mu(S_{k_{j+1},R}) \le CR\mu(B_R)^{1-\frac{1}{p_0}}\Delta_{k_j,k_{j+1}}^{\frac{1}{p_0}-\frac{1}{p}}\left(\int_{S_{k_j,\lambda\sigma R}} H\left(x,\frac{S-k_j}{R}\right)\mathrm{d}\mu\right)^{\frac{1}{p}}$$
$$\le CR\mu(B_R)^{1-\frac{1}{p_0}}\Delta_{k_j,k_{j+1}}^{\frac{1}{p_0}-\frac{1}{p}}\frac{S-k_j}{R}\left(\int_{S_{k_j,\lambda\sigma R}} H(x,1)\mathrm{d}\mu\right)^{\frac{1}{p}} \quad \text{(for all } j \ge n_0 \text{ by (3.5) and (5.4))}$$
$$\le C(S-k_j)\mu(B_R)^{1-\frac{1}{p_0}+\frac{1}{p}}\Delta_{k_j,k_{j+1}}^{\frac{1}{p_0}-\frac{1}{p}},$$

in the previous inequality, we took into account that $\int_{S_{k_j,\lambda\sigma R}} H(x,1)\mathrm{d}\mu \le c\mu(B_R)$ by (3.7). So,

$$\mu(S_{k_{j+1},R})^{\frac{p_0p}{p-p_0}} \le C\mu(B_R)^{\frac{p_0p}{p-p_0}-1}\Delta_{k_j,k_{j+1}} \quad \text{(for all } j \ge n_0)$$

that implies

$$\sum_{j=n_0}^{n-1}\mu(S_{k_{j+1},R})^{\frac{p_0p}{p-p_0}} \le C\mu(B_R)^{\frac{p_0p}{p-p_0}-1}\Delta_{k_{n_0},k_n} \le C\mu(B_R)^{\frac{p_0p}{p-p_0}}.$$

Since $k_j \uparrow S$, we obtain $\mu(S_{k_n,R}) \le \mu(S_{n_0,R})$ whenever $n \ge n_0$, and hence we can use the previous inequality to obtain

$$(n-n_0)\mu(S_{k_n,R})^{\frac{p_0p}{p-p_0}} \le C\mu(B_R)^{\frac{p_0p}{p-p_0}} \le C\mu(\Omega)^{\frac{p_0p}{p-p_0}} \le C. \tag{5.5}$$

This implies that $\mu(S_{k_n,R})$ goes to zero as $n \to +\infty$, and so

$$\lim_{k\to S^-}\mu(S_{k,R}) = 0.$$

Hence, the conclusion of the theorem is reached. □

Remark 5.3. We note that it is not restrictive to suppose that $\mu(S_{k,R}) \le (1-\gamma)\mu(B_R)$ for all $k \in \mathbb{R}$ and $R > 0$ in Lemma 5.2. Indeed, if $\gamma \in (0, 2^{-1}]$, then $\mu(S_{k,R}) > (1-\gamma)\mu(B_R)$ implies

$$\mu(\{x \in B_R : -u(x) > -k\}) \le (1-\gamma)\mu(B_R),$$

and since $-u$ satisfies (3.17) for every $k \in \mathbb{R}$ too, we just replace u by $-u$.

Lemma 5.4. *Let $u \in N^{1,1}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ with $H(\cdot, g_u) \in L^1_{\mathrm{loc}}(\Omega)$ satisfying* (3.17) *together with $-u$ for every $k \in \mathbb{R}$, $R > 0$ such that $B_R \Subset \Omega$ and $\omega \ge \omega(u,R)$. Then there exists $\nu \in (0,1)$ depending only on the data, but independent of ω, such that if for some $\varepsilon \in (0,1)$, we have*

$$\mu(S_{s(u,R)-\varepsilon\omega,R}) \le \nu\mu(B_R), \tag{5.6}$$

then

$$u \le s(u,R) - \frac{\varepsilon\omega}{2} \quad \text{a.e. in } B_{R/2}. \tag{5.7}$$

Proof. The proof is an adaptation to the metric context of Proposition 4.1 of [8], p. 354, given in the Euclidean setting with $p = q$. Let $\{B_{\rho_j}\}$ denote a sequence of nested balls concentric to B_R for $j \ge 0$, where $\rho_j \coloneqq R(1+2^{-j})/2$, and set $s_j \coloneqq (\rho_j + \rho_{j+1})/2$. We can introduce suitable cut-off Lipschitz functions η_j such that $\eta_j \equiv 1$ on $B_{\rho_{j+1}}$ and the support of η_j is contained in B_{s_j}. Now, we consider the levels

$$k_j = s(u,R) - \frac{\varepsilon\omega}{2} - \frac{\varepsilon\omega}{2^{j+1}}$$

satisfying the inequality

$$\left[\frac{\varepsilon\omega}{2^{j+2}}\right]^p \mu(S_{k_{j+1},\rho_{j+1}}) = (k_{j+1}-k_j)^p \mu(S_{k_{j+1},\rho_{j+1}}) \le \int_{S_{k_{j+1},\rho_{j+1}}} (u-k_j)^p \mathrm{d}\mu \le \int_{S_{k_j,s_j}} [\eta_j(u-k_j)]^p \mathrm{d}\mu. \tag{5.8}$$

Note that it is not restrictive to assume $\varepsilon \le R\omega^{-1}$ in (5.6). This implies

$$\frac{u-k_j}{R} \le \frac{\varepsilon\omega}{R} \le 1 \quad \text{on } S_{k_j,\rho_j}. \tag{5.9}$$

By (3.5) and (3.7), it follows that

$$\int_{S_{k_j,\rho_j}} H\left(x, \frac{u-k_j}{R}\right)\mathrm{d}\mu \le \left(\frac{\varepsilon\omega}{R}\right)^p \int_{S_{k_j,\rho_j}} H(x,1)\mathrm{d}\mu \le C\left(\frac{\varepsilon\omega}{R}\right)^p \mu(S_{k_j,\rho_j}) \text{ for all } j \ge 0. \tag{5.10}$$

Now, let p_0 as in (3.1) if $p \le N_\mu$ or (3.2) if $p > N_\mu$. We first note that $\eta_j(u-k_j) \in N_0^{1,1}(B_{s_j})$, then by Hölder inequality, Remark 3.1, and inequality (3.20) with $\rho = s_j$ and $R = \rho_j$, we obtain the following estimate:

$$\begin{aligned}\int_{S_{k_j,s_j}} [\eta_j(u-k_j)]^p \mathrm{d}\mu &\le \left(\int_{B_{s_j}} [\eta_j(u-k_j)]_+^{p_0^*} \mathrm{d}\mu\right)^{\frac{p}{p_0^*}} \mu(S_{k_j,s_j})^{1-\frac{p}{p_0^*}} \\ &\le CR^p \mu(S_{k_j,s_j})^{1-\frac{p}{p_0^*}} \int_{B_{s_j}} g^p_{[\eta_j(u-k_j)]_+} \mathrm{d}\mu \\ &\le CR^p \mu(S_{k_j,s_j})^{1-\frac{p}{p_0^*}} \int_{B_{s_j}} H(x, g_{[\eta_j(u-k_j)]_+}) \mathrm{d}\mu \\ &\le CR^p 2^{qj} \mu(S_{k_j,\rho_j})^{1-\frac{p}{p_0^*}} \int_{S(k_j,\rho_j)} H\left(x, \frac{u-k_j}{R}\right)\mathrm{d}\mu \\ &\le C2^{qj}(\varepsilon\omega)^p \mu(S_{k_j,\rho_j})^{1+\gamma} \quad \text{by (5.10)},\end{aligned}$$

where $\gamma = 1 - p/p_0^* > 0$. If we combine this estimate with the inequality (5.8), then we deduce that

$$\mu(S_{k_{j+1},\rho_{j+1}}) \le C2^{qj}2^{p(j+2)}\mu(S_{k_j,\rho_j})^{1+\gamma} \le C4^{qj}\mu(S_{k_j,\rho_j})^{1+\gamma}. \tag{5.11}$$

Now, using (2.2), we obtain that for some $c > 0$,

$$c\frac{\mu(B_{\rho_{j+1}})}{\mu(B_{\rho_j})^{1+\gamma}} \ge 1,$$

and so we have

$$\frac{\mu(S_{k_{j+1},\rho_{j+1}})}{\mu(B_{\rho_{j+1}})} \le C(4^q)^j \left(\frac{\mu(S_{k_j,\rho_j})}{\mu(B_{\rho_j})}\right)^{1+\gamma}.$$

We compact the notation putting $\Lambda_j := \frac{\mu(S_{k_j,\rho_j})}{\mu(B_{\rho_j})}$, then we obtain a recursive estimate in the form

$$\Lambda_{j+1} \le C(4^q)^j \Lambda_j^{1+\gamma}, \tag{5.12}$$

for a constant $C \equiv C(C_S)$ and for every integer $j \ge 0$. We can apply Lemma 4.1 and deduce that the aforementioned recursion satisfies

$$\lim_{j\to+\infty} \Lambda_j = 0,$$

provided that

$$\Lambda_0 = \frac{\mu(S_{s(u,R)-\varepsilon\omega,R})}{\mu(B_R)} \le C^{-\frac{1}{\gamma}}(4^q)^{-\frac{1}{\gamma^2}}. \tag{5.13}$$

Consequently, for every $\nu \in (0,1)$ such that $\nu \le C^{-\frac{1}{\gamma}}(4^q)^{-\frac{1}{\gamma^2}}$, we have $\Lambda_j \to 0$, and this ensures that (5.7) holds. □

Remark 5.5. Let $R > 0$ and $\lambda \in (1,2]$ such that $B_{\lambda\sigma R} \Subset \Omega$, $u \in N^{1,1}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ with $H(\cdot, g_u) \in L^1_{\mathrm{loc}}(\Omega)$ satisfying (3.17) together with $-u$ for every $k \in \mathbb{R}$, and let $2k_0 = s(u,\lambda\sigma R) + i(u,\lambda\sigma R)$. We note that it is not restrictive to suppose that $\mu(S_{k_0,R}) \le (1-\gamma)\mu(B_R)$ for some $0 < \gamma < 1$ (Remark 5.3). By Lemma 5.2, we say that

$$\lim_{k\to s(u,\lambda\sigma R)^-} \mu(S_{k,R}) = 0.$$

So, for some $\omega \ge \omega(u,\lambda\sigma R)$, we can choose $\varepsilon \in (0,1)$ such that (5.6) holds, and hence (5.7) holds too. Then, we obtain

$$s(u,\lambda\sigma R/4) \le s(u,\lambda\sigma R) - \frac{\varepsilon}{2}\omega(u,\lambda\sigma R).$$

Now

$$-i(u,\lambda\sigma R/4) \le -i(u,\lambda\sigma R).$$

Adding these inequalities, we obtain that

$$\omega(u,\lambda\sigma R/4) \le \theta\omega(u,\lambda\sigma R), \quad \text{where } \theta = 1 - \varepsilon/2. \tag{5.14}$$

Clearly, the inequality 5.14 holds whenever u is a K-quasiminimizer to (1.2), as u or $-u$ satisfies the hypotheses of Lemmas 5.1, 5.2, and 5.4.

Finally, using the previous results, we obtain the local Hölder inequality for the K-quasiminimizers to (1.2).

Theorem 5.6. *If $u \in N^{1,1}(\Omega)$ is a K-quasiminimizer of the functional $\mathcal{H}(\cdot,\Omega)$ defined in* (1.2), *under the assumptions* (1.3) *and* (1.4), *then there exists $\beta \in (0,1)$ such that for all $R > 0$ and $\lambda \in (1,2]$ such that $B_{2\lambda\sigma R} \Subset \Omega$, we have*

$$|u(x) - u(y)| \le C\left(\frac{|x-y|}{R}\right)^{\beta} \|u\|_{L^\infty(B_{\lambda\sigma R})} \tag{5.15}$$

for all $x, y \in B_R$.

Proof. By Remark 5.5, we know that for all $R > 0$ and $\lambda \in (1,2]$ such that $B_{\lambda\sigma R} \Subset \Omega$ holds (5.14) for some $\theta \in (1/2, 1)$. So, we have

$$\omega(u,\lambda\sigma R_{j+1}) \le \theta^{j+1}\omega(u,\lambda\sigma R) \quad \text{for all } j \ge 0, \tag{5.16}$$

where $R_j \coloneqq R/4^j$. Now, for all $\rho < R$, there exists $j \ge 0$ such that $R_{j+1} \le \rho < R_j$. It follows that $4^{-(j+1)} \le \rho R^{-1}$, and hence, we have

$$\omega(u,\rho) \le \omega(u,\lambda\sigma R_j) \le 4^{\beta}\left(\frac{\rho}{R}\right)^{\beta} \quad \omega(u,\lambda\sigma R), \tag{5.17}$$

where $\beta = -\log_4\theta$. The local Hölder inequality for u in (5.15) now follows from (5.17). The proof is complete. □

6 Local higher integrability

We apply the Sobolev-Poincaré and Caccioppoli type results established in previous Section 3 to deal with local higher integrability of the function $H(\cdot, g_u)$, where $u \in N^{1,1}_{\mathrm{loc}}(\Omega)$ is a K-quasiminimizer to our $\mathcal{H}$-energy integral (see (1.2)). The aforementioned results are combined with the Gehring's lemma (refer to Section 1, too) to establish Theorem 6.2. Precisely, we use the following version of the Gehring's lemma.

Theorem 6.1. [19, Theorem 3.2] *Fixed $s_0, s_1 > 1$, let $g \in L^s_{\mathrm{loc}}(X)$ and $f \in L^{s_1}_{\mathrm{loc}}(X)$ be non-negative functions, with $s_0 \le s \le s_1$, and suppose that we can find $b > 1$ satisfying the inequality*

$$\fint_{B_R} g^s \mathrm{d}\mu \le b\left(\fint_{B_{\sigma R}} g \mathrm{d}\mu\right)^s + b \fint_{B_{\sigma R}} f^s \mathrm{d}\mu$$

for all $B_R \subset B_{\sigma R} \subset X$ and some $\sigma > 1$. Then, we can find $\varepsilon_0 \equiv \varepsilon_0(s_0, s_1, C_\mu, \sigma, b) > 0$ such that $g \in L^t_{\mathrm{loc}}(X)$ for $t \in [s, s + \varepsilon_0)$, and we have

$$\left(\fint_{B_R} g^t \mathrm{d}\mu\right)^{\frac{1}{t}} \le C\left(\fint_{B_{\sigma R}} g^s \mathrm{d}\mu\right)^{\frac{1}{s}} + C\left(\fint_{B_{\sigma R}} f^t \mathrm{d}\mu\right)^{\frac{1}{t}}$$

for $C \equiv C(s_0, s_1, C_\mu, \sigma, b)$.

We have the following inner local higher integrability theorem.

Theorem 6.2. *There exist a constant $C \equiv C(C_S, p_0, \sigma) > 0$ and a positive integrability exponent $\delta_0 \equiv \delta_0(C_S, p_0, \sigma) > 0$ such that if $u \in N^{1,1}_{\mathrm{loc}}(\Omega)$ is a K-quasiminimizer of the integral functional* (1.2) *under assumptions* (1.3) *and* (1.4), *then $H(\cdot, g_u) \in L^{1+\delta_0}_{\mathrm{loc}}(\Omega)$ and*

$$\left(\fint_{B_R} H(x, g_u)^{1+\delta_0} \mathrm{d}\mu\right)^{\frac{1}{1+\delta_0}} \le C \fint_{B_{\sigma R}} [H(x, g_u) + 1] \mathrm{d}\mu, \tag{6.1}$$

for all B_R such that $B_{\sigma R} \Subset \Omega$, that is a bounded open subset of X.

Proof. The inequalities (3.20), (3.21) with $\rho = R/2$ and $k = u_{B_R}$, and Remark 2.3 lead us to the following estimate:

$$\begin{aligned}
\fint_{B_{R/2}} H(x, g_u) \mathrm{d}\mu &\le C \fint_{B_R} H\left(x, \frac{u - u_{B_R}}{R}\right) \mathrm{d}\mu \quad \text{(we use (3.5))} \\
&\le C_S \left(\fint_{B_{\sigma R}} H(x, g_u)^\theta \mathrm{d}\mu\right)^{\frac{1}{\theta}} \\
&\qquad \text{(by Sobolev-Poincaré inequality (3.14) with } \theta = \frac{p_0}{p}\text{)}.
\end{aligned} \tag{6.2}$$

Now, we consider the test functions

$$g(x) = \begin{cases} H(x, g_u)^\theta & \text{if } x \in \Omega, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f(x) = 0 \quad \text{for all } x \in X.$$

Further, we choose $s_0 \in (1, \frac{1}{\theta})$, $s_1 = \frac{1}{\theta} + 1$ and $s = \frac{1}{\theta}$. Clearly, $g \in L^s_{\text{loc}}(X)$ and $f \in L^{s_1}_{\text{loc}}(X)$; hence, we are in position to apply the Gehring's lemma. It follows that we can find $\varepsilon_0 > 0$ such that $g \in L^t_{\text{loc}}(X)$ for $t \in [s, s + \varepsilon_0)$, and therefore, we have

$$\left(\fint_{B_R} g^t \mathrm{d}\mu\right)^{\frac{1}{t}} \le C\left(\fint_{B_{\sigma R}} g^s \mathrm{d}\mu\right)^{\frac{1}{s}}$$

for $C \equiv C(C_S, p_0, \sigma) > 0$. Let $0 < \delta_0 < \theta\ \varepsilon_0$, then we have

$$\left(\fint_{B_R} H(x, g_u)^{1+\delta_0} \mathrm{d}\mu\right)^{\frac{1}{1+\delta_0}} \le C \fint_{B_{\sigma R}} [H(x, g_u) + 1] \mathrm{d}\mu,$$

which implies that $H(\cdot, g_u)$ has higher integrability in the ball B_R. Now let $\Omega_0 \Subset \Omega$, so that we can cover it with a finite number of balls B_{R_i} $(i = 1, \dots, n)$ such that $B_{\sigma R_i} \Subset \Omega$ for each $i = 1, \dots, n$ and satisfying the assumptions of the first part of the proof. Hence, we have

$$\left(\fint_{\Omega_0} H(x, g_u)^{1+\delta_0} \mathrm{d}\mu\right)^{\frac{1}{1+\delta_0}} \le C \sum_{i=1}^{n} \fint_{B_{\sigma R_i}} [H(x, g_u) + 1] \mathrm{d}\mu < +\infty,$$

for some $C > 0$. This ensures that $H(\cdot, g_u) \in L^{1+\delta_0}_{\text{loc}}(\Omega)$. By appealing to the Hölder inequality, it can be shown that (6.1) remains true whenever δ_0 is replaced by any value $\delta \in (0, \delta_0)$. The result is now established. □

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