Research article

Anisotropic elliptic equations with gradient-dependent lower order terms and L^1 data[†]

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Abstract: We prove the existence of a weak solution for a general class of Dirichlet anisotropic elliptic problems such as $\mathcal{A}u + \Phi(x, u, \nabla u) = \mathfrak{B}u + f$ in Ω, where Ω is a bounded open subset of $\mathbb{R}^N$ and $f \in L^1(\Omega)$ is arbitrary. The principal part is a divergence-form nonlinear anisotropic operator $\mathcal{A}$, the prototype of which is $\mathcal{A}u = -\sum_{j=1}^{N} \partial_j(|\partial_j u|^{p_j-2}\partial_j u)$ with $p_j > 1$ for all $1 \le j \le N$ and $\sum_{j=1}^{N}(1/p_j) > 1$. As a novelty in this paper, our lower order terms involve a new class of operators $\mathfrak{B}$ such that $\mathcal{A} - \mathfrak{B}$ is bounded, coercive and pseudo-monotone from $W_0^{1,\vec{p}}(\Omega)$ into its dual, as well as a gradient-dependent nonlinearity Φ with an "anisotropic natural growth" in the gradient and a good sign condition.

Keywords: nonlinear anisotropic elliptic equations; Leray–Lions operators; pseudo-monotone operators; lower order terms; summable data

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To Professor Neil S. Trudinger, on the occasion of his 80th birthday, with admiration and gratitude.

1. Introduction and main results

1.1. Setting of the problem

In their famous book [34], Gilbarg and Trudinger captured the astonishing achievements in the theory of nonlinear elliptic second order partial differential equations. For recent developments of fully nonlinear equations and their applications to optimal transportation and conformal geometry, see e.g., Trudinger [41, 42].

A quasilinear operator is not always the differential of a functional of the Calculus of Variations. What makes it possible to go further than the Calculus of Variations in the convex case is the abstract concept of monotone operator and, more generally, of pseudo-monotone operator. Several papers [10, 11, 13, 15, 16] deal with nonlinear elliptic problems in a bounded open subset Ω of $\mathbb{R}^N$ involving coercive, bounded, continuous and pseudo-monotone Leray–Lions type operators from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$, where $1 < p < \infty$ and $p' = p/(p-1)$ is the conjugate exponent of p. The prototype model of such an operator is the p-Laplacian $\Delta_p u = \text{div}\,(|\nabla u|^{p-2}\nabla u)$. The techniques developed in these papers accommodate for a lower-order term $g(x, u, \nabla u)$ with a "natural growth" in the gradient $|\nabla u|$ and without any restriction of its growth in $|u|$. Either $f \in L^1(\Omega)$ or $h \in W^{-1,p'}(\Omega)$ could be included because of the "sign-condition" on g (that is, $g(x,t,\xi)\,t \geq 0$ for a.e. $x \in \Omega$ and all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$). For related works, we refer to [3, 4, 12, 29, 30, 35].

In this paper, we expand the above research program into the anisotropic arena by providing a suitable general framework under which for *every* $f \in L^1(\Omega)$ we prove the existence of a weak solution to Dirichlet anisotropic elliptic problems such as

$$\begin{cases} \mathcal{A}u + \Phi(x, u, \nabla u) + \Theta(x, u, \nabla u) = \mathfrak{B}u + f \quad \text{in } \Omega, \\ u \in W_0^{1,\vec{p}}(\Omega), \quad \Phi(x, u, \nabla u) \in L^1(\Omega). \end{cases} \tag{1.1}$$

Here, and henceforth, Ω is a bounded, open subset of $\mathbb{R}^N$ ($N \geq 2$). We impose no smoothness condition on $\partial\Omega$. Without loss of generality, we assume throughout that

$$1 < p_j \leq p_{j+1} < \infty \text{ for every } 1 \leq j \leq N-1 \quad \text{and} \quad p < N, \tag{1.2}$$

where $p = N/\sum_{j=1}^N (1/p_j)$ is the harmonic mean of $p_1, \ldots, p_N$. Let $\nabla u = (\partial_1 u, \ldots, \partial_N u)$ be the gradient of u. Let $W_0^{1,\vec{p}}(\Omega)$ be the closure of $C_c^\infty(\Omega)$ (the set of smooth functions with compact support in Ω) with respect to the norm $\|u\|_{W_0^{1,\vec{p}}(\Omega)} = \sum_{j=1}^N \|\partial_j u\|_{L^{p_j}(\Omega)}$. We use $W^{-1,\vec{p}'}(\Omega)$ to denote the dual of $W_0^{1,\vec{p}}(\Omega)$ and $\langle \cdot, \cdot \rangle$ for the duality between $W^{-1,\vec{p}'}(\Omega)$ and $W_0^{1,\vec{p}}(\Omega)$. The prototype for $\mathcal{A}$ is the anisotropic $\vec{p}$-Laplacian, namely,

$$\mathcal{A}u = -\sum_{j=1}^N \partial_j(|\partial_j u|^{p_j-2}\partial_j u), \tag{1.3}$$

(see (1.7) and (1.8)). The model for Φ in (1.1) is as follows

$$\Phi(u, \nabla u) = \left(\sum_{j=1}^N |\partial_j u|^{p_j} + 1\right)|u|^{m-2}u + \sum_{j=1}^N \mathfrak{b}_j |\partial_j u|^{q_j} |u|^{\theta_j-2}u, \tag{1.4}$$

where $\mathfrak{b}_j \geq 0$ and $0 \leq q_j < p_j$, while $m, \theta_j > 1$ for all $1 \leq j \leq N$ are arbitrary (see (1.9) and (1.10)). We assume throughout that $\Theta(x,t,\xi), \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function (that is, measurable on Ω for every $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and continuous in t, ξ for a.e. $x \in \Omega$) and there exists a constant $C_\Theta > 0$ such that

$$|\Theta(x,t,\xi)| \leq C_\Theta \quad \text{for a.e. } x \in \Omega \text{ and for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N. \tag{1.5}$$

Furthermore, our problem (1.1) features a new class of operators $\mathfrak{B}$ as follows.

Definition 1.1. Let (1.7) and (1.8) hold. By $\mathfrak{BC}$ we denote the class of all *bounded* operators $\mathfrak{B}$ from $W_0^{1,\vec{p}}(\Omega)$ into $W^{-1,\vec{p}'}(\Omega)$ satisfying the following two properties:

(P_1) The operator $\mathcal{A} - \mathfrak{B}$ from $W_0^{1,\vec{p}}(\Omega)$ into $W^{-1,\vec{p}'}(\Omega)$ is *coercive* (see Definition 2.2).

(P_2) If $u_\ell \rightharpoonup u$ and $v_\ell \rightharpoonup v$ (weakly) in $W_0^{1,\vec{p}}(\Omega)$ as $\ell \to \infty$, then

$$\lim_{\ell \to \infty} \langle \mathfrak{B}u_\ell, v_\ell \rangle = \langle \mathfrak{B}u, v \rangle.$$

Let $\mathfrak{BC}_+$ be the class of operators in $\mathfrak{BC}$ satisfying the extra condition

(P_3) *For $\nu_0 > 0$ in the coercivity condition of (1.8) and each $k > 0$, it holds*

$$\nu_0 \sum_{j=1}^N \|\partial_j u\|_{L^{p_j}(\Omega)}^{p_j} - \langle \mathfrak{B}u, T_k u \rangle \to \infty \text{ as } \|u\|_{W_0^{1,\vec{p}}(\Omega)} \to \infty. \tag{1.6}$$

We use T_k for the truncation at height k, see (1.16). Unlike $\mathcal{A}$, the operator $-\mathfrak{B}$ is not coercive in general. Our assumption (P_2) is reminiscent of (iii) in the Hypothesis (II) of Theorem 1 in the celebrated paper [37] by Leray and Lions. Every operator satisfying (P_2) is strongly continuous (see Lemma 2.6) and pseudo-monotone (cf. [44, p. 586]).

In Example 1.2 we use that $p^* = Np/(N-p)$ is the critical exponent for the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^r(\Omega)$ (see Remark A.2 in the Appendix). For any $r > 1$, let $r' = r/(r-1)$.

Example 1.2. Let $F \in L^{(p^*)'}(\Omega)$ and $h, \widetilde{h} \in W^{-1,\vec{p}'}(\Omega)$ be arbitrary. Let $\rho, \alpha_k \in \mathbb{R}$ for $0 \leq k \leq 4$. For every $u \in W_0^{1,\vec{p}}(\Omega)$, we define

(1) $\mathfrak{B}u = h$;

(2) $\mathfrak{B}u = F + \rho |u|^{\vartheta - 2}u$ with $1 < \vartheta < p$ if $\rho > 0$ and $1 < \vartheta < p^*$ if $\rho < 0$;

(3) $\mathfrak{B}u = \left(\alpha_0 + \alpha_1\|u\|_{L^r(\Omega)}^{\mathfrak{b}_1} + \alpha_2|\langle \widetilde{h}, u \rangle|^{\mathfrak{b}_2}\right)(\alpha_3 h + \alpha_4 F)$, where $r \in [1, p^*)$; we take $\mathfrak{b}_1 \in (0, p/p_1')$ and $\mathfrak{b}_2 \in (0, p_1 - 1)$ if $\alpha_3 \neq 0$; $\mathfrak{b}_1 \in (0, p - 1)$ and $\mathfrak{b}_2 \in (0, p_1/p')$ if $\alpha_3 = 0$;

(4) $\mathfrak{B}u = -\sum_{j=1}^N \partial_j \left(\beta_j(x) + |u|^{\sigma_j - 1}u\right)$, where $\beta_j \in L^{p_j'}(\Omega)$ and $0 < \sigma_j < p/p_j'$ for every $1 \leq j \leq N$.

In each of these situations, $\mathfrak{B}$ belongs to the class $\mathfrak{BC}_+$.

1.2. Our assumptions

Let (1.2) and (1.5) hold. The anisotropic $\vec{p}$-Laplacian in (1.3) is the prototype for a coercive, bounded, continuous and pseudo-monotone operator $\mathcal{A} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ in divergence form $\mathcal{A}u = -\sum_{j=1}^N \partial_j(A_j(x, u, \nabla u))$, that is,

$$\langle \mathcal{A}u, v \rangle = \sum_{j=1}^N \int_\Omega A_j(x, u, \nabla u) \, \partial_j v \, dx \quad \text{for every } u, v \in W_0^{1,\vec{p}}(\Omega). \tag{1.7}$$

• For each $1 \leq j \leq N$, let $A_j(x, t, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function and assume that there exist constants $\nu_0, \nu > 0$ and a nonnegative function $\eta_j \in L^{p_j'}(\Omega)$ such that for a.e. $x \in \Omega$, for all

$(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and every $\widehat{\xi} \in \mathbb{R}^N$, we have

$$
\left.
\begin{aligned}
& \sum_{i=1}^{N} A_i(x, t, \xi) \xi_i \geq \nu_0 \sum_{i=1}^{N} |\xi_i|^{p_i} \qquad\qquad\qquad \text{[coercivity]}, \\
& \sum_{i=1}^{N} \left(A_i(x, t, \xi) - A_i(x, t, \widehat{\xi}) \right) \left(\xi_i - \widehat{\xi}_i \right) > 0 \quad \text{if } \xi \neq \widehat{\xi} \quad \text{[monotonicity]}, \\
& |A_j(x, t, \xi)| \leq \nu \left[\eta_j(x) + |t|^{p^*/p_j'} + \left(\sum_{i=1}^{N} |\xi_i|^{p_i} \right)^{1/p_j'} \right] \qquad \text{[growth condition]}.
\end{aligned}
\right\}
\tag{1.8}
$$

We note that in the growth condition in (1.8), we take the greatest exponent for $|t|$ from the viewpoint of the anisotropic Sobolev inequalities. This requires modifying the standard proof of pseudo-monotonicity of $\mathcal{A}$ (see Lemma 2.7).

• Suppose that $\Phi(x, t, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function and there exist a nonnegative function $c \in L^1(\Omega)$ and a continuous nondecreasing function $\phi : \mathbb{R} \to \mathbb{R}^+$ such that for a.e. $x \in \Omega$ and for all $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$,

$$
\Phi(x, t, \xi) t \geq 0 \quad \text{[sign-condition]}, \quad |\Phi(x, t, \xi)| \leq \phi(|t|) \left(\sum_{j=1}^{N} |\xi_j|^{p_j} + c(x) \right).
\tag{1.9}
$$

For Theorem 1.3 (*ii*), we further assume that there exist constants $\tau, \gamma > 0$ such that

$$
|\Phi(x, t, \xi)| \geq \gamma \sum_{j=1}^{N} |\xi_j|^{p_j} \quad \text{for all } |t| \geq \tau, \text{ a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^N.
\tag{1.10}
$$

1.3. Main results

By a *solution* of (1.1) we mean any function $u \in W_0^{1,\vec{p}}(\Omega)$ such that $\Phi(x, u, \nabla u) \in L^1(\Omega)$ and, for every $v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$,

$$
\langle \mathcal{A}u, v \rangle + \int_\Omega \Phi(x, u, \nabla u) v \, dx + \int_\Omega \Theta(x, u, \nabla u) v \, dx = \langle \mathcal{B}u, v \rangle + \int_\Omega f v \, dx.
\tag{1.11}
$$

Under the assumptions in Section 1.2, the main advance in this paper is the following.

Theorem 1.3. *Let* (1.2)*,* (1.5)*,* (1.8) *and* (1.9) *hold.*

  (*i*) *If* $f = 0$ *in* (1.1)*, then* (1.1) *has a solution* U *for every* $\mathcal{B}$ *in the class* $\mathcal{BC}$*. Moreover,* $\Phi(x, U, \nabla U) U \in L^1(\Omega)$ *and* (1.11) *holds with* $u = v = U$*.*
  (*ii*) *If* (1.10) *is satisfied, then* (1.1) *has at least a solution for every* $f \in L^1(\Omega)$ *and* $\mathcal{B}$ *in the class* $\mathcal{BC}_+$*.*

Theorem 1.3 is new even when transposed to isotropic Leray–Lions type operators $\mathcal{A}$ from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$. This is due to the introduction of $\mathcal{B}$ in (1.1), which adds extra difficulties. Were a solution of (1.1) to exist in $W_0^{1,\vec{p}}(\Omega)$, then we would expect it to be *unbounded*. This was observed by Bensoussan, Boccardo and Murat [11] for isotropic nonlinear elliptic equations involving $h \in W^{-1,p'}(\Omega)$

and Leray–Lions type operators of the p-Laplacian type. Hence, the regularizing effect that otherwise Φ would bring to the solutions in $W_0^{1,\vec{p}}(\Omega)$ is countered by the presence of $\mathfrak{B}$ in our class $\mathfrak{BC}$.

We stress that, without the term Φ, one cannot expect to find solutions of (1.1) in $W_0^{1,\vec{p}}(\Omega)$ for every $f \in L^1(\Omega)$. In the isotropic case, this observation was made by Boccardo and Gallouët [16]. We leverage the gradient-dependent lower-order term $\Phi(x, u, \nabla u)$ with an "anisotropic natural growth" in the gradient and a *good* sign condition (see (1.9) and (1.10)) to prove the *a priori* estimates contained in Lemmata 3.1 and 4.1, and hence obtain the existence of solutions of (1.1) in $W_0^{1,\vec{p}}(\Omega)$.

For Theorem 1.3 (*ii*) we encounter two obstacles: a low summability for f and, on the other hand, the unrestricted growth of Φ with respect to $|u|$. Previously mentioned works in the isotropic case provide ways to surmount one problem at a time. The function $f \in L^1(\Omega)$ can surely be approximated by $L^\infty(\Omega)$-functions f_ε in the sense that $|f_\varepsilon| \leq |f|$ and $f_\varepsilon \to f$ a.e. in Ω as $\varepsilon \to 0$. Also Φ could be replaced by a "nice" function Φ_ε, preserving the properties of Φ, but gaining boundedness, see (3.1). However, as it was pointed out by Bensoussan and Boccardo [10] in the isotropic case, one cannot deal with both approximations for f and Φ *simultaneously*. For the approximate problems involving both Φ_ε and f_ε, we would not be able to obtain that the solutions u_ε are uniformly bounded in $W_0^{1,\vec{p}}(\Omega)$ with respect to ε. For the above reason, we need to consider $f = 0$ first and prove Theorem 1.3 (*i*), which is a crucial step in establishing the second assertion of Theorem 1.3, but at the same time of independent interest.

The techniques and results we obtain here provide the means to address other types of lower order terms than $f \in L^1(\Omega)$, yet maintaining the class $\mathfrak{BC}$ of operators $\mathfrak{B}$. We briefly mention possible developments. It is natural to ask what happens when $\mathfrak{b}_j$ in (1.4) is negative for $1 \leq j \leq N$. Then, the sign-condition on Φ in (1.9) breaks down. Since we impose no restriction on the growth of Φ with respect to $|u|$, the current paper lays the foundation for dealing with potentially singular lower order terms with no sign restriction. The approximation of such terms is afforded by our inclusion in (1.1) of the term Θ. The approximate problems become of the type (1.1) for which we gain existence of solutions via our Theorem 1.3 (*i*). It is essential that we can take the solution itself as a test function. This fact can be exploited to obtain *a priori* estimates for the solutions and pass to the limit. Such an analysis goes beyond the scope of this paper and will be carried out elsewhere (see [19]).

Our work is also motivated by the various applications of anisotropic elliptic and parabolic partial differential equations to the mathematical modelling of physical and mechanical processes. Such equations provide, for instance, the mathematical models for the dynamics of fluids in anisotropic media when the conductivities of the media are distinct in different directions. The term $\mathfrak{B}$ allows us to model a large class of situations, including the ones when the diffusion/absorption balance is subject to suitable conditions (see [6, Chapter 1]). Anisotropic equations also appear in biology as a model for the propagation of epidemic diseases in heterogeneous domains [9].

With a rapidly growing literature on anisotropic problems, several questions have been resolved on the existence, uniqueness and regularity of weak solutions (see, for instance, [1, 2, 5, 7, 8, 14, 22–25, 28, 31–33, 38]). Many difficulties arise in passing from the isotropic setting to the anisotropic one since some fundamental tools available for the former (such as the strong maximum principle, see [43]) cannot be extended to the latter.

We end this brief overview by observing that, in the case of variable exponents, that is when $p_j = p_j(x)$, there are many applications to electrorheological fluids, thermorheological fluids, elastic materials, and image restoration (see, for example, [17, 18, 26, 27] and the references therein).

1.4. Sketch of the main ideas in the proof of Theorem 1.3

We remark that because of Φ, even when $f = 0$, we cannot directly apply the theory of pseudo-monotone operators to prove the existence claim in Theorem 1.3 (i). To overcome this difficulty, we consider the approximate problem

$$
\begin{cases}
\mathcal{A}u_\varepsilon + \Phi_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) + \Theta(x, u_\varepsilon, \nabla u_\varepsilon) = \mathcal{B}u_\varepsilon & \text{in } \Omega, \\
u_\varepsilon \in W_0^{1,\vec{p}}(\Omega)
\end{cases}
\tag{1.12}
$$

for which we obtain the existence of a solution u_ε as a consequence of our Theorem 2.1 in Section 2. Indeed, $\Phi_\varepsilon + \Theta$ satisfies the same type of assumption as Θ in (1.5), that is, there exists a constant $C_\varepsilon > 0$ such that $|(\Phi_\varepsilon + \Theta)(x, t, \xi)| \leq C_\varepsilon$ for a.e. $x \in \Omega$ and all $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$. Thus, by Theorem 2.1, for every $\varepsilon > 0$, the approximate problem (1.12) has a solution $u_\varepsilon \in W_0^{1,\vec{p}}(\Omega)$. In Lemma 3.1 we prove *a priori* estimates in $W_0^{1,\vec{p}}(\Omega)$ for the solutions u_ε, which (up to a subsequence) converge weakly to some U in $W_0^{1,\vec{p}}(\Omega)$ and a.e. in Ω as $\varepsilon \to 0$.

We point out that in Section 5, we will be able to show that, up to a subsequence,

$$
u_\varepsilon \to U \text{ (strongly) in } W_0^{1,\vec{p}}(\Omega) \text{ as } \varepsilon \to 0.
\tag{1.13}
$$

We achieve this by combining and extending techniques from the isotropic case in [10] and [13] to establish in Lemma 3.2 that, up to a subsequence of u_ε, we have

$$
\nabla u_\varepsilon \to \nabla U \text{ a.e. in } \Omega \text{ and } T_k(u_\varepsilon) \to T_k(U) \text{ (strongly) in } W_0^{1,\vec{p}}(\Omega) \text{ as } \varepsilon \to 0
\tag{1.14}
$$

for every integer $k \geq 1$, where $T_k(\cdot)$ is given in (1.16). Then, we can pass to the limit as $\varepsilon \to 0$ in the weak formulation of the solution u_ε and obtain that U is a solution of (1.1) with $f = 0$ (see Subsection 3.2).

Generally speaking, the proof of Theorem 1.3 (ii), which we give in Section 4, follows a similar course with that of Theorem 1.3 (i) in Section 3. But there are some modifications that we outline below. We approximate $f \in L^1(\Omega)$ by $L^\infty(\Omega)$-functions f_ε and we apply Theorem 1.3 (i) to obtain a solution U_ε for the problem

$$
\begin{cases}
\mathcal{A}U_\varepsilon + \Phi(x, U_\varepsilon, \nabla U_\varepsilon) + \Theta(x, U_\varepsilon, \nabla U_\varepsilon) = \mathcal{B}U_\varepsilon + f_\varepsilon & \text{in } \Omega, \\
U_\varepsilon \in W_0^{1,\vec{p}}(\Omega), \quad \Phi(x, U_\varepsilon, \nabla U_\varepsilon) \in L^1(\Omega).
\end{cases}
\tag{1.15}
$$

We emphasize that unlike in (1.12), we have Φ (and not Φ_ε) in (1.15). Because of this reason, coupled with the introduction of f_ε, we need the extra assumption (1.10) and to choose $\mathcal{B}$ in the class $\mathfrak{BC}_+$ to obtain that $\{U_\varepsilon\}_\varepsilon$ is uniformly bounded in $W_0^{1,\vec{p}}(\Omega)$ with respect to ε (see Lemma 4.1 for details). Then, extracting a subsequence, U_ε tends to some U_0 weakly in $W_0^{1,\vec{p}}(\Omega)$ and a.e. in Ω. With an almost identical argument, we gain the counterpart of (1.14), namely, up to a subsequence, $\nabla U_\varepsilon \to \nabla U_0$ a.e. in Ω and $T_k(U_\varepsilon) \to T_k(U_0)$ (strongly) in $W_0^{1,\vec{p}}(\Omega)$ as $\varepsilon \to 0$ for every integer $k \geq 1$. To conclude the proof of Theorem 1.3 (ii), it remains to pass to the limit in the weak formulation of U_ε. The change appearing here compared with the corresponding argument in Subsection 3.2 is the strong convergence of $\Phi(x, U_\varepsilon, \nabla U_\varepsilon)$ to $\Phi(x, U_0, \nabla U_0)$ in $L^1(\Omega)$. For the latter, we adapt an argument from [13]. For details, we refer to Lemma 4.3 in Subsection 4.3.

Structure of this paper. In Section 2 we prove an existence result (Theorem 2.1), which gives the existence of a solution u_ε of (1.12) for every $\varepsilon > 0$. We dedicate Sections 3 and 4 to the proof of Theorem 1.3 (*i*) and Theorem 1.3 (*ii*), respectively. In Section 5 we make further comments on Theorem 1.3 (*i*) by proving the strong convergence in (1.13). In the Appendix we include some facts used in the paper and, for completeness, prove the anisotropic counterparts of well-known isotropic convergence results, see Lemmata A.4 and A.5. These will be used in the proof of Lemmata 2.7 and 3.2, respectively.

Notation. For $k > 0$, we let $T_k : \mathbb{R} \to \mathbb{R}$ stand for the truncation at height k, that is,

$$T_k(s) = s \quad \text{if } |s| \le k, \quad T_k(s) = k\,\frac{s}{|s|} \quad \text{if } |s| > k. \tag{1.16}$$

Moreover, we define $G_k : \mathbb{R} \to \mathbb{R}$ by

$$G_k(s) = s - T_k(s) \quad \text{for every } s \in \mathbb{R}. \tag{1.17}$$

In particular, we have $G_k = 0$ on $[-k, k]$ and $t\,G_k(t) \ge 0$ for every $t \in \mathbb{R}$.
For every $u \in W_0^{1,\vec{p}}(\Omega)$ and for a.e. $x \in \Omega$, we define

$$\widehat{\Phi}(u)(x), = \Phi(x, u(x), \nabla u(x)), \quad \widehat{\Theta}(u)(x), = \Theta(x, u(x), \nabla u(x)),$$
$$\widehat{A_j}(u)(x) = A_j(x, u(x), \nabla u(x)) \quad \text{for every } 1 \le j \le N.$$

We set $\vec{p} = (p_1, p_2, \dots, p_N)$ and $\vec{p}\,' = (p_1', p_2', \dots, p_N')$.
As usual, χ_ω stands for the characteristic function of a set $\omega \subset \mathbb{R}^N$.

2. An existence result

Throughout this section, we assume (1.2), (1.5), and (1.8), besides $\mathfrak{B}$ belonging to the class $\mathfrak{BC}$. Here, our aim is to prove the existence of a solution to the following problem

$$\begin{cases} \mathcal{A}u + \Theta(x, u, \nabla u) = \mathfrak{B}u & \text{in } \Omega, \\ u \in W_0^{1,\vec{p}}(\Omega). \end{cases} \tag{2.1}$$

By a solution of (2.1), we mean a function $u \in W_0^{1,\vec{p}}(\Omega)$ such that

$$\langle \mathcal{A}u, v \rangle + \int_\Omega \Theta(x, u, \nabla u)\, v\, dx - \langle \mathfrak{B}u, v \rangle = 0 \quad \text{for every } v \in W_0^{1,\vec{p}}(\Omega). \tag{2.2}$$

Theorem 2.1. *Problem* (2.1) *admits at least a solution.*

We establish Theorem 2.1 via the theory of pseudo-monotone operators. Before giving the proof of Theorem 2.1 in Subsection 2.1, we recall a few concepts that we need in the sequel (see, for example, [20] and [44, p. 586]).

Definition 2.2. An operator $\mathcal{P} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is called

(a_1) *monotone (strictly monotone)* if $\langle \mathcal{P}u - \mathcal{P}v, u - v \rangle \geq 0$ for every $u, v \in W_0^{1,\vec{p}}(\Omega)$ (with equality if and only if $u = v$);

(a_2) *pseudo-monotone* if whenever $u_\ell \rightharpoonup u$ (weakly) in $W_0^{1,\vec{p}}(\Omega)$ as $\ell \to \infty$ and $\limsup_{\ell\to\infty}\langle \mathcal{P}u_\ell, u_\ell - u \rangle \leq 0$, we get that $\langle \mathcal{P}u, u - w \rangle \leq \liminf_{\ell\to\infty}\langle \mathcal{P}u_\ell, u_\ell - w \rangle$ for all $w \in W_0^{1,\vec{p}}(\Omega)$;

(a_3) *strongly continuous** if $u_\ell \rightharpoonup u$ (weakly) in $W_0^{1,\vec{p}}(\Omega)$ as $\ell \to \infty$ implies that $\mathcal{P}u_\ell \to \mathcal{P}u$ in $W^{-1,\vec{p}'}(\Omega)$ as $\ell \to \infty$;

(a_4) *coercive* if $\langle \mathcal{P}u, u \rangle / \|u\|_{W_0^{1,\vec{p}}(\Omega)} \to \infty$ as $\|u\|_{W_0^{1,\vec{p}}(\Omega)} \to \infty$;

(a_5) *of M type†* if $u_\ell \rightharpoonup u$ (weakly) in $W_0^{1,\vec{p}}(\Omega)$ as $\ell \to \infty$, together with $\mathcal{P}u_\ell \rightharpoonup g$ (weakly) in $W^{-1,\vec{p}'}(\Omega)$ as $\ell \to \infty$ and $\limsup_{\ell\to\infty}\langle \mathcal{P}u_\ell, u_\ell \rangle \leq \langle g, u \rangle$, imply that $g = \mathcal{P}u$ and $\langle \mathcal{P}u_\ell, u_\ell \rangle \to \langle g, u \rangle$ as $\ell \to \infty$.

Proposition 2.3. *Every strongly continuous operator $\mathcal{P} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is pseudo-monotone. Every bounded operator $\mathcal{P} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ of M type is pseudo-monotone. The sum of two pseudo-monotone operators is pseudo-monotone.*

2.1. Proof of Theorem 2.1

We immediately observe from (1.5) that the operator $\mathcal{P}_\Theta : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is bounded, where we define

$$\langle \mathcal{P}_\Theta(u), v \rangle := \int_\Omega \widehat{\Theta}(u)\, v\, dx \quad \text{for every } u, v \in W_0^{1,\vec{p}}(\Omega). \tag{2.3}$$

In view of (2.2), the existence of a solution to (2.1) follows whenever the operator $\mathcal{A} + \mathcal{P}_\Theta - \mathfrak{B} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is surjective. Since $W_0^{1,\vec{p}}(\Omega)$ is a real, reflexive, and separable Banach space, it is known that $\mathcal{A} + \mathcal{P}_\Theta - \mathfrak{B} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is surjective whenever it is bounded, coercive and pseudo-monotone (see, for instance, [44, p. 589]). In Lemma 2.5, we establish the boundedness and coercivity of $\mathcal{A} + \mathcal{P}_\Theta - \mathfrak{B}$, whereas its pseudo-monotonicity is concluded in Corollary 2.8.

For the reader's convenience and to make our presentation self-contained, we give all the details about the pseudo-monotonicity of $\mathcal{A} + \mathcal{P}_\Theta - \mathfrak{B} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$. These computations could be of interest also in the corresponding isotropic case, when, to our best knowledge, only very special instances of $\mathfrak{B}$ have been considered and the details are usually scattered in the literature.

The property (P_2) ensures that $\pm\mathfrak{B} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is strongly continuous (see Lemma 2.6) and, hence, pseudo-monotone by Proposition 2.3. As the sum of two pseudo-monotone operators is pseudo-monotone, to prove that $\mathcal{A} + \mathcal{P}_\Theta - \mathfrak{B} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is pseudo-monotone, it suffices to show that $\mathcal{A} + \mathcal{P}_\Theta : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is pseudo-monotone. The proof of the latter is more involved, see Lemma 2.7. In view of Proposition 2.3 and Lemma 2.4, it is enough to show that $\mathcal{A} + \mathcal{P}_\Theta$ is an operator of M type. We proceed with the details.

Lemma 2.4. *The operator $\mathcal{A} + \mathcal{P}_\Theta : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is bounded, coercive and continuous.*

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* Strongly continuous operators are also referred to as *completely continuous* (see, for instance, Showalter [39, p. 36]).
† Some authors (see, for example, Le Dret [36, p. 232]) use the terminology *sense 1 pseudomonotone* instead of M type.

Proof. The boundedness of the operator $\mathcal{A}+\mathcal{P}_\Theta : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is a consequence of the growth condition of A_j in (1.8), coupled with (1.5). The coercivity of $\mathcal{A}+\mathcal{P}_\Theta$ follows readily from (1.5) and the coercivity assumption in (1.8). Moreover, by Hölder's inequality and the continuity of the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, we find a positive constant C such that, for every $u_1, u_2 \in W_0^{1,\vec{p}}(\Omega)$,

$$\|(\mathcal{A} + \mathcal{P}_\Theta)(u_1) - (\mathcal{A} + \mathcal{P}_\Theta)(u_2)\|_{W^{-1,\vec{p}'}(\Omega)}$$

$$\leq \sup_{\substack{v\in W_0^{1,\vec{p}}(\Omega), \\ \|v\|_{W_0^{1,\vec{p}}(\Omega)}\leq 1}} \Big(\sum_{j=1}^N \int_\Omega |\widehat{A}_j(u_1) - \widehat{A}_j(u_2)||\partial_j v|\, dx + \int_\Omega |\widehat{\Theta}(u_1) - \widehat{\Theta}(u_2)||v|\, dx \Big)$$

$$\leq \sum_{j=1}^N \|\widehat{A}_j(u_1) - \widehat{A}_j(u_2)\|_{L^{p'_j}(\Omega)} + C\, \|\widehat{\Theta}(u_1) - \widehat{\Theta}(u_2)\|_{L^{(p^*)'}(\Omega)}.$$

We get the continuity of $\mathcal{A} + \mathcal{P}_\Theta : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ by showing the following.

Claim: The mappings $\widehat{\Theta} : W_0^{1,\vec{p}}(\Omega) \to L^{(p^*)'}(\Omega)$ and $\widehat{A}_j : W_0^{1,\vec{p}}(\Omega) \to L^{p'_j}(\Omega)$ are continuous for each $1 \leq j \leq N$.

Proof of the Claim. Let $1 \leq j \leq N$ be arbitrary. By the growth condition of A_j in (1.8), there exist a constant $C > 0$ and a nonnegative function $\eta_j \in L^{p'_j}(\Omega)$ such that

$$|\widehat{A}_j(u)|^{p'_j} \leq C\left(\eta_j^{p'_j} + |u|^{p^*} + \sum_{i=1}^N |\partial_i u|^{p_i} \right) \in L^1(\Omega) \tag{2.4}$$

for all $u \in W_0^{1,\vec{p}}(\Omega)$. Since the embeddings $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and $L^\infty(\Omega) \hookrightarrow L^{(p^*)'}(\Omega)$ are continuous, from (2.4) and (1.5), we infer that $\widehat{A}_j : W_0^{1,\vec{p}}(\Omega) \to L^{p'_j}(\Omega)$ and $\widehat{\Theta} : W_0^{1,\vec{p}}(\Omega) \to L^{(p^*)'}(\Omega)$ are well-defined. To prove the continuity of these mappings, we let $u_n \to u$ (strongly) in $W_0^{1,\vec{p}}(\Omega)$ as $n \to \infty$. Hence, $u_n \to u$ (strongly) in $L^{p^*}(\Omega)$ and $\partial_i u_n \to \partial_i u$ (strongly) in $L^{p_i}(\Omega)$ as $n \to \infty$ for every $1 \leq i \leq N$. Now, using (2.4) with u_n instead of u, we obtain that $\{|\widehat{A}_j(u_n)|^{p'_j}\}_{n\geq 1}$ is uniformly integrable over Ω. By passing to a subsequence $\{u_{n_k}\}_{k\geq 1}$ of $\{u_n\}$, we have $u_{n_k} \to u$ and $\nabla u_{n_k} \to \nabla u$ a.e. in Ω as $k \to \infty$. Since A_j and Θ are Carathéodory functions, we have $\widehat{\Theta}(u_{n_k}) \to \widehat{\Theta}(u)$ and $\widehat{A}_j(u_{n_k}) \to \widehat{A}_j(u)$ a.e. in Ω as $k \to \infty$. Then, by (1.5) and the Dominated Convergence Theorem, $\widehat{\Theta}(u_{n_k}) \to \widehat{\Theta}(u)$ in $L^{(p^*)'}(\Omega)$. By Vitali's Theorem, we see that $\widehat{A}_j(u_{n_k}) \to \widehat{A}_j(u)$ in $L^{p'_j}(\Omega)$ as $k \to \infty$. Since the limits $\widehat{\Theta}(u)$ and $\widehat{A}_j(u)$ are independent of the subsequence $\{u_{n_k}\}_{k\geq 1}$, we conclude that $\widehat{\Theta}(u_n) \to \widehat{\Theta}(u)$ in $L^{(p^*)'}(\Omega)$ and $\widehat{A}_j(u_n) \to \widehat{A}_j(u)$ in $L^{p'_j}(\Omega)$ as $n \to \infty$.

This completes the proof of the Claim and of Lemma 2.4. □

Lemma 2.5. *The operator* $\mathcal{A} + \mathcal{P}_\Theta - \mathfrak{B} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ *is bounded and coercive.*

Proof. Using Lemma 2.4 and Definition 1.1, we find that $\mathcal{A} + \mathcal{P}_\Theta - \mathfrak{B}$ is a bounded operator from $W_0^{1,\vec{p}}(\Omega)$ into $W^{-1,\vec{p}'}(\Omega)$. We now show that it is also coercive, namely,

$$\frac{\langle \mathcal{A}u + \mathcal{P}_\Theta(u) - \mathfrak{B}u, u \rangle}{\|u\|_{W_0^{1,\vec{p}}(\Omega)}} \to \infty \quad \text{as } \|u\|_{W_0^{1,\vec{p}}(\Omega)} \to \infty. \tag{2.5}$$

Using (1.5) and the continuity of the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^1(\Omega)$, we find a constant $C > 0$ such that $\langle \mathcal{P}_\Theta(u), u \rangle \geq -C\|u\|_{W_0^{1,\vec{p}}(\Omega)}$ for every $u \in W_0^{1,\vec{p}}(\Omega)$. Then, by the coercivity property of $\mathcal{A} - \mathcal{B}$, we readily conclude (2.5). $\qquad\square$

Lemma 2.6. *Every operator $\mathcal{B} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ satisfying (P_2) is strongly continuous.*

Proof. Let $u_\ell \rightharpoonup u$ (weakly) in $W_0^{1,\vec{p}}(\Omega)$ as $\ell \to \infty$. We show that $\mathcal{B}u_\ell \to \mathcal{B}u$ in $W^{-1,\vec{p}'}(\Omega)$ as $\ell \to \infty$. Assume by contradiction that there exist $\varepsilon_0 > 0$ and a subsequence of $\{u_\ell\}$ (relabeled $\{u_\ell\}$) such that

$$\sup_{\substack{v \in W_0^{1,\vec{p}}(\Omega), \\ \|v\|_{W_0^{1,\vec{p}}(\Omega)} \leq 1}} |\langle \mathcal{B}u_\ell - \mathcal{B}u, v \rangle| > \varepsilon_0 \quad \text{for every } \ell \geq 1.$$

Hence, there also exists $\{v_\ell\}$ in $W_0^{1,\vec{p}}(\Omega)$ with $\|v_\ell\|_{W_0^{1,\vec{p}}(\Omega)} \leq 1$ such that

$$|\langle \mathcal{B}u_\ell - \mathcal{B}u, v_\ell \rangle| > \varepsilon_0 \quad \text{for all } \ell \geq 1. \tag{2.6}$$

By the boundedness of $\{v_\ell\}$ in $W_0^{1,\vec{p}}(\Omega)$, up to a subsequence, $v_\ell \rightharpoonup v$ (weakly) in $W_0^{1,\vec{p}}(\Omega)$ as $\ell \to \infty$. Since $\mathcal{B}u \in W^{-1,\vec{p}'}(\Omega)$, we have $\langle \mathcal{B}u, v_\ell \rangle \to \langle \mathcal{B}u, v \rangle$ as $\ell \to \infty$. Hence, from (P_2) we find that $|\langle \mathcal{B}u_\ell - \mathcal{B}u, v_\ell \rangle| \to 0$ as $\ell \to \infty$, which is in contradiction with (2.6). Thus, $\mathcal{B}$ is strongly continuous, completing the proof. $\qquad\square$

Lemma 2.7. *The operator $\mathcal{A} + \mathcal{P}_\Theta : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is pseudo-monotone.*

Proof. Since the operator $\mathcal{A} + \mathcal{P}_\Theta$ is bounded, it is enough to show that it is of M type (see Proposition 2.3). To this end, suppose that there exist $u, \{u_\ell\}_{\ell \geq 1}$ in $W_0^{1,\vec{p}}(\Omega)$ and $g \in W^{-1,\vec{p}'}(\Omega)$ such that

$$u_\ell \rightharpoonup u \text{ (weakly) in } W_0^{1,\vec{p}}(\Omega) \text{ as } \ell \to \infty, \tag{2.7}$$

$$(\mathcal{A} + \mathcal{P}_\Theta)(u_\ell) \rightharpoonup g \text{ (weakly) in } W^{-1,\vec{p}'}(\Omega) \text{ as } \ell \to \infty, \tag{2.8}$$

$$\limsup_{\ell \to \infty} \langle (\mathcal{A} + \mathcal{P}_\Theta)(u_\ell), u_\ell \rangle \leq \langle g, u \rangle. \tag{2.9}$$

We prove that

$$g = (\mathcal{A} + \mathcal{P}_\Theta)(u), \tag{2.10}$$

$$\langle (\mathcal{A} + \mathcal{P}_\Theta)(u_\ell), u_\ell \rangle \to \langle g, u \rangle \quad \text{as } \ell \to \infty. \tag{2.11}$$

We first show that (2.11) holds. From (2.7) and the compactness of the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^p(\Omega)$ (see Remark A.2), we obtain that, up to a subsequence,

$$u_\ell \to u \text{ strongly in } L^p(\Omega) \text{ and a.e. in } \Omega. \tag{2.12}$$

Moreover, using (2.4) with u replaced by u_ℓ, we get that $\widehat{A_j}(u_\ell)$ is bounded in $L^{p'_j}(\Omega)$ for every $1 \leq j \leq N$. Hence, in view of (1.5), there exist $\mu \in L^{p'}(\Omega)$ and $g_j \in L^{p'_j}(\Omega)$ for $1 \leq j \leq N$ so that, up to a further subsequence of $\{u_\ell\}$ (denoted by $\{u_\ell\}$), we have

$$\widehat{\Theta}(u_\ell) \rightharpoonup \mu \quad \text{(weakly) in } L^{p'}(\Omega) \quad \text{and} \quad \widehat{A_j}(u_\ell) \rightharpoonup g_j \quad \text{(weakly) in } L^{p'_j}(\Omega) \tag{2.13}$$

as $\ell \to \infty$ for every $1 \le j \le N$. Thus, by the reflexivity of $W_0^{1,\vec{p}}(\Omega)$ and (2.8), we get

$$\langle g, v \rangle = \lim_{\ell \to \infty} \langle (\mathcal{A} + \mathcal{P}_\Theta)(u_\ell), v \rangle = \sum_{j=1}^{N} \int_{\Omega} g_j \, \partial_j v \, dx + \int_{\Omega} \mu \, v \, dx \tag{2.14}$$

for every $v \in W_0^{1,\vec{p}}(\Omega)$. From (2.12) and (2.13), we infer that

$$\lim_{\ell \to \infty} \int_{\Omega} \widehat{\Theta}(u_\ell) \, u_\ell \, dx = \int_{\Omega} \mu \, u \, dx. \tag{2.15}$$

From (2.9), (2.14) and (2.15), we obtain that

$$\limsup_{\ell \to \infty} \langle (\mathcal{A} + \mathcal{P}_\Theta)(u_\ell), u_\ell \rangle = \limsup_{\ell \to \infty} \left(\sum_{j=1}^{N} \int_{\Omega} \widehat{A}_j(u_\ell) \, \partial_j u_\ell \, dx + \int_{\Omega} \widehat{\Theta}(u_\ell) \, u_\ell \, dx \right)$$
$$\le \langle g, u \rangle = \sum_{j=1}^{N} \int_{\Omega} g_j \, \partial_j u \, dx + \int_{\Omega} \mu \, u \, dx, \tag{2.16}$$

that is,

$$\limsup_{\ell \to \infty} \sum_{j=1}^{N} \int_{\Omega} \widehat{A}_j(u_\ell) \, \partial_j u_\ell \, dx \le \sum_{j=1}^{N} \int_{\Omega} g_j \, \partial_j u \, dx. \tag{2.17}$$

In light of (2.15)–(2.17), we conclude (2.11) by showing that

$$\liminf_{\ell \to \infty} \sum_{j=1}^{N} \int_{\Omega} \widehat{A}_j(u_\ell) \, \partial_j u_\ell \, dx \ge \sum_{j=1}^{N} \int_{\Omega} g_j \, \partial_j u \, dx. \tag{2.18}$$

The proof of (2.18) is a bit different from the classical one in the isotropic case since in our growth condition on A_j in (1.8), we have taken the greatest exponent for $|t|$ from the viewpoint of the anisotropic Sobolev inequalities. Let us emphasize what is new compared with the classical proof. Let $1 \le j \le N$ be arbitrary. Since $u_\ell \to u$ a.e. in Ω and A_j is a Carathéodory function, we see that

$$A_j(x, u_\ell, \nabla u) \to A_j(x, u, \nabla u) \quad \text{a.e. in } \Omega. \tag{2.19}$$

The growth condition in (1.8) gives a constant $C > 0$ and a nonnegative function $\eta_j \in L^{p'_j}(\Omega)$ such that

$$|A_j(x, u_\ell, \nabla u)|^{p'_j} \le C \left(\eta_j^{p'_j} + |u_\ell|^{p^*} + \sum_{i=1}^{N} |\partial_i u|^{p_i} \right) \tag{2.20}$$

for every $\ell \ge 1$. Because the power of $|u_\ell|$ in the right-hand side of (2.20) is p^*, the critical exponent, the compactness of the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ fails, in general. Hence, we cannot claim anymore that $\{|A_j(x, u_\ell, \nabla u)|^{p'_j}\}_{\ell \ge 1}$ is uniformly integrable over Ω. Thus, we cannot apply Vitali's theorem to deduce the strong convergence of $A_j(x, u_\ell, \nabla u)$ to $A_j(x, u, \nabla u)$ in $L^{p'_j}(\Omega)$ as $\ell \to \infty$. However, if we fix $k \ge 1$, then by the growth condition in (1.8), we infer that

$$\{|A_j(x, u_\ell, \nabla u)|^{p'_j} \chi_{\{|u_\ell| \le k\}}\}_{\ell \ge 1} \quad \text{is uniformly integrable over } \Omega.$$

Then, since $\chi_{\{|u_\ell|\le k\}} \to \chi_{\{|u|\le k\}}$ as $\ell \to \infty$, from (2.19) and Vitali's theorem, we get

$$A_j(x, u_\ell, \nabla u)\chi_{\{|u_\ell|\le k\}} \to A_j(x, u, \nabla u)\chi_{\{|u|\le k\}} \text{ strongly in } L^{p'_j}(\Omega) \text{ as } \ell \to \infty. \tag{2.21}$$

We return to the proof of (2.18) with modifications suggested by (2.21). By the Dominated Convergence Theorem, we obtain (2.18) by showing that for every integer $k \ge 1$,

$$\liminf_{\ell \to \infty} \sum_{j=1}^{N} \int_\Omega \widehat{A}_j(u_\ell)\, \partial_j u_\ell \, dx \ge \sum_{j=1}^{N} \int_\Omega g_j\,(\partial_j u)\chi_{\{|u|\le k\}}\, dx. \tag{2.22}$$

Proof of (2.22). Fix an integer $k \ge 1$. The coercivity condition in (1.8) yields that

$$\sum_{j=1}^{N} \widehat{A}_j(u_\ell)\, \partial_j u_\ell \ge \sum_{j=1}^{N} \widehat{A}_j(u_\ell)\, (\partial_j u_\ell)\chi_{\{|u_\ell|\le k\}}. \tag{2.23}$$

For the right-hand side of (2.23), we use the monotonicity condition in (1.8), that is,

$$\begin{aligned}
\sum_{j=1}^{N} \widehat{A}_j(u_\ell)\, (\partial_j u_\ell)\chi_{\{|u_\ell|\le k\}} \ge {} & \sum_{j=1}^{N} \widehat{A}_j(u_\ell)\, (\partial_j u)\chi_{\{|u_\ell|\le k\}} \\
& + \sum_{j=1}^{N} A_j(x, u_\ell, \nabla u)\, (\partial_j u_\ell - \partial_j u)\chi_{\{|u_\ell|\le k\}}.
\end{aligned} \tag{2.24}$$

Let $1 \le j \le N$ be arbitrary. By the Dominated Convergence Theorem, we have $(\partial_j u)\chi_{\{|u_\ell|\le k\}} \to (\partial_j u)\chi_{\{|u|\le k\}}$ strongly in $L^{p_j}(\Omega)$ as $\ell \to \infty$. Recall from (2.13) that $\widehat{A}_j(u_\ell) \rightharpoonup g_j$ (weakly) in $L^{p'_j}(\Omega)$ as $\ell \to \infty$. Hence we have

$$\widehat{A}_j(u_\ell)\, (\partial_j u)\chi_{\{|u_\ell|\le k\}} \to g_j\,(\partial_j u)\chi_{\{|u|\le k\}} \text{ strongly in } L^1(\Omega) \text{ as } \ell \to \infty. \tag{2.25}$$

Since $\partial_j u_\ell \rightharpoonup \partial_j u$ (weakly) in $L^{p_j}(\Omega)$ as $\ell \to \infty$, using (2.21), we gain the following

$$A_j(x, u_\ell, \nabla u)\, (\partial_j u_\ell - \partial_j u)\chi_{\{|u_\ell|\le k\}} \to 0 \quad \text{strongly in } L^1(\Omega). \tag{2.26}$$

In light of (2.25) and (2.26), we see that

$$\sum_{j=1}^{N} \int_\Omega \widehat{A}_j(u_\ell)\, (\partial_j u)\chi_{\{|u_\ell|\le k\}} + \sum_{j=1}^{N} \int_\Omega A_j(x, u_\ell, \nabla u)\, (\partial_j u_\ell - \partial_j u)\chi_{\{|u_\ell|\le k\}}$$

converges as $\ell \to \infty$ to the right-hand side of (2.22). Using this convergence, jointly with the inequalities in (2.23) and (2.24), we conclude the proof of (2.22).

As mentioned above, from (2.22) we obtain (2.18). Inequalities (2.17) and (2.18) ensure that

$$\lim_{\ell \to \infty} \sum_{j=1}^{N} \int_\Omega \widehat{A}_j(u_\ell)\, \partial_j u_\ell \, dx = \sum_{j=1}^{N} \int_\Omega g_j\, \partial_j u \, dx. \tag{2.27}$$

It remains to establish (2.10). From (2.25)–(2.27), we get

$$\sum_{j=1}^{N} \int_{\Omega} \left[A_j(x, u_\ell, \nabla u_\ell) - A_j(x, u_\ell, \nabla u) \right] (\partial_j u_\ell - \partial_j u) \chi_{\{|u_\ell| \le k\}} \, dx \to 0 \quad \text{as } \ell \to \infty. \tag{2.28}$$

By (2.28) and the monotonicity condition in (1.8), we infer that

$$\sum_{j=1}^{N} \left[A_j(x, u_\ell, \nabla u_\ell) - A_j(x, u_\ell, \nabla u) \right] (\partial_j u_\ell - \partial_j u) \to 0 \text{ a.e in } \{|u_\ell| \le k\} \text{ as } \ell \to \infty. \tag{2.29}$$

For $z, v, w \in W_0^{1,\vec{p}}(\Omega)$ and a.e. $x \in \Omega$, we define $\mathcal{D}_z(v, w)(x)$ as follows

$$\mathcal{D}_z(v, w)(x) = \sum_{j=1}^{N} \left[A_j(x, z(x), \nabla v(x)) - A_j(x, z(x), \nabla w(x)) \right] \partial_j(v - w)(x) \tag{2.30}$$

(see Subsection A.1 in the Appendix). By a standard diagonal argument, we can find a subsequence of $\{u_\ell\}$ (still denoted by $\{u_\ell\}$) such that the convergence in (2.29) holds for every $k \ge 1$. This implies that

$$\mathcal{D}_{u_\ell}(u_\ell, u) = \sum_{j=1}^{N} \left[A_j(x, u_\ell, \nabla u_\ell) - A_j(x, u_\ell, \nabla u) \right] (\partial_j u_\ell - \partial_j u) \to 0 \text{ a.e. in } \Omega \text{ as } \ell \to \infty.$$

Thus, by Lemma A.4 in the Appendix, up to a subsequence, $\nabla u_\ell \to \nabla u$ a.e. in Ω as $\ell \to \infty$. Since Φ and A_j (with $1 \le j \le N$) are Carathéodory functions, we find that $\widehat{\Theta}(u_\ell) \to \widehat{\Theta}(u)$ and $\widehat{A}_j(u_\ell) \to \widehat{A}_j(u)$ a.e. in Ω as $\ell \to \infty$. Using this fact, jointly with (2.13), we obtain that $\mu = \widehat{\Theta}(u)$ and $g_j = \widehat{A}_j(u)$ for every $1 \le j \le N$. From (2.14) we conclude that

$$\langle g, v \rangle = \sum_{j=1}^{N} \int_{\Omega} \widehat{A}_j(u) \, \partial_j v \, dx + \int_{\Omega} \widehat{\Theta}(u) \, v \, dx = \langle \mathcal{A}u, v \rangle + \langle \mathcal{P}_\Theta(u), v \rangle$$

for every $v \in W_0^{1,\vec{p}}(\Omega)$. This proves that $g = (\mathcal{A} + \mathcal{P}_\Theta) u$, namely, (2.10) holds.

In conclusion, by satisfying the M type condition in Definition 2.2, the operator $\mathcal{A} + \mathcal{P}_\Theta$ turns out to be pseudo-monotone. $\qquad \square$

Corollary 2.8. *The operator* $\mathcal{A} + \mathcal{P}_\Theta - \mathfrak{B} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ *is pseudo-monotone.*

Proof. The claim follows from Lemmata 2.6 and 2.7, jointly with Proposition 2.3. $\qquad \square$

3. Proof of the first assertion in Theorem 1.3

Here, we assume (1.2), (1.5), (1.8) and (1.9), whereas $\mathfrak{B}$ belongs to the class $\mathfrak{BC}$. For every $\varepsilon > 0$, we define $\Phi_\varepsilon(x, t, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ as follows

$$\Phi_\varepsilon(x, t, \xi) = \frac{\Phi(x, t, \xi)}{1 + \varepsilon |\Phi(x, t, \xi)|} \tag{3.1}$$

for a.e. $x \in \Omega$ and all $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$. For $\varepsilon > 0$ fixed, Φ_ε satisfies the same properties as Φ, that is, the sign-condition and the growth condition in (1.9). Moreover, Φ_ε becomes a bounded function, namely, for a.e. $x \in \Omega$ and every $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$,

$$\Phi_\varepsilon(x, t, \xi)\, t \geq 0, \quad |\Phi_\varepsilon(x, t, \xi)| \leq \min\{|\Phi(x, t, \xi)|, 1/\varepsilon\}. \tag{3.2}$$

We consider approximate problems to (1.1) with $f = 0$ and Φ replaced by Φ_ε, that is,

$$\begin{cases} \mathcal{A}u_\varepsilon + \Phi_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) + \Theta(x, u_\varepsilon, \nabla u_\varepsilon) = \mathfrak{B}u_\varepsilon & \text{in } \Omega, \\ u_\varepsilon \in W_0^{1,\vec{p}}(\Omega). \end{cases} \tag{3.3}$$

As in Theorem 2.1, by a solution of (3.3), we mean a function $u_\varepsilon \in W_0^{1,\vec{p}}(\Omega)$ such that

$$\sum_{j=1}^N \int_\Omega \widehat{A}_j(u_\varepsilon)\, \partial_j v \, dx + \int_\Omega \widehat{\Phi}_\varepsilon(u_\varepsilon)\, v \, dx + \int_\Omega \widehat{\Theta}(u_\varepsilon)\, v \, dx = \langle \mathfrak{B}u_\varepsilon, v \rangle \tag{3.4}$$

for every $v \in W_0^{1,\vec{p}}(\Omega)$, where for convenience we define

$$\widehat{\Phi}_\varepsilon(u_\varepsilon)(x) = \Phi_\varepsilon(x, u_\varepsilon(x), \nabla u_\varepsilon(x)) \quad \text{for a.e. } x \in \Omega.$$

Lemma 3.1. *For every $\varepsilon > 0$, there exists a solution u_ε to (3.3). Moreover, we have:*
   (a) *For a positive constant C, independent of ε, it holds*

$$\|u_\varepsilon\|_{W_0^{1,\vec{p}}(\Omega)} + \int_\Omega \widehat{\Phi}_\varepsilon(u_\varepsilon)\, u_\varepsilon \, dx \leq C. \tag{3.5}$$

   (b) *There exists $U \in W_0^{1,\vec{p}}(\Omega)$ such that, up to a subsequence of $\{u_\varepsilon\}$,*

$$u_\varepsilon \rightharpoonup U \text{ (weakly) in } W_0^{1,\vec{p}}(\Omega) \quad \text{and} \quad u_\varepsilon \to U \text{ a.e. in } \Omega \text{ as } \varepsilon \to 0. \tag{3.6}$$

Proof. Let $\varepsilon > 0$ be arbitrary. From (3.2), we see that $\Phi_\varepsilon + \Theta$ satisfies the same assumptions as Θ in Section 2. So, Theorem 2.1 applies with $\mathcal{P}_\Theta$ replaced by $\mathcal{P}_{\Theta,\varepsilon}$, where

$$\langle \mathcal{P}_{\Theta,\varepsilon}(u), v \rangle := \int_\Omega \left(\widehat{\Theta}(u) + \widehat{\Phi}_\varepsilon(u) \right) v \, dx \quad \text{for every } u, v \in W_0^{1,\vec{p}}(\Omega).$$

This means that (3.3) admits at least a solution $u_\varepsilon \in W_0^{1,\vec{p}}(\Omega)$ for every $\varepsilon > 0$.

   (a) By taking $v = u_\varepsilon$ in (3.4), we derive that

$$\langle \mathcal{A}u_\varepsilon + \mathcal{P}_\Theta(u_\varepsilon) - \mathfrak{B}u_\varepsilon, u_\varepsilon \rangle + \int_\Omega \widehat{\Phi}_\varepsilon(u_\varepsilon)\, u_\varepsilon \, dx = 0. \tag{3.7}$$

Moreover, since $\mathfrak{B}$ is a bounded operator from $W_0^{1,\vec{p}}(\Omega)$ into its dual, it follows that for some constant $C_0 > 0$, we have

$$\|\mathfrak{B}u_\varepsilon\|_{W^{-1,\vec{p}'}(\Omega)} \leq C_0 \quad \text{for every } \varepsilon > 0.$$

Using (1.5), the coercivity condition in (1.8) and Young's inequality, we infer that for every $\delta > 0$, there exists a constant $C_\delta > 0$ such that

$$
\langle \mathcal{A}u_\varepsilon + \mathcal{P}_\Theta(u_\varepsilon) - \mathfrak{B}u_\varepsilon, u_\varepsilon \rangle \geq \nu_0 \sum_{j=1}^{N} \|\partial_j u_\varepsilon\|_{L^{p_j}(\Omega)}^{p_j} - (C_0 + C_\Theta) \|u_\varepsilon\|_{W_0^{1,\vec{p}}(\Omega)}
$$

$$
\geq (\nu_0 - \delta) \sum_{j=1}^{N} \|\partial_j u_\varepsilon\|_{L^{p_j}(\Omega)}^{p_j} - C_\delta
$$

(3.8)

for every $\varepsilon > 0$. Thus, using (3.7) and (3.8), jointly with (3.2), we arrive at

$$
(\nu_0 - \delta) \sum_{j=1}^{N} \|\partial_j u_\varepsilon\|_{L^{p_j}(\Omega)}^{p_j} \leq (\nu_0 - \delta) \sum_{j=1}^{N} \|\partial_j u_\varepsilon\|_{L^{p_j}(\Omega)}^{p_j} + \int_\Omega \widehat{\Phi}_\varepsilon(u_\varepsilon)\, u_\varepsilon \, dx \leq C_\delta.
$$

By choosing $\delta \in (0, \nu_0)$, we readily conclude the assertion of (3.5).

(b) From (3.5) and the reflexivity of $W_0^{1,\vec{p}}(\Omega)$, we infer that, up to a subsequence, u_ε converges weakly to some U in $W_0^{1,\vec{p}}(\Omega)$. Then, we conclude (3.6) by using Remark A.2 in the Appendix, which implies that, up to a subsequence, $u_\varepsilon \to U$ (strongly) in $L^q(\Omega)$ if $q \in [1, p^*)$ and $u_\varepsilon \to U$ a.e. in Ω as $\varepsilon \to 0$. $\qquad\square$

For the remainder of this section, u_ε and U have the same meaning as in Lemma 3.1.

3.1. Strong convergence of $T_k(u_\varepsilon)$

We recall the notation introduced in (2.30), and for simplicity, instead of $\mathcal{D}_{u_\varepsilon}(T_k(u_\varepsilon), T_k(U))(x)$, we write $\mathcal{D}_{\varepsilon,k}(x)$, that is,

$$
\mathcal{D}_{\varepsilon,k}(x) = \sum_{j=1}^{N} \Big[A_j(x, u_\varepsilon, \nabla T_k(u_\varepsilon)) - A_j(x, u_\varepsilon, \nabla T_k(U)) \Big] \partial_j(T_k(u_\varepsilon) - T_k(U)). \tag{3.9}
$$

Lemma 3.2. *There exists a subsequence of $\{u_\varepsilon\}$, relabeled $\{u_\varepsilon\}$, such that*

$$
\nabla u_\varepsilon \to \nabla U \text{ a.e. in } \Omega \text{ and } T_k(u_\varepsilon) \to T_k(U) \text{ (strongly) in } W_0^{1,\vec{p}}(\Omega) \text{ as } \varepsilon \to 0 \tag{3.10}
$$

for every integer $k \geq 1$.

Proof. Recall that $\{u_\varepsilon\}$ satisfies (3.6) in Lemma 3.1. By a standard diagonal argument, it suffices to show that for every integer $k \geq 1$, there exists a subsequence $\{u_\varepsilon\}$ (depending on k and relabeled $\{u_\varepsilon\}$) satisfying

$$
\nabla T_k(u_\varepsilon) \to \nabla T_k(U) \text{ a.e. in } \Omega \quad \text{and} \quad T_k(u_\varepsilon) \to T_k(U) \text{ (strongly) in } W_0^{1,\vec{p}}(\Omega). \tag{3.11}
$$

Moreover, in light of Lemma A.5 in the Appendix, we conclude (3.11) by showing that, for every integer $k \geq 1$, there exists a subsequence of $\{u_\varepsilon\}$ (depending on k and relabeled $\{u_\varepsilon\}$) such that

$$
\mathcal{D}_{\varepsilon,k} \to 0 \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{3.12}
$$

Let $k \geq 1$ be fixed. Clearly, the monotonicity assumption in (1.8) yields that $\mathcal{D}_{\varepsilon,k} \geq 0$ a.e. in Ω. Hence, to prove (3.12), it suffices to show that (up to a subsequence of $\{u_\varepsilon\}$),

$$\limsup_{\varepsilon \to 0} \int_\Omega \mathcal{D}_{\varepsilon,k}(x) \, dx \leq 0. \tag{3.13}$$

We define $z_{\varepsilon,k}$ as follows

$$z_{\varepsilon,k} := T_k(u_\varepsilon) - T_k(U).$$

We observe that

$$\partial_j z_{\varepsilon,k} \chi_{\{|u_\varepsilon| \geq k\}} = -\partial_j T_k(U) \chi_{\{|u_\varepsilon| \geq k\}} = -\partial_j U \chi_{\{|u_\varepsilon| \geq k\}} \chi_{\{|U| < k\}}.$$

Moreover, we see that

$$\chi_{\{|u_\varepsilon| \geq k\}} \chi_{\{|U| < k\}} \to 0 \quad \text{a.e. in } \Omega \text{ as } \varepsilon \to 0. \tag{3.14}$$

By the Dominated Convergence Theorem, for every $1 \leq j \leq N$, we have

$$\partial_j U \chi_{\{|u_\varepsilon| \geq k\}} \chi_{\{|U| < k\}} \to 0 \quad \text{(strongly) in } L^{p_j}(\Omega) \text{ as } \varepsilon \to 0. \tag{3.15}$$

On the other hand, from the growth condition on A_j in (1.8) and the *a priori* estimates in Lemma 3.1, we infer that $\{A_j(x, u_\varepsilon, \nabla T_k(u_\varepsilon))\}_\varepsilon$ and $\{A_j(x, u_\varepsilon, \nabla T_k(U))\}_\varepsilon$ are bounded in $L^{p'_j}(\Omega)$ and, hence, up to a subsequence of $\{u_\varepsilon\}$, they converge weakly in $L^{p'_j}(\Omega)$ for each $1 \leq j \leq N$. This, jointly with (3.15), gives that

$$\Xi_{j,\varepsilon,k}(x) = \left[A_j(x, u_\varepsilon, \nabla T_k(u_\varepsilon)) - A_j(x, u_\varepsilon, \nabla T_k(U)) \right] \partial_j U \chi_{\{|u_\varepsilon| \geq k\}} \chi_{\{|U| < k\}}$$

converges to 0 in $L^1(\Omega)$ as $\varepsilon \to 0$ for every $1 \leq j \leq N$. It follows that

$$\int_\Omega \mathcal{D}_{\varepsilon,k}(x) \chi_{\{|u_\varepsilon| \geq k\}} \, dx = -\sum_{j=1}^N \int_\Omega \Xi_{j,\varepsilon,k}(x) \, dx \to 0 \quad \text{as } \varepsilon \to 0.$$

Thus, to conclude (3.13), it remains to show that

$$\limsup_{\varepsilon \to 0} \int_\Omega \mathcal{D}_{\varepsilon,k}(x) \chi_{\{|u_\varepsilon| < k\}} \, dx \leq 0. \tag{3.16}$$

Proof of (3.16). We define $\varphi_\lambda : \mathbb{R} \to \mathbb{R}$ as follows

$$\varphi_\lambda(t) = t \exp\left(\lambda t^2\right) \quad \text{for every } t \in \mathbb{R}.$$

We choose $\lambda = \lambda(k) > 0$ large such that $4\nu_0^2 \lambda > \phi^2(k)$, where ϕ appears in the growth assumption on Φ, see (1.9). This choice of λ ensures that for every $t \in \mathbb{R}$

$$\lambda t^2 - \frac{\phi(k)}{2\nu_0}|t| + \frac{1}{4} > 0 \quad \text{and, hence,} \quad \varphi'_\lambda(t) - \frac{\phi(k)}{\nu_0}|\varphi_\lambda(t)| > \frac{1}{2}. \tag{3.17}$$

For $v \in W_0^{1,\vec{p}}(\Omega)$, we define

$$\mathcal{E}_{\varepsilon,k}(v) = \sum_{j=1}^N \int_\Omega A_j(x, u_\varepsilon, \nabla v) \partial_j z_{\varepsilon,k} \left[\varphi'_\lambda(z_{\varepsilon,k}) - \frac{\phi(k)}{\nu_0}|\varphi_\lambda(z_{\varepsilon,k})| \right] \chi_{\{|u_\varepsilon| < k\}} \, dx.$$

Returning to the definition of $\mathcal{D}_{\varepsilon,k}$ in (3.9) and using (3.17), we arrive at

$$\frac{1}{2} \int_{\Omega} \mathcal{D}_{\varepsilon,k}(x) \chi_{\{|u_\varepsilon|<k\}} \, dx \leq \mathcal{E}_{\varepsilon,k}(T_k(u_\varepsilon)) - \mathcal{E}_{\varepsilon,k}(T_k(U)). \tag{3.18}$$

Since $T_k(u_\varepsilon) = u_\varepsilon$ on the set $\{|u_\varepsilon| < k\}$, in light of (3.18), we complete the proof of (3.16) by showing that

$$\lim_{\varepsilon \to 0} \mathcal{E}_{\varepsilon,k}(T_k(U)) = 0, \tag{3.19}$$

$$\limsup_{\varepsilon \to 0} \mathcal{E}_{\varepsilon,k}(u_\varepsilon) \leq 0. \tag{3.20}$$

Proof of (3.19). For each $1 \leq j \leq N$, the growth condition in (1.8) gives a nonnegative function $F_j \in L^{p'_j}(\Omega)$ such that on the set $\{|u_\varepsilon| < k\}$, we have $|A_j(x, u_\varepsilon, \nabla T_k(U))| \leq F_j$ for every $\varepsilon > 0$. Since $|z_{\varepsilon,k}| \leq 2k$, we can find a constant $C_k > 0$ such that

$$\left| \varphi'_\lambda(z_{\varepsilon,k}) - \frac{\phi(k)}{\nu_0} |\varphi_\lambda(z_{\varepsilon,k})| \right| \leq C_k.$$

On the other hand, for each $1 \leq j \leq N$, we have

$$\partial_j z_{\varepsilon,k} \chi_{\{|u_\varepsilon|<k\}} = \partial_j z_{\varepsilon,k} + \partial_j U \chi_{\{|U|<k\}} \chi_{\{|u_\varepsilon|\geq k\}}.$$

This, together with (3.15) and the weak convergence of $\partial_j z_{\varepsilon,k}$ to 0 in $L^{p_j}(\Omega)$ as $\varepsilon \to 0$, implies that $\partial_j z_{\varepsilon,k} \chi_{\{|u_\varepsilon|<k\}}$ converges weakly to 0 in $L^{p_j}(\Omega)$ as $\varepsilon \to 0$. Hence, we have

$$|\mathcal{E}_{\varepsilon,k}(T_k(U))| \leq C_k \sum_{j=1}^{N} \int_{\Omega} F_j |\partial_j z_{\varepsilon,k}| \chi_{\{|u_\varepsilon|<k\}} \, dx \to 0 \quad \text{as } \varepsilon \to 0,$$

which proves (3.19).

Proof of (3.20). From (3.6), we have

$$z_{\varepsilon,k} \to 0 \text{ a.e. in } \Omega \text{ and } z_{\varepsilon,k} \rightharpoonup 0 \text{ (weakly) in } W_0^{1,\vec{p}}(\Omega) \text{ as } \varepsilon \to 0.$$

Since $|z_{\varepsilon,k}| \leq 2k$ a.e. in Ω, we get $\varphi_\lambda(z_{\varepsilon,k}) \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$. Moreover,

$$\varphi_\lambda(z_{\varepsilon,k}) \to 0 \text{ a.e. in } \Omega \text{ and } \varphi_\lambda(z_{\varepsilon,k}) \rightharpoonup 0 \text{ (weakly) in } W_0^{1,\vec{p}}(\Omega) \text{ as } \varepsilon \to 0. \tag{3.21}$$

Observe that $u_\varepsilon z_{\varepsilon,k} \geq 0$ on the set $\{|u_\varepsilon| \geq k\}$, which gives that

$$\widehat{\Phi}_\varepsilon(u_\varepsilon) \varphi_\lambda(z_{\varepsilon,k}) \chi_{\{|u_\varepsilon|\geq k\}} \geq 0.$$

Thus, by testing (3.4) with $v = \varphi_\lambda(z_{\varepsilon,k})$, we obtain that

$$\langle \mathcal{A}u_\varepsilon, \varphi_\lambda(z_{\varepsilon,k}) \rangle + \int_{\Omega} \widehat{\Phi}_\varepsilon(u_\varepsilon) \varphi_\lambda(z_{\varepsilon,k}) \chi_{\{|u_\varepsilon|<k\}} \, dx \leq \langle \mathcal{B}u_\varepsilon, \varphi_\lambda(z_{\varepsilon,k}) \rangle - \int_{\Omega} \widehat{\Theta}(u_\varepsilon) \varphi_\lambda(z_{\varepsilon,k}) \, dx. \tag{3.22}$$

To simplify exposition, we now introduce some notation:

$$X_k(\varepsilon) = \phi(k) \int_\Omega \left[\frac{1}{\nu_0} \sum_{j=1}^N \widehat{A}_j(u_\varepsilon) \, \partial_j(T_k U) + c(x) \right] |\varphi_\lambda(z_{\varepsilon,k})| \chi_{\{|u_\varepsilon|<k\}} \, dx,$$

$$Y_k(\varepsilon) = \sum_{j=1}^N \int_\Omega \widehat{A}_j(u_\varepsilon) \, \partial_j U \, \varphi'_\lambda(z_{\varepsilon,k}) \chi_{\{|U|<k\}} \chi_{\{|u_\varepsilon|\geq k\}} \, dx.$$

We rewrite the first term in the left-hand side of (3.22) as follows

$$\langle \mathscr{A} u_\varepsilon, \varphi_\lambda(z_{\varepsilon,k}) \rangle = \sum_{j=1}^N \int_\Omega \widehat{A}_j(u_\varepsilon) \partial_j z_{\varepsilon,k} \, \varphi'_\lambda(z_{\varepsilon,k}) \chi_{\{|u_\varepsilon|<k\}} \, dx - Y_k(\varepsilon). \tag{3.23}$$

The coercivity condition in (1.8) and the growth condition of Φ in (1.9) imply that

$$|\widehat{\Phi}_\varepsilon(u_\varepsilon)| \chi_{\{|u_\varepsilon|<k\}} \leq \phi(k) \left[\frac{1}{\nu_0} \sum_{j=1}^N \widehat{A}_j(u_\varepsilon) \partial_j u_\varepsilon + c(x) \right] \chi_{\{|u_\varepsilon|<k\}}. \tag{3.24}$$

In the right-hand side of (3.24) we replace $\partial_j u_\varepsilon$ by $\partial_j z_{\varepsilon,k} + \partial_j T_k(U)$, then we multiply the inequality by $|\varphi_\lambda(z_{\varepsilon,k})|$ and integrate over Ω with respect to x. It follows that the second term in the left-hand side of (3.22) is at least

$$-\frac{\phi(k)}{\nu_0} \sum_{j=1}^N \int_\Omega \widehat{A}_j(u_\varepsilon) \partial_j z_{\varepsilon,k} \, |\varphi_\lambda(z_{\varepsilon,k})| \chi_{\{|u_\varepsilon|<k\}} \, dx - X_k(\varepsilon).$$

Using this fact, as well as (3.23), in (3.22), we see that $\mathcal{E}_{\varepsilon,k}(u_\varepsilon)$ satisfies the estimate

$$\mathcal{E}_{\varepsilon,k}(u_\varepsilon) \leq X_k(\varepsilon) + Y_k(\varepsilon) + \langle \mathfrak{B} u_\varepsilon, \varphi_\lambda(z_{\varepsilon,k}) \rangle - \int_\Omega \widehat{\Theta}(u_\varepsilon) \, \varphi_\lambda(z_{\varepsilon,k}) \, dx. \tag{3.25}$$

To conclude the proof of (3.20), it suffices to show that each term in the right-hand side of (3.25) converges to 0 as $\varepsilon \to 0$. Recall that $\varphi_\lambda(z_{\varepsilon,k}) \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ satisfies (3.21). Thus, using (1.5) and the property (P_2) of $\mathfrak{B}$, we get that the third, as well as the fourth, term in the right-hand side of (3.25) converges to zero as $\varepsilon \to 0$.

We next look at $X_k(\varepsilon)$. In view of the pointwise convergence in (3.21) and $c \in L^1(\Omega)$, we infer from the Dominated Convergence Theorem that

$$c(x)|\varphi_\lambda(z_{\varepsilon,k})| \chi_{\{|u_\varepsilon|<k\}} \to 0 \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{3.26}$$

Next, up to a subsequence of $\{u_\varepsilon\}$, we find that $\widehat{A}_j(u_\varepsilon)$ converges weakly in $L^{p'_j}(\Omega)$ as $\varepsilon \to 0$ for every $1 \leq j \leq N$ using the boundedness of $\widehat{A}_j : W_0^{1,\vec{p}}(\Omega) \to L^{p'_j}(\Omega)$ (see Lemma 2.4). Hence, $\sum_{j=1}^N \widehat{A}_j(u_\varepsilon) \partial_j U$ converges in $L^1(\Omega)$ as $\varepsilon \to 0$. Then, there exists a nonnegative function $F \in L^1(\Omega)$ (independent of ε) such that, up to a subsequence of $\{u_\varepsilon\}$, we have

$$\left| \sum_{j=1}^N \widehat{A}_j(u_\varepsilon) \partial_j U \right| \leq F \quad \text{a.e. in } \Omega \text{ for every } \varepsilon > 0. \tag{3.27}$$

We can now again use the Dominated Convergence Theorem to conclude that

$$\sum_{j=1}^{N} \widehat{A}_j(u_\varepsilon)\, \partial_j T_k(U)\, |\varphi_\lambda(z_{\varepsilon,k})| \chi_{\{|u_\varepsilon|<k\}} \to 0 \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{3.28}$$

From (3.26) and (3.28), we find that $\lim_{\varepsilon\to 0} X_k(\varepsilon) = 0$. Since $|\varphi'_\lambda(z_{\varepsilon,k})|$ is bounded above by a constant independent of ε (but dependent on k), we can use a similar argument, based on (3.14) and (3.27), to obtain that, up to a subsequence of $\{u_\varepsilon\}$, $\lim_{\varepsilon\to 0} Y_k(\varepsilon) = 0$. This ends the proof of the convergence to zero of the right-hand side of (3.25) as $\varepsilon \to 0$. Consequently, the proof of (3.20), and thus of (3.16), is complete. $\qquad\square$

3.2. Passing to the limit

From now on, the meaning of $\{u_\varepsilon\}_\varepsilon$ is given by Lemma 3.2. Using Lemma 3.1, we prove in Lemma 3.4 that U is a solution of (1.1) with $f = 0$ and, moreover, U satisfies all the properties stated in Theorem 1.3 (*i*). Besides (3.10), the other fundamental property that allows us to pass to the limit as $\varepsilon \to 0$ in (3.4) for every $v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ is the following convergence

$$\widehat{\Phi}_\varepsilon(u_\varepsilon) \to \widehat{\Phi}(U) \text{ (strongly) in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{3.29}$$

The proof of (3.29) is the main objective of our next result.

Lemma 3.3. *We have* $\widehat{\Phi}(U), \widehat{\Phi}(U)\, U \in L^1(\Omega)$ *and* (3.29) *holds.*

Proof. From the pointwise convergence $u_\varepsilon \to U$ and $\nabla u_\varepsilon \to \nabla U$ a.e. in Ω as $\varepsilon \to 0$, jointly with the fact that $\Phi(x,t,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, we infer that $\widehat{\Phi}(u_\varepsilon) \to \widehat{\Phi}(U)$ and $\widehat{\Phi}_\varepsilon(u_\varepsilon)\, u_\varepsilon \to \widehat{\Phi}(U)\, U$ a.e. in Ω as $\varepsilon \to 0$. Using this fact and that $\{\widehat{\Phi}_\varepsilon(u_\varepsilon)\, u_\varepsilon\}_\varepsilon$ is a sequence of nonnegative functions that is uniformly bounded in $L^1(\Omega)$ with respect to ε (from Lemma 3.1), by Fatou's Lemma we conclude that

$$\widehat{\Phi}(U)\, U \in L^1(\Omega).$$

This and the growth condition in (1.9) yield that $\widehat{\Phi}(U) \in L^1(\Omega)$. Indeed, for any $M > 0$, on the set $\Omega \cap \{|U| \le M\}$, we have $|\widehat{\Phi}(U)| \le \phi(M)\left(\sum_{j=1}^{N} |\partial_j U|^{p_j} + c(x)\right) \in L^1(\Omega)$. In turn, on the set $\Omega \cap \{|U| > M\}$, it holds $|\widehat{\Phi}(U)| \le M^{-1}\, \widehat{\Phi}(U)\, U \in L^1(\Omega)$.

To finish the proof of Lemma 3.3, it remains to establish (3.29).

Proof of (3.29). Since $\widehat{\Phi}_\varepsilon(u_\varepsilon) \to \widehat{\Phi}(U)$ a.e. in Ω as $\varepsilon \to 0$ and $\widehat{\Phi}(U) \in L^1(\Omega)$, by Vitali's Theorem, it suffices to show that $\{\widehat{\Phi}_\varepsilon(u_\varepsilon)\}_\varepsilon$ is uniformly integrable over Ω. We next check this fact. For every $M > 0$, we define

$$D_{\varepsilon,M} := \{|u_\varepsilon| \le M\} \quad \text{and} \quad E_{\varepsilon,M} := \{|u_\varepsilon| > M\}.$$

For every $x \in D_{\varepsilon,M}$, using the growth condition of Φ in (1.9), we find that

$$|\widehat{\Phi}_\varepsilon(u_\varepsilon)(x)| \le |\widehat{\Phi}(u_\varepsilon)(x)| \le \phi(M)\left(\sum_{j=1}^{N} |\partial_j T_M(u_\varepsilon)|^{p_j} + c(x)\right),$$

with $c \in L^1(\Omega)$. Let ω be any measurable subset of Ω. It follows that

$$\int_{\omega \cap D_{\varepsilon,M}} |\widehat{\Phi}_\varepsilon(u_\varepsilon)| \, dx \le \phi(M) \left(\sum_{j=1}^{N} \|\partial_j(T_M u_\varepsilon)\|_{L^{p_j}(\omega)}^{p_j} + \int_\omega c(x) \, dx \right).$$

On the other hand, using (3.5) in Lemma 3.1, we see that

$$\int_{\omega \cap E_{\varepsilon,M}} |\widehat{\Phi}_\varepsilon(u_\varepsilon)| \, dx \le \frac{1}{M} \int_{\omega \cap E_{\varepsilon,M}} \widehat{\Phi}_\varepsilon(u_\varepsilon) \, u_\varepsilon \, dx \le \frac{C}{M},$$

where $C > 0$ is a constant independent of ε and ω. Consequently, we find that

$$\int_\omega |\widehat{\Phi}_\varepsilon(u_\varepsilon)| \, dx \le \phi(M) \left(\sum_{j=1}^{N} \|\partial_j(T_M u_\varepsilon)\|_{L^{p_j}(\omega)}^{p_j} + \int_\omega c(x) \, dx \right) + \frac{C}{M}. \tag{3.30}$$

Lemma 3.2 yields that $\partial_j T_M(u_\varepsilon) \to \partial_j T_M(U)$ (strongly) in $L^{p_j}(\Omega)$ as $\varepsilon \to 0$ for every $1 \le j \le N$. Since $c \in L^1(\Omega)$, from (3.30) we get the uniform integrability of $\{\widehat{\Phi}_\varepsilon(u_\varepsilon)\}_\varepsilon$ over Ω. We end the proof of (3.29) by Vitali's Theorem. $\square$

By Lemma 3.3, to finish the proof of Theorem 1.3 (*i*), we need to show the following.

Lemma 3.4. *The function U is a solution to (1.1) with $f = 0$ and, moreover, (1.11) holds for $v = u = U$.*

Proof. Fix $v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ arbitrary. Since u_ε is a solution of (3.3), we have

$$\sum_{j=1}^{N} \int_\Omega \widehat{A}_j(u_\varepsilon) \, \partial_j v \, dx + \int_\Omega \widehat{\Phi}_\varepsilon(u_\varepsilon) \, v \, dx + \int_\Omega \widehat{\Theta}(u_\varepsilon) \, v \, dx = \langle \mathcal{B} u_\varepsilon, v \rangle. \tag{3.31}$$

By Lemma 3.3, the second term in the left-hand side of (3.31) converges to $\int_\Omega \widehat{\Phi}(U) v$ as $\varepsilon \to 0$, whereas the right-hand side of (3.31) converges to $\langle \mathcal{B} U, v \rangle$ based on the weak convergence of u_ε to U in $W_0^{1,\vec{p}}(\Omega)$ as $\varepsilon \to 0$. Using (3.6) and (3.10), we find that

$$\widehat{\Theta}(u_\varepsilon) \to \widehat{\Theta}(U) \quad \text{and} \quad \widehat{A}_j(u_\varepsilon) \to \widehat{A}_j(U) \text{ a.e. in } \Omega \text{ for } 1 \le j \le N. \tag{3.32}$$

Thus, in light of (1.5), and the Dominated Convergence Theorem, we obtain that

$$\int_\Omega \widehat{\Theta}(u_\varepsilon) \, v \, dx \to \int_\Omega \widehat{\Theta}(U) \, v \, dx \quad \text{as } \varepsilon \to 0.$$

Since $\{\widehat{A}_j(u_\varepsilon)\}_\varepsilon$ is uniformly bounded in $L^{p'_j}(\Omega)$ with respect to ε, we observe from (3.32) that (up to a subsequence) $\widehat{A}_j(u_\varepsilon) \rightharpoonup \widehat{A}_j(U)$ (weakly) in $L^{p'_j}(\Omega)$ as $\varepsilon \to 0$ for each $1 \le j \le N$. It follows that

$$\sum_{j=1}^{N} \int_\Omega \widehat{A}_j(u_\varepsilon) \, \partial_j v \, dx \to \sum_{j=1}^{N} \int_\Omega \widehat{A}_j(U) \, \partial_j v \, dx \quad \text{as } \varepsilon \to 0.$$

By letting $\varepsilon \to 0$ in (3.31), we conclude that

$$\sum_{j=1}^{N} \int_\Omega \widehat{A}_j(U) \, \partial_j v \, dx + \int_\Omega \widehat{\Phi}(U) \, v \, dx + \int_\Omega \widehat{\Theta}(U) \, v \, dx = \langle \mathcal{B} U, v \rangle \tag{3.33}$$

for every $v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$. Hence, U is a solution to (1.1) with $f = 0$.

It remains to prove (1.11) for $v = u = U$. Since U may not be in $L^\infty(\Omega)$, we cannot directly use $v = U$ in (3.33). Nevertheless, for every $k > 0$, we have $T_k(U) \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$. Hence, by taking $v = T_k(U)$ in (3.33), we have

$$\langle \mathcal{A}U, T_k(U) \rangle + \int_\Omega \widehat{\Phi}(U) \, T_k(U) \, dx + \int_\Omega \widehat{\Theta}(U) \, T_k(U) \, dx = \langle \mathcal{B}U, T_k(U) \rangle. \tag{3.34}$$

Notice that $\|T_k(U)\|_{W_0^{1,\vec{p}}(\Omega)} \le \|U\|_{W_0^{1,\vec{p}}(\Omega)}$ for all $k > 0$. Moreover, $\partial_j(T_k(U)) \to \partial_j U$ a.e. in Ω as $k \to \infty$, for every $1 \le j \le N$, so that $T_k(U) \rightharpoonup U$ (weakly) in $W_0^{1,\vec{p}}(\Omega)$ as $k \to \infty$. Since $\mathcal{A}U$ and $\mathcal{B}U$ belong to $W^{-1,\vec{p}'}(\Omega)$, it follows that

$$\lim_{k \to \infty} \langle \mathcal{A}U, T_k(U) \rangle = \langle \mathcal{A}U, U \rangle \quad \text{and} \quad \lim_{k \to \infty} \langle \mathcal{B}U, T_k(U) \rangle = \langle \mathcal{B}U, U \rangle.$$

Recalling that $\widehat{\Phi}(U) \, U \in L^1(\Omega)$ and (1.5) holds, from the Dominated Convergence Theorem, we can pass to the limit $k \to \infty$ in (3.34) to conclude the proof. $\qquad\square$

4. Proof of the second assertion in Theorem 1.3

Suppose for the moment only (1.2), (1.5), (1.8), and (1.9). Let $\mathcal{B}$ be in the class $\mathfrak{BC}$. Overall, to prove Theorem 1.3 (*ii*), we follow similar arguments to those developed for proving Theorem 1.3 (*i*) in Section 3. But there are several differences that appear when introducing a function $f \in L^1(\Omega)$ in the equation in (1.1). We first approximate f by a "nice" function $f_\varepsilon \in L^\infty(\Omega)$ with the properties that

$$|f_\varepsilon| \le |f| \text{ a.e. in } \Omega \quad \text{and} \quad f_\varepsilon \to f \text{ a.e. in } \Omega \text{ as } \varepsilon \to 0. \tag{4.1}$$

Then, by the Dominated Convergence Theorem, we find that

$$f_\varepsilon \to f \text{ (strongly) in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{4.2}$$

For example, for every $\varepsilon > 0$, we could take $f_\varepsilon(x) = f(x)/(1 + \varepsilon|f(x)|)$ for a.e. $x \in \Omega$. This approximation is done so that we can apply Theorem 1.3 (*i*) for the problem generated by (1.1) with f_ε in place of f. Then such an approximate problem admits at least a solution U_ε, namely,

$$\begin{cases} \mathcal{A}U_\varepsilon + \widehat{\Phi}(U_\varepsilon) + \widehat{\Theta}(U_\varepsilon) = \mathcal{B}U_\varepsilon + f_\varepsilon & \text{in } \Omega, \\ U_\varepsilon \in W_0^{1,\vec{p}}(\Omega), \quad \widehat{\Phi}(U_\varepsilon) \in L^1(\Omega). \end{cases} \tag{4.3}$$

To see this, we observe that $\mathcal{B}_\varepsilon : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ belongs to the class $\mathfrak{BC}$, where

$$\langle \mathcal{B}_\varepsilon u, v \rangle = \langle \mathcal{B}u, v \rangle + \int_\Omega f_\varepsilon \, v \, dx \quad \text{for every } u, v \in W_0^{1,\vec{p}}(\Omega). \tag{4.4}$$

By Theorem 1.3 (*i*) applied for $\mathcal{B}_\varepsilon$ instead of $\mathcal{B}$, we obtain a solution U_ε for (4.3). Thus,

$$\sum_{j=1}^N \int_\Omega \widehat{A}_j(U_\varepsilon) \, \partial_j v \, dx + \int_\Omega \widehat{\Phi}(U_\varepsilon) \, v \, dx + \int_\Omega \widehat{\Theta}(U_\varepsilon) \, v \, dx = \langle \mathcal{B}U_\varepsilon, v \rangle + \int_\Omega f_\varepsilon \, v \, dx \tag{4.5}$$

for every $v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$. However, unlike Theorem 1.3 (*i*), to obtain that U_ε is uniformly bounded in $W_0^{1,\vec{p}}(\Omega)$ with respect to ε, we need the following:

(i) $\mathfrak{B}$ to satisfy the extra condition (P_3), that is, $\mathfrak{B}$ is chosen in the class $\mathfrak{BC}_+$;

(ii) the additional hypothesis (1.10), which we recall below:

there exist positive constants τ and γ such that for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^N$

$$|\Phi(x, t, \xi)| \geq \gamma \sum_{j=1}^{N} |\xi_j|^{p_j} \quad \text{for all } |t| \geq \tau. \tag{4.6}$$

Without any loss of generality, we can assume $\tau > 0$ large such that $\tau\gamma \geq \nu_0$, where ν_0 appears in the coercivity condition of (1.8).

For the rest of this section, besides (1.2), (1.5), (1.8) and (1.9), we also assume (i) and (ii) above. To avoid repetition, we understand that all the computations in Section 3 are done here replacing u_ε, U and Φ_ε by U_ε, U_0 and Φ, respectively. We only stress the differences that appear compared with the developments in Section 3.

4.1. A priori *estimates*

In Lemma 3.1 we gave *a priori* estimates for the solution u_ε to (3.3), corresponding to the problem (1.1) with $f = 0$ and Φ_ε instead of Φ. We next get *a priori* estimates for U_ε solving (4.3), that is, (1.1) with f_ε instead of f.

Lemma 4.1. *Let U_ε be a solution to (4.3).*

(a) For a positive constant C, independent of ε, we have

$$\|U_\varepsilon\|_{W_0^{1,\vec{p}}(\Omega)} + \int_\Omega |\widehat{\Phi}(U_\varepsilon)| \, dx \leq C. \tag{4.7}$$

(b) There exists $U_0 \in W_0^{1,\vec{p}}(\Omega)$ such that, up to a subsequence of $\{U_\varepsilon\}$,

$$U_\varepsilon \rightharpoonup U_0 \text{ (weakly) in } W_0^{1,\vec{p}}(\Omega), \quad U_\varepsilon \to U_0 \text{ a.e. in } \Omega \text{ as } \varepsilon \to 0. \tag{4.8}$$

Proof. (a) The choice of f_ε gives that $\|f_\varepsilon\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$. Let $\tau > 0$ be as in (4.6). We have $\partial_j T_\tau(U_\varepsilon) = \chi_{\{|U_\varepsilon|<\tau\}} \partial_j U_\varepsilon$ a.e. in Ω for every $1 \leq j \leq N$. We now define

$$K_{\tau,\varepsilon} := \sum_{j=1}^{N} \int_\Omega \widehat{A_j}(U_\varepsilon) \, \partial_j U_\varepsilon \chi_{\{|U_\varepsilon|<\tau\}} \, dx + \tau \int_\Omega |\widehat{\Phi}(U_\varepsilon)| \chi_{\{|U_\varepsilon|\geq\tau\}} \, dx - \langle \mathfrak{B}U_\varepsilon, T_\tau(U_\varepsilon) \rangle.$$

By taking $v = T_\tau(U_\varepsilon) \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ in (4.5) and using the sign-condition of Φ in (1.9), we obtain that

$$K_{\tau,\varepsilon} \leq \tau \left(\|f\|_{L^1(\Omega)} + C_\Theta \operatorname{meas}(\Omega) \right). \tag{4.9}$$

By virtue of (4.6) and the coercivity condition in (1.8), we see that

$$\nu_0 \sum_{j=1}^{N} \int_\Omega |\partial_j U_\varepsilon|^{p_j} \chi_{\{|U_\varepsilon|<\tau\}} \, dx + \tau\gamma \sum_{j=1}^{N} \int_\Omega |\partial_j U_\varepsilon|^{p_j} \chi_{\{|U_\varepsilon|\geq\tau\}} \, dx - \langle \mathfrak{B}U_\varepsilon, T_\tau(U_\varepsilon) \rangle \leq K_{\tau,\varepsilon}.$$

By our choice of τ, we have $\tau\gamma > \nu_0$. Then, the above estimates lead to

$$\nu_0 \sum_{j=1}^{N} \int_{\Omega} |\partial_j U_\varepsilon|^{p_j}\, dx - \langle \mathfrak{B} U_\varepsilon, T_\tau(U_\varepsilon)\rangle \le \tau\left(\|f\|_{L^1(\Omega)} + C_\Theta \operatorname{meas}(\Omega)\right).$$

This fact, jointly with the property (P_3), gives the boundedness of $\{U_\varepsilon\}_{\varepsilon>0}$ in $W_0^{1,\vec{p}}(\Omega)$. Since $\mathfrak{B}$ is a bounded operator from $W_0^{1,\vec{p}}(\Omega)$ into its dual, we have $|\langle \mathfrak{B} U_\varepsilon, T_\tau(U_\varepsilon)| \le C_1$, where C_1 is a positive constant independent of ε. Using (4.9), we find that

$$\int_{\Omega} |\widehat{\Phi}(U_\varepsilon)| \chi_{\{|U_\varepsilon|\ge\tau\}}\, dx \le C_1 \tau^{-1} + \|f\|_{L^1(\Omega)} + C_\Theta \operatorname{meas}(\Omega) = C_2. \tag{4.10}$$

Now, using the growth condition of Φ in (1.9), we obtain a positive constant C_3 such that $\int_{\Omega} |\widehat{\Phi}(U_\varepsilon)| \chi_{\{|U_\varepsilon|\le\tau\}}\, dx \le C_3$ for every $\varepsilon > 0$. This completes the proof of (4.7).

(*b*) The assertion in (4.8) follows from (4.7) (see the proof of (*b*) in Lemma 3.1). $\qquad\square$

4.2. Strong convergence of $T_k(U_\varepsilon)$

The game plan is closely related to that in Subsection 3.1. As mentioned before, when adapting the calculations, we need to replace u_ε, U and $\mathfrak{B}$ in Section 3 by U_ε, U_0 and $\mathfrak{B}_\varepsilon$, respectively. The counterpart of Lemma 3.2 holds so that we obtain the following.

Lemma 4.2. *There exists a subsequence of $\{U_\varepsilon\}_\varepsilon$, relabeled $\{U_\varepsilon\}_\varepsilon$, such that*

$$\nabla U_\varepsilon \to \nabla U_0 \text{ a.e. in } \Omega \text{ and } T_k(U_\varepsilon) \to T_k(U_0) \text{ (strongly) in } W_0^{1,\vec{p}}(\Omega) \text{ as } \varepsilon \to 0$$

for every positive integer k.

Proof. The computations in Subsection 3.1 can be carried out with Φ instead of Φ_ε since the upper bounds used for $|\Phi_\varepsilon|$ were derived from those satisfied by $|\Phi|$ and the sign-condition of Φ is the same as for Φ_ε (see (3.2)). A small change arises in the proof of (3.20) because of the introduction of f_ε in (4.3). Using the definition of $\mathfrak{B}_\varepsilon$ in (4.4), the inequalities in (3.22) and (3.25) must be read with $\mathfrak{B}_\varepsilon$ instead of $\mathfrak{B}$. We note that $\langle \mathfrak{B}_\varepsilon U_\varepsilon, \varphi_\lambda(z_{\varepsilon,k})\rangle$ is the sum between $\langle \mathfrak{B} U_\varepsilon, \varphi_\lambda(z_{\varepsilon,k})\rangle$ and $\int_\Omega f_\varepsilon\, \varphi_\lambda(z_{\varepsilon,k})\, dx$. The latter term, like the former, converges to 0 as $\varepsilon \to 0$. The new claim regarding the convergence to zero of $\int_\Omega f_\varepsilon\, \varphi_\lambda(z_{\varepsilon,k})\, dx$ follows from the Dominated Convergence Theorem using (4.1), $|\varphi_\lambda(z_{\varepsilon,k})| \le 2k \exp(4\lambda k^2)$ and $\varphi_\lambda(z_{\varepsilon,k}) \to 0$ a.e. in Ω as $\varepsilon \to 0$. The remainder of the proof of (3.20) carries over easily to our setting. $\qquad\square$

4.3. Passing to the limit

We aim to pass to the limit as $\varepsilon \to 0$ in (4.5) to obtain that U_0 is a solution to (1.1). Since f_ε satisfies (4.2) and $U_\varepsilon \rightharpoonup U_0$ (weakly) in $W_0^{1,\vec{p}}(\Omega)$ as $\varepsilon \to 0$, we readily have the convergence of the right-hand side of (4.5) to $\langle \mathfrak{B} U_0, v\rangle + \int_\Omega f v\, dx$ for every $v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$. Moreover, because of the convergence $\nabla U_\varepsilon \to \nabla U_0$ a.e. in Ω, we can use the same argument as in Lemma 3.4 to deduce that, as $\varepsilon \to 0$,

$$\int_{\Omega} \widehat{\Theta}(u_\varepsilon) v\, dx \to \int_{\Omega} \widehat{\Theta}(U_0) v\, dx, \quad \sum_{j=1}^{N} \int_{\Omega} \widehat{A}_j(U_\varepsilon)\, \partial_j v\, dx \to \sum_{j=1}^{N} \int_{\Omega} \widehat{A}_j(U_0)\, \partial_j v\, dx$$

for every $v \in W_0^{1,\vec{p}}(\Omega)$. What is here different compared with Subsection 3.2 is the proof of the convergence

$$\widehat{\Phi}(U_\varepsilon) \to \widehat{\Phi}(U_0) \quad \text{(strongly) in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{4.11}$$

To prove that U_0 is a solution of (1.1), it remains to justify (4.11). Instead of Lemma 3.3, we establish the following.

Lemma 4.3. *We have $\widehat{\Phi}(U_0) \in L^1(\Omega)$ and (4.11) holds.*

Proof. From Lemma 4.2, the pointwise convergence in (4.8) and the continuity of $\Phi(x, \cdot, \cdot)$ in the last two variables, we infer that $|\widehat{\Phi}(U_\varepsilon)| \to |\widehat{\Phi}(U_0)|$ a.e. in Ω as $\varepsilon \to 0$. Then, (4.7) and Fatou's Lemma ensure that $\widehat{\Phi}(U_0) \in L^1(\Omega)$.

Proof of (4.11). We will use Vitali's Theorem. To this end, we need to show that $\{\widehat{\Phi}(U_\varepsilon)\}_\varepsilon$ is uniformly integrable over Ω. We can only partially imitate the proof of the uniform integrability of $\{\widehat{\Phi}_\varepsilon(u_\varepsilon)\}_\varepsilon$ in Lemma 3.3. Fix $M > 1$ arbitrary. For any measurable subset ω of Ω, using the growth condition of Φ in (1.9), we find that

$$\int_\omega |\widehat{\Phi}(U_\varepsilon)| \chi_{\{|U_\varepsilon| \leq M\}} \, dx \leq \phi(M) \left(\sum_{j=1}^N \|\partial_j T_M(U_\varepsilon)\|_{L^{p_j}(\omega)}^{p_j} + \|c\|_{L^1(\omega)} \right). \tag{4.12}$$

Since $\partial_j T_M(U_\varepsilon) \to \partial_j T_M(U_0)$ (strongly) in $L^{p_j}(\Omega)$ as $\varepsilon \to 0$ for every $1 \leq j \leq N$ and $c \in L^1(\Omega)$, we see that the right-hand side of (4.12) is as small as desired uniformly in ε when the measure of ω is small. We next bound from above $\int_\omega |\widehat{\Phi}(U_\varepsilon)| \chi_{\{|U_\varepsilon| > M\}} \, dx$. This is where the modification appears since we don't have anymore that $\{\widehat{\Phi}(U_\varepsilon) U_\varepsilon\}_\varepsilon$ is uniformly bounded in $L^1(\Omega)$ with respect to ε. We adapt an approach from [13]. In (4.5) we take

$$v = T_1(G_{M-1}(U_\varepsilon)) \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega).$$

Then, using (1.5), the coercivity condition in (1.8) and the sign-condition of Φ in (1.9), we obtain the estimate

$$\int_\Omega |\widehat{\Phi}(U_\varepsilon)| \chi_{\{|U_\varepsilon| > M\}} \, dx \leq \int_\Omega (|f_\varepsilon| + C_\Theta) \chi_{\{|U_\varepsilon| \geq M-1\}} \, dx + |\langle \mathfrak{B} U_\varepsilon, T_1(G_{M-1}(U_\varepsilon)) \rangle|. \tag{4.13}$$

Now, up to a subsequence of $\{U_\varepsilon\}$, from (4.8), we have

$$T_1(G_{M-1}(U_\varepsilon)) \rightharpoonup T_1(G_{M-1}(U_0)) \text{ (weakly) in } W_0^{1,\vec{p}}(\Omega) \text{ as } \varepsilon \to 0.$$

Using this in (4.13), jointly with (4.1) and the property (P_2) for $\mathfrak{B}$, we find that

$$\limsup_{\varepsilon \to 0} \int_\Omega |\widehat{\Phi}(U_\varepsilon)| \chi_{\{|U_\varepsilon| > M\}} \, dx \leq \int_\Omega (|f| + C_\Theta) \chi_{\{|U_0| \geq M-1\}} \, dx + |\langle \mathfrak{B} U_0, T_1(G_{M-1}(U_0)) \rangle|.$$

Recall that $f \in L^1(\Omega)$. Since $\partial_j T_1(G_{M-1}(U_0)) = \chi_{\{M-1 < |U_0| < M\}} \partial_j U_0$ a.e. in Ω for every $1 \leq j \leq N$, from the above inequality, we infer that

$$\int_\omega |\widehat{\Phi}(U_\varepsilon)| \chi_{\{|U_\varepsilon| > M\}} \, dx$$

is small, uniformly in ε and ω, when M is sufficiently large. Thus, using the comments after (4.12), we conclude the uniform integrability of $\{\widehat{\Phi}(U_\varepsilon)\}_\varepsilon$ over Ω. The proof of Lemma 4.3 is complete. $\qquad \square$

By letting $\varepsilon \to 0$ in (4.5), we conclude that U_0 is a solution of (1.1). This ends the proof of Theorem 1.3 *(ii)*. $\qquad \square$

5. Strong convergence of u_ε in Theorem 1.3 (*i*)

We show that in the setting of Theorem 1.3 (*i*), up to a subsequence of $\{u_\varepsilon\}$, not only the assertions of Lemma 3.2 hold, but also the strong convergence in (1.13), that is

$$u_\varepsilon \to U \text{ (strongly) in } W_0^{1,\vec{p}}(\Omega) \text{ as } \varepsilon \to 0. \tag{5.1}$$

Lemma 5.1. *Up to a subsequence of $\{u_\varepsilon\}_\varepsilon$, relabeled $\{u_\varepsilon\}_\varepsilon$, we have (5.1).*

Proof. For every $k \geq 1$, we define

$$L_k := v_0^{-1} \left[|\langle \mathfrak{B}U, G_k(U) \rangle| + C_\Theta \|G_k(U)\|_{L^1(\Omega)} \right]. \tag{5.2}$$

We next show that, up to a subsequence of $\{u_\varepsilon\}$, we have

$$\limsup_{\varepsilon \to 0} \|G_k(u_\varepsilon)\|_{W_0^{1,\vec{p}}(\Omega)} \leq \sum_{j=1}^N L_k^{1/p_j}. \tag{5.3}$$

Proof of (5.3). Let $k \geq 1$ be a fixed integer. Since $G_k(u_\varepsilon) = u_\varepsilon - T_k(u_\varepsilon)$ and $\partial_j T_k(u_\varepsilon) = \partial_j u_\varepsilon \chi_{\{|u_\varepsilon| < k\}}$ for every $1 \leq j \leq N$, from the coercivity assumption in (1.8), we see that

$$\langle \mathcal{A}u_\varepsilon, G_k(u_\varepsilon) \rangle = \sum_{j=1}^N \int_{\{|u_\varepsilon| > k\}} \widehat{A}_j(u_\varepsilon) \, \partial_j u_\varepsilon \, dx$$

$$\geq v_0 \sum_{j=1}^N \int_{\{|u_\varepsilon| > k\}} |\partial_j u_\varepsilon|^{p_j} \, dx = v_0 \sum_{j=1}^N \|\partial_j G_k(u_\varepsilon)\|_{L^{p_j}(\Omega)}^{p_j}.$$

Using (3.2) and $t\, G_k(t) \geq 0$ for every $t \in \mathbb{R}$, we observe that $G_k(t)\, \widehat{\Phi}_\varepsilon(t) \geq 0$ for all $t \in \mathbb{R}$. Then, by testing (3.4) with $v = G_k(u_\varepsilon)$ and using (1.5), we find that

$$\langle \mathcal{A}u_\varepsilon, G_k(u_\varepsilon) \rangle \leq \langle \mathcal{A}u_\varepsilon, G_k(u_\varepsilon) \rangle + \int_\Omega G_k(u_\varepsilon) \, \widehat{\Phi}_\varepsilon(u_\varepsilon) \, dx$$

$$\leq |\langle \mathfrak{B}u_\varepsilon, G_k(u_\varepsilon) \rangle| + C_\Theta \int_\Omega |G_k(u_\varepsilon)| \, dx.$$

From (3.6), the boundedness of $\{u_\varepsilon\}$ in $W_0^{1,\vec{p}}(\Omega)$ and Remark A.2, we can pass to a subsequence of $\{u_\varepsilon\}$ (relabeled $\{u_\varepsilon\}$) such that, as $\varepsilon \to 0$, we have

$$T_k(u_\varepsilon) \to T_k(U) \text{ a.e. in } \Omega \text{ and } T_k(u_\varepsilon) \rightharpoonup T_k(U) \text{ (weakly) in } W_0^{1,\vec{p}}(\Omega),$$

$$G_k(u_\varepsilon) \to G_k(U) \text{ a.e. in } \Omega \text{ and } G_k(u_\varepsilon) \rightharpoonup G_k(U) \text{ (weakly) in } W_0^{1,\vec{p}}(\Omega),$$

$$G_k(u_\varepsilon) \to G_k(U) \text{ strongly in } L^r(\Omega) \text{ with } 1 \leq r < p^*.$$

Hence, using the property (P_2), we derive that

$$\lim_{\varepsilon \to 0} \langle \mathfrak{B}u_\varepsilon, G_k(u_\varepsilon) \rangle = \langle \mathfrak{B}U, G_k(U) \rangle \quad \text{and} \quad \lim_{\varepsilon \to 0} \|G_k(u_\varepsilon)\|_{L^1(\Omega)} = \|G_k(U)\|_{L^1(\Omega)}.$$

Consequently, for every $1 \leq j \leq N$, we have

$$\limsup_{\varepsilon \to 0} \|\partial_j(G_k(u_\varepsilon))\|_{L^{p_j}(\Omega)} \leq \left(v_0^{-1} \left[|\langle \mathcal{B}U, G_k(U) \rangle| + C_\Theta \|G_k(U)\|_{L^1(\Omega)} \right]\right)^{1/p_j} = L_k^{1/p_j}.$$

This establishes the inequality in (5.3).

Recall that $\{u_\varepsilon\}_\varepsilon$ stands for a sequence $\{u_{\varepsilon_\ell}\}_{\ell \geq 1}$ with $\varepsilon_\ell \searrow 0$ as $\ell \to \infty$. By Lemma 3.1 and (5.3), as well as from the proof of Lemma 3.2, we get that for any given integer $k \geq 1$, there exists a subsequence of $\{u_\varepsilon\}_\varepsilon$ that depends on k, say $\{u_{\varepsilon_\ell}^{(k)}\}_{\ell \geq 1}$, for which (5.3) and (3.11) hold with $u_{\varepsilon_\ell}^{(k)}$ in place of $\{u_\varepsilon\}$. This means that

$$\limsup_{\ell \to \infty} \|G_k(u_{\varepsilon_\ell}^{(k)})\|_{W_0^{1,\vec{p}}(\Omega)} \leq \sum_{j=1}^N L_k^{1/p_j}, \qquad \lim_{\ell \to \infty} \|T_k(u_{\varepsilon_\ell}^{(k)}) - T_k(U)\|_{W_0^{1,\vec{p}}(\Omega)} = 0. \tag{5.4}$$

We proceed inductively with respect to k, at each step $(k + 1)$ selecting the subsequence $\{u_{\varepsilon_\ell}^{(k+1)}\}_{\ell \geq 1}$ from $\{u_{\varepsilon_\ell}^{(k)}\}_{\ell \geq 1}$, the subsequence of $\{u_\varepsilon\}$ with the properties in (5.4). Then, $\{u_{\varepsilon_\ell}^{(\ell)}\}_{\ell \geq k}$ is a subsequence of $\{u_{\varepsilon_\ell}^{(j)}\}_{\ell \geq 1}$ for every $1 \leq j \leq k$. Hence, by a standard diagonal argument, there exists a subsequence of $\{u_\varepsilon\}_\varepsilon$, that is, $\{u_{\varepsilon_\ell}^{(\ell)}\}_\ell$, relabeled $\{u_\varepsilon\}_\varepsilon$, such that (5.3) and (3.11) hold for every $k \geq 1$, namely

$$\limsup_{\varepsilon \to 0} \|G_k(u_\varepsilon)\|_{W_0^{1,\vec{p}}(\Omega)} \leq \sum_{j=1}^N L_k^{1/p_j}, \qquad \lim_{\varepsilon \to 0} \|T_k(u_\varepsilon) - T_k(U)\|_{W_0^{1,\vec{p}}(\Omega)} = 0. \tag{5.5}$$

Using the weak convergence of $G_k(u_\varepsilon)$ to $G_k(U)$ in $W_0^{1,\vec{p}}(\Omega)$ as $\varepsilon \to 0$, we see that

$$\|G_k(U)\|_{W_0^{1,\vec{p}}(\Omega)} \leq \liminf_{\varepsilon \to 0} \|G_k(u_\varepsilon)\|_{W_0^{1,\vec{p}}(\Omega)} \leq \sum_{j=1}^N L_k^{1/p_j}. \tag{5.6}$$

We now complete the proof of (5.1). From the definition of G_k in (1.17), we find that

$$\|u_\varepsilon - U\|_{W_0^{1,\vec{p}}(\Omega)} \leq \|G_k(u_\varepsilon)\|_{W_0^{1,\vec{p}}(\Omega)} + \|G_k(U)\|_{W_0^{1,\vec{p}}(\Omega)} + \|T_k(u_\varepsilon) - T_k(U)\|_{W_0^{1,\vec{p}}(\Omega)}.$$

Then, in view of (5.5) and (5.6), for every $k \geq 1$, we obtain that

$$\limsup_{\varepsilon \to 0} \|u_\varepsilon - U\|_{W_0^{1,\vec{p}}(\Omega)} \leq 2 \sum_{j=1}^N L_k^{1/p_j}. \tag{5.7}$$

Remark that L_k (defined in (5.2)) converges to 0 as $k \to \infty$ since $G_k(U) \rightharpoonup 0$ (weakly) in $W_0^{1,\vec{p}}(\Omega)$ and $G_k(U) \to 0$ (strongly) in $L^1(\Omega)$ as $k \to \infty$. Hence, by letting $k \to \infty$ in (5.7), we obtain (5.1). $\qquad \square$

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Conflict of interest

The authors declare no conflict of interest.

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A. Appendix

In this section, we prove some convergence results that have been used in Sections 2 and 3, respectively. We assume (1.2) and (1.8).

We first recall an anisotropic Sobolev inequality for the case $p < N$, see [40].

Lemma A.1. *Let $N \geq 2$ be an integer. If* (1.2) *holds, then there exists a constant $\mathcal{S} = \mathcal{S}(N, \vec{p}) > 0$ such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq \mathcal{S} \prod_{j=1}^{N} \|\partial_j u\|_{L^{p_j}(\mathbb{R}^N)}^{1/N} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^N).$$

Remark A.2. Let Ω be a bounded, open subset of $\mathbb{R}^N$ ($N \geq 2$). If (1.2) holds, then using a density argument and the arithmetic-geometric mean inequality, we find that

$$\|u\|_{L^{p^*}(\Omega)} \leq \mathcal{S} \prod_{j=1}^{N} \|\partial_j u\|_{L^{p_j}(\Omega)}^{1/N} \leq \frac{\mathcal{S}}{N} \|u\|_{W_0^{1,\vec{p}}(\Omega)} \quad \text{for all } u \in W_0^{1,\vec{p}}(\Omega). \tag{A.1}$$

Moreover, by Hölder's inequality, the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^s(\Omega)$ is continuous for every $s \in [1, p^*]$ and compact for every $s \in [1, p^*)$.

Remark A.3. Note that if $\Omega \subset \mathbb{R}^N$ is an open bounded domain with Lipschitz boundary and (1.2) holds, then the "true" critical exponent is p_∞, the maximum between p^* and p_N. Indeed, Fragalà, Gazzola and Kawohl [31] showed that the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for every $r \in [1, p_\infty]$ and compact if $r \in [1, p_\infty)$.

A.1. Notation

For v, w and $\{u_\varepsilon\}_\varepsilon$ in $W_0^{1,\vec{p}}(\Omega)$ and for a.e. $x \in \Omega$, we define

$$\mathcal{D}_{u_\varepsilon}(v, w)(x) = \sum_{j=1}^{N} \left[A_j(x, u_\varepsilon(x), \nabla v(x)) - A_j(x, u_\varepsilon(x), \nabla w(x)) \right] \partial_j(v - w)(x),$$

$$H_{u_\varepsilon}(v, w)(x) = \sum_{j=1}^{N} A_j(x, u_\varepsilon(x), \nabla v(x)) \, \partial_j w(x). \tag{A.2}$$

Hence, $\mathcal{D}_{u_\varepsilon}(v, w) = H_{u_\varepsilon}(v, v) - H_{u_\varepsilon}(v, w) - H_{u_\varepsilon}(w, v) + H_{u_\varepsilon}(w, w)$. The monotonicity assumption in (1.8) gives that $\mathcal{D}_{u_\varepsilon}(v, w) \geq 0$ a.e. in Ω, whereas the coercivity condition in (1.8) yields that $H_{u_\varepsilon}(v, v) \geq v_0 \sum_{j=1}^{N} |\partial_j v|^{p_j}$, where $v_0 > 0$. We thus find that

$$\mathcal{D}_{u_\varepsilon}(v, w) \geq v_0 \sum_{j=1}^{N} |\partial_j v|^{p_j} - |H_{u_\varepsilon}(v, w)| - |H_{u_\varepsilon}(w, v)|. \tag{A.3}$$

Here, we establish Lemma A.4, which is invoked in the proof of Lemma 2.7. Further, we prove Lemma A.5, which is useful in the proof of Theorem 1.3 (*i*) in Section 3. To prove Lemmata A.4 and A.5, we adapt an argument from [15, Lemma 5], the proof of which goes back to Browder [21]. As previously often recalled, by Remark A.2, whenever

$$u_\varepsilon \rightharpoonup u \text{ (weakly) in } W_0^{1,\vec{p}}(\Omega) \text{ as } \varepsilon \to 0, \tag{A.4}$$

we can pass to a subsequence (always relabeled $\{u_\varepsilon\}$) such that

$$u_\varepsilon \to u \text{ strongly in } L^r(\Omega) \text{ if } r \in [1, p^*) \text{ and } u_\varepsilon \to u \text{ a.e. in } \Omega. \tag{A.5}$$

A.2. Some convergence results

Lemma A.4. *Let u, $\{u_\varepsilon\}_\varepsilon$ be in $W_0^{1,\vec{p}}(\Omega)$ such that (A.4) holds. If $\mathcal{D}_{u_\varepsilon}(u_\varepsilon, u) \to 0$ a.e. in Ω as $\varepsilon \to 0$, then, up to a subsequence, $\nabla u_\varepsilon \to \nabla u$ a.e. in Ω as $\varepsilon \to 0$.*

Proof. Let Z be a subset of Ω with $\operatorname{meas}(Z) = 0$ such that for every $x \in \Omega \setminus Z$, we have $|u(x)| < \infty$, $|\nabla u(x)| < \infty$, $|\eta_j(x)| < \infty$ for all $1 \le j \le N$, as well as

$$u_\varepsilon(x) \to u(x), \quad \mathcal{D}_{u_\varepsilon}(u_\varepsilon, u)(x) \to 0 \text{ as } \varepsilon \to 0, \tag{A.6}$$

where η_j are the functions appearing in the growth condition in (1.8). For every $x \in \Omega \setminus Z$, we claim that

$$\{|\nabla u_\varepsilon(x)|\}_\varepsilon \text{ is uniformly bounded with respect to } \varepsilon. \tag{A.7}$$

Proof of (A.7). We fix $x \in \Omega \setminus Z$. In view of (A.3), we have

$$\mathcal{D}_{u_\varepsilon}(u_\varepsilon, u)(x) \ge \nu_0 \sum_{j=1}^N |\partial_j u_\varepsilon(x)|^{p_j} - |H_{u_\varepsilon}(u_\varepsilon, u)(x)| - |H_{u_\varepsilon}(u, u_\varepsilon)(x)|. \tag{A.8}$$

By Young's inequality, for every $\delta > 0$, there exists $C_\delta > 0$ such that

$$|H_{u_\varepsilon}(u_\varepsilon, u)(x)| \le \sum_{j=1}^N \left(\delta |A_j(x, u_\varepsilon, \nabla u_\varepsilon)|^{p'_j} + C_\delta |\partial_j u(x)|^{p_j} \right),$$

$$\tag{A.9}$$

$$|H_{u_\varepsilon}(u, u_\varepsilon)(x)| \le \sum_{j=1}^N \left(\delta |\partial_j u_\varepsilon(x)|^{p_j} + C_\delta |A_j(x, u_\varepsilon, \nabla u)|^{p'_j} \right).$$

We use the growth condition in (1.8) to bound from above the right-hand side of each inequality in (A.9). Then, from (A.8), there exist positive constants C and $\widehat{C_\delta}$, both independent of ε (with $\widehat{C_\delta}$ depending on δ), such that

$$\mathcal{D}_{u_\varepsilon}(u_\varepsilon, u)(x) \ge (\nu_0 - C\delta) \sum_{j=1}^N |\partial_j u_\varepsilon(x)|^{p_j} - \widehat{C_\delta}\, \mathfrak{g}_{u_\varepsilon}(u)(x), \tag{A.10}$$

where $\mathfrak{g}_{u_\varepsilon}(u)(x) = \sum_{j=1}^N \eta_j^{p'_j}(x) + |u_\varepsilon(x)|^{p^*} + \sum_{j=1}^N |\partial_j u(x)|^{p_j}$. Using (A.6) and choosing $\delta \in (0, \nu_0/C)$, from (A.10) we conclude (A.7).

Proof of Lemma A.4 concluded. Let $x \in \Omega \setminus Z$ be arbitrary. Define $\xi_\varepsilon = \nabla u_\varepsilon(x)$ and $\xi = \nabla u(x)$. To show that $\xi_\varepsilon \to \xi$ as $\varepsilon \to 0$, it is enough to prove that any accumulation point of ξ_ε, say ξ^*, coincides with ξ. From (A.7), we have $|\xi^*| < \infty$. By (A.6) and the continuity of $A_j(x, \cdot, \cdot)$ with respect to the last two variables, we find that

$$\mathcal{D}_{u_\varepsilon}(u_\varepsilon, u)(x) \to \sum_{j=1}^N \left[A_j(x, u(x), \xi^*) - A_j(x, u(x), \xi) \right](\xi_j^* - \xi_j) \quad \text{as } \varepsilon \to 0.$$

This, jointly with (A.6) and the monotonicity condition in (1.8), gives that $\xi^* = \xi$. This ends the proof since $x \in \Omega \setminus Z$ is arbitrary and $\operatorname{meas}(Z) = 0$. $\qquad\square$

Lemma A.5. *Let $k \geq 1$ be a fixed integer. Let u, $\{u_\varepsilon\}_\varepsilon$ be in $W_0^{1,\vec{p}}(\Omega)$ such that (A.4) holds. Suppose that, up to a subsequence of $\{u_\varepsilon\}$ (depending on k and relabeled $\{u_\varepsilon\}$)*

$$\mathcal{D}_{u_\varepsilon}(T_k(u_\varepsilon), T_k(u)) \to 0 \quad \text{in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{A.11}$$

Then, up to a subsequence of $\{u_\varepsilon\}$, as $\varepsilon \to 0$, we have

$$\nabla T_k(u_\varepsilon) \to \nabla T_k(u) \text{ a.e. in } \Omega, \tag{A.12}$$
$$T_k(u_\varepsilon) \to T_k(u) \text{ (strongly) in } W_0^{1,\vec{p}}(\Omega). \tag{A.13}$$

Proof. By (A.4) and (A.11), up to a subsequence of $\{u_\varepsilon\}$, we have (A.5), as well as $\mathcal{D}_{u_\varepsilon}(T_k(u_\varepsilon), T_k(u)) \to 0$ a.e. in Ω as $\varepsilon \to 0$. Let Z be a subset of Ω as in the proof of Lemma A.4, where $\mathcal{D}_{u_\varepsilon}(T_k(u_\varepsilon), T_k(u)))$ replaces $\mathcal{D}_{u_\varepsilon}(u_\varepsilon, u)$. We follow the same argument as in Lemma A.4 with the obvious modifications suggested by the above replacement. Then, for every $x \in \Omega \setminus Z$, we obtain

$$\mathcal{D}_{u_\varepsilon}(T_k(u_\varepsilon), T_k(u))(x) \geq \nu_0 \sum_{j=1}^N |\partial_j T_k(u_\varepsilon)(x)|^{p_j}$$
$$- |H_{u_\varepsilon}(T_k(u_\varepsilon), T_k(u))(x)| - |H_{u_\varepsilon}(T_k(u), T_k(u_\varepsilon))(x)|. \tag{A.14}$$

This leads to $\{|\nabla T_k(u_\varepsilon)(x)|\}_\varepsilon$ being uniformly bounded with respect to ε and we also obtain (A.12).

We conclude the proof of Lemma A.5 by showing (A.13). From (A.12), we see that $\{|\partial_j T_k(u_\varepsilon) - \partial_j T_k(u)|^{p_j}\}_\varepsilon$ is a sequence of nonnegative integrable functions, converging to 0 a.e. on Ω. Thus, by Vitali's Theorem, we obtain that $\partial_j T_k(u_\varepsilon) \to \partial_j T_k(u)$ in $L^{p_j}(\Omega)$ as $\varepsilon \to 0$ for every $1 \leq j \leq N$ by proving that

$$\left\{ \sum_{j=1}^N |\partial_j T_k(u_\varepsilon)|^{p_j} \right\}_\varepsilon \quad \text{is uniformly integrable over } \Omega. \tag{A.15}$$

The claim of (A.15) follows from (A.11) and (A.14) whenever $\{H_{u_\varepsilon}(T_k(u_\varepsilon), T_k(u))\}_\varepsilon$ and $\{H_{u_\varepsilon}(T_k(u), T_k(u_\varepsilon))\}_\varepsilon$ converge in $L^1(\Omega)$ as $\varepsilon \to 0$. We next establish that

$$H_{u_\varepsilon}(T_k(u_\varepsilon), T_k(u)) \to \sum_{j=1}^N A_j(x, u, \nabla T_k(u)) \partial_j T_k(u) \quad \text{in } L^1(\Omega) \text{ as } \varepsilon \to 0,$$
$$H_{u_\varepsilon}(T_k(u), T_k(u_\varepsilon)) \to \sum_{j=1}^N A_j(x, u, \nabla T_k(u)) \partial_j T_k(u) \quad \text{in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{A.16}$$

Proof of (A.16). Let $1 \leq j \leq N$ be arbitrary. We see that $\{A_j(x, u_\varepsilon, \nabla T_k(u_\varepsilon))\}_\varepsilon$ is bounded in $L^{p_j'}(\Omega)$ from the growth condition in (1.8) and the boundedness of $\{u_\varepsilon\}_\varepsilon$ in $W_0^{1,\vec{p}}(\Omega)$ and, hence, in $L^{p^*}(\Omega)$. Moreover, $A_j(x, u_\varepsilon, \nabla T_k(u_\varepsilon)) \to A_j(x, u, \nabla T_k(u))$ a.e. in Ω as $\varepsilon \to 0$ using (A.12), the convergence $u_\varepsilon \to u$ a.e. in Ω (from (A.5)) and the continuity of $A_j(x, \cdot, \cdot)$ in the last two variables. Thus, up to a subsequence of $\{u_\varepsilon\}$, we infer that $A_j(x, u_\varepsilon, \nabla T_k(u_\varepsilon)) \rightharpoonup A_j(x, u, \nabla T_k(u))$ (weakly) in $L^{p_j'}(\Omega)$ as $\varepsilon \to 0$. This proves the first convergence in (A.16). We now prove the second one.

Using (A.12) and the continuity properties of A_j, as $\varepsilon \to 0$,

$$A_j(x, u_\varepsilon, \nabla T_k(u)) \partial_j T_k(u_\varepsilon) \to A_j(x, u, \nabla T_k(u)) \partial_j T_k(u) \text{ a.e. in } \Omega \tag{A.17}$$

for each $1 \le j \le N$. Observe that $\{\chi_{\{|u_\varepsilon|<k\}}|A_j(x, u_\varepsilon, \nabla T_k(u))|^{p'_j}\}_\varepsilon$ is uniformly integrable over Ω (from the growth condition of A_j in (1.8)) and $\partial_j T_k(u_\varepsilon) = \chi_{\{|u_\varepsilon|<k\}} \partial_j u_\varepsilon$. Thus, since $\{\partial_j u_\varepsilon\}_\varepsilon$ is bounded in $L^{p_j}(\Omega)$, it follows from Hölder's inequality that $\{A_j(x, u_\varepsilon, \nabla T_k(u)) \partial_j T_k(u_\varepsilon)\}_\varepsilon$ is uniformly integrable over Ω for each $1 \le j \le N$. From (A.17) and Vitali's Theorem, we reach the claim of (A.16). $\qquad\square$

From Lemma A.5 and a standard diagonal argument, we obtain the following.

Corollary A.6. *Let* (A.4) *and* (A.11) *hold. Then, there exists a subsequence of* $\{u_\varepsilon\}_\varepsilon$*, relabeled* $\{u_\varepsilon\}_\varepsilon$*, such that* $\nabla u_\varepsilon \to \nabla u$ *a.e. in* Ω *and* $T_k(u_\varepsilon) \to T_k(u)$ *(strongly) in* $W_0^{1,\vec{p}}(\Omega)$ *as* $\varepsilon \to 0$ *for every integer* $k \ge 1$.