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Cones of cycles on blowups of $(\mathbb{P}^1)^n$

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Abstract

We study cones of pseudoeffective cycles on the blow up of $(\mathbb{P}^1)^n$ at points in very general position, proving some results concerning their structure. In particular, we show that in some cases they turn out to be generated by exceptional classes and fiber classes relatively to the projections onto a smaller number of copies of projective lines.

Keywords Blowing-ups · Cones of effective and nef cycles · Mori dream spaces

Mathematics Subject Classification Primary 14C25; Secondary 14M07 · 14E99

1 Introduction

Cones of curves and divisors of an algebraic variety X are closely related to the birational geometry of X and they have played a central role since the work of Mori in 1980s.

More recently, the interest in cycles of intermediate dimensions has increased, and there has been significant progress in understanding the cones of such cycles (see for instance [4, 5, 10]). Nevertheless, the number of examples in which cones of effective cycles have been explicitly computed is relatively small. The most significant results concern blowups of projective space $\mathbb{P}^n$ at sets of points [3]. In this case, if the number r of points is small, the blow up X is a toric variety, so that the pseudoeffective cone $\overline{\mathrm{Eff}}_k(X)$ of k-dimensional cycles is polyhedral for any $1 \le k \le n-1$, and generated by classes of torus-invariant subvarieties (see for instance [11]). These are linear classes, i.e. classes of strict transforms of k-dimensional linear spaces containing some of the points, and k-dimensional linear spaces contained in the exceptional divisors. Starting from this observation, the authors show that the cone $\overline{\mathrm{Eff}}_k(X)$ is still generated by these linear classes even if the number of points is increased beyond the toric range, and in particular they give bounds on $k,\ r$ and n that guarantee this property.

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In this paper we consider the analogous problem for the blow up X_r^n of $(\mathbb{P}^1)^n$ at a set of r very general points. Blowing up a small number of points (up to 2) in $(\mathbb{P}^1)^n$ we still have a toric variety, so that the cone $\overline{\mathrm{Eff}}_k(X_r^n)$ is generated by classes of torus-invariant subvarieties. It turns out that these are the linear spaces in the exceptional divisors and the strict transforms of the fibers of the projections $(\mathbb{P}^1)^n \to (\mathbb{P}^1)^{n-k}$ (see Proposition 3.4). We then say that the cone $\overline{\mathrm{Eff}}_k(X_r^n)$ is *fiber-generated* (Definition 3.1).

Therefore, a natural problem is to investigate the values of n, k and r for which $\overline{\mathrm{Eff}}_k(X_r^n)$ is fiber-generated. If so, the latter is in particular rational polyhedral, as well as the dual cone $\mathrm{Nef}^k(X_r^n)$ of codimension k nef cycles, thus giving more classes of varieties as in Problem 6.9 in [4].

Our first result concerns the "extremal" cases, i.e. $k = n-1$ or $k = 1$, when it is possible to give a complete characterization.

Theorem 1 *Let n be any integer greater than one.*

1. *(Corollary 3.6) $\overline{\mathrm{Eff}}_{n-1}(X_n^r)$ is fiber-generated if and only if $r \leq 2$.*
2. *(Theorem 3.7) $\overline{\mathrm{Eff}}_1(X_n^r)$ is fiber-generated if and only if $r \leq n!$.*

The analogous result for the blow-up Y_r^n of $\mathbb{P}^n$ at r general points is the following: $\overline{\mathrm{Eff}}_{n-1}(Y_r^n)$ is linearly generated if and only if $r \leq n+2$ ([3, Thm. 2.7]); $\overline{\mathrm{Eff}}_1(Y_r^n)$ is linearly generated if and only if $r \leq 2^n$ ([3, Prop. 4.1]).

The theorem above suggests that the number r of points that can be blown up in order to have fiber-generated cones decreases as the dimension k of the cycles increases. This is confirmed by the following result, which gives a bound for any $1 \leq k \leq n-1$ (the characterization given for $k = 1$ shows that in general this bound is not sharp).

Theorem 2 *(Theorem 3.9) For any $1 \leq k \leq n-1$, the pseudoeffective cone $\overline{\mathrm{Eff}}_k(X_n^r)$ is fiber-generated if $r \leq n-k+1$.*

On the negative side, we prove that for any n and k, if we take a sufficiently large number r of points, the cone $\overline{\mathrm{Eff}}_k(X_r^n)$ is not fiber-generated.

Proposition 1 *(Proposition 4.1) The cone $\overline{\mathrm{Eff}}_k(X_r^n)$ is not fiber-generated for $r > (n-k+1)!$.*

For $k = 1$ or $n-1$ the above bound is sharp, while for the other values of k we are not able to check the sharpness. We remark that when $n = 4$, combining Theorem 2 with the above proposition, only three cases remain open, i.e. $\overline{\mathrm{Eff}}_2(X_r^4)$, for $r = 4, 5, 6$. In Proposition 4.2 we will deal with the case $r = 4$, showing that $\overline{\mathrm{Eff}}_2(X_4^4)$ is still fiber-generated.

Finally we recall that if X is a *Mori dream space* (see [7]), then the cone $\overline{\mathrm{Eff}}_{n-1}(X)$ is polyhedral, but in general the cones $\overline{\mathrm{Eff}}_k(X)$ are not necessarily polyhedral for $1 \leq k \leq n-2$ (see for instance [4, Ex. 6.2]). Nevertheless, in [3] it is proved that if X is the blow up of $\mathbb{P}^n$ at r general points, in all the cases in which X is Mori dream, $\overline{\mathrm{Eff}}_k(X)$ turns out to be polyhedral for any k. Therefore it makes sense to consider the analogous question in our case.

Question 3 *If X_r^n is a Mori dream space, is the cone $\overline{\mathrm{Eff}}_k(X_r^n)$ polyhedral for any $1 \leq k \leq n-1$?*

We close our paper with some comments about the above question, showing that our results give a partial answer.

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2 Preliminaries

Let X be a smooth projective variety of dimension n. Let $Z_k(X)$ denote the group of algebraic cycles of dimension k on X. We define the space of numerical classes of k-cycles

$$N_k(X) := (Z_k(X)/\equiv) \otimes \mathbb{R}$$

where $\equiv$ is the numerical equivalence. For any $1 \leq k \leq n-1$, $N_k(X)$ is a finite-dimensional vector space, and intersection gives a perfect pairing

$$N_k(X) \times N_{n-k}(X) \to \mathbb{R}.$$

Given a k-dimensional subvariety $Z \subseteq X$, we write $[Z]$ to denote its class in $N_k(X)$. We recall that if Y and Z are subvarieties of X of dimension k and $n-k$ respectively, and $Y \cap Z$ is a finite set, then $[Y] \cdot [Z] \geq 0$.

A class $\omega \in Z_k(X)$ is called *effective* if there exist subvarieties $Z_1, \ldots, Z_s \subseteq X$ and real positive numbers $\alpha_1, \ldots, \alpha_s$ such that $\omega = \sum_i \alpha_i [Z_i]$. We denote by $\overline{\mathrm{Eff}}_k(X) \subseteq N_k(X)$ the *pseudoeffective cone* of k-dimensional classes, i.e. the cone containing the (limits of) classes of k-dimensional effective cycles.

A class $\omega \in Z_k(X)$ is called *nef* if $\omega \cdot [Y] \geq 0$, for any subvariety $Y \subseteq X$ of codimension k, or equivalenty $\omega \cdot \alpha \geq 0$, for any effective class $\alpha \in Z_{n-k}(X)$. We denote by $\mathrm{Nef}_k(X)$ the *nef cone*, i.e. the cone of nef k-dimensional classes. If we write $\mathrm{Nef}^k(X)$ we refer to codimension k nef classes.

Let now Γ be a set of r distinct points $q_1, \ldots, q_r$ in $(\mathbb{P}^1)^n$. In what follows, we will denote by $\pi : X^n_\Gamma \to (\mathbb{P}^1)^n$ the blow up of $(\mathbb{P}^1)^n$ at the points in Γ, with exceptional divisors $E_1, \ldots, E_r$. We recall that the points q_i are in *very general position* (or equivalently they are *very general*) if their configuration lies in the complement of a countable union of proper configurations. If this is the case, we will simply write X^n_r instead of X^n_Γ. Given a subset $I \subseteq \{1, \ldots, n\}$ of cardinality $|I| = n-k$ we denote by $p_I : (\mathbb{P}^1)^n \to (\mathbb{P}^1)^{n-k}$ the projection defined by

$$((x_1 : y_1), \ldots, (x_n : y_n)) \mapsto ((x_i : y_i) \mid i \in I).$$

Moreover we set $H_I \in \mathrm{N}_k(X^n_\Gamma)$ to be the class of the pull back of a fiber of p_I via the blow-up π and $E_{j,k} \in \mathrm{N}_k(X^n_\Gamma)$ to be the class of a linear space of dimension k in the exceptional divisor E_j. Therefore, for any $1 \leq k \leq n-1$, the vector space $\mathrm{N}_k(X^n_\Gamma)$ has dimension $\binom{n}{k} + r$, with basis

$$\{H_I \mid |I| = n-k\} \cup \{E_{j,k} \mid 1 \leq j \leq r\}.$$

When $|I| = 1$ we will simply write H_i instead of $H_{\{i\}}$.

Let Y be a k-dimensional irreducible subvariety in X^n_Γ. Reasoning as in [3, Lem. 2.3] we have that the following holds:

- if $Y \subseteq E_j$ for some $1 \leq j \leq r$, then $[Y] = b_j E_{j,k}$ for some $b_j > 0$;
- if Y is not contained in any exceptional divisor then the numerical class of Y is given by $[Y] = \sum_{|I|=n-k} a_I H_I - \sum_{j=1}^r b_j E_{j,k}$, with $a_I, b_j \geq 0$.

Remark 2.1 From the intersection formulas on $(\mathbb{P}^1)^n$ and from the fact that every E_j is isomorphic to $\mathbb{P}^{n-1}$, we have that for any $I, J \subseteq \{1, \ldots, n\}$ and $1 \leq j \leq r$,

$$H_I \cdot H_J = \begin{cases} H_{I \cup J}, & \text{if } I \cap J = \emptyset \\ 0, & \text{otherwise,} \end{cases} \qquad (-1)^{n-k+1} E_j^{n-k} = E_{j,k}, \quad E_j \cdot E_{j,k} = -E_{j,k-1}.$$

In particular, when $|I| = n-k$ and $|J| = k$ we deduce that

$$H_I \cdot H_J = \begin{cases} 1, \text{ if } I \cup J = \{1, \ldots, n\} \\ 0, \text{ otherwise,} \end{cases} \qquad H_I \cdot E_{j,n-k} = 0, \qquad E_{j,n-k} \cdot E_{j,k} = -1$$

Therefore, if we choose a suitable order on the generators, the matrix of the pairing $N_k(X^n_\Gamma) \times N_{n-k}(X^n_\Gamma) \to \mathbb{R}$ is the diagonal join of I_N and $-I_r$, where $N = \binom{n}{k}$.

We recall that for any $n \geq 2$, given n points $p_1, \ldots, p_n \in \mathbb{P}^n$ in very general position there exists an isomorphism in codimension 1

$$\varphi : \mathrm{Bl}_{p_1 \ldots p_n} \mathbb{P}^n \dashrightarrow \mathrm{Bl}_{q_1} (\mathbb{P}^1)^n. \tag{1}$$

Therefore, for any $r \geq 1$ we have an isomorphism in codimension 1 between the blow up of $\mathbb{P}^n$ at $n+r-1$ very general points and the blow up X^n_r of $(\mathbb{P}^1)^n$ at r very general points. If we denote by $\mathcal{H}, \mathcal{E}_1, \ldots, \mathcal{E}_{n+r-1}$, a basis for the Picard group of the blow up of $\mathbb{P}^n$, the isomorphism on the Picard groups induced by φ can be described as follows (see [8])

$$\begin{cases} \mathcal{H} \mapsto \sum_{i=1}^n H_i - (n-1)E_1 & \\ \mathcal{E}_i \mapsto H_{n+1-i} - E_1 & \text{for } 1 \leq i \leq n \\ \mathcal{E}_i \mapsto E_{i-n+1} & \text{for } i \geq n+1. \end{cases}$$

In particular, the pseudoeffective cone $\overline{\mathrm{Eff}}_{n-1}(X^n_r)$ is isomorphic to the pseudoeffective cone of the blow up of $\mathbb{P}^n$ at $n+r-1$ very general points.

Since φ and its birational inverse contract varieties of codimension ≥ 2, they can not be used in order to study the cones $\overline{\mathrm{Eff}}_k(X^n_r)$ when $k \leq n-2$.

For the purposes of what follows, we briefly recall the definition of multiplicity of a subvariety in an ambient variety, as well as that of intersection multiplicity along a subvariety.

Let M be a smooth ambient variety of dimension n. Assume Z is a subvariety of M. Then *the multiplicity* $e_Z(M)$ *of* M *along* Z is the coefficient of the class $[Z]$ in the (total) Segre class $s(Z, M)$, i.e. the Segre class of the normal cone $C_Z M$ to Z in M. As explained in [6, Ex. 4.3.1], this definition is equivalent to that of the multiplicity of the local ring $\mathcal{O}_{Z,M}$ given by Samuel. Since $\mathcal{O}_{Z,M}$ can be viewed as the stalk of the structure sheaf $\mathcal{O}_M$ at the generic point of Z, the multiplicity of M along Z can also be computed as the multiplicity of M at the generic point of Z.

Let $Z_1, \ldots, Z_r$ be pure-dimensional subschemes of M with

$$m = \sum_{j=1}^r \dim(Z_j) - (r-1)n \geq 0.$$

If Z is an m-dimensional smooth irreducible component of the intersection $\cap_{j=1}^r Z_j$, let $\pi : \tilde{M} \to M$ be the blow-up of M along Z. Denote by $j : E \to \tilde{M}$ the embedding of the exceptional divisor E into $\tilde{M}$. Moreover, let $\eta : E \to Z$ be the projective bundle morphism of E onto Z, and $\tilde{Z}_j$ the blow-up of Z_j along the intersection $Z_j \cap Z$. Then Example 12.4.4. in [6] states that

$$i(Z, Z_1 \cdot \ldots \cdot Z_r; M) = \prod_{j=1}^r e_Z(Z_j) + q,$$

where $i(Z, Z_1 \cdot \ldots \cdot Z_r; M)$ is the intersection multiplicity as defined in [6, Ex. 8.2.1] and q is the coefficient of $[Z]$ in the cycle $\eta_*(j^*(\tilde{Z}_1 \cdot \ldots \cdot \tilde{Z}_r)) \in A_*(Z)$.

We are now going to prove a technical result about the blow-up X_1^n of $(\mathbb{P}^1)^n$ at one point q_1. In what follows we will denote simply by L the 1-dimensional fiber with class $H_{\{2,\dots,n\}} - E_{1,1}$.

Lemma 2.2 *If we denote by W_s the class $H_2 + \cdots + H_n - sE_1$, then for any $1 \le s \le n-1$ the following hold.*

(i) *The base locus of the linear system $|W_s|$ is the union of fibers with classes $H_I - E_{1,s}$, for $I \subseteq \{2,\dots,n\}$ and $|I| = n - s$.*
(ii) *The general divisor in $|W_s|$ has multiplicity s along the curve L.*
(iii) *If $s \le n-2$, the intersection of a general divisor in $|W_s|$ with the base locus of $|W_{s+1}|$ is the base locus of $|W_s|$.*

Proof First of all observe that the linear system $|H_2 + \cdots + H_n|$ corresponds to $\mathcal{O}(0, 1, \dots, 1)$ on $(\mathbb{P}^1)^n$. A basis for this linear system is given by the 2^{n-1} monomials

$$\left\{ \prod_{i \in I, j \in I^c} x_j y_i, \mid I \subseteq \{2,\dots,n\} \right\}$$

where we set $I^c := \{2,\dots,n\} \setminus I$. For any integer $1 \le s \le n-1$, the subsystem $|W_s| = |H_2 + \cdots + H_n - sE_1|$ corresponds to $\mathcal{O}(0, 1, \dots, 1) \otimes \mathcal{I}_{q_1}^s$ (i.e. the sections vanishing with multiplicity at least s at q_1). We can suppose that the point that we blow up is $q_1 := ((1:0), \dots, (1:0)) \in (\mathbb{P}^1)^n$, so that the system $|W_s|$ has the following monomial basis

$$\left\{ \prod_{i \in I, j \in I^c} x_j y_i, \mid I \subseteq \{2,\dots,n\},\ |I| \ge s \right\}. \tag{2}$$

Let us prove (i). Observe that the monomial $y_2 \cdots y_n$ appears in a basis for $|W_s|$, for any $1 \le s \le n-1$. In order for this monomial to vanish, it must be $y_j = 0$ for some $j \in \{2,\dots,n\}$. We substitute in the monomial $x_j \prod_{i \neq j} y_i$ and since x_j can not vanish, we get that at least another y_i vanishes. Iterating the above reasoning, in order to make all the monomials vanish, at least $n - s$ variables y_l must vanish, which gives the statement.

Let us prove (ii). Since the ideal $\mathcal{I}_L$ of the curve L is $(y_2, \dots, y_n)$, it is enough to observe that every monomial in (2) belongs to the power $\mathcal{I}_L^s$, and there is at least one monomial that does not belong to $\mathcal{I}_L^{s+1}$.

Finally, since by (i) we know the base locus of $|W_{s+1}|$, in order to prove (iii) it suffices to show that a general element in $|W_s|$ intersects a fiber $H_I - E_{1,s+1}$, with $|I| = n - s - 1$, along the fibers $H_{I \cup \{j\}} - E_{1,s}$, for $j \notin I$. Indeed the fiber $H_I - E_{1,s+1}$ can be described by the vanishing of all the variables y_i, for $i \in I$. If we substitute in a defining equation for W_s, all the monomials but $\prod_{i \in I,\ j \in I^c} x_i y_j$ vanish. Since $x_i \neq 0$ for every $i \in I$, the vanishing of this monomial gives the statement. □

Let us consider the blow-up $\sigma : \tilde{X} \to X_1^n$ along L, with exceptional divisor E. Since the normal bundle of L in X_1^n is $\mathcal{O}(-1)^{n-1}$, E is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^{n-2}$. By abuse of notation, in what follows we will denote simply by H_i the pull-back $\sigma^* H_i$ and by E_1 the pull-back $\sigma^* E_1$.

Lemma 2.3 *For any $1 \le s \le n-1$, let $\tilde{W}_s$ be the strict transform of W_s. The restriction of $|\tilde{W}_s|$ to E cuts on a general $\mathbb{P}^{n-2}$ the linear system of hypersurfaces of degree s, with $n-1$ general points of multiplicity $s-1$.*

Proof Since we are blowing up $(\mathbb{P}^1)^n$ at the invariant point q_1, and then along the invariant curve L, $\tilde{X}$ is a toric variety. The fan $\Sigma \subseteq \mathbb{Q}^n$ of $(\mathbb{P}^1)^n$ has $2n$ rays, generated by $\pm e_i$, for $1 \leq i \leq n$. We can suppose that the point q_1 corresponds to the n-dimensional cone $\langle -e_i \mid i \in \{1, \dots, n\}\rangle$, while L corresponds to the $(n-1)$-dimensional cone $\langle -e_i \mid i \in \{2, \dots, n\}\rangle$. Therefore the fan $\tilde{\Sigma}$ of the blow up $\tilde{X}$ is obtained from Σ by a stellar subdivision of the above two cones with respect to the rays $\rho_1 := -\sum_{i=1}^n e_i$ and $\rho_2 := -\sum_{i=2}^n e_i$ respectively. Each ray corresponds to an invariant divisor, and to a variable in the Cox ring $\mathbb{C}[T_1, S_1, \dots, T_n, S_n, U_1, U_2]$ of $\tilde{X}$, as follows:

$$\begin{array}{rll} e_i \Rightarrow H_i & & \Rightarrow T_i,\ \ i = 1, \dots, n \\ -e_1 \Rightarrow H_1 - E_1 & & \Rightarrow S_1 \\ -e_i \Rightarrow H_i - E_1 - E & & \Rightarrow S_i,\ \ i = 2, \dots, n \\ \rho_1 \Rightarrow E_1 - E & & \Rightarrow U_1 \\ \rho_2 \Rightarrow E & & \Rightarrow U_2. \end{array}$$

We can now describe the sections of $|\tilde{W}_s| = |\sum_{i=2}^n H_i - sE_1 - sE|$ in Cox coordinates as

$$\left\{ \prod_{i \in I, j \in I^c} T_j S_i (U_1 U_2)^{|I|-s} \mid I \subseteq \{2, \dots, n\},\ |I| \geq s \right\}. \tag{3}$$

Let us consider the restriction of these sections to the exceptional divisor $E = \mathbb{P}^1 \times \mathbb{P}^{n-2}$. The fan of E lies inside $\mathbb{Q}^{n-1}$ and can be obtained taking the images of the cones containing ρ_2, via the quotient by ρ_2. The quotient map can be described via the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 1 \\ 0 & 0 & -1 & \dots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 1 \\ 0 & 0 & 0 & \dots & -1 & 1 \end{pmatrix}.$$

The rays of the cones containing ρ_2 are $e_1, \rho_1, -e_2, \dots, -e_n$, so that their images via the quotient are

$$\varepsilon_1,\ -\varepsilon_1,\ \varepsilon_2,\ \dots,\ \varepsilon_{n-1},\ -\sum_{i=2}^{n-1} \varepsilon_i,$$

where $\varepsilon_1, \dots, \varepsilon_{n-1}$ is the canonical basis of $\mathbb{Q}^{n-1}$. Moreover, if we substitute $U_2 = 0$ in (3), we obtain the restriction of the sections of $|\tilde{W}_s|$ to E, namely

$$\left\{ \prod_{i \in I, j \in I^c} T_j S_i \mid I \subseteq \{2, \dots, n\},\ |I| = s \right\},$$

corresponding to $\{\prod_{i \in I} z_i \mid |I| = s\}$ on $\mathbb{P}^{n-2}$, with variables $z_2, \dots, z_n$. These squarefree monomials are a basis of the linear system of hypersurfaces of degree s, having multiplicity at least $s-1$ at the $n-1$ fundamental points of $\mathbb{P}^{n-2}$.

□

Lemma 2.4 *Let $v = (a_1, \dots, a_N, -b_1, \dots, -b_r) \in \mathbb{Z}^{N+r}$ be a vector satisfying the following inequalities:*

1. *$a_i \geq 0$ and $b_j \geq 0$, for any $1 \leq i \leq N$ and $1 \leq j \leq r$;*

2. $\sum_{i=1}^{N} a_i \geq \sum_{j=1}^{r} b_j$.

Then v is a non-negative linear combination of e_j and $e_i - e_j$, for $1 \leq i \leq N$ and $N+1 \leq j \leq N+r$.

Proof We argue by induction on $b := \sum_{j=1}^{r} b_j$. If $b = 0$ we have that

$$v = (a_1, \dots, a_N, 0 \dots, 0) = \sum_{i=0}^{N} a_i(e_i - e_{N+1}) + \left(\sum_{i=1}^{N} a_i\right) e_{N+1}$$

and since $a_i \geq 0$, the claim follows. Let us suppose the statement to be true for $b > 0$ and let us consider $v := (a_1, \dots, a_N, -b_1 \dots, -b_r) \in \mathbb{Z}^{N+r}$ such that $\sum_{j=1}^{r} b_j = b + 1$. There exist at least two indexes $1 \leq i \leq N$ and $1 \leq j \leq r$ such that $a_i, b_j > 0$. We can write $v = (a_1, \dots, a_i - 1, \dots, a_N, -b_1, \dots, -(b_j - 1), \dots, -b_r) + (e_i - e_{N+j})$. We conclude by the induction hypothesis. □

3 Fiber-generated effective cones

In this section we first give the definition of fiber generated pseudoeffective cone and, after that, we present the main results of our work.

Definition 3.1 For any $1 \leq k \leq n-1$ we define the *cone of fibers* $\mathrm{CF}_k(X^n_\Gamma) \subseteq \overline{\mathrm{Eff}}_k(X^n_\Gamma)$ to be the cone generated by the classes

$$\begin{array}{ll} H_I - E_{i,k}, & |I| = n-k,\ 1 \leq i \leq r \\ E_{i,k}, & 1 \leq i \leq r \end{array}$$

i.e. the k-dimensional fibers through the points and the classes of k-dimensional linear spaces in the exceptional divisors. If the equality $\overline{\mathrm{Eff}}_k(X^n_\Gamma) = \mathrm{CF}_k(X^n_\Gamma)$ holds, we say that the pseudoeffective cone $\overline{\mathrm{Eff}}_k(X^n_\Gamma)$ is *fiber-generated*.

Below, you find two different characterizations of fiber-generation: the first is geometric and depends on the nefness of a specific class; the second is numeric and depends on the nonnegativity of a combination of suitable integers.

Lemma 3.2 *Let $\Gamma = \{q_1, \dots, q_r\}$ be a set of distinct points of $(\mathbb{P}^1)^n$. For any $1 \leq k \leq n-1$, the pseudoeffective cone $\overline{\mathrm{Eff}}_k(X^n_\Gamma)$ is fiber-generated if and only if the class*

$$\sum_{|I|=k} H_I - \sum_{j=1}^{r} E_{j,n-k}$$

is nef.

Proof First of all observe that $\overline{\mathrm{Eff}}_k(X^n_\Gamma)$ is fiber-generated if and only if the nef cone $\mathrm{Nef}_{n-k}(X^n_\Gamma)$ coincides with the dual cone of $\mathrm{CF}_k(X^n_\Gamma)$. Using the matrix of the intersection product (see Remark 2.1), we can see that the dual of $\mathrm{CF}_k(X^n_\Gamma)$ has the following $\binom{n}{k} + 2^r - 1$ extremal rays

$$H_I \quad \text{for } |I| = k, \qquad \text{and} \qquad \sum_{|I|=k} H_I - \sum_{j \in S} E_{j,n-k}, \quad \text{for } S \subseteq \{1, \dots, r\},$$

so that $\overline{\mathrm{Eff}}_k(X^n_\Gamma)$ is fiber-generated if and only if all the above classes are nef. Observe that any class H_I is nef since it is the intersection of the nef divisors H_i, for $i \in I$ (see for instance [5]). We now claim that any class of the form $\sum_{|I|=k} H_I - \sum_{j\in S} E_{j,n-k}$, with $S \subseteq \{1,\dots,r\}$ is nef if and only if the class $\sum_{|I|=k} H_I - \sum_{j=1}^r E_{j,n-k}$ is. Indeed, given any k-dimensional subvariety $V \subseteq X^n_\Gamma$, not contained in an exceptional divisor, we have $[V] = \sum_{|J|=n-k} a_J H_J - \sum_{j=1}^r b_j E_{j,k} \in \overline{\mathrm{Eff}}_k(X^n_\Gamma)$, where $a_J, b_j \geq 0$. For any $S \subseteq \{1,\dots,r\}$ we can write

$$[V]\cdot\left(\sum_{|I|=k} H_I - \sum_{j\in S} E_{j,n-k}\right) = [V]\cdot\left(\sum_{|I|=k} H_I - \sum_{j=1}^{r} E_{j,n-k}\right) + \sum_{j\notin S} b_j.$$

We conclude observing that if $\sum_{|I|=k} H_I - \sum_{j=1}^r E_{j,n-k}$ is nef then the above intersection products are all non-negative. □

Lemma 3.3 *The pseudoeffective cone $\overline{\mathrm{Eff}}_k(X^n_\Gamma)$ is fiber-generated if and only if for any k-dimensional variety Y, not contained in an exceptional divisor and having class $[Y] = \sum_{|I|=n-k} a_I H_I - \sum_{j=1}^r b_j E_{j,k}$, the inequality $\sum_{|I|=n-k} a_I \geq \sum_{j=1}^r b_j$ holds.*

Proof Observe that if Y is not contained in an exceptional divisor, then $[Y] = \sum_{|I|=n-k} a_I H_I - \sum_{j=1}^r b_j E_{j,k}$, with $a_I, b_j \geq 0$. If the inequality $\sum_{|I|=n-k} a_I \geq \sum_{j=1}^r b_j$ holds, then we can apply Lemma 2.4 with $N = \binom{n}{k}$. We conclude that $[Y]$ is a non-negative linear combination of $E_{j,k}$ and $H_I - E_{j,k}$, for $|I| = n-k$ and $1 \leq j \leq r$, so that $\overline{\mathrm{Eff}}_k(X^n_\Gamma)$ is fiber-generated.

On the other hand, if $\overline{\mathrm{Eff}}_k(X^n_\Gamma)$ is fiber-generated, then by Lemma 3.2 the class $\sum_{|J|=k} H_J - \sum_{j=1}^r E_{j,n-k}$ is nef. Taking the intersection product of the latter with $[Y]$ we get the inequality in the statement. □

Our first result is that if we blow up few very general points, then all the pseudoeffective cones are fiber-generated.

Proposition 3.4 *If $r \leq 2$, the cone $\overline{\mathrm{Eff}}_k(X^n_r)$ is fiber-generated for any $n \geq 2$ and $1 \leq k \leq n-1$.*

Proof If $r \leq 2$, the blow up X^n_r is a toric variety. In order to prove the statement it is enough to deal with the case $r = 2$. By [9, Prop. 3.1], for any $1 \leq k \leq n-1$, the cone $\overline{\mathrm{Eff}}_k(X^n_2)$ is generated by the classes of invariant k-cycles, i.e. the classes of cycles corresponding to the cones of codimension k in the fan of X^n_2.

We recall that the fan $\Sigma \subseteq \mathbb{Q}^n$ of $(\mathbb{P}^1)^n$ has $2n$ rays, generated by $\pm e_i$, for $1 \leq i \leq n$. For any $0 \leq k \leq n-1$ there are $2^{n-k} \cdot \binom{n}{k}$ cones of codimension k in Σ, namely

$$\langle \pm e_i \mid i \in I\rangle, \quad |I| = n-k.$$

If we blow up two points $q_1, q_2 \in (\mathbb{P}^1)^n$, we can suppose that they correspond to the n-dimensional cones $C_1 = \langle e_i \mid i \in \{1,\dots,n\}\rangle$ and $C_2 = \langle -e_i \mid i \in \{1,\dots,n\}\rangle$ respectively. Therefore the fan Σ' of the blow up X^n_2 is obtained from Σ by a stellar subdivision of C_1 and C_2 with respect to the rays $\rho := \sum_i e_i$ and $-\rho$ respectively, so that it has $2n+2$ rays. For any $k > 1$ the cones of codimension k in Σ' can be described as the cones of codimension k of Σ, together with the cones of the form $\langle e_i, \rho \mid i \in I\rangle$ and $\langle -e_i, -\rho \mid i \in I\rangle$, for all the subsets $I \subseteq \{1,\dots,n\}$ such that $|I| = n-k-1$. We summarize these cones and the corresponding cycles in the following table.

$$
\begin{aligned}
\langle e_i \mid i \in I\rangle &\Leftrightarrow H_I - E_{1,k}, \ |I| = n-k,\\
\langle -e_i \mid i \in I\rangle &\Leftrightarrow H_I - E_{2,k}, \ |I| = n-k,\\
\langle e_i, \rho \mid i \in I\rangle &\Leftrightarrow E_{1,k}, \qquad\quad |I| = n-k-1,\\
\langle -e_i, -\rho \mid i \in I\rangle &\Leftrightarrow E_{2,k}, \qquad\quad |I| = n-k-1,\\
\langle e_i, -e_j \mid i \in I,\ j \in J\rangle &\Leftrightarrow H_{I\cup J}, \qquad\ |I| + |J| = n-k,\ I \cap J = \emptyset.
\end{aligned}
$$

We conclude by observing that the classes above are exactly the generators of the cone of fibers $\mathrm{CF}_k(X^n_2)$. $\square$

Remark 3.5 If $\Gamma = \{q_1, q_2\}$ consists of 2 points lying on a s-dimensional fiber F of a projection $p_J\colon (\mathbb{P})^n \to (\mathbb{P})^{n-s}$, with $|J| = n-s$, the variety X^n_Γ is still toric and we can reason in the same way. In particular $\overline{\mathrm{Eff}}_k(X^n_\Gamma)$ has the the same generators as before, for $k < s$, while for $k \geq s$ the cone $\overline{\mathrm{Eff}}_k(X^n_\Gamma)$ is generated by the following classes:

$$
\begin{aligned}
&H_I - E_{1,k},\ H_I - E_{2,k},\ |I| = n-k,\ I \not\subset J\\
&H_I - E_{1,k} - E_{2,k}, \qquad |I| = n-k,\ I \subset J\\
&E_{1,k},\ E_{2,k}.
\end{aligned}
$$

We immediately get the following characterization in the case of cycles of codimension 1.

Corollary 3.6 *For any $n \geq 2$ the cone $\overline{\mathrm{Eff}}_{n-1}(X^n_r)$ is fiber-generated if and only if $r \leq 2$.*

Proof By Proposition 3.4 we only need to show that if $r \geq 3$, the cone $\overline{\mathrm{Eff}}_{n-1}(X^n_r)$ is not fiber-generated. The class $H_1 + H_2 - E_1 - E_2 - E_3$ corresponds to a hyperplane of $\mathbb{P}^n$ passing through 4 points via the map φ described in (1), and in particular it is effective. We conclude observing that $H_1 + H_2 - E_1 - E_2 - E_3$ does not belong to the cone of fibers $\mathrm{CF}_{n-1}(X^n_2)$. $\square$

In the case of curves, it is possible to give a sharp bound for fiber generation, by means of the following.

Theorem 3.7 *For any $n \geq 2$, the cone $\overline{\mathrm{Eff}}_1(X^n_r)$ is fiber-generated if and only if $r \leq n!$.*

Proof We argue as in the proof of [3, Prop. 4.1]. Let us consider the embedding $(\mathbb{P}^1)^n \subseteq \mathbb{P}^{2^n-1}$ and let us take n general hyperplane sections $W_1, \ldots, W_n$. Since the W_i are general, $\Gamma := W_1 \cap \cdots \cap W_n$ consists of $n!$ distinct points of $(\mathbb{P}^1)^n$. We claim that the class $D := H_1 + \cdots + H_n - E_1 - \cdots - E_{n!}$ is nef on X^n_Γ. Indeed, given any curve C on X^n_Γ, not contained in an exceptional divisor, there exists at least one W_i whose proper transform does not contain C. Since the class of W_i is D, we get $D \cdot C \geq 0$ and the claim follows. Arguing by semicontinuity we deduce that the class $H_1 + \cdots + H_n - E_1 - \cdots - E_{n!}$ is nef on $X^n_{n!}$ too.

Therefore, if $r \leq n!$, the class $H_1 + \cdots + H_n - E_1 - \cdots - E_r$ is nef on X^n_r, so that, by Lemma 3.2, $\overline{\mathrm{Eff}}(X^n_r)$ is fiber-generated. If otherwise $r > n!$, the above class is not nef since its n-th power is negative, and we conclude again by means of Lemma 3.2. $\square$

Going back to any $1 \leq k \leq n-1$, in order to prove our general bound (Theorem 2) we need the following results about the blow up X^n_1 of $(\mathbb{P}^1)^n$ at one point q_1. We recall that L is the curve with class $H_{\{2,\ldots,n\}} - E_{1,1}$ (i.e. the strict transform of the fiber of the projection $p_{2,\ldots,n}\colon (\mathbb{P}^1)^n \to (\mathbb{P}^1)^{n-1}$, passing through q_1).

Lemma 3.8 *Let Y be a purely k-dimensional subvariety on the blow up X_1^n of $(\mathbb{P}^1)^n$ at a point q_1, with class $[Y] = \sum_{|I|=n-k} a_I H_I - b_1 E_{1,k}$. If $b_1 - \sum_{1\in I} a_I = \beta > 0$, then the curve L is contained in Y with multiplicity at least β.*

Proof We recall that for any $1 \le s \le n-1$, we set $W_s := \sum_{i=2}^n H_i - sE_1$. We have

$$\begin{aligned} W_1 \cdot W_2 \ \cdots \ W_k \cdot Y &= ((\textstyle\sum_{i=2}^n H_i)^k - (k!)E_{1,n-k}) \cdot Y \\ &= k! \left(\sum_{\substack{I\subseteq\{2,\dots,n\} \\ |I|=k}} H_I - E_{1,n-k} \right) \cdot Y \\ &= k! \left(\sum_{1\in I} a_I - b_1 \right) \\ &= -k!\beta < 0. \end{aligned}$$

By Lemma 2.2, the general variety V in the intersection product $W_1 \ \cdots \ W_{k-1}$ has codimension $k-1$, as the divisors W_s are general. For the same reason the intersection $V \cap Y$ has dimension 1 and it contains L as an irreducible component. As explained in [6, p. 118], the intersection multiplicity $i(L, V \cdot Y; X_1^n)$ is the coefficient of L in the intersection class $V \cdot Y$.

By Lemma 2.2 (iii), L is the only irreducible component of $V \cap Y$ contained in the base locus of $|W_k|$, so that the intersection product of W_k with every irreducible component but L is nonnegative. We then have the following inequality

$$-k!\beta = W_k \cdot (V \cdot Y) \ge i(L, V \cdot Y; X_1^n)(W_k \cdot L),$$

and since $W_k \cdot L = -k$, we deduce that

$$i(L, V \cdot Y; X_1^n) \ge (k-1)!\beta. \tag{4}$$

Let us consider the commutative diagram

$$\begin{array}{ccc} E & \overset{j}{\hookrightarrow} & \tilde{X} \\ \eta\downarrow & & \downarrow\sigma \\ L & \hookrightarrow & X_1^n \end{array}$$

where σ is the blowing up of L, while η is the projective bundle morphism. By [6, Ex. 12.4.4], we have

$$i(L, V \cdot Y; X_1^n) = i(L, W_1 \ \cdots \ W_k \cdot Y; X_1^n) = e_L(W_1) \ \cdots \ e_L(W_{k-1}) \cdot e_L(Y) + q,$$

where q is the coefficient of $[L]$ in the cycle $\eta_*(j^*(\tilde{V} \cdot \tilde{Y}))$. We claim that in our case $q = 0$, so that, since by Lemma 2.2 (ii), $e_L(W_s) = s$ for any $s \ge 1$, substituting in (4) we get the statement, $e_L(Y) \ge \beta$.

In order to prove the claim, we first recall that E is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^{n-2}$. The pullbacks $j^*(\tilde{V})$ and $j^*(\tilde{Y})$ cut on a general $\mathbb{P}^{n-2}$ two cycles of codimension $k-1$ and $n-k$ respectively. We conclude by showing that these two cycles do not intersect, so that the pushforward $\eta_*(j^*(\tilde{V} \cdot \tilde{Y}))$ is 0-dimensional, and in particular the coefficient q of $[L]$ must be 0. Indeed, by Lemma 2.3, $j^*(\tilde{W}_s)$ cuts on $\mathbb{P}^{n-2}$ the linear system of hypersurfaces of degree s, with $n-1$ points of multiplicity $s-1$. This implies that $j^*(\tilde{V})$ does not intersect the $(k-3)$-dimensional base locus of $j^*(\tilde{W}_{k-1})$ (which is the union of $(k-3)$-dimensional linear spaces). Hence $j^*(\tilde{V})$ and $j^*(\tilde{Y})$ do not intersect on a general $\mathbb{P}^{n-2}$.

□

We are now able to give an upper bound that works for any n and k (even if it is not sharp in general).

Theorem 3.9 *The cone* $\overline{\mathrm{Eff}}_k(X^n_r)$ *is fiber-generated for any* $r \le n-k+1$.

Proof By Theorem 3.7 and Proposition 3.4 the statement is true when $k=1$ or $k=n-1$, so that in what follows we can assume $2 \le k \le n-2$. We proceed by induction on n as in [3, Thm. 4.3], the base of induction being $n=2$. Assume the theorem is true for $\overline{\mathrm{Eff}}_k(X^m_r)$, for any $k < m < n$. Let $\Gamma \subseteq (\mathbb{P}^1)^n$ be a set consisting of $r-1$ very general points $q_1,\dots,q_{r-1}$ in a fiber F of the projection morphism $p_{\{1\}} : (\mathbb{P}^1)^n \to \mathbb{P}^1$, and a very general point q_r not contained in F. Notice that $F \simeq (\mathbb{P}^1)^{n-1}$. We denote by F' be the proper transform of F in X^n_Γ, the blow up of $(\mathbb{P}^1)^n$ along Γ, so that the class of F' in X^n_Γ is $H_1 - \sum_{j=1}^{r-1} E_j$.

Let Y be an irreducible k-dimensional subvariety of X^n_Γ, not contained in an exceptional divisor, so that

$$[Y] = \sum_{|I|=n-k} a_I H_I - \sum_{j=1}^{r} b_j E_{j,k},$$

where a_I, b_j are nonnegative integers. By Proposition 3.3 we need to prove that

$$\sum_{|I|=n-k} a_I \ge \sum_{j=1}^{r} b_j. \tag{5}$$

We distinguish two cases.

(i) Y is contained in F'. Since the points $q_1,\dots,q_{r-1}$ are very general in F we have that $F' \simeq X^{n-1}_{r-1}$. By the induction hypothesis, $\overline{\mathrm{Eff}}_k(X^{n-1}_{r-1})$ is fiber-generated, so that inequality (5) holds.

(ii) Y is not contained in F'. In this case we set $Z := Y \cap F'$, so that Z is an effective cycle of dimension $k-1$ in X^{n-1}_{r-1}. The 1-dimensional fiber of the projection $\pi_{\{2,\dots,n\}} : (\mathbb{P}^1)^n \to (\mathbb{P}^1)^{n-1}$, passing through the point q_r intersects F in a unique point $\bar{q}_r$. The blow up of F' at the point corresponding to $\bar{q}_r$ coincides with the blow-up X^{n-1}_r of F along the r very general points $q_1,\dots,q_{r-1},\bar{q}_r$. By Lemma 3.8, Z has multiplicity $\bar{b}_r \ge \beta = \max(0, b_r - \sum_{1\in I} a_I)$ at $\bar{q}_r$. Therefore the strict transform Z' of Z is a $(k-1)$-dimensional effective cycle in X^{n-1}_r, with class

$$[Z'] = \sum_{1\notin I} a_I H_I - \sum_{j=1}^{r-1} b_j E_{j,k-1} - \bar{b}_r E_{r,k-1},$$

(with a slight abuse of notation we are considering the classes H_I of $N_{k-1}(X^n_r)$, such that $1 \in I$, as classes in $N_{k-1}(X^{n-1}_r)$). Notice that $r \le (n-1)-(k-1)+1$, so the inductive procedure can be applied. Therefore, $\overline{\mathrm{Eff}}_{k-1}(X^{n-1}_r)$ is fiber-generated and the coefficients of $[Z']$ satisfy the inequality $\sum_{1\notin I} a_I \ge \sum_{j=1}^{r-1} b_j + \bar{b}_r$. We can then write

$$0 \le \sum_{1\notin I} a_I - \sum_{j=1}^{r-1} b_j - \bar{b}_r \le \sum_{1\notin I} a_I - \sum_{j=1}^{r-1} b_j - \left(b_r - \sum_{1\in I} a_I\right) = \sum a_I - \sum_{j=1}^{r} b_j,$$

as required. Hence we can apply Lemma 2.4 and deduce that $\overline{\mathrm{Eff}}_k(X^n_\Gamma)$ is fiber-generated. Reasoning by semicontinuity as in [3, Cor. 2.3], we conclude that $\overline{\mathrm{Eff}}_k(X^n_r)$ is fiber-generated too. □

4 Other results

In this section we first show that for any n and k, if we blow up a sufficiently big number r of points, the cone $\overline{\mathrm{Eff}}_k(X^n_r)$ is not fiber-generated. As a consequence, we consider the particular case of $\overline{\mathrm{Eff}}_2(X^4_4)$, and finally we discuss Question 3.

Proposition 4.1 *The cone $\overline{\mathrm{Eff}}_k(X^n_r)$ is not fiber-generated for $r > (n-k+1)!$.*

Proof If $k = 1$ or $k = n-1$, the assertion follows from Theorem 3.7 and Corollary 3.6, so we can restrict to the case $2 \le k \le n-2$. Let us fix the projection $\pi_J : (\mathbb{P}^1)^n \to (\mathbb{P}^1)^{n-k+1}$, where $J = \{1, \ldots, n-k+1\}$, and let X^{n-k+1}_r be the blow up of $(\mathbb{P}^1)^{n-k+1}$ at the points $q'_1, \ldots, q'_r$, images of $q_1, \ldots, q_r$ via π_J. Since by hypothesis $r > (n-k+1)!$, by Theorem 3.7 the cone $\overline{\mathrm{Eff}}_1(X^{n-k+1}_r)$ is not fiber-generated, so that by Proposition 3.3 there exists a curve class $[C] = \sum_I a_I H_I - \sum_{j=1}^r b_j E_{j,1}$, where $I \subseteq J$, and $|I| = n-k$, satisfying the inequality

$$\sum_I a_I < \sum_{j=1}^r b_j.$$

Observe that the image C' of C in $(\mathbb{P}^1)^{n-k+1}$ is a curve passing through q'_j with multiplicity at least b_j, for any $1 \le j \le r$. The fiber product $Z := (\mathbb{P}^1)^{k-1} \times C \subseteq (\mathbb{P}^1)^n$ is a k-dimensional variety having multiplicity at least b_j at each point q_j, so that its class can be written as $[Z] = \sum_I \alpha_I H_I - \sum_{j=1}^r b_j E_{j,k}$, with $I \subseteq \{1, \ldots, n\}$ and $|I| = n-k$. Since Z is the product of $(\mathbb{P}^1)^{k-1}$ and C, we have that

$$\alpha_I = \begin{cases} a_I & \text{if } I \subseteq J, \\ 0 & \text{otherwise} \end{cases}$$

and in particular the coefficients satisfy $\sum_I \alpha_I = \sum_I a_I < \sum_{j=1}^r b_j$. Since $[Z]$ belongs to the cone $\overline{\mathrm{Eff}}_k(X^n_r)$, by Proposition 3.3 we conclude that the latter is not fiber-generated. □

As we pointed out in the Introduction, when $n = 4$, Theorem 2 and the above proposition show that only three cases remain open, i.e. $\overline{\mathrm{Eff}}_2(X^4_r)$, for $r = 4, 5, 6$. We are now going to deal with the first of these cases.

Proposition 4.2 *The cone $\overline{\mathrm{Eff}}_2(X^4_4)$ is fiber-generated.*

Proof Let us consider two divisors D_1 and D_2 having classes

$$[D_1] = H_1 + H_2 + H_3 + H_4 - 2E_1 - 2E_2 - E_3 - E_4,$$
$$[D_2] = H_1 + H_2 + H_3 + H_4 - E_1 - E_2 - 2E_3 - 2E_4,$$

i.e. D_1 (resp. D_2) can be described as the intersection of X with a hyperplane passing through q_3 and q_4 (resp. q_1 and q_2) and tangent to X at q_1 and q_2 (resp. at q_3 and q_4). We have that $\dim |D_1| = 3$ and its base locus is 1-dimensional. Indeed, with the help of computer algebra system MAGMA [1] it is possible to see that it is the union of the 8 fibers through q_1 and q_2 and 2 rational normal curves of degree 4. Since none of the components of the base locus is contained in D_2, we have that $D_1|_{D_2}$ has only finitely many base points and in particular it is nef.

Let us consider now a surface Y with class $\sum_{|I|=2} a_I H_I - \sum b_k E_{k,2}$. If Y is not contained in D_2, its restriction to D_2 is an effective 1-cycle, so that its intersection product with $D_1|_{D_2}$

is non-negative. Therefore we can write

$$0 \leq Y|_{D_2} \cdot D_1|_{D_2} = 2\left(\sum_{|I|=2} a_I - \sum_{k=1}^{4} b_k\right)$$

and by Lemma 2.4 we deduce that $[Y]$ lies in the cone spanned by fiber classes.

If Y is not contained in D_1 we can conclude in the same way, so that we only have to consider the case in which $Y \subseteq D_1 \cap D_2$. Since the latter intersection is irreducible, we have $Y = D_1 \cap D_2$, so that $a_I = 2$ for any I such that $|I| = 2$, and $b_k = 2$ for any $1 \leq k \leq 4$. Therefore $\sum a_I - \sum b_k = 4 > 0$ and again we can apply Lemma 2.4. □

Let us go back now to Question 3. As proved in [2], X_r^n is a Mori dream space exactly in the following cases:

n	2	3	4	≥ 5
$r \leq$	7	6	5	4

Clearly, if $n = 2$ there is nothing to prove, while if $n = 3$ an affirmative answer to Question 3 follows from Theorem 3.7. If $n = 4$, by the same theorem and Proposition 4.2, the only open case is $k = 2$, $r = 5$. Finally, if $n \geq 5$ and $r \leq 4$, the bound of Theorem 3.9 implies that for any $k \leq n-3$ the pseudoeffective cone $\overline{\mathrm{Eff}}_k(X_r^n)$ is fiber-generated, so that in particular it is polyhedral. Therefore the only open case is $k = n-2$ and $r = 4$. We remark that, as proved in Lemma 3.2, $\overline{\mathrm{Eff}}_k(X_r^n)$ is fiber-generated if and only if the numerical class in Lemma 3.2 is nef. Therefore, it is natural to ask the following.

Question 4.3 *Is the numerical class $\sum_{|I|=n-2} H_I - \sum_{j=1}^{r} E_{j,2}$ nef when $n = 4$, $r = 5$ or $n \geq 5$, $r = 4$?*

An affirmative answer to this question would allow us to answer Question 3 too (see the Introduction). We hope to return to this problem in the next future.

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