MULTIPLE SOLUTIONS FOR PARAMETRIC DOUBLE PHASE DIRICHLET PROBLEMS

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Abstract. We consider a parametric double phase Dirichlet problem. Using variational tools together with suitable truncation and comparison techniques, we show that for all parametric values $\lambda > \lambda^*$ the problem has at least three nontrivial solutions, two of which have constant sign. Also we identify the critical parameter λ^* precisely in terms of the spectrum of the q-Laplacian.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear parametric Dirichlet problem

$$(P_\lambda) \qquad \begin{cases} -\text{div } (a(z)|\nabla u|^{p-2}\nabla u) - \Delta_q u = \lambda |u|^{q-2}u - f(z,u) \text{ in } \Omega, \\ u\big|_{\partial\Omega} = 0, \ \lambda > 0, \ 1 < q < p. \end{cases}$$

So, this problem is driven by a differential operator which is the sum of a weighted p-Laplacian and of a q-Laplacian. The weight function $a : \overline{\Omega} \to \mathbb{R}_+$ is Lipschitz continuous. This operator is related to the so-called two-phase integral functional defined by

$$\vartheta_{pq}(u) = \int_\Omega [a(z)|\nabla u|^p + |\nabla u|^q]\, dz.$$

The integrand in this integral functional is given by

$$\eta(z,y) = a(z)|y|^p + |y|^q \quad \text{for all } (z,y) \in \overline{\Omega} \times \mathbb{R}^N.$$

Since the weight $a(\cdot)$ is not bounded away from zero, this integrand exhibits unbalanced growth, namely we have

$$|y|^q \le \eta(z,y) \le c_0[1 + |y|^p] \quad \text{for all } z \in \Omega, \text{ all } y \in \mathbb{R}^N, \text{ with } c_0 > 0.$$

Such functionals were first investigated in the context of problems of the calculus of variations and of elasticity theory by Marcellini [10, 11] and Zhikov [18, 19]. The study of equations in which the energy density changes its ellipticity and growth properties according to the point of the domain, has been revived more recently by Mingione and coworkers, who in a series of remarkable papers produced local regularity results (see [1, 2, 5, 6]). However, no global regularity results are yet available for two phase elliptic problems and this fact does not allow the application of many of the tools and techniques used in the study of (p,q)-equations (see, for example, Papageorgiou-Vetro-Vetro [14]). Neverthless, using variational methods and suitable truncation and comparison techniques, we show that if $\widehat{\lambda}_2(q)$ is the second eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$ and $\lambda > \widehat{\lambda}_2(q)$, then problem (P_λ) admits at least three nontrivial solutions. The

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conditions on the perturbation $f(z,x)$ are minimal. We require that $f(z,x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z,x)$ is continuous) and $f(z,\cdot)$ exhibits $(p-1)$-superlinear growth near $\pm\infty$ and $(q-1)$-sublinear growth near 0. So, the reaction of problem (P_λ) is of logistic type and more precisely of the equidiffusive kind (see [14]).

Existence and multiplicity results for different types of double phase equations were proved recently by Cencelj-Rădulescu-Repovš [3], Colasuonno-Squassina [4], Liu-Dai [9] and Rădulescu [16].

2. Mathematical Background - Auxiliary results

The unbalanced growth of the integrand $\eta(z,y)$ requires a different function space framework for problem (P_λ), which is provided by the Musielak-Orlicz-Sobolev spaces.

Consider the Carathéodory function $\eta : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\eta(z,x) = a(z)x^p + x^q \quad \text{for all } z \in \Omega, \text{ all } x \geq 0,\ 1 < q < p.$$

This is a "generalized N-function" (see Musielak [12]) and we have

$$\eta(z,2x) \leq 2^p \eta(z,x) \quad \text{for all } z \in \Omega, \text{ all } x \geq 0.$$

This is the so-called Δ_2-property (see Musielak [12], p. 52). We introduce the modular function

$$\rho_\eta(u) = \int_\Omega \eta(z,|u|)dz.$$

Then the Musielak-Orlicz space $L^\eta(\Omega)$ is defined by

$$L^\eta(\Omega) = \{u : \Omega \to \mathbb{R} \text{ is measurable and } \rho_\eta(u) < +\infty\}.$$

We endow $L^\eta(\Omega)$ with the Luxemburg norm given by

$$\|u\|_\eta = \inf\left\{\lambda > 0 \,:\, \rho_\eta\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

This is a separable, reflexive Banach space. Also we introduce the weighted Lebesgue space $L^p_a(\Omega)$ defined by

$$L^p_a(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ is measurable and } \|u\|_{a,p} = \left[\int_\Omega a(z)|u|^p dz\right]^{1/p} < +\infty\right\}.$$

We have

$$L^p(\Omega) \hookrightarrow L^\eta(\Omega) \hookrightarrow L^p_a(\Omega) \cap L^q(\Omega).$$

The modular function $\rho_\eta(\cdot)$ and the Luxemburg norm $\|\cdot\|_\eta$ are closely related.

Proposition 1. *(a) If $u \neq 0$, then $\|u\|_\eta = \lambda$ if and only if $\rho_\eta(\frac{u}{\lambda}) = 1$.*
(b) $\|u\|_\eta < 1$ (resp. > 1, $= 1$) if and only if $\rho_\eta(u) < 1$ (resp. > 1, $= 1$).
(c) $\|u\|_\eta < 1$ implies $\|u\|^p_\eta \leq \rho_\eta(u) \leq \|u\|^q_\eta$.
(d) $\|u\|_\eta > 1$ implies $\|u\|^q_\eta \leq \rho_\eta(u) \leq \|u\|^p_\eta$.
(e) $\|u\|_\eta \to 0$ if and only if $\rho_\eta(u) \to 0$.
(f) $\|u\|_\eta \to +\infty$ if and only if $\rho_\eta(u) \to +\infty$.

Using the Musielak-Orlicz spaces, we can define the Musielak-Orlicz-Sobolev spaces. So, we set

$$W^{1,\eta}(\Omega) = \{u \in L^\eta(\Omega) \,:\, |\nabla u| \in L^\eta(\Omega)\}.$$

We furnish this space with the following norm

$$\|u\| = \|u\|_\eta + \|\nabla u\|_\eta \quad \text{for all } u \in W^{1,\eta}(\Omega).$$

Also we set $W_0^{1,\eta}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|}$. Both $W^{1,\eta}(\Omega)$ and $W_0^{1,\eta}(\Omega)$ are separable reflexive Banach spaces.

By q^* we denote the Sobolev critical exponent corresponding to q, that is

$$q^* = \begin{cases} \dfrac{Nq}{N-q} & \text{if } q < N, \\ +\infty & \text{if } N \le q. \end{cases}$$

Proposition 2. *(a) $W_0^{1,\eta}(\Omega) \hookrightarrow W_0^{1,r}(\Omega)$ continuously for $1 \le r \le q$.*
(b) If $q \ne N$, then $W_0^{1,\eta}(\Omega) \hookrightarrow L^r(\Omega)$ continuously for $1 \le r \le q^*$,
if $q = N$, then $W_0^{1,\eta}(\Omega) \hookrightarrow L^r(\Omega)$ continuously for $1 \le r < +\infty$.
(c) If $q \le N$, then $W_0^{1,\eta}(\Omega) \hookrightarrow L^r(\Omega)$ compactly for $1 \le r < q^*$,
if $q > N$, then $W_0^{1,\eta}(\Omega) \hookrightarrow L^\infty(\Omega)$ compactly.

Remark 1. Of course the same embeddings remain valid if we replace $W_0^{1,\eta}(\Omega)$ by $W^{1,\eta}(\Omega)$.

For the standard Sobolev space $W_0^{1,q}(\Omega)$ we have the Poincaré inequality (see Papageorgiou-Rădulescu-Repovš [13], Theorem 1.8.1, p. 43). A similar result is also valid for $W_0^{1,\eta}(\Omega)$ provided we impose a restriction on the exponents q, p which implies that the two exponents can not be far apart (see Colasuonno-Squassina [4], Proposition 2.8).

Proposition 3. *If $a : \overline{\Omega} \to \mathbb{R}$ is Lipschitz continuous and $\dfrac{p}{q} < 1 + \dfrac{1}{N}$, then $\|u\| \le \widehat{c}\,\|\nabla u\|_\eta$ for some $\widehat{c} > 0$, all $u \in W_0^{1,\eta}(\Omega)$.*

Remark 2. Note that $q\left(1 + \dfrac{1}{N}\right) < q^*$. So, the condition $\dfrac{p}{q} < 1 + \dfrac{1}{N}$ implies that $p < q^*$.

In the sequel the following conditions will be in effect.

H_0: $a : \overline{\Omega} \to \mathbb{R}_+$ is Lipschitz continuous and $\dfrac{p}{q} < 1 + \dfrac{1}{N}$.

In what follows to simplify things we assume that $1 < q < p < N$ which is the interesting case. Also by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(W_0^{1,\eta}(\Omega), W_0^{1,\eta}(\Omega)^*)$.

Definition 1. Let $u^* \in W_0^{1,\eta}(\Omega)^*$. We say that $u^* \ge 0$ if

$$\langle u^*, h \rangle \ge 0 \quad \text{for all } h \in W_0^+,$$

where $W_0^+ = \{h \in W_0^{1,\eta}(\Omega) : h(z) \ge 0 \text{ for a.a. } z \in \Omega\}$.

The next result can be viewed as a kind of weak maximum principle for double phase equations. In the proof we follow closely Zhang [17] (see also Pucci-Serrin [15], p. 108).

Proposition 4. *If hypotheses H_0 hold, $\widehat{\xi} \in L^\infty(\Omega)$, $\widehat{\xi}(z) \ge 0$ for a.a. $z \in \overline{\Omega}$, $\widehat{\xi} \ne 0$ and $u \in W_0^+$, $u \ne 0$ satisfies*

$$-div\,\left(a(z)|\nabla u|^{p-2}\nabla u\right) - \Delta_q u + \widehat{\xi}(z)u^{p-1} \ge 0 \quad \text{in } W_0^{1,\eta}(\Omega)^*,$$

then for every $K \subseteq \Omega$ compact, we can find $c_K > 0$ such that $u(z) \ge c_K > 0$ for a.a. $z \in K$.

Proof. First we assume that $u \in C^1(\Omega)$.

Suppose that the Proposition is not true. We can find $z_1, z_2 \in \Omega$ and an open ball $B_{2\rho}(z_2) = \{z \in \Omega : |z - z_2| < 2\rho\}$ such that $B_{2\rho}(z_2) \subset\subset \Omega$ (that is, $\overline{B}_{2\rho}(z_2) \subseteq \Omega$) and

$$z_1 \in \partial B_{2\rho}(z_2), \quad u(z_1) = 0, \quad u\Big|_{B_{2\rho}(z_2)} > 0. \tag{1}$$

Fixing z_1 and allowing z_2 to vary, we see that we can have $2\rho = |z_1 - z_2|$ arbitrarily small.

From (1) and since by hypothesis $u \geq 0$, we see that $z_1 \in \Omega$ is a minimizer of $u(\cdot)$. Therefore

$$\nabla u(z_1) = 0. \tag{2}$$

We set

$$b = \min\left[u(z) \, : \, z \in \partial B_\rho(z_2)\right].$$

On account of (1) we have $b > 0$. If $\rho \to 0^+$, then z_2 collapses to z_1 (which we have fixed) and so we have

$$b \to 0^+ \text{ and } \frac{b}{\rho} \to 0^+ \quad \text{(by L'Hôpital's rule; see (1) and (2))}. \tag{3}$$

Now we introduce the open annulus

$$D = \{z \in \Omega : \rho < |z - z_2| < 2\rho\}.$$

Since $a(\cdot)$ is Lipschitz continuous, it is differentiable almost everywhere on Ω (Rademacher's theorem, see Gasiński-Papageorgiou [8], p. 56). So, we can define

$$m = \sup\{|\nabla a(z)| \, : \, z \in D\}.$$

We set

$$\vartheta = -\ln\frac{b}{\rho} + \frac{N-1}{\rho} + 2m \tag{4}$$

and consider the function

$$v(t) = \frac{b\left(e^{\frac{\vartheta t}{q-1}} - 1\right)}{e^{\frac{\vartheta \rho}{q-1}} - 1} \quad \text{for all } t \in [0, \rho]. \tag{5}$$

Clearly if $\rho > 0$ is small, then

$$0 < v(t), v'(t) < 1 \quad \text{for all } t \in [0, \rho] \text{ (see (3))}. \tag{6}$$

Also, from (5) we see that

$$v''(t) = \frac{\vartheta}{q-1} v'(t) \quad \text{for all } t \in (0, \rho). \tag{7}$$

For simplicity we take $z_2 = 0$ and then set $r = |z|$ and $t = 2\rho - r$. For $t \in [0, \rho]$ and $r \in [\rho, 2\rho]$ we define

$$\begin{aligned} & y(r) = v(2\rho - r) = v(t), \\ \Rightarrow \quad & y'(r) = -v'(t), \quad y''(r) = v''(t). \end{aligned}$$

We set

$$y(z) = y(r) \quad \text{for all } z \in D, \, |z| = r.$$

Then, we have

$$\operatorname{div}\left(a(z)|\nabla y|^{p-2}\nabla y + |\nabla y|^{q-2}\nabla y\right)$$

$$
\begin{aligned}
&= |\nabla y|^{p-2}(\nabla a, \nabla y)_{\mathbb{R}^N} + a(z)\Delta_p y + \Delta_q y \\
&= -v'(t)^{p-1}\sum_{k=1}^{N}\frac{\partial a}{\partial z_k}\frac{z_k}{r} + a(z)\left[(p-1)v'(t)^{p-2}v''(t) - \frac{N-1}{r}v'(t)^{p-1}\right] \\
&\quad + (q-1)v'(t)^{q-2}v''(t) - \frac{N-1}{r}v'(t)^{q-1} \\
&\geq a(z)\left[\frac{p-1}{q-1}\vartheta - \frac{N-1}{r}\right]v'(t)^{p-1} + \left[\vartheta - \frac{N-1}{r} - m\right]v'(t)^{q-1} \quad \text{(see (7))} \\
&\geq a(z)\left[\vartheta - \frac{N-1}{r} - m\right]v'(t)^{p-1} + \left[\vartheta - \frac{N-1}{r} - m\right]v'(t)^{q-1} \\
&\quad \text{(since } p > q \text{ and } \vartheta > 0 \text{ for } \rho > 0 \text{ small, see (3), (4))} \\
&\geq \left[\vartheta - \frac{N-1}{r} - m\right](a(z)+1)v'(t)^{q-1} \quad \text{(see (6) and recall } q < p) \\
&\geq \left(-\ln\frac{b}{\rho}\right)(a(z)+1)v'(t)^{q-1} \\
&\geq \widehat{\xi}(z)v'(t)^{p-1} \quad \text{for } \rho > 0 \text{ small.}
\end{aligned}
$$

Therefore we have

$$
-\mathrm{div}\left(\left(a(z)|\nabla y|^{p-2}\nabla y\right) + |\nabla y|^{q-2}\nabla y\right) + \widehat{\xi}(z)y^{p-1} \leq 0 \quad \text{in } W_0^{1,\eta}(\Omega)^*.
$$

Note that $y \leq u$ on ∂D. So, by the weak comparison principle (see Pucci-Serrin [15], Theorem 3.4.1, p. 61 for the space $W_0^{1,\eta}(\Omega)$), we have

$$
y(z) \leq u(z) \quad \text{for all } z \in D.
$$

Then we have

$$
\lim_{s\to 0^+}\frac{u(z_1 + s(z_2 - z_1)) - u(z_1)}{s} \geq \lim_{s\to 0}\frac{y(z_1 + s(z_2 - z_1)) - y(z_1)}{s} = v'(0) > 0,
$$

a contradiction since $\nabla u(z_1) = 0$. Therefore $u\Big|_{\Omega} > 0$.

Now we consider the general case $u \in W_0^{1,\eta}(\Omega)$.

Let $\Omega_0 = \{z \in \Omega \ : \ \text{for some } \varepsilon > 0, \text{ we have } B_\varepsilon(z) \subseteq \Omega,\ u\Big|_{B_\varepsilon(z)} = 0 \text{ a.e.}\}$. We set $\Omega_+ = \Omega \setminus \Omega_0$.

Let $z_0 \in \Omega_+$. We can find $\varepsilon > 0$ such that $\overline{B}_{2\varepsilon}(z_0) \subseteq \Omega$ and u is not identically zero on $\partial B_{2\varepsilon}(z_0)$. We consider the following nonlinear boundary value problem

$$
\begin{cases}
-\mathrm{div}\ (a(z)|\nabla y|^{p-2}\nabla y) - \Delta_q y + \widehat{\xi}(z)|y|^{p-2}y = 0 \quad \text{on } B_{2\varepsilon}(z_0), \\
y\Big|_{\partial B_{2\varepsilon}(z_0)} = u\Big|_{\partial B_{2\varepsilon}(z_0)}.
\end{cases}
$$

Evidently this problem has a unique solution y and from Baroni-Colombo-Mingione [2], Theorem 1.1, we know that $y \in C^{1,\alpha}_{loc}(B_{2\varepsilon}(z_0))$ with $\alpha \in (0,1)$. Since $y\Big|_{\partial B_{2\varepsilon}(z_0)} = u\Big|_{\partial B_{2\varepsilon}(z_0)} \geq 0$, from the weak comparison principle we have $y(z) \geq 0$ for a.a. $z \in B_{2\varepsilon}(z_0)$. Then from the first part of the proof, we have $y(z) > 0$ for all $z \in B_{2\varepsilon}(z_0)$. Thus $y(z) \geq c > 0$ for all $z \in \overline{B}_\varepsilon(z_0)$, hence $u(z) \geq c$ for a.a. $z \in \overline{B}_\varepsilon(z_0)$.

Now we show that $\Omega_0 = \emptyset$. Proceeding indirectly, suppose that $\Omega_0 \neq \emptyset$. Note that Ω_0 is open. So, we can find $z_0 \in \Omega_+ \cap \partial\Omega_0$. From what we did above, we know that we can find $\varepsilon > 0$ such that $u\Big|_{B_\varepsilon(z_0)} > 0$ a.e.. Hence $u\Big|_{B_\varepsilon(z_0)\cap\Omega_0} > 0$ a.e., a contradiction to the definition of Ω_0.

Finally let $K \subseteq \Omega$ be compact. We can find balls $\{B_i\}_{i=1}^m$ such that $K \subseteq \bigcup_{k=1}^m B_k$ and $u\Big|_{B_k} \geq c_k > 0$ a.e. Set $c_K = \min\{c_k\}_{k=1}^m > 0$. Then

$$u(z) \geq c_K > 0 \quad \text{for a.a. } z \in K.$$

$\square$

Next we recall some basic facts about the spectrum of $(-\Delta_q, W_0^{1,q}(\Omega))$ (see Gasiński-Papageorgiou [8]).

So, we consider the following nonlinear eigenvalue problem

$$-\Delta_q u(z) = \widehat{\lambda}|u(z)|^{q-2}u(z) \text{ in } \Omega,\ u\Big|_{\partial\Omega} = 0. \tag{8}$$

We say that $\widehat{\lambda} \in \mathbb{R}$ is an "eigenvalue" of $(-\Delta_q, W_0^{1,q}(\Omega))$, if problem (8) admits a nontrivial solution $\widehat{u} \in W_0^{1,q}(\Omega)$, which is an "eigenfunction" corresponding to the eigenvalue $\widehat{\lambda}$. By $\widehat{\sigma}(q)$ we denote the set of eigenvalues of $(-\Delta_q, W_0^{1,q}(\Omega))$. This set is closed and has a smallest element $\widehat{\lambda}_1(q)$ with the following properties:

- $\widehat{\lambda}_1(q) > 0$.
- $\widehat{\lambda}_1(q)$ is isolated (that is, there exists $\varepsilon > 0$ such that $(\widehat{\lambda}_1(q), \widehat{\lambda}_1(q)+\varepsilon)\cap\widehat{\sigma}(q) = \emptyset$).
- $\widehat{\lambda}_1(q)$ is simple (that is, if u, v are eigenfunctions corresponding to $\widehat{\lambda}_1(q)$, then $u = \zeta v$ for some $\zeta \in \mathbb{R} \setminus \{0\}$).

We have a variational characterization of this eigenvalue, namely

$$\widehat{\lambda}_1(q) = \inf\left\{\frac{\|\nabla u\|_q^q}{\|u\|_q^q} \,:\, u \in W_0^{1,q}(\Omega),\, u \neq 0\right\}. \tag{9}$$

In (9) the infimum is realized on the one dimensional eigenspace corresponding to $\widehat{\lambda}_1(q) > 0$. Moreover, from (9) we see that the elements of this eigenspace have fixed sign. In fact $\widehat{\lambda}_1(q) > 0$ is the only eigenvalue with eigenfunctions of fixed sign. All other eigenvalues have eigenfunctions which are nodal (sign changing). By $\widehat{u}_1 = \widehat{u}_1(q)$ we denote the L^q-normalized (that is, $\|\widehat{u}_1\|_q = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1(q) > 0$. The nonlinear regularity theory and the nonlinear maximum principle imply that $\widehat{u}_1(q) \in \operatorname{int} C_+^0$, where C_+^0 is the positive (order) cone of $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u\Big|_{\partial\Omega} = 0\}$. We have

$$C_+^0 = \left\{u \in C_0^1(\overline{\Omega}) \,:\, u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\right\},$$
$$\operatorname{int} C_+^0 = \left\{u \in C_+^0 \,:\, u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} < 0\right\},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. Since $\widehat{u}_1(q) \in \operatorname{int} C_+^0$, we have $\widehat{u}_1(q) \in W_0^{1,\eta}(\Omega)$. Since $\widehat{\lambda}_1(q) > 0$ is isolated and $\widehat{\sigma}(q)$ is closed, the second eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$ is well-defined by

$$\widehat{\lambda}_2^*(q) = \inf\{\lambda \,:\, \lambda \text{ is an eigenvalue of } (-\Delta_q, W_0^{1,q}(\Omega)),\ \lambda > \widehat{\lambda}_1(q)\}.$$

The Ljusternik-Schnirelmann minimax scheme gives a whole strictly increasing sequence of distinct eigenvalues $\{\widehat{\lambda}_k(q)\}_{k\geq 1}$ of $(-\Delta_q, W_0^{1,q}(\Omega))$ such that $\widehat{\lambda}_k(q) \to +\infty$. If $N = 1$ (ordinary differential equation) or if $p = 2$ (linear eigenvalue problem), then this sequence exhausts $\widehat{\sigma}(q)$. In general we do not know if this is the case. However, we know that $\widehat{\lambda}_2^*(q) = \widehat{\lambda}_2(q)$ (that is, the second eigenvalue coincides with the second Ljusternik-Schnirelmann eigenvalue). The Ljusternik-Schnirelmann scheme provides a minimax expression (variational characterization) for $\widehat{\lambda}_2(q)$. However, this minimax characterization of $\widehat{\lambda}_2(q)$ is not convenient for our purpose. Instead we will use an alternative one due to Cuesta-de Figueiredo-Gossez [7].

So, let $\widehat{\Gamma} = \{\widehat{\gamma} \in C([-1,1], M) : \widehat{\gamma}(-1) = -\widehat{u}_1(q), \widehat{\gamma}(1) = \widehat{u}_1(q)\}$ with $M = W_0^{1,q}(\Omega) \cap \partial B_1^{L^q}$ and $\partial B_1^{L^q} = \{u \in L^q(\Omega) : \|u\|_q = 1\}$.

Proposition 5. $\widehat{\lambda}_2(q) = \inf\limits_{\widehat{\gamma}\in\widehat{\Gamma}} \max\limits_{-1\leq t\leq 1} \|\nabla\widehat{\gamma}(t)\|_q^q$.

Finally let us fix our notation. If $x \in \mathbb{R}$, then $x^{\pm} = \max\{\pm x, 0\}$. For $u \in W_0^{1,\eta}(\Omega)$, we set $u^{\pm}(z) = u(z)^{\pm}$ for all $z \in \Omega$ and we have

$$u^{\pm} \in W_0^{1,\eta}(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^-.$$

Also by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. If X is a Banach space and $\varphi \in C^1(X)$, by K_φ we denote the critical set of φ, that is

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

Moreover, if $\vartheta \in \mathbb{R}$, then

$$\varphi^\vartheta = \{u \in X : \varphi(u) \leq \vartheta\} \text{ and } K_\varphi^\vartheta = \{u \in K_\varphi : \varphi(u) = \vartheta\}.$$

3. Three Nontrivial Solutions

We introduce the conditions on the perturbation $f(z,x)$.

H_1 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z,x)| \leq \widehat{a}(z)[1 + |x|^{r-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\widehat{a} \in L^\infty(\Omega)$, $p < r < q^*$;

(ii) $\lim\limits_{x\to\pm\infty} \dfrac{f(z,x)}{|x|^{p-2}x} = +\infty$ uniformly for a.a. $z \in \Omega$;

(iii) $\lim\limits_{x\to 0} \dfrac{f(z,x)}{|x|^{q-2}x} = 0$ uniformly for a.a. $z \in \Omega$.

First we establish the existence of nontrivial solutions of constant sign.

Proposition 6. *If hypotheses H_0, H_1 hold and $\lambda > \widehat{\lambda}_1(q)$, then problem (P_λ) has at least two nontrivial constant sign solutions $u_0 \in W_0^+$ and $v_0 \in (-W_0^+)$ such that for every $K \subseteq \Omega$ compact we have $u_0(z) \geq c_K > 0$ for a.a. $z \in K$, $v_0(z) \leq -\widehat{c}_K < 0$ for a.a. $z \in K$.*

Proof. On account of hypothesis $H_1(ii)$, given any $\beta > 0$, we can find $M_1 = M_1(\beta) \geq 1$ such that

$$f(z,x)x \geq \beta|x|^p \geq \beta|x|^q \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M_1 \geq 1 \text{ (recall } q < p). \tag{10}$$

Let $\beta = \lambda > \widehat{\lambda}_1(q)$ and for some $\tau_0 > M_1$ we set

$$\overline{u}(z) = \tau_0 \text{ and } \underline{u}(z) = -\tau_0 \quad \text{for all } z \in \overline{\Omega}.$$

Using these constants, we truncate the reaction of problem (P_λ). So, we introduce the Carathéodory functions $g_\lambda^\pm(z,x)$ defined by

$$g_\lambda^+(z,x)=\begin{cases}\lambda(x^+)^{q-1}-f(z,x^+) & \text{if } x\le \overline{u}(z)=\tau_0,\\ \lambda\tau_0^{q-1}-f(z,\tau_0) & \text{if } \overline{u}(z)=\tau_0<x.\end{cases}\tag{11}$$

$$g_\lambda^-(z,x)=\begin{cases}\lambda|\tau_0|^{q-2}(-\tau_0)-f(z,-\tau_0) & \text{if } x\le \underline{u}(z)=-\tau_0,\\ -\lambda(x^-)^{q-1}-f(z,-x^-) & \text{if } \underline{u}(z)=-\tau_0<x.\end{cases}\tag{12}$$

We set $G_\lambda^\pm(z,x)=\int_0^x g_\lambda^\pm(z,s)ds$ and consider the C^1-functionals $\widehat{\varphi}_\lambda^\pm : W_0^{1,\eta}(\Omega)\to\mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda^\pm(u)=\frac{1}{p}\int_\Omega a(z)|\nabla u|^p dz+\frac{1}{q}\|\nabla u\|_q^q-\int_\Omega G_\lambda^\pm(z,u)dz \quad \text{for all } u\in W_0^{1,\eta}(\Omega).$$

We have

$$\widehat{\varphi}_\lambda^\pm(u)\ge \frac{1}{p}\rho_\eta(|\nabla u|)-c_1 \quad \text{for some } c_1>0 \text{ (see (10) and recall } p>q).$$

If $\|u\|\to+\infty$, then $\|\nabla u\|_\eta\to+\infty$ (by the Poincaré inequality, see Proposition 3) and so $\rho_\eta(|\nabla u|)\to+\infty$ by Proposition 1(f). Therefore $\widehat{\varphi}_\lambda^+(\cdot)$ is coercive. Also, using Proposition 2(b) we see that $\widehat{\varphi}_\lambda^+(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_0\in W_0^{1,\eta}(\Omega)$ such that

$$\widehat{\varphi}_\lambda^+(u_0)=\inf\left[\widehat{\varphi}_\lambda^+(u)\,:\,u\in W_0^{1,\eta}(\Omega)\right].\tag{13}$$

Let $F(z,x)=\int_0^x f(z,s)ds$. According to hypothesis $H_1(iii)$ given $\varepsilon>0$ we can find $\delta_0\in(0,1)$ such that

$$F(z,x)\ge-\frac{\varepsilon}{q}|x|^q \quad \text{for a.a. } z\in\Omega,\text{ all } |x|\le\delta_0.\tag{14}$$

Consider the principal eigenfunction $\widehat{u}_1=\widehat{u}_1(q)\in \operatorname{int} C_+^0$. We can find $t\in(0,1)$ small such that $t\widehat{u}_1(z)\in[0,\delta_0]$ for all $z\in\overline{\Omega}$. We have

$$\widehat{\varphi}_\lambda^+(t\widehat{u}_1)\le\frac{t^p}{p}\int_\Omega a(z)|\nabla\widehat{u}_1|^p dz+\frac{t^q}{q}\left[\widehat{\lambda}_1(q)+\varepsilon-\lambda\right]\tag{15}$$

(see (10), (13) and recall that $\|\widehat{u}_1\|_q=1$).

Since $\lambda>\widehat{\lambda}_1(q)$, we choose $\varepsilon\in(0,\lambda-\widehat{\lambda}_1(q))$. Then from (15) and by choosing $t\in(0,1)$ even smaller if necessary, we will have

$$\begin{aligned}&\widehat{\varphi}_\lambda^+(t\widehat{u}_1)<0,\\ \Rightarrow\ &\widehat{\varphi}_\lambda^+(u_0)<0=\widehat{\varphi}_\lambda^+(0)\quad\text{(see (12))},\\ \Rightarrow\ &u_0\neq 0.\end{aligned}$$

From (13) we have

$$\begin{aligned}&\int_\Omega a(z)|\nabla u_0|^{p-2}(\nabla u_0,\nabla h)_{\mathbb{R}^N}dz+\int_\Omega|\nabla u_0|^{q-2}(\nabla u_0,\nabla h)_{\mathbb{R}^N}dz\\ &=\int_\Omega \widehat{g}_\lambda^+(z,u_0)h dz\quad\text{for all } h\in W_0^{1,\eta}(\Omega).\end{aligned}\tag{16}$$

In (16) first we choose $h = -u_0^- \in W_0^{1,\eta}(\Omega)$. Then

$$\begin{aligned}&\int_\Omega a(z)|\nabla u_0^-|^p dz + \|\nabla u_0^-\|_q^q = 0 \quad \text{(see (11))},\\ \Rightarrow\quad & u_0 \geq 0,\ u_0 \neq 0.\end{aligned}$$

Next in (15) we choose $h = (u_0 - \tau_0)^+ \in W_0^{1,\eta}(\Omega)$. Then

$$\begin{aligned}&\int_\Omega a(z)|\nabla u_0|^{p-2}(\nabla u_0, \nabla(u_0-\tau_0)^+)_{\mathbb{R}^N} dz + \int_\Omega |\nabla u_0|^{q-2}(\nabla u_0, \nabla(u_0-\tau_0)^+)_{\mathbb{R}^N} dz\\ &= \int_\Omega \left[\lambda \tau_0^{q-1} - f(z,\tau_0)\right](u_0-\tau_0)^+ dz \quad \text{(see (11))}\\ &\leq \int_\Omega [\lambda - \beta]\, \tau_0^{q-1}(u_0-\tau_0)^+ dz \quad \text{(see (10))}\\ &= 0 \quad (\text{since } \beta = \lambda),\\ \Rightarrow\quad & u_0 \leq \tau_0.\end{aligned}$$

So, we have proved that

$$u_0 \in [0,\tau_0] = \{u \in W_0^{1,\eta}(\Omega) \,:\, 0 \leq u(z) \leq \tau_0 \text{ for a.a. } z \in \Omega\}, u_0 \neq 0. \tag{17}$$

From (16), (17) and (11), we infer that u_0 is a positive solution for problem (P_λ), $\lambda > \widehat{\lambda}_1(q)$. Proposition 4 implies that for all $K \subseteq \Omega$ compact, there exists $c_K > 0$ such that

$$u_0(z) \geq c_K > 0 \quad \text{for a.a. } z \in K.$$

Similarly, using this time the functional $\widehat{\varphi}_\lambda^-(\cdot)$ and (12), we produce a negative solution $v_0 \in W_0^{1,\eta}(\Omega)$ of (P_λ), $\lambda > \widehat{\lambda}_1(q)$. Again for every $K \subseteq \Omega$ compact, we can find $\widehat{c}_K > 0$ such that

$$v_0(z) \leq -\widehat{c}_K < 0 \quad \text{for a.a. } z \in K.$$

□

We consider the energy functional $\varphi_\lambda : W_0^{1,\eta}(\Omega) \to \mathbb{R}$ for problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{p}\int_\Omega a(z)|\nabla u|^p dz + \frac{1}{q}\|\nabla u\|_q^q + \int_\Omega F(z,u)dz - \frac{\lambda}{q}\|u\|_q^q \quad \text{for all } u \in W_0^{1,\eta}(\Omega),$$

(recall $F(z,x) = \int_0^x f(z,s)ds$).

Let u_0, v_0 be the two nontrivial constant sign solutions of (P_λ) produced in Proposition 6.

Proposition 7. *If hypotheses H_0, H_1 hold and $\lambda > \widehat{\lambda}_1(q)$, then u_0 and v_0 are local minimizers of φ_λ.*

Proof. Consider the set

$$\overline{D}_\rho = \{u \in W_0^{1,\eta}(\Omega) \,:\, \|(u_0 - u)^-\| \leq \rho\} \quad \text{with } \rho > 0.$$

Using Proposition 2(c) we see that $\varphi_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. Also, it is clear from hypothesis $H_1\,(ii)$ that $\varphi_\lambda(\cdot)$ is coercive.

We consider the minimization problem

$$\inf\left[\varphi_\lambda(u) \,:\, u \in \overline{D}_\rho\right].$$

On account of the coercivity of $\varphi_\lambda(\cdot)$, a minimizing sequence for this problem is bounded. So, from the reflexivity of $W_0^{1,\eta}(\Omega)$ and the Eberlein-Smulian theorem, it is relatively sequentially weakly compact. This in conjunction with the sequential weak lower semicontinuity of φ_λ, imply that we can find $\widehat{u}_\rho \in \overline{D}_\rho$ such that

$$\begin{aligned} & \varphi_\lambda(\widehat{u}_\rho) = \inf\left[\varphi_\lambda(u) \,:\, u \in \overline{D}\rho\right], \\ \Rightarrow \quad & \langle \varphi_\lambda'(\widehat{u}_\rho), h\rangle \geq 0 \quad \text{for all } h \in W_0^+. \end{aligned} \tag{18}$$

In (18) we choose $h = \widehat{u}_\rho^- \in W_0^{1,\eta}(\Omega)$. Then

$$\begin{aligned} & -\int_\Omega a(z)|\nabla \widehat{u}_\rho^-|^p dz - \|\nabla \widehat{u}_\rho^-\|_q^q + \lambda\|\widehat{u}_\rho^-\|_q^q + \int_\Omega f(z, -\widehat{u}_\rho^-)\widehat{u}_\rho^- dz \geq 0, \\ \Rightarrow \quad & \int_\Omega a(z)|\nabla \widehat{u}_\rho^-|^p dz + \left[\|\nabla \widehat{u}_\rho^-\|_q^q - \lambda\|\widehat{u}_\rho^-\|_q^q\right] - \int_\Omega f(z, -\widehat{u}_\rho^-)\widehat{u}_\rho^- dz \leq 0. \end{aligned} \tag{19}$$

Hypotheses H_1 imply that

$$f(z,x)x \geq -c_2|x|^q \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_2 > 0. \tag{20}$$

Using (20) in (19), we obtain

$$\|\nabla \widehat{u}_\rho^-\|_q^q - (\lambda + c_2)\|\widehat{u}_\rho^-\|_q^q \leq 0. \tag{21}$$

Let $\widehat{\Omega}_\rho = \{z \in \Omega \,:\, \widehat{u}_\rho(z) < 0\}$. From Proposition 6 we know that $u_0(z) > 0$ for a.a. $z \in \Omega$. Therefore $|\widehat{\Omega}_\rho|_N \to 0$ as $\rho \to 0^+$. For any $\Omega' \subseteq \Omega$ open and any $u \in W_0^{1,q}(\Omega')$ we have

$$\begin{aligned} & \int_{\Omega'} |u|^q dz \leq |\Omega'|_N^{\frac{q}{N}} \left[\int_{\Omega'} |u|^{q^*} dz\right]^{\frac{N-q}{N}} \quad \text{(by Hölder's inequality)} \\ & \leq c_3 |\Omega'|_N^{\frac{q}{N}} \int_{\Omega'} |\nabla u|^q dz \quad \text{for some } c_3 > 0 \text{ (by the Sobolev embedding theorem).} \end{aligned} \tag{22}$$

Recall that

$$\begin{aligned} & \widehat{\lambda}_1(\widehat{\Omega}_\rho, q) = \inf\left\{\frac{\int_{\widehat{\Omega}_\rho} |\nabla u|^q dz}{\int_{\widehat{\Omega}_\rho} |u|^q dz} \,:\, u \in W_0^{1,q}(\widehat{\Omega}_\rho),\, u \neq 0\right\}, \\ \Rightarrow \quad & \widehat{\lambda}_1(\widehat{\Omega}_\rho, q) \to +\infty \quad \text{as } \rho \to 0^+ \\ & \text{(use (22) with } \Omega' = \widehat{\Omega}_\rho \text{ and recall that } |\widehat{\Omega}_\rho|_N \to 0 \text{ as } \rho \to 0^+\text{).} \end{aligned}$$

Therefore for $\rho \in (0,1)$ small, we will have

$$\widehat{\lambda}_1(\widehat{\Omega}_\rho, q) > \lambda + c_2.$$

So, from (21) we have

$$\|\nabla \widehat{u}_\rho^-\|_q^q < \widehat{\lambda}_1(\widehat{\Omega}_\rho, q)\|\widehat{u}_\rho^-\|_q^q \quad \text{if } \widehat{u}_\rho^- \neq 0,$$

which contradicts (9). Therefore

$$\begin{aligned} & \widehat{u}_\rho \geq 0 \quad \text{for } \rho \in (0,1) \text{ small}, \\ \Rightarrow \quad & \varphi_\lambda(u_0) \leq \varphi_\lambda(\widehat{u}_\rho), \\ \Rightarrow \quad & \varphi_\lambda(u_0) = \varphi_\lambda(\widehat{u}_\rho) = \min_{\overline{D}_\rho} \varphi_\lambda \quad \text{for } \rho \in (0,1) \text{ small}, \\ \Rightarrow \quad & u_0 \text{ is a local minimizer of } \varphi_\lambda \text{ (since } \operatorname{int} \overline{D}_\rho \neq \emptyset\text{).} \end{aligned}$$

Similarly for the negative solution $v_0 \in -W_0^+$. $\square$

Using this proposition and a minimax argument, we can generate a third nontrivial solution for problem (P_λ) when $\lambda > \widehat{\lambda}_2(q)$.

Proposition 8. *If hypotheses H_0, H_1 hold and $\lambda > \widehat{\lambda}_2(q)$, then problem (P_λ) admits a third nontrivial solution $y_0 \in W_0^{1,\eta}(\Omega)$.*

Proof. Let $u_0 \in W_0^+$ and $v_0 \in -W_0^+$ be the two nontrivial constant sign solutions of (P_λ) produced in Proposition 6. Without any loss of generality we assume that

$$\varphi_\lambda(v_0) \leq \varphi_\lambda(u_0).$$

The reasoning is similar if the opposite inequality holds. As usual we assume that K_{φ_λ} is finite or otherwise we already have a whole sequence of nontrivial solutions of (P_λ). From Proposition 7 we know that $u_0 \in W_0^+$ is a local minimizer of φ_λ. So, by Theorem 5.7.6, p. 449, of Papageorgiou-Rădulescu-Repovš [13], we can find $\rho \in (0,1)$ small such that

$$\varphi_\lambda(v_0) \leq \varphi_\lambda(u_0) < \inf\,[\varphi_\lambda(u) \,:\, \|u-u_0\| = \rho] = m_\lambda, \quad \|v_0 - u_0\| > \rho. \tag{23}$$

Also, hypotheses $H_1\,(i),(ii)$ imply that

$$\begin{aligned} & \varphi_\lambda(\cdot) \text{ is coercive,} \\ \Rightarrow\ & \varphi_\lambda(\cdot) \text{ satisfies the PS-condition} \\ & \text{(see Papageorgiou-Rădulescu-Repovš [13], Proposition 5.1.15, p. 369).} \end{aligned} \tag{24}$$

Then (23) and (24) permit the use of the mountain pass theorem. So, we can find $y_0 \in W_0^{1,\eta}(\Omega)$ such that

$$y_0 \in K_{\varphi_\lambda} \text{ and } m_\lambda \leq \varphi_\lambda(y_0) = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi_\lambda(\gamma(t)), \tag{25}$$

where $\Gamma = \{\gamma \in C([0,1], W_0^{1,\eta}(\Omega)) \,:\, \gamma(0) = v_0,\, \gamma(1) = u_0\}$. From (23) and (25) it follows that

$$y_0 \notin \{v_0, u_0\} \text{ and it is a solution of } (P_\lambda).$$

We need to show that $y_0 \neq 0$. On account of (25), if we can generate a path $\gamma_* \in \Gamma$ such that $\varphi_\lambda\Big|_{\gamma_*} < 0 = \varphi_\lambda(0)$, then $y_0 \neq 0$.

Recall (see Section 2) that we have

$$\begin{aligned} &\partial B_1^{L^q} = \{u \in L^q(\Omega) \,:\, \|u\|_q = 1\}, \\ &M = W_0^{1,q}(\Omega) \cap \partial B_1^{L^q}, \\ &\widehat{\Gamma} = \{\widehat{\gamma} \in C([-1,1], M) \,:\, \widehat{\gamma}(-1) = -\widehat{u}_1(q),\, \widehat{\gamma}(1) = \widehat{u}_1(q)\}. \end{aligned}$$

We also introduce

$$\begin{aligned} &M_c = M \cap C_0^1(\overline{\Omega}) \quad \text{(endowed with the } C_0^1(\overline{\Omega})\text{-topology)}, \\ &\widehat{\Gamma}_c = \{\widehat{\gamma} \in C([-1,1], M_c) \,:\, \widehat{\gamma}(-1) = -\widehat{u}_1(q),\, \widehat{\gamma}(1) = \widehat{u}_1(q)\}, \end{aligned}$$

(recall that $\widehat{u}_1(q) \in \operatorname{int} C_+^0$).

<u>Claim:</u> $\widehat{\Gamma}_c$ is dense in $\widehat{\Gamma}$.

Let $\widehat{\gamma} \in \widehat{\Gamma}$ and $\varepsilon > 0$. We introduce the multifunction $L_\varepsilon : [-1,1] \to 2^{C_0^1(\overline{\Omega})}$ defined by

$$L_\varepsilon(t) = \begin{cases} \{u \in C_0^1(\overline{\Omega}) \,:\, \|u - \widehat{\gamma}(t)\|_{1,q} < \varepsilon\} & \text{if } -1 < t < 1, \\ \{\pm\widehat{u}_1(q)\} & \text{if } t = \pm 1, \end{cases}$$

(by $\|\cdot\|_{1,q}$ we denote the norm of $W_0^{1,q}(\Omega)$; by the Poincaré inequality we have $\|u\|_{1,q} = \|\nabla u\|_q$ for all $u \in W_0^{1,q}(\Omega)$).

Clearly $L_\varepsilon(\cdot)$ has nonempty and convex values. For $t \in (-1,1)$ the values of $L_\varepsilon(\cdot)$ are open sets, while for $t = \pm 1$ are singletons. The continuity of $\widehat{\gamma}(\cdot)$ implies that the multifunction $L_\varepsilon(\cdot)$ is lsc (see [13], Proposition 2.5.4, p. 101). Therefore we can apply the Michael selection theorem (see Papageorgiou-Rădulescu-Repovš [13], Theorem 2.5.17, p. 106) and find a continuous path $\widehat{\gamma}_\varepsilon : [-1,1] \to C_0^1(\Omega)$ such that $\widehat{\gamma}_\varepsilon \in L_\varepsilon(t)$ for all $t \in [-1,1]$.

Now let $\varepsilon_n = \dfrac{1}{n}$, $n \in \mathbb{N}$ and let $\{\widehat{\gamma}_n = \widehat{\gamma}_{\varepsilon_n}\}_{n\geq 1} \subseteq C([-1,1], C_0^1(\overline{\Omega}))$ be as above. We have

$$\|\widehat{\gamma}_n(t) - \widehat{\gamma}(t)\|_{1,q} < \frac{1}{n} \text{ for } t \in (-1,1) \text{ and } \widehat{\gamma}_n(\pm 1) = \pm \widehat{u}_1(q) \text{ for all } n \in \mathbb{N}. \tag{26}$$

Since $\widehat{\gamma}(t) \in \partial B_1^{L^q}$ for all $t \in [-1,1]$, from (26) it is clear that we may assume that $\|\widehat{\gamma}_n(t)\|_q \neq 0$ for all $t \in [-1,1]$ and all $n \in \mathbb{N}$. We set

$$\widehat{\gamma}_n^*(t) = \frac{\widehat{\gamma}_n(t)}{\|\widehat{\gamma}_n(t)\|_q} \quad \text{for all } t \in [-1,1], \text{ all } n \in \mathbb{N}. \tag{27}$$

Then we have

$$\widehat{\gamma}_n^* \in C([-1,1], M_c) \text{ and } \widehat{\gamma}_n^*(\pm 1) = \pm \widehat{u}_1(q) \text{ for all } n \in \mathbb{N}.$$

From (27) and (26), we have

$$\begin{aligned} \|\widehat{\gamma}_n^*(t) - \widehat{\gamma}(t)\|_{1,q} &\leq \|\widehat{\gamma}_n^*(t) - \widehat{\gamma}_n(t)\|_{1,q} + \|\widehat{\gamma}_n(t) - \widehat{\gamma}(t)\|_{1,q} \\ &\leq \frac{|1 - \|\widehat{\gamma}_n(t)\|_q|}{\|\widehat{\gamma}_n(t)\|_q} \|\widehat{\gamma}_n(t)\|_{1,q} + \frac{1}{n} \quad \text{for all } t \in [-1,1], \text{ all } n \in \mathbb{N}. \end{aligned} \tag{28}$$

We have

$$\begin{aligned} &\max_{-1\leq t\leq 1} |1 - \|\widehat{\gamma}_n(t)\|_q| \\ &= \max_{-1\leq t\leq 1} |\|\widehat{\gamma}(t)\|_q - \|\widehat{\gamma}_n(t)\|_q| \quad (\text{since } \widehat{\gamma}(t) \in M \text{ for all } t \in [-1,1]) \\ &\leq \max_{-1\leq t\leq 1} \|\widehat{\gamma}(t) - \widehat{\gamma}_n(t)\|_q \\ &\leq c_4 \max_{-1\leq t\leq 1} \|\widehat{\gamma}(t) - \widehat{\gamma}_n(t)\|_{1,q} \quad \text{for some } c_4 > 0 \ (\text{since } W_0^{1,q}(\Omega) \hookrightarrow L^q(\Omega)) \\ &\leq \frac{c_4}{n} \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Using this in (28), we infer that

$$\widehat{\gamma}_n^* \to \widehat{\gamma} \text{ in } C([-1,1], M) \text{ with } \widehat{\gamma}_n^* \in C([-1,1], M_c) \text{ for all } n \in \mathbb{N}.$$

This proves the Claim.

Then the Claim and Proposition 5 imply that given $\varepsilon > 0$, we can find $\widehat{\gamma}_0 \in \widehat{\Gamma}_c$ such that

$$\max_{-1\leq t\leq 1} \|\nabla \widehat{\gamma}_0(t)\|_q^q \leq \widehat{\lambda}_2(q) + \varepsilon. \tag{29}$$

For this $\varepsilon > 0$, we choose $\delta_0 > 0$ as in (14). Since $\widehat{\gamma}_0([-1,1]) \subseteq C_0^1(\overline{\Omega})$ is compact, we can find $\widehat{\vartheta} \in (0,1)$ small such that

$$|\widehat{\vartheta}\widehat{\gamma}_0(t)(z)| \leq \delta_0 \quad \text{for all } z \in \overline{\Omega}, \text{ all } t \in [-1,1]. \tag{30}$$

Then we have

$$\varphi_\lambda(\widehat{\vartheta}\widehat{\gamma}_0(t)) \le \frac{\widehat{\vartheta}^p}{p}\int_\Omega a(z)|\nabla\widehat{\gamma}_0(t)|^p dz + \frac{\widehat{\vartheta}^q}{q}\left[\widehat{\lambda}_2(q) + 2\varepsilon - \lambda\right]$$

for all $t \in [-1,1]$ (see (14), (29), (30)).

But by hypothesis $\lambda > \widehat{\lambda}_2(q)$. So, if we choose $\varepsilon \in \left(0, \frac{1}{2}(\lambda - \widehat{\lambda}_2(q))\right)$, then we see that for $\widehat{\gamma}_0^* = \widehat{\vartheta}\widehat{\gamma}_0$ (which is a continuous path in $C_0^1(\overline{\Omega}) \subseteq W_0^{1,\eta}(\Omega)$ connecting $-\widehat{\vartheta}\widehat{u}_1(q)$ with $\widehat{\vartheta}\widehat{u}_1(q)$), we have

$$\varphi_\lambda\Big|_{\widehat{\gamma}_0^*} < 0. \tag{31}$$

We consider the positive truncation of the reaction, namely the Carathéodory function

$$k_\lambda^+(z,x) = \lambda(x^+)^{q-1} - f(z,x^+).$$

We set $K_\lambda^+(z,x) = \int_0^x k_\lambda^+(z,s)ds$ and consider the C^1-functional $\varphi_\lambda^+ : W_0^{1,\eta}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda^+(u) = \frac{1}{p}\int_\Omega a(z)|\nabla u|^p dz + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega K_\lambda^+(z,u)dz \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

It is easy to see that

$$K_{\varphi_\lambda^+} \subseteq W_0^+. \tag{32}$$

So, we may assume that

$$K_{\varphi_\lambda^+} = \{0, u_0\}, \tag{33}$$

otherwise we already have a third nontrivial solution of (P_λ) which in fact is positive (see (32)).

Let $a = \varphi_\lambda^+(u_0) = \varphi_\lambda(u_0) < 0$ (see the proof of Proposition 6). Also $\varphi_\lambda^+(0) = 0$. So, there are no critical values of φ_λ^+ in $(a,0)$ and $K_{\varphi_\lambda^+}^a = \{u_0\}$ (see (33)). The functional $\varphi_\lambda^+(\cdot)$ is coercive, hence it satisfies the PS-condition. So, we can use the Second Deformation Theorem (see Papageorgiou-Rădulescu-Repovš [13], Theorem 5.3.12, p. 386) and find a deformation $h : [0,1] \times ((\varphi_\lambda^+)^0 \setminus \{0\}) \to (\varphi_\lambda^+)^0$ such that

$$h(0,u) = u \quad \text{for all } u \in ((\varphi_\lambda^+)^0 \setminus \{0\}), \tag{34}$$

$$h(1,u) = u_0 \quad \text{for all } u \in ((\varphi_\lambda^+)^0 \setminus \{0\}), \tag{35}$$

$$\varphi_\lambda^+(h(t,u)) \le \varphi_\lambda^+(h(s,u)) \quad \text{for all } 0 \le s \le t \le 1, \text{ all } u \in ((\varphi_\lambda^+)^0 \setminus \{0\}). \tag{36}$$

We set $\widehat{\gamma}_+^*(t) = h(t, \widehat{\vartheta}\widehat{u}_1(q))^+$ for all $t \in [0,1]$. This is a continuous path in $W_0^{1,\eta}(\Omega)$ which connects $\widehat{\vartheta}\widehat{u}_1(q)$ (see (34)) with u_0 (see (35)). Also we have

$$\begin{aligned}
\varphi_\lambda(\widehat{\gamma}_+^*(t)) &= \varphi_\lambda^+(\widehat{\gamma}_+^*(t)) \quad (\text{since } \varphi_\lambda\Big|_{W_0^+} = \varphi_\lambda^+\Big|_{W_0^+}) \\
&= \varphi_\lambda^+(h(t, \widehat{\vartheta}\widehat{u}_1(q))^+) \\
&\le \varphi_\lambda^+(\widehat{\vartheta}\widehat{u}_1(q)) \quad (\text{see (36), (34)}) \\
&= \varphi_\lambda(\widehat{\vartheta}\widehat{u}_1(q)) \quad (\text{since } \widehat{u}_1(q) \in \operatorname{int} C_+^0) \\
&< 0 \quad \text{for all } t \in [0,1] \text{ (see (31))},
\end{aligned}$$

$$\Rightarrow \quad \varphi_\lambda\Big|_{\widehat{\gamma}_+^*} < 0. \tag{37}$$

Similarly we produce a continuous path $\widehat{\gamma}_-^*$ in $W_0^{1,\eta}(\Omega)$ connecting $-\widehat{\vartheta}\widehat{u}_1(q)$ with v_0 and such that

$$\varphi_\lambda\Big|_{\widehat{\gamma}_-^*} < 0. \tag{38}$$

In this case we consider the negative truncation of the reaction, namely the Carathéodory function

$$k_\lambda^-(z,x) = \lambda|x^-|^{q-2}(-x^-) - f(z,-x^-).$$

We set $K_\lambda^-(z,x) = \int_0^x k_\lambda^-(z,s)ds$ and then work with the C^1-functional $\varphi_\lambda^- : W_0^{1,\eta}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda^-(u) = \frac{1}{p}\int_\Omega a(z)|\nabla u|^p dz + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega K_\lambda^-(z,u)dz \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

We concatenate paths $\widehat{\gamma}_-^*$, $\widehat{\gamma}_0^*$ and $\widehat{\gamma}_+^*$ and generate a continuous path γ_* in $W_0^{1,\eta}(\Omega)$ connecting v_0 with u_0 (that is, $\gamma_* \in \Gamma$) such that

$$\varphi_\lambda\Big|_{\gamma_*} < 0 \quad \text{(see (31), (37), (38))},$$

$$\Rightarrow \quad y_0 \neq 0 \text{ and so it is the third nontrivial solution of } (P_\lambda).$$

□

Summarizing we can state the following multiplicity theorem for problem (P_λ).

Theorem 1. *If hypotheses H_0, H_1 hold and $\lambda > \widehat{\lambda}_2(q)$, then problem (P_λ) has at least three nontrivial solutions $u_0 \in W_0^+$, $v_0 \in -W_0^+$, $y_0 \in W_0^{1,\eta}(\Omega)$; moreover for all $K \subseteq \Omega$ compact we have $u_0(z) \geq c_K > 0$, $v_0(z) \leq -\widehat{c}_K < 0$ for a.a. $z \in K$.*

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