CARDINAL INVARIANTS OF CELLULAR-LINDELÖF SPACES

ANGELO BELLA AND SANTI SPADARO

ABSTRACT. A space X is said to be *cellular-Lindelöf* if for every cellular family \mathcal{U} there is a Lindelöf subspace L of X which meets every element of \mathcal{U} . Cellular-Lindelöf spaces generalize both Lindelöf spaces and spaces with the countable chain condition. Solving questions of Xuan and Song, we prove that every cellular-Lindelöf monotonically normal space is Lindelöf and that every cellular-Lindelöf space with a regular G_{δ} -diagonal has cardinality at most 2^c. We also prove that every normal cellular-Lindelöf first-countable space has cardinality at most continuum under $2^{<\mathfrak{c}} = \mathfrak{c}$ and that every normal cellular-Lindelöf space with a G_{δ} -diagonal of rank 2 has cardinality at most continuum.

1. INTRODUCTION

Two of the most important cardinal inequalities regarding Hausdorff topological spaces are Arhangel'skii's Theorem (see [1] and [11], for example) stating that the cardinality of every Lindelöf first-countable space does not exceed the continuum and the Hajnal-Juhász inequality (see [12]), which in the countable case says that every first-countable space with the countable chain condition has cardinality at most continuum.

The weak Lindelöf property is a common generalization of the Lindelöf property and the countable chain condition which may be used to state a common strengthening of Arhangel'skii's Theorem and of the Hajnal-Juhász inequality within the realm of normal spaces.

Definition 1. A space X is weakly Lindelöf if for every open cover \mathcal{U} of X there is a countable subcollection $\mathcal{V} \subset \mathcal{U}$ such that $X \subset \bigcup \mathcal{V}$.

It can be readily seen that every Lindelöf space is weakly Lindelöf. It's also easy to see that every open cover which does not have a countable subcollection with a dense union can be refined to an uncountable

²⁰⁰⁰ Mathematics Subject Classification. Primary: 54A25, 54D20; Secondary: 54E99, 54D55.

Key words and phrases. Cardinal inequality, Lindelöf, Arhangel'skii Theorem, elementary submodel, cellular-Lindelöf, ccc.

cellular family, that is an uncountable family of pairwise disjoint nonempty open sets. So the countable chain condition implies the weak Lindelöf property.

Theorem 2. (Bell, Ginsburg and Woods, [3]) Every normal weakly Lindelöf first-countable space has cardinality at most continuum.

The problem of whether normality can be relaxed to regularity in the above theorem is still open.

However, there are (see [3]) Hausdorff non-regular examples of weakly Lindelöf first-countable spaces of arbitrarily large cardinality, so one cannot expect to find a common generalization to Arhangel'skii's Theorem and the Hajnal-Juhász inequality by using the weak Lindelöf property.

In [4] we proposed another common generalization of the ccc and the Lindelöf property.

Definition 3. A space is called cellular-Lindelöf if for every cellular family \mathcal{U} there is a Lindelöf subspace L of X such that $U \cap L \neq \emptyset$, for every $U \in \mathcal{U}$.

We noted that every cellular-Lindelöf first-countable space has cardinality at most $2^{\mathfrak{c}}$ and asked whether the bound could be improved from $2^{\mathfrak{c}}$ to \mathfrak{c} .

Question 1. [4] Let X be a first-countable cellular-Lindelöf space. Is $|X| \leq \mathfrak{c}$?

A positive answer would lead to a common generalization of the Arhangel'skii and Hajnal-Juhász inequalities.

In this note we prove that, under $2^{<\mathfrak{c}} = \mathfrak{c}$, every normal cellular-Lindelöf first countable space has cardinality at most continuum, thus partially solving Question 1. We also prove, solving a question of Xuan and Song from [19], that every monotonically normal cellular-Lindelöf space is Lindelöf. As a byproduct, we obtain that Question 1 has a positive answer in ZFC for the class of monotonically normal spaces. We also prove that every normal cellular-Lindelöf space with a rank 2 diagonal has cardinality at most continuum. This gives a partial answer to a question from [18].

In our proofs we will sometimes use elementary submodels of the structure $(H(\mu), \epsilon)$. We believe that they make closing off arguments more transparent and concise. We encourage readers who have not done so already to acquaint themselves with Dow's survey [7]. Recall that $H(\mu)$ is the set of all sets whose transitive closure has cardinality smaller than μ . When μ is regular uncountable, $H(\mu)$ is known to

 $\mathbf{2}$

satisfy all axioms of set theory, except the power set axiom. We say, informally, that a formula is satisfied by a set S if it is true when all bounded quantifiers are restricted to S. A set $M \subset H(\mu)$ is said to be an elementary submodel of $H(\mu)$ (and we write $M \prec H(\mu)$) if a formula with parameters in M is satisfied by $H(\mu)$ if and only if it is satisfied by M.

The downward Löwenheim-Skolem theorem guarantees that for every $S \subset H(\mu)$, there is an elementary submodel $M \prec H(\mu)$ such that $|M| \leq |S| \cdot \omega$ and $S \subset M$. This theorem is sufficient for many applications, but it is often useful (especially in cardinal bounds for topological spaces) to have the following closure property. We say that M is κ closed if for every $S \subset M$ such that $|S| \leq \kappa$ we have $S \in M$. For every countable set $S \subset H(\mu)$ there is always a κ -closed elementary submodel $M \prec H(\mu)$ such that $|M| = 2^{\kappa}$ and $S \subset M$.

The following theorem is also used often: let $M \prec H(\mu)$ such that $\kappa + 1 \subset M$ and $S \in M$ be such that $|S| \leq \kappa$. Then $S \subset M$.

All spaces under consideration are assumed to be Hausdorff. Undefined notions can be found in [8] for topology and [14] for set theory. Our notation regarding cardinal functions mostly follows [12].

2. When is a cellular-Lindelöf space Lindelöf?

It is apparent from the definition that every ccc space is cellular-Lindelöf and every Lindelöf space is cellular-Lindelöf. The converses to either of the previous two implications do not hold as can be shown by simple examples distinguishing the ccc and the Lindelöf property.

Moreover, the weak Lindelöf property does not imply the cellular-Lindelöfness. In [18] Xuan and Song even provided an example of a weakly Lindelöf Moore space which is not cellular-Lindelöf. However, the question about the existence of a cellular-Lindelöf non-weakly Lindelöf space is still open.

We will prove that the cellular-Lindelöf property and the Lindelöfness are equivalent for monotonically normal spaces. This solves Questions 4.11–4.13 from [19].

Given a topological space X we indicate with $\mathcal{U}(X)$ the set of all pairs (x, U), where U is open and $x \in U$.

Definition 4. A topological space (X, τ) is called monotonically normal if there exists an operator $H : \mathcal{U}(X) \to \tau$ with the following properties:

(1) $x \in H(x, U) \subset U$, for every $(x, U) \in \mathcal{U}(X)$.

(2) If $H(x, U) \cap H(y, V) \neq \emptyset$ then $x \in V$ or $y \in U$.

Monotonically normal spaces generalize both metric spaces and linearly ordered spaces. Moreover, monotone normality is a hereditary property, so even every GO space (i.e., a subspace of a linearly ordered space) is monotonically normal (see [10] and [5]).

The proof of Theorem 6 is a variation on the proof of Theorem 3.17 from [13]. We're going to need a characterization of paracompactness for monotonically normal spaces due to Balogh and Rudin.

Lemma 5. (Balogh and Rudin [2]) A monotonically normal space X is paracompact if and only if it does not contain a closed subset homeomorphic to a stationary subset of a regular uncountable cardinal.

Theorem 6. Let X be a monotonically normal cellular-Lindelöf space. Then X is Lindelöf.

Proof. As noted by the authors of [19], the space X has countable extent. Therefore, to show that X is Lindelöf, it is sufficient to prove that X is paracompact. If the space X were not paracompact, it would contain a closed copy S of a stationary set of some regular uncountable cardinal κ . Let D be the set of all isolated points of S. By the nature of the topology on S we can choose, for every $x \in X$ a neighbourhood U_x of x such that $|U_x \cap D| < \kappa$.

For every $x \in D$, choose an open set V_x such that $V_x \cap D = \{x\}$. Note that $\mathcal{V} = \{H(x, H(x, V_x)) : x \in D\}$ is a cellular family, and hence there is a Lindelöf subspace L of X such that $L \cap V \neq \emptyset$, for every $V \in \mathcal{V}$. Since D has cardinality κ , the family \mathcal{V} also has cardinality κ and since κ is a regular uncountable cardinal and L is Lindelöf, Lmust contain a complete accumulation point for the family \mathcal{V} , that is a point $p \in L$ such that $\{V \in \mathcal{V} : O \cap V \neq \emptyset\}$ has cardinality κ , for every open neighbourhood O of p. Clearly $p \notin H(x, V_x)$ for every $x \in D$, or otherwise $H(x, V_x)$ would be an open neighbourhood of p which meets only one element of the family \mathcal{V} . So if $H(p, U_p) \cap H(x, H(x, V_x)) \neq \emptyset$, for some $x \in D$, we must have $x \in U_p$, by the second property of a monotone normality operator.

It follows that $\{x \in D : H(x, H(x, V_x)) \cap H(p, U_p) \neq \emptyset\} \subset D \cap U_p$, and since the latter set has cardinality smaller than κ , the point p is not a complete accumulation point for the family \mathcal{V} , which is a contradiction.

Corollary 7. Let X be a GO space. Then X is Lindelöf if and only if X is cellular-Lindelöf.

Corollary 8. Let X be a LOTS. Then X is Lindelöf if and only if X is cellular-Lindelöf.

4

3. The cardinality of cellular-Lindelöf first-countable spaces

In this section we will establish a cardinal inequality for cellular-Lindelöf spaces and find a class of spaces where cellular-Lindelöf implies weakly Lindelöf, under CH. The core of the argument of both results is the following lemma, which roughly says that a cellular-Lindelöf space is close to being weakly Lindelöf for covers of small size.

Lemma 9. Let X be a cellular-Lindelöf space. Let \mathcal{U} be an open cover of X having cardinality continuum. Then there is a subcollection $\mathcal{V} \subset \mathcal{U}$ of cardinality smaller than the continuum such that $X \subset \bigcup \mathcal{V}$.

Proof. Suppose the statement is false and let $\{U_{\alpha} : \alpha < \mathfrak{c}\}$ be an enumeration of \mathcal{U} . Then we can find a strictly increasing sequence of ordinals $\{\alpha_{\beta} : \beta < \mathfrak{c}\}$ such that $V_{\beta} = U_{\alpha_{\beta+1}} \setminus \bigcup \{U_{\gamma} : \gamma \leq \alpha_{\beta}\}$ is a non-empty open set, for every $\beta < \mathfrak{c}$. But then the cellular-Lindelöf property implies the existence of a Lindelöf subspace L of X such that $L \cap V_{\beta} \neq \emptyset$, for every $\beta < \mathfrak{c}$. Since \mathcal{U} is an open cover of the Lindelöf space L, there must be an ordinal $\beta < \mathfrak{c}$ such that $L \subset \bigcup \{U_{\gamma} : \gamma \leq \alpha_{\beta}\}$, but this contradicts $L \cap V_{\beta} \neq \emptyset$ and we are done.

The following theorem gives a partial positive answer to Question 4 from [4].

Theorem 10. $(2^{<\mathfrak{c}} = \mathfrak{c})$ Let X be a normal sequential cellular-Lindelöf space such that $\chi(X) \leq \mathfrak{c}$. Then $|X| \leq \mathfrak{c}$

Proof. Let M be an elementary submodel of $H(\theta)$, where θ is a regular large enough cardinal, such that $X \in M$, $\mathfrak{c} + 1 \subset M$, M is closed under sequence of cardinality less than \mathfrak{c} and $|M| = \mathfrak{c}$. The sequentiality of X implies that $X \cap M$ is a closed subspace of X.

We claim that $X \subset M$. Suppose by contradiction that there is a point $p \in X \setminus M$. By regularity we can find an open set $U \subset X$ such that $X \cap M \subset U$ and $p \notin \overline{U}$. Use $\chi(X) \leq \mathfrak{c}$ and $\mathfrak{c} + 1 \subset M$ to choose, for every $x \in X \cap M$, an open neighbourhood U_x of x such that $U_x \subset U$ and $U_x \in M$. Let $V = \bigcup \{U_x : x \in X \cap M\}$ and note that $X \cap M$ and $X \setminus V$ are disjoint closed sets, so there are disjoint open sets G_1 and G_2 such that $X \cap M \subset G_1$ and $X \setminus V \subset G_2$. Now $\{U_x : x \in X \cap M\} \cup \{G_2\}$ is an open cover of X having cardinality continuum, so by Lemma 9 there must be a set $C \subset X \cap M$ of cardinality less than continuum such that $X \subset \bigcup \{U_x : x \in X \cap M\} \cup \overline{G_2}$. But $\overline{G_2} \cap X \cap M = \emptyset$, so we actually have $X \cap M \subset \bigcup \{U_x : x \in C\}$. But the fact that M is closed under $< \mathfrak{c}$ -sequences implies that $C \in M$, so:

$$M \models X \subset \overline{\bigcup\{U_x : x \in C\}}$$

Therefore, by elementarity, we can say that:

$$H(\theta) \models X \subset \overline{\bigcup\{U_x : x \in C\}}$$

But the above formula contradicts the fact that $p \notin \overline{U}$. So $X \subset M$, which implies $|X| \leq |M| \leq 2^{\aleph_0}$.

The following corollary is a consequence of Theorem 6 and gives another partial positive answer to Question 4 from [4].

Corollary 11. Every monotonically normal cellular-Lindelöf first-countable space has cardinality at most continuum.

Corollary 12. (CH) Every normal first-countable cellular-Lindelöf space is weakly Lindelöf.

Proof. Let X be a normal first-countable cellular-Lindelöf space. By Theorem 10, the space X has cardinality at most $\mathfrak{c} = \omega_1$, so an argument similar to the one proving Lemma 9 shows that X is weakly Lindelöf.

4. Cellular-Lindelöf spaces with G_{δ} -diagonals

Recall that a space X is said to have a G_{δ} -diagonal iff its diagonal is a countable intersection of open sets in X^2 . This is equivalent to the existence of a sequence of open covers $\{\mathcal{U}_n : n < \omega\}$ of the space X such that $\bigcap \{St(x, \mathcal{U}_n) : n < \omega\} = \{x\}$, for every $x \in X$.

While Lindelöf spaces with a G_{δ} -diagonal have cardinality at most continuum, there is no bound on the cardinality of cellular-Lindelöf spaces with a G_{δ} -diagonal. Indeed, Shakmatov [15] and Uspenskii [16] constructed examples of arbitrarily large ccc spaces with a G_{δ} -diagonal.

Some cardinality restrictions can be obtained by using certain strengthenings of the notion of a G_{δ} -diagonal. A space X has a G_{δ} -diagonal of rank 2 if there exists a sequence $\{\mathcal{U}_n : n < \omega\}$ of open covers of X such that $\bigcap \{St(St(x,\mathcal{U}_n),\mathcal{U}_n) : n < \omega\} = \{x\}$ for every $x \in X$.

In [19], Xuan and Song proved that every cellular-Lindelöf space with a G_{δ} -diagonal of rank 2 has cardinality at most 2^c. We prove that in the presence of normality the bound can be improved.

Theorem 13. Let X be a normal cellular-Lindelöf space X with a G_{δ} -diagonal of rank 2. Then $|X| \leq \mathfrak{c}$.

Proof. Let $\{\mathcal{U}_n : n < \omega\}$ be a sequence of open covers witnessing that X has a G_{δ} -diagonal of rank 2 and suppose by contradiction that $|X| > \mathfrak{c}$. Set $F_n = \{\{x, y\} \in [X]^2 : St(x, \mathcal{U}_n) \cap St(y, \mathcal{U}_n) = \emptyset\}$. Since $[X]^2 = \bigcup\{F_n : n < \omega\}$, by the Erdős-Rado theorem there is an uncountable set $S \subset X$ and an integer n_0 such that $[S]^2 \subset F_{n_0}$. Since \mathcal{U}_{n_0} is an open cover of X, the set S is closed. Therefore, we may pick an open set V such that $S \subset V \subset \overline{V} \subset \bigcup\{St(x, \mathcal{U}_{n_0}) : x \in S\}$. The family $\mathcal{U} = \{St(x, \mathcal{U}_{n_0}) \cap V : x \in S\}$ consists of pairwise disjoint non-empty open sets and thus there is a Lindelöf subspace L of X which meets every member of \mathcal{U} . But then $\{U \cap L : U \in \mathcal{U}\}$ is an uncountable discrete family in L and that is a contradiction.

One of the referees noted that the above theorem also follows from the main result of [17].

The following theorem answers Question 5.5 from [19].

Theorem 14. Every cellular-Lindelöf space with a regular G_{δ} -diagonal has cardinality bounded by 2^c.

Proof. Every Lindelöf space with a G_{δ} -diagonal has cardinality at most continuum, and hence every cellular-Lindelöf space with a G_{δ} -diagonal has cellularity at most continuum. Now, a space with cellularity at most continuum and a regular G_{δ} -diagonal has cardinality at most 2^c by Theorem 4.2 of [9] (see also [6]).

5. Open Problems

The topic of cellular-Lindelöf spaces is still in its infancy and several intriguing questions remain open about them. Besides Question 1, these are the main ones:

Question 2. Let X be a cellular-Lindelöf first-countable regular space. Is $|X| \leq \mathfrak{c}$?

Theorem 6 and Corollary 11 give partial positive answers to the above question.

Question 3. Let X be a cellular-Lindelöf first-countable normal space. Is $|X| \leq \mathfrak{c}$ in ZFC?

Theorem 6 shows that the above question has a positive answer under $2^{<\mathfrak{c}} = \mathfrak{c}$.

Question 4. Is there a cellular-Lindelöf non-weakly Lindelöf (regular) space?

Question 5. Is every normal cellular-Lindelöf first-countable space weakly Lindelöf in ZFC? Corollary 12 shows that the above question has a positive answer under CH.

If the following question had a positive answer, then, in view of Corollary 12, the cellular-Lindelöf and the weak Lindelöf property would be equivalent for the class of normal first-countable spaces under CH.

Question 6. Is every normal first-countable weakly Lindelöf space cellular-Lindelöf under CH?

Question 7. [19] Is there a normal weakly Lindelöf non-cellular-Lindelöf space?

Question 8. Let X be a cellular-Lindelöf regular space with a G_{δ} diagonal of rank 2. Is $|X| \leq \mathfrak{c}$?

By Theorem 13 the above question has a positive answer for the class of normal spaces.

Question 9. Let X be a cellular-Lindelöf space with a regular G_{δ} -diagonal. Is $|X| \leq \mathfrak{c}$?

References

- A. Arhangel'skii, On the cardinality of bicompacta satisfying the first axiom of countability, Soviet Math. Dokl. 10 (1969), 951–955.
- [2] Z. Balogh and M.E. Rudin, Monotone normality, Topology Appl. 47 (1992), 115–127.
- [3] M. Bell, J. Ginsburg and G. Woods, Cardinal inequalities for topological spaces involving the weak Lindelöf number, Pacific J. Math. 79 (1978), 37–45.
- [4] A. Bella and S. Spadaro, On the cardinality of almost discretely Lindelöf spaces, Monatsh. Math. 186 (2018), 345–353.
- [5] C.R. Borges, A study of monotonically normal spaces, Proc. Amer. Math. Soc. 38 (1973), 211–214.
- [6] R. Buzyakova, Cardinalities of ccc-spaces with regular G_{δ} -diagonals, Topology Appl. **153** (2006), 1696–1698.
- [7] A. Dow, An introduction to applications of elementary submodels to topology, Topology Proc. 13 (1988), no. 1, 17–72.
- [8] R. Engelking, General Topology, PWN, Warsaw, 1977.
- [9] I. Gotchev, Cardinalities of weakly Lindelöf spaces with regular G_{κ} -diagonals, preprint, arXiv:1504.01785.
- [10] R.W. Heath, D.J. Lutzer and P.L. Zenor, Monotonically normal spaces, Trans. Amer. Math. Soc. 178 (1973), 481–493.
- [11] R. E. Hodel, Arhangelskii's solution to Alexandroff's problem, Topology Appl. 153 (2006), 2199–2217.
- [12] I. Juhász, Cardinal Functions in Topology Ten Years Later, Math. Centre Tracts 123, 1980, Amsterdam.
- [13] I. Juhász, V.V. Tkachuk and R.G. Wilson, Weakly linearly Lindelöf monotonically normal spaces are Lindelöf, Studia Sci. Math. Hungarica 54:4 (2017), 523–535.

8

- [14] K. Kunen, Set Theory, Studies in Logic, n. 34, College Publications, London, 2011.
- [15] D. Shakhmatov, No upper bound for cardinalities of Tychonoff C.C.C. spaces with a G_{δ} diagonal exist (an answer to J. Ginsburg and R.G. Woods' question), Comment. Math. Univ. Carolinae **25** (1984), 731–746..
- [16] V. Uspenskii, A large F_{σ} -discrete Fréchet space having the Souslin property, Comment. Math. Univ. Carolinae **25** (1984), 257–260.
- [17] W.-F. Xuan and Y.-K. Song, Cardinalities of DCCC normal spaces with a rank 2 diagional, Math. Bohem. 141 (2016), 457–461.
- [18] W.-F. Xuan and Y.-K. Song, On cellular-Lindelöf spaces, Bull. Iran. Math. Soc. 44 (2018), 1485–1491.
- [19] W.-F. Xuan and Y.-K. Song, A study of cellular-Lindelöf spaces, Topology Appl. 251 (2019), 1–9.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF CATANIA, CITTÁ UNIVERSITARIA, VIALE A. DORIA 6, 95125 CATANIA, ITALY *E-mail address*: bella@dmi.unict.it

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF CATANIA, CITTÁ UNIVERSITARIA, VIALE A. DORIA 6, 95125 CATANIA, ITALY *E-mail address*: santidspadaro@gmail.com