SOLUTIONS WITH SIGN INFORMATION FOR NONLINEAR ROBIN PROBLEMS WITH NO GROWTH RESTRICTION ON REACTION

NIKOLAOS S. PAPAGEORGIOU, CALOGERO VETRO, FRANCESCA VETRO

Abstract. We consider a parametric nonlinear Robin problem driven by a nonhomogeneous differential operator. The reaction is a Carathéodory function which is only locally defined (that is, the hypotheses concern only its behaviour near zero). The conditions on the reaction are minimal. Using variational tools together with truncation, perturbation and comparison techniques and critical groups, we show that for all small values of the parameter $\lambda > 0$, the problem has at least three nontrivial smooth solutions, two of constant sign and the third nodal.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear parametric Robin problem

$$(P_\lambda) \qquad \begin{cases} -\operatorname{div} a(\nabla u(z)) + \xi(z)|u(z)|^{p-2}u(z) = \lambda f(z,u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega,\ \lambda > 0,\ 1 < p < +\infty. \end{cases}$$

In this problem $a : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous and strictly monotone map (thus $a(\cdot)$ is maximal monotone too), which satisfies certain other regularity and growth conditions listed in hypotheses H(a) below. These hypotheses form a general framework, which incorporates many differential operators of interest (see the Examples in Section 2). The potential function $\xi \in L^\infty(\Omega)$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$. In the reaction (right hand side of (P_λ)), $\lambda > 0$ is a parameter and $f(z,x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z,x)$ is continuous). The distinguishing feature of our work, is that we do not impose any global growth condition on $f(z,\cdot)$. We only assume that $f(z,\cdot)$ is $(q-1)$-superlinear near 0 with $1 < q < p$ and also that $f(z,\cdot)$ is locally $L^\infty(\Omega)$-bounded. Hence the conditions on $f(z,\cdot)$ are minimal and so the setting of problem (P_λ) is general. In the boundary condition, $\dfrac{\partial u}{\partial n_a}$ denotes the conormal derivative corresponding to the map $a(\cdot)$. The boundary condition is interpreted via the nonlinear Green's identity (see Gasiński-Papageorgiou [2], p. 210) and if $u \in C^1(\overline{\Omega})$, then

$$\frac{\partial u}{\partial n_a} = (a(\nabla u), n)_{\mathbb{R}^N},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The boundary coefficient $\beta \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$. The case $\beta \equiv 0$ which corresponds to the Neumann problem, is also included in our setting.

Key words and phrases. Nonlinear regularity theory, nonlinear maximum principle, extremal constant sign solutions, nodal solutions, critical groups.

2010 Mathematics Subject Classification: 35J20, 35J60, 58E05.

Corresponding author: Francesca Vetro (francescavetro@tdtu.edu.vn).

Under these general conditions on the data of the problem, we show that for all $\lambda > 0$ small, problem (P_λ) has at least three nontrivial smooth solutions, all with sign information (two solutions have constant sign and the third is nodal (that is, the solution is sign changing). Our work here complements those of Papageorgiou-Winkert [17] and of Guarnotta-Marano-Papageorgiou [6], which have a nonparametric reaction of arbitrary growth with two zeros of constant sign. So, the reaction $f(z,\cdot)$ is forced to have an oscillatory behavior near zero. No such condition is used in this work. We also mention the recent work of Papageorgiou-Rădulescu-Repovš [15], who impose a symmetry condition on $f(z,\cdot)$ and produce a whole sequence of distinct nodal solutions converging to zero in $C^1(\overline{\Omega})$.

2. Mathematical Background - Hypotheses

The main spaces that we will use in the analysis of problem (P_λ) are the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the "boundary" Lebesgue spaces $L^s(\partial\Omega)$, $1 \leq s \leq +\infty$. By $\|\cdot\|$ we denote the norm of $W^{1,p}(\Omega)$ defined by

$$\|u\| = \left[\|u\|_p^p + \|\nabla u\|_p^p\right]^{1/p} \quad \text{for all } u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is ordered with positive cone $C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$D_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure on $\partial\Omega$, we can define in the usual way the Lebesgue spaces $L^s(\partial\Omega)$, $1 \leq s \leq +\infty$. From the theory of Sobolev spaces, we know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u\Big|_{\partial\Omega} \quad \text{for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

The trace map is not surjective. In fact $\operatorname{im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega)$ with $\dfrac{1}{p'} + \dfrac{1}{p} = 1$ and $\ker\gamma_0 = W_0^{1,p}(\Omega)$. Moreover, the trace map $\gamma_0(\cdot)$ is compact into $L^s(\partial\Omega)$ for all $1 \leq s < \dfrac{(N-1)p}{N-p}$ if $p < N$ and into $L^s(\partial\Omega)$ for all $1 \leq s < +\infty$ if $p \geq N$. In what follows, for the sake of notational economy, we drop the use of the trace map $\gamma_0(\cdot)$. All restrictions of Sobolev functions on $\partial\Omega$, are understood in the sense of traces.

Now let $l \in C^1(0,+\infty)$ which satisfies

$$0 < \widehat{c} \leq \frac{tl'(t)}{l(t)} \leq c_0 \text{ and } c_1 t^{p-1} \leq l(t) \leq c_2[t^{s-1} + t^{p-1}] \tag{1}$$

for all $t > 0$, some $c_1, c_2 > 0$ and $1 \leq s < p$.

The hypotheses on the map $a(\cdot)$, are the following:

H(a): $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0,+\infty)$, $t \to ta_0(t)$ is strictly increasing on $(0,+\infty)$, $\lim_{t\to 0^+} ta_0(t) = 0$ and $\lim_{t\to 0^+} \dfrac{ta_0'(t)}{a_0(t)} > -1$;

(ii) $|\nabla a(y)| \leq c_3 \dfrac{l(|y|)}{|y|}$ for all $y \in \mathbb{R}^N \setminus \{0\}$, some $c_3 > 0$;

(iii) $(\nabla a(y)\xi,\xi)_{\mathbb{R}^N} \geq \dfrac{l(|y|)}{|y|}|\xi|^2$ for all $y \in \mathbb{R}^N \setminus \{0\}$, all $\xi \in \mathbb{R}^N$;

(iv) if $G_0(t) = \int_0^t a_0(s)sds$, $t > 0$, then there exist $1 < \tau < q < p$ such that

$$c_4 t^p \leq a_0(t)t^2 - \tau G_0(t) \text{ for all } t > 0, \text{ some } c_4 > 0,$$

$$\limsup_{t\to 0^+} \frac{qG_0(t)}{t^q} \leq c^* < +\infty.$$

Remark 1. Hypotheses H(a) (i), (ii), (iii) are dictated by the nonlinear regularity theory of Lieberman [8] and the nonlinear maximum principle of Pucci-Serrin [18]. Hypothesis H(a) (iv) serves the needs of our problem. It is mild and it is satisfied in all cases of interest (see the examples below).

From these hypotheses we see that $G_0(\cdot)$ is strictly increasing and strictly convex. We set $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. Evidently $G(\cdot)$ is convex and we have

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\}.$$

The convexity of $G(\cdot)$ and since $G(0) = 0$, imply that

$$G(y) \leq (a(y), y)_{\mathbb{R}^N} \quad \text{for all } y \in \mathbb{R}^N. \tag{2}$$

From (1) and hypotheses H(a), we easily deduce the following lemma which summarizes the main properties of the map $a(\cdot)$ (see Papageorgiou-Rădulescu [11]).

Lemma 1. *If hypotheses H(a) (i), (ii), (iii) hold, then*

(a) *$a(\cdot)$ is continuous and strictly monotone (hence maximal monotone too);*
(b) *$|a(y)| \leq c_5[|y|^{s-1} + |y|^{p-1}]$ for all $y \in \mathbb{R}^N$, some $c_5 > 0$;*
(c) *$(a(y), y)_{\mathbb{R}^N} \geq \dfrac{c_1}{p-1}|y|^p$ for all $y \in \mathbb{R}^N$.*

This lemma and (2) lead to the following growth estimates for the primitive $G(\cdot)$.

Corollary 1. *If hypotheses H(a) (i), (ii), (iii) hold, then $\dfrac{c_1}{p(p-1)}|y|^p \leq G(y) \leq c_6[1 + |y|^p]$ for all $y \in \mathbb{R}^N$, some $c_6 > 0$.*

The examples that follow show that the framework provided by hypotheses H(a) is general and includes many cases of interest.

Examples 1. *Let $a : \mathbb{R}^N \to \mathbb{R}^N$ be given as:*

(a) *$a(y) = |y|^{p-2}y$, $1 < p < +\infty$. This map corresponds to the well-known p-Laplace differential operator $\Delta_p u = div\,(|\nabla u|^{p-2}\nabla u)$ for all $u \in W^{1,p}(\Omega)$.*
(b) *$a(y) = |y|^{p-2}y + |y|^{q-2}y$, $1 < q < p < +\infty$. This map corresponds to the (p,q)-Laplace differential operator $\Delta_p u + \Delta_q u$ for all $u \in W^{1,p}(\Omega)$.*

   Such operators arise in the mathematical models of various physical processes. Recently there have been several existence and multiplicity results for such equations. An informative survey of these results with several relevant references can be found in the paper of Marano-Mosconi [9].
(c) *$a(y) = [1 + |y|^2]^{\frac{p-2}{2}}y$, $1 < p < +\infty$. This map corresponds to the extended capillary differential operator.*

(d) $a(y) = |y|^{p-2}y\left[1+\dfrac{1}{1+|y|^p}\right]$, $1 < p < +\infty$. *This map corresponds to the differential operator* $u \to \Delta_p u + div\left(\frac{1}{1+|\nabla u|^p}|\nabla u|^{p-2}\nabla u\right)$ *which arises in problems of plasticity theory (see Fuchs-Li* [1]*).*

The hypotheses on the potential function $\xi(\cdot)$ and the boundary coefficient are the following:

H(ξ): $\xi \in L^\infty(\Omega)$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$.
H(β): $\beta \in C^{0,\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$, $\beta(z) \geq 0$ for all $z \in \partial\Omega$.
H$_0$: $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

Remark 2. These hypotheses include also the Neumann problem, which corresponds to the case $\beta \equiv 0$.

Consider the C^1-functional $\gamma : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\gamma(u) = \int_\Omega pG(\nabla u)dz + \int_\Omega \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

Using Lemma 4.11 of Mugnai-Papageorgiou [10] (for the case $\xi \not\equiv 0$) and Proposition 2.4 of Gasiński-Papageorgiou [5] (for the case $\beta \not\equiv 0$), together with Corollary 1, we infer that

$$\gamma(u) \geq c_7\|u\|^p \quad \text{for some } c_7 > 0, \text{ all } u \in W^{1,p}(\Omega). \tag{3}$$

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear operator defined by

$$\langle A(u), h\rangle = \int_\Omega (a(\nabla u), \nabla h)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W^{1,p}(\Omega).$$

From Gasiński-Papageorgiou [3] (Proposition 3.5), we have:

Proposition 1. *If hypotheses H(a) (i), (ii), (iii) hold, then* $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ *is continuous, monotone (hence maximal monotone too) and of type* (S_+)*, that is,* $u_n \xrightarrow{w} u$ *in* $W^{1,p}(\Omega)$ *and* $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leq 0$ *imply* $u_n \to u$ *in* $W^{1,p}(\Omega)$*.*

Let X be a Banach space and $\varphi \in C^1(X, \mathbb{R})$, $c \in \mathbb{R}$. We introduce the following sets:

$$K_\varphi = \{u \in X : \varphi'(u) = 0\} \quad \text{(the critical set of } \varphi\text{)},$$
$$\varphi^c = \{u \in X : \varphi(u) \leq c\} \quad \text{(the c-sublevel set of } \varphi\text{)}.$$

Given a topological pair (Y_1, Y_2) such that $Y_2 \subseteq Y_1 \subseteq X$, by $H_k(Y_1, Y_2)$, $k \in \mathbb{N}_0$, we denote the k^{th}-relative singular homology group with integer coefficients. Let $u \in K_\varphi$ be isolated and $\varphi(u) = c$. Then, the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

with U being a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$ (isolating neighborhood of u). The excision property of singular homology implies that this definition is independent of the choice of the isolating neighborhood U.

Next let us introduce the basic notation used in this paper. For every $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then given $u \in W^{1,p}(\Omega)$, we define $u^\pm(z) = u(z)^\pm$ for all $z \in \Omega$. We know that

$$u^\pm \in W^{1,p}(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^-.$$

Given a measurable function $k : \Omega \times \mathbb{R} \to \mathbb{R}$ (for example, a Carathéodory function), by $N_k(\cdot)$ we denote the Nemytskii (superposition) operator corresponding to k, defined by $N_k(u)(\cdot)=k(\cdot,u(\cdot))$. Evidently $z\to N_k(u)(z)$ is measurable on Ω. If $u,v\in W^{1,p}(\Omega)$ and $u\le v$, then by $[u,v]$ we denote the order interval in $W^{1,p}(\Omega)$ defined by

$$[u,v]=\{h\in W^{1,p}(\Omega): u(z)\le h(z)\le v(z) \text{ for a.a. } z\in\Omega\}.$$

Now we introduce the hypotheses on the reaction $f(z,x)$.

$H(f)$: $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $f(z,0)=0$ for a.a. $z\in\Omega$ and

- (i) for every $\rho>0$, there exists $a_\rho\in L^\infty(\Omega)_+$ such that $|f(z,x)|\le a_\rho(z)$ for a.a. $z\in\Omega$, all $|x|\le\rho$;
- (ii) with $q\in(\tau,p)$ as in hypothesis H(a) (iv), we have $\lim_{x\to 0}\dfrac{f(z,x)}{|x|^{q-2}x}=+\infty$ uniformly for a.a. $z\in\Omega$;
- (iii) with $\tau\in(1,q)$ as in hypothesis H(a) (iv) and $F(z,x)=\int_0^x f(z,s)ds$, we have $0\le\liminf_{x\to 0}\dfrac{\tau F(z,x)-f(z,x)x}{|x|^p}$ uniformly for a.a. $z\in\Omega$;
- (iv) for every $\rho>0$, there exists $\widehat{\xi}_\rho>0$ such that for a.a. $z\in\Omega$ the function $x\to f(z,x)+\widehat{\xi}_\rho|x|^{p-2}x$ is nondecreasing on $[-\rho,\rho]$.

Remark 3. We stress that no global growth condition is imposed on $f(z,\cdot)$. Also, note that no sign condition is assumed.

3. Solutions of Constant Sign

In what follows by $\mathcal{L}_+$ (resp. $\mathcal{L}_-$) we denote the set of all parameters $\lambda>0$ such that problem (P_λ) admits positive (resp. negative) solutions. Also by $\mathcal{S}_\lambda^+$ (resp. $\mathcal{S}_\lambda^-$) we denote the corresponding set of positive (resp. negative) solutions of problem (P_λ).

Proposition 2. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, H_0 and $H(f)$ hold, then*

(a) $\mathcal{L}_+\neq\emptyset$, $\mathcal{L}_-\neq\emptyset$ *and* $S_\lambda^+\subseteq D_+$, $S_\lambda^-\subseteq -D_+$;

(b) *If* $\lambda\in\mathcal{L}_+$ *(resp.* $\lambda\in\mathcal{L}_-$*),* $0<\eta<\lambda$ *and* $u_\lambda\in S_\lambda^+$ *(resp.* $v_\lambda\in\mathcal{S}_\lambda^-$*), then* $\eta\in\mathcal{L}_+$ *(resp.* $\eta\in\mathcal{L}_-$*) and there exists* $u_\eta\in S_\eta^+$ *(resp.* $v_\eta\in\mathcal{S}_\eta^-$*) such that* $u_\eta\le u_\lambda$ *(resp.* $v_\lambda\le v_\eta$*).*

Proof. (a) We start by considering the following auxiliary nonlinear Robin problem

$$\begin{cases} -\operatorname{div} a(\nabla u(z))+\xi(z)|u(z)|^{p-2}u(z)=1 \quad \text{in } \Omega,\\ \dfrac{\partial u}{\partial n_a}+\beta(z)|u|^{p-2}u=0 \quad \text{on } \partial\Omega,\ u>0. \end{cases} \tag{4}$$

On account of (3) and using the nonlinear regularity theory of Lieberman [8] and the nonlinear maximum principle of Pucci-Serrin [18], problem (4) admits a unique solution $\widetilde{u}\in D_+$.

Let $\widetilde{\lambda}_+=\dfrac{1}{\|N_f(\widetilde{u})\|_\infty}$ (see hypothesis H(f) (i)). Then we have

$$-\operatorname{div} a(\nabla\widetilde{u}(z))+\xi(z)\widetilde{u}(z)^{p-1}=1\ge\widetilde{\lambda}_+f(z,\widetilde{u}(z)) \quad \text{for a.a. } z\in\Omega \tag{5}$$

(see Papageorgiou-Rădulescu [12]).

We introduce the Carathéodory function $\widetilde{k}_+(z,x)$ defined by

$$\widetilde{k}_+(z,x)=\begin{cases}\widetilde{\lambda}_+f(z,x^+) & \text{if } x\le\widetilde{u}(z),\\ \widetilde{\lambda}_+f(z,\widetilde{u}(z)) & \text{if } \widetilde{u}(z)<x.\end{cases} \tag{6}$$

We set $\widetilde{K}_+(z,x) = \int_0^x \widetilde{k}_+(z,s)ds$ and consider the C^1-functional $\widetilde{\psi}_+ : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widetilde{\psi}_+(u) = \frac{1}{p}\gamma(u) - \int_\Omega \widetilde{K}_+(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (3) and (6) it is clear that $\widetilde{\psi}_+(\cdot)$ is coercive. Also, using the Sobolev embedding theorem and the compactness of the trace map, we see that $\widetilde{\psi}_+(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\widetilde{\psi}_+(u_0) = \inf[\widetilde{\psi}_+(u) : u \in W^{1,p}(\Omega)]. \tag{7}$$

Fix $\overline{u} \in D_+$ with $\|\overline{u}\|_q = 1$. On account of hypotheses H(a) (iv), H(f) (ii), given $c_0^* > c^*$ and $\eta > c_0^*\|\nabla\overline{u}\|_q^q$, we can find $\delta \in (0, \min_{\overline{\Omega}} \widetilde{u})$ (recall that $\widetilde{u} \in D_+$) such that

$$G(y) \le \frac{c_0^*}{q}|y|^q \text{ for all } |y| \le \delta \text{ and } \widetilde{\lambda}_+ F(z,x) \ge \frac{\eta}{q}x^q \text{ for a.a. } z \in \Omega, \text{ all } 0 \le x \le \delta. \tag{8}$$

Choose $t \in (0,1)$ small such that

$$t|\nabla\overline{u}(z)| \le \delta \text{ and } 0 < t\overline{u}(z) \le \delta \quad \text{for all } z \in \overline{\Omega}. \tag{9}$$

Using (8) and (9) and since $\|\overline{u}\|_q = 1$, we have

$$\begin{aligned} & \widetilde{\psi}_+(t\overline{u}) \le \frac{c_0^* t^q}{q}\|\nabla\overline{u}\|_q^q - \frac{\eta t^q}{q} = \frac{t^q}{q}\left[c_0^*\|\nabla\overline{u}\|_q^q - \eta\right] < 0, \\ \Rightarrow\quad & \widetilde{\psi}_+(u_0) < 0 = \widetilde{\psi}_+(0) \quad \text{(see (7))}, \\ \Rightarrow\quad & u_0 \ne 0. \end{aligned}$$

From (7) we have

$$\begin{aligned} & \widetilde{\psi}_+'(u_0) = 0, \\ \Rightarrow\quad & \langle A(u_0), h\rangle + \int_\Omega \xi(z)|u_0|^{p-2}u_0 h dz + \int_{\partial\Omega} \beta(z)|u_0|^{p-2}u_0 h d\sigma = \int_\Omega \widetilde{k}_+(z,u_0)hdz \end{aligned} \tag{10}$$

for all $h \in W^{1,p}(\Omega)$.

In (10) first we choose $h = -u_0^- \in W^{1,p}(\Omega)$. Using (3) and (6), we obtain

$$\begin{aligned} & c_7\|u_0^-\|^p \le 0, \\ \Rightarrow\quad & u_0 \ge 0,\ u_0 \ne 0. \end{aligned}$$

Next in (10) we choose $h = (u_0 - \widetilde{u})^+ \in W^{1,p}(\Omega)$. We have

$$\begin{aligned} & \langle A(u_0), (u_0-\widetilde{u})^+\rangle + \int_\Omega \xi(z)u_0^{p-1}(u_0-\widetilde{u})^+dz + \int_{\partial\Omega}\beta(z)u_0^{p-1}(u_0-\widetilde{u})^+d\sigma \\ & = \int_\Omega \widetilde{\lambda}_+ f(z,\widetilde{u})(u_0-\widetilde{u})^+dz \quad \text{(see (6))} \\ & \le \langle A(\widetilde{u}), (u_0-\widetilde{u})^+\rangle + \int_\Omega \xi(z)\widetilde{u}^{p-1}(u_0-\widetilde{u})^+dz + \int_{\partial\Omega}\beta(z)\widetilde{u}^{p-1}(u_0-\widetilde{u})^+d\sigma \quad \text{(see (5))}, \end{aligned}$$

$\Rightarrow \quad u_0 \le \widetilde{u}.$

So, we have proved that

$$u_0 \in [0, \widetilde{u}],\ u_0 \ne 0. \tag{11}$$

From (6), (10), (11) it follows that u_0 is a positive solution of problem $(P_{\widetilde{\lambda}_+})$ and we have

$$-\operatorname{div} a(\nabla u_0(z)) + \xi(z)u_0(z)^{p-1} = \widetilde{\lambda}_+ f(z, u_0(z)) \quad \text{for a.a. } z \in \Omega \text{ (see [12])}. \tag{12}$$

From Proposition 7 of Papageorgiou-Rădulescu [13], we have $u_0 \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [8] implies that $u_0 \in C_+ \setminus \{0\}$. Let $\rho = \|u_0\|_\infty$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis H(f) (iv). We have

$$\widetilde{\lambda}_+[f(z,x) + \widehat{\xi}_\rho x^{p-1}] \geq 0 \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \rho. \tag{13}$$

From (12) and (13), we obtain

$$\begin{aligned} &\operatorname{div} a(\nabla u_0(z)) \leq [\|\xi\|_\infty + \widetilde{\lambda}_+ \widehat{\xi}_\rho] u_0(z)^{p-1} \quad \text{for a.a. } z \in \Omega, \\ \Rightarrow\ & u_0 \in D_+ \quad \text{(see Pucci-Serrin [18] (pp. 111, 120))}. \end{aligned}$$

So, we conclude that $\widetilde{\lambda}_+ \in \mathcal{L}_+$, that is, $\mathcal{L}_+ \neq \emptyset$ and $S^+_\lambda \subseteq D_+$ for all $\lambda \in \mathcal{L}_+$.

For the negative solution we consider the auxiliary problem (4) with -1 as forcing term (instead of 1). Evidently $\widetilde{v} = -\widetilde{u} \in -D_+$ is the unique solution of this new auxiliary nonlinear Robin problem. Using $\widetilde{v} \in -D_+$ and reasoning as above with $\widetilde{\lambda}_- = \dfrac{1}{\|N_f(\widetilde{v})\|_\infty}$, we produce a negative solution $v_0 \in W^{1,p}(\Omega)$ of problem $(P_{\widetilde{\lambda}_-})$. In fact we have $v_0 \in [\widetilde{v}, 0] \cap (-D_+)$.

Therefore we have $\widetilde{\lambda}_- \in \mathcal{L}_-$, hence $\mathcal{L}_- \neq \emptyset$ and $S^-_\lambda \subseteq -D_+$ for all $\lambda \in \mathcal{L}_-$.

(b) Now let $\lambda \in \mathcal{L}_+$, $0 < \eta < \lambda$ and $u_\lambda \in S^+_\lambda \subseteq D_+$. For $\rho = \|u_\lambda\|_\infty$, let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis H(f) (iv). We introduce the Carathéodory function $e^+_\eta(z,x)$ defined by

$$e^+_\eta(z,x) = \begin{cases} \eta[f(z,x^+) + \widehat{\xi}_\rho (x^+)^{p-1}] & \text{if } x \leq u_\lambda(z), \\ \eta[f(z,u_\lambda(z)) + \widehat{\xi}_\rho u_\lambda(z)^{p-1}] & \text{if } u_\lambda(z) < x. \end{cases} \tag{14}$$

We set $E^+_\eta(z,x) = \int_0^x e^+_\eta(z,s)ds$ and consider the C^1-functional $\widehat{\varphi}^+_\eta : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}^+_\eta(u) = \frac{1}{p}\gamma(u) + \frac{\eta\widehat{\xi}_\rho}{p}\|u\|^p_p - \int_\Omega E^+_\eta(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (3) and (14) it is clear that $\widehat{\varphi}^+_\eta(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, as before by minimization, we produce $u_\eta \in W^{1,p}(\Omega)$ a minimizer of $\widehat{\varphi}^+_\eta(\cdot)$ such that

$$\begin{aligned} & u_\eta \in [0, u_\lambda],\ u_\eta \neq 0, \\ \Rightarrow\ & u_\eta \in S^+_\eta \subseteq D_+ \text{ and so } \eta \in \mathcal{L}_+ \text{ and } u_\eta \leq u_\lambda. \end{aligned}$$

In a similar fashion, if $\lambda \in \mathcal{L}_-$, $\eta \in (0, \lambda)$ and $v_\lambda \in S^-_\lambda \subseteq -D_+$, then we show that $\eta \in \mathcal{L}_-$ and we produce $v_\eta \in S^-_\eta \subseteq -D_+$ such that $v_\lambda \leq v_\eta$. ☐

Remark 4. Part (b) of the above proposition implies that $\mathcal{L}_+, \mathcal{L}_- \subseteq (0, +\infty)$ are intervals, with left end $0 \notin \mathcal{L}_+, \mathcal{L}_-$.

From Papageorgiou-Rădulescu-Repovš [14] (see the proof of Proposition 7), we know that S^+_λ is downward directed ($\lambda \in \mathcal{L}_+$), that is, if $u_1, u_2 \in S^+_\lambda$, then there exists $u \in S^+_\lambda$ such that $u \leq u_1$, $u \leq u_2$ and S^-_λ is upward directed ($\lambda \in \mathcal{L}_-$), that is, if $v_1, v_2 \in S^-_\lambda$, then there exists $v \in S^-_\lambda$ such that $v_1 \leq v$, $v_2 \leq v$.

Next we prove the existence of extremal constant sign solutions for problem (P_λ), that is, we will show the existence of a smallest positive solution $u_*^\lambda \in D_+$ ($u_*^\lambda \leq u$ for all $u \in S_\lambda^+$) and of a biggest negative solution $v_*^\lambda \in -D_+$ ($v \leq v_*^\lambda$ for all $v \in S_\lambda^-$). These extremal solutions will be used in Section 4 to produce a nodal solution.

Proposition 3. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, H_0, $H(f)$ hold, then for every $\lambda \in \mathcal{L}_+$ problem (P_λ) has a smallest positive solution $u_*^\lambda \in D_+$ and for every $\lambda \in \mathcal{L}_-$ problem (P_λ) has a biggest negative solution $v_*^\lambda \in -D_+$.*

Proof. Let $\lambda \in \mathcal{L}_+$. The set $S_\lambda^+ \subseteq D_+$ is downward directed. So, invoking Lemma 3.10, p. 178, of Hu-Papageorgiou [7], we can find $\{u_n\}_{n\geq 1} \subseteq S_\lambda^+$ decreasing such that

$$\inf S_\lambda^+ = \inf_{n\geq 1} u_n.$$

We have

$$\langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h d\sigma = \int_\Omega \lambda f(z, u_n) h dz \tag{15}$$

for all $h \in W^{1,p}(\Omega)$, all $n \in \mathbb{N}$,

$$0 \leq u_n \leq u_1 \in D_+ \quad \text{for all } n \in \mathbb{N}. \tag{16}$$

If in (15) we choose $h = u_n \in W^{1,p}(\Omega)$ and we use (13), (16) and hypothesis H(f) (i) we infer that

$$\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.} \tag{17}$$

Then from (15), (17) and Proposition 7 of Papageorgiou-Rădulescu [13], we can find $c_8 > 0$ such that

$$u_n \in L^\infty(\Omega) \text{ and } \|u_n\|_\infty \leq c_8 \quad \text{for all } n \in \mathbb{N}.$$

The nonlinear regularity theory of Lieberman [8] implies that there exist $\alpha \in (0, 1)$ and $c_9 > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \text{ and } \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq c_9 \quad \text{for all } n \in \mathbb{N}. \tag{18}$$

Recalling that $C^{1,\alpha}(\overline{\Omega})$ is embedded compactly in $C^1(\overline{\Omega})$, from (18) and the monotonicity of $\{u_n\}_{n\geq 1}$, we infer that

$$u_n \to u_*^\lambda \text{ in } C^1(\overline{\Omega}) \text{ as } n \to +\infty. \tag{19}$$

On account of hypothesis H(f) (ii), we can find $\delta > 0$ such that

$$\lambda f(z, x) \geq \lambda x^{q-1} \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta. \tag{20}$$

We consider the following auxiliary nonlinear Robin problem

$$\begin{cases} -\operatorname{div} a(\nabla u(z)) + \xi(z)|u(z)|^{p-2}u(z) = \lambda |u(z)|^{q-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega,\ u > 0. \end{cases} \tag{21}$$

Since $q < p$, we can easily see that for every $\lambda > 0$, problem (P_λ) has a unique solution $y_\lambda \in D_+$.

Suppose that $u_*^\lambda = 0$ (see (19)). Then we can find $n_0 \in \mathbb{N}$ such that

$$0 < u_n(z) \leq \delta \quad \text{for all } z \in \overline{\Omega}, \text{ all } n \geq n_0,$$

$$\Rightarrow \quad \lambda f(z, u_n(z)) \geq \lambda u_n(z)^{q-1} \quad \text{for a.a. } z \in \Omega, \text{ all } n \geq n_0 \text{ (see (20))}. \tag{22}$$

We introduce the Carathéodory function $\vartheta_\lambda(z,x)$ defined by

$$\vartheta_\lambda(z,x)=\begin{cases}\lambda(x^+)^{q-1} & \text{if } x\le u_n(z),\\ \lambda u_n(z)^{q-1} & \text{if } u_n(z)<x,\end{cases}\quad \text{with } n\ge n_0. \tag{23}$$

Let $\Theta_\lambda(z,x)=\int_0^x\vartheta_\lambda(z,s)ds$ and consider the C^1-functional $\mu_\lambda: W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\mu_\lambda(u)=\frac{1}{p}\gamma(u)-\int_\Omega\Theta_\lambda(z,u)dz\quad\text{for all } u\in W^{1,p}(\Omega).$$

As before using (3), (23) and the direct method of the calculus of variations we can find $\widehat{y}_\lambda\in W^{1,p}(\Omega)$ such that

$$\begin{aligned}&\mu_\lambda(\widehat{y}_\lambda)=\inf[\mu_\lambda(u):u\in W^{1,p}(\Omega)]<0=\mu_\lambda(0)\quad(\text{since } q<p),\\ \Rightarrow\quad &\widehat{y}_\lambda\ne 0.\end{aligned}$$

Also, we have

$$\begin{aligned}&\mu'_\lambda(\widehat{y}_\lambda)=0,\\ \Rightarrow&\langle A(\widehat{y}_\lambda),h\rangle+\int_\Omega\xi(z)|\widehat{y}_\lambda|^{p-2}\widehat{y}_\lambda h dz+\int_{\partial\Omega}\beta(z)|\widehat{y}_\lambda|^{p-2}\widehat{y}_\lambda h d\sigma=\int_\Omega\vartheta_\lambda f(z,\widehat{y}_\lambda)hdz\end{aligned}\tag{24}$$

for all $h\in W^{1,p}(\Omega)$. In (24) first we choose $h=-y_\lambda^-\in W^{1,p}(\Omega)$. Then

$$\begin{aligned}&c_7\|y_\lambda^-\|^p\le 0,\quad(\text{see (3) and (23)}),\\ \Rightarrow\quad&\widehat{y}_\lambda\ge 0,\ \widehat{y}_\lambda\ne 0.\end{aligned}$$

Also in (24) we choose $h=(\widehat{y}_\lambda-u_n)^+\in W^{1,p}(\Omega)$. Then

$$\begin{aligned}&\langle A(\widehat{y}_\lambda),(\widehat{y}_\lambda-u_n)^+\rangle+\int_\Omega\xi(z)\widehat{y}_\lambda^{p-1}(\widehat{y}_\lambda-u_n)^+dz+\int_{\partial\Omega}\beta(z)\widehat{y}_\lambda^{p-1}(\widehat{y}_\lambda-u_n)^+d\sigma\\ &=\int_\Omega\lambda u_n^{q-1}(\widehat{y}_\lambda-u_n)^+dz\quad(\text{see (23)})\\ &\le\int_\Omega\lambda f(z,u_n)(\widehat{y}_\lambda-u_n)^+dz\quad(\text{see (22)})\\ &=\langle A(u_n),(\widehat{y}_\lambda-u_n)^+\rangle+\int_\Omega\xi(z)u_n^{p-1}(\widehat{y}_\lambda-u_n)^+dz+\int_{\partial\Omega}\beta(z)u_n^{p-1}(\widehat{y}_\lambda-u_n)^+d\sigma\\ &\hspace{25em}(\text{since } u_n\in S_\lambda^+),\\ \Rightarrow\quad&\widehat{y}_\lambda\le u_n\quad\text{for all } n\ge n_0.\end{aligned}$$

Therefore we have

$$\begin{aligned}&\widehat{y}_\lambda\in[0,u_n],\ n\ge n_0,\ \widehat{y}_\lambda\ne 0,\\ \Rightarrow\quad&\widehat{y}_\lambda=y_\lambda\quad(\text{see (23), (24)}),\\ \Rightarrow\quad&y_\lambda\le u_n\quad\text{for all } n\ge n_0,\\ \Rightarrow\quad&y_\lambda\le u_*^\lambda\text{ and so } u_*^\lambda\ne 0,\end{aligned}$$

which contradicts our hypothesis that $u_*^\lambda=0$. Therefore $u_*^\lambda\ne 0$ and by passing to the limit as $n\to+\infty$ in (15) and using (19), we infer that

$$u_*^\lambda\in S_\lambda^+\subseteq D_+\text{ and } u_*^\lambda=\inf S_\lambda^+.$$

A similar argument produces a maximal negative solution $v_*^\lambda \in -D_+$. In this case since $S_\lambda^- \subseteq -D_+$ ($\lambda \in \mathcal{L}_-$) is upward directed, we can find an increasing sequence $\{v_n\}_{n\geq 1} \subseteq S_\lambda^- \subseteq -D_+$ such that $\sup_{n\geq 1} v_n = \sup S_\lambda^-$ (see [7]). $\square$

4. Nodal Solutions

In this section, we use the extremal constant sign solutions of Proposition 3, to produce a nodal (sign changing) solution for problem (P_λ) ($\lambda \in \mathcal{L}_+ \cap \mathcal{L}_-$). The idea is simple. We focus on the order interval $[v_*^\lambda, u_*^\lambda]$ and we look for a nontrivial solution y_λ in that order interval. If $y_\lambda \neq u_*^\lambda$, $y_\lambda \neq v_*^\lambda$, then on account of the extremality of u_*^λ and v_*^λ, the solution y_λ will be nodal.

So, let $\lambda \in \mathcal{L}_+ \cap \mathcal{L}_-$ and let $u_*^\lambda \in D_+$ and $v_*^\lambda \in -D_+$ be the two extremal constant sign solutions from Proposition 3. We introduce the Carathéodory function $\widehat{f}_\lambda(z,x)$ defined by

$$\widehat{f}_\lambda(z,x) = \begin{cases} \lambda f(z, v_*^\lambda(z)) & \text{if } x < v_*^\lambda(z), \\ \lambda f(z,x) & \text{if } v_*^\lambda(z) \leq x \leq u_*^\lambda(z), \\ \lambda f(z, u_*^\lambda(z)) & \text{if } u_*^\lambda(z) < x. \end{cases} \tag{25}$$

Also we consider the positive and negative truncations of $\widehat{f}_\lambda(z,\cdot)$, that is, the Carathéodory functions

$$\widehat{f}_\lambda^+(z,x) = \widehat{f}_\lambda(z,x^+) \text{ and } \widehat{f}_\lambda^-(z,x) = \widehat{f}_\lambda(z,-x^-). \tag{26}$$

We set $\widehat{F}_\lambda(z,x) = \int_0^x \widehat{f}_\lambda(z,s)ds$ and $\widehat{F}_\lambda^\pm(z,x) = \int_0^x \widehat{f}_\lambda^\pm(z,s)ds$ and then introduce the C^1-functionals $\widehat{\varphi}_\lambda, \widehat{\varphi}_\lambda^\pm : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda(u) = \frac{1}{p}\gamma(u) - \int_\Omega \widehat{F}_\lambda(z,u)dz,$$
$$\widehat{\varphi}_\lambda^\pm(u) = \frac{1}{p}\gamma(u) - \int_\Omega \widehat{F}_\lambda^\pm(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Using (25), (26) and the fact that $u_*^\lambda \in D_+$ and $v_*^\lambda \in -D_+$ are the extremal constant sign solutions of (P_λ) ($\lambda \in \mathcal{L}_+ \cap \mathcal{L}_-$), we easily obtain the following result concerning the critical sets of $\widehat{\varphi}_\lambda$ and of $\widehat{\varphi}_\lambda^\pm$.

Proposition 4. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, H_0, $H(f)$ hold and $\lambda \in \mathcal{L}_+ \cap \mathcal{L}_-$, then $K_{\widehat{\varphi}_\lambda} \subseteq [v_*^\lambda, u_*^\lambda] \cap C^1(\overline{\Omega})$, $K_{\widehat{\varphi}_\lambda^+} = \{0, u_*^\lambda\}$, $K_{\widehat{\varphi}_\lambda^-} = \{0, v_*^\lambda\}$.*

Next we compute the critical groups of $\widehat{\varphi}_\lambda$ ($\lambda \in \mathcal{L}_+ \cap \mathcal{L}_-$) at the origin. We will use this computation to distinguish the solution we will produce from the trivial one. Our result here extends Proposition 4.1 of Papageorgiou-Winkert [17] and Lemma 3.4 of Guarnotta-Marano-Papageorgiou [6]. Our proof is based on their proofs. The result is in fact of independent interest, since it determines the critical groups of a functional with general concave nonlinearity near zero.

In what follows let $\widehat{m}_\lambda = \max\{\|u_*^\lambda\|_\infty, \|v_*^\lambda\|_\infty\}$.

Proposition 5. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, H_0, $H(f)$ hold and $\lambda \in \mathcal{L}_+ \cap \mathcal{L}_-$, then $C_k(\widehat{\varphi}_\lambda, 0) = 0$ for all $k \in \mathbb{N}_0$.*

Proof. Let $r > p$ and $\eta > 0$. On account of hypotheses H(f) (i), (ii), we have

$$\widehat{F}_\lambda(z,x) \geq \lambda[\eta|x|^q - c_{10}|x|^r] \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [-\widehat{m}_\lambda, \widehat{m}_\lambda], \text{ some } c_{10} > 0. \tag{27}$$

Also from hypothesis H(a) (iv) and Corollary 1, we have

$$G(y) \leq c_{11}[|y|^q + |y|^p] \quad \text{for all } y \in \mathbb{R}^N, \text{ some } c_{11} > 0. \tag{28}$$

Let $u \in W^{1,p}(\Omega)$, $u \neq 0$ and $t > 0$. Using (27) and (28), we have

$$\begin{aligned}\widehat{\varphi}_\lambda(tu) \leq\; & c_{11}\left[t^q\|\nabla u\|_q^q + t^p\|\nabla u\|_p^p\right] + \frac{t^p}{p}\int_\Omega \xi(z)|u|^p dz + \frac{t^p}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma \\ & - \lambda\eta t^q\|u\|_q^q + \lambda c_{10}t^r\|u\|_r^r.\end{aligned}$$

Since $q < p$ and $\eta > 0$ is arbitrary, we can find $t^* \in (0,1)$ such that

$$\widehat{\varphi}_\lambda(tu) < 0 \quad \text{for all } t \in (0, \lambda^*).$$

Let $\widehat{t}_1 = \sup\{t \in [0,1] : \widehat{\varphi}_\lambda(tu) < 0\}$ and $C_\lambda = \{t \in [0,1] : \widehat{\varphi}_\lambda(tu) \geq 0\}$. We set

$$\widehat{t}_2 = \begin{cases} \inf C_\lambda & \text{if } C_\lambda \neq \emptyset, \\ 1 & \text{if } C_\lambda = \emptyset. \end{cases} \tag{29}$$

We will show that $\widehat{t}_1 = \widehat{t}_2$. First we show that $\widehat{t}_1 \leq \widehat{t}_2$. Arguing by contradiction suppose that $\widehat{t}_2 < \widehat{t}_1$. From hypotheses H(f) (i), (iii), we see that given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

(30)
$$\tau\widehat{F}_\lambda(z,x) - \widehat{f}_\lambda(z,x) \geq -\varepsilon|x|^p - c_\varepsilon|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [-\widehat{m}_\lambda, \widehat{m}_\lambda] \text{ (see (25))}.$$

Suppose that for some $t_0 \in (0,1)$, we have $\widehat{\varphi}_\lambda(t_0u) = 0$. Then

$$\begin{aligned} t_0\frac{d}{dt}\widehat{\varphi}_\lambda(tu)\Big|_{t=t_0} &= \langle\widehat{\varphi}'_\lambda(t_0u), t_0u\rangle \quad \text{(by the chain rule)} \\ &= \langle\widehat{\varphi}'_\lambda(t_0u), t_0u\rangle - \tau\widehat{\varphi}_\lambda(t_0u) \quad (\text{recall } \widehat{\varphi}_\lambda(t_0u) = 0) \\ &= \langle\widehat{\varphi}'_\lambda(y), y\rangle - \tau\widehat{\varphi}_\lambda(y) \quad (\text{with } y = t_0u) \\ &\geq [c_{12} - \varepsilon]\|y\|^p - \widehat{c}_\varepsilon\|y\|^r \quad \text{for some } c_{12}, \widehat{c}_\varepsilon > 0 \text{ (see (3) and (30))}. \end{aligned}$$

Choosing $\varepsilon \in (0, c_{12})$, we obtain

$$t_0\frac{d}{dt}\widehat{\varphi}_\lambda(tu)\Big|_{t=t_0} \geq c_{13}\|y\|^p - \widehat{c}_\varepsilon\|y\|^r \quad \text{for some } c_{13} > 0 \text{ and with } y = t_0u.$$

Since $p < r$, if $\rho \in (0,1]$ is small and $0 < \|u\| \leq \rho$, then

$$t_0\frac{d}{dt}\widehat{\varphi}_\lambda(tu)\Big|_{t=t_0} > 0 \quad \text{with } t_0 \in (0,1) \text{ such that } \widehat{\varphi}_\lambda(t_0u) = 0. \tag{31}$$

But from (29), we have

$$\widehat{\varphi}_\lambda(t_2u) = 0,$$

$$\Rightarrow \quad \widehat{\varphi}_\lambda(tu) > 0 \quad \text{for all } t \in (\widehat{t}_2, \widehat{t}_2 + \delta] \text{ with } \delta < \widehat{t}_1 - \widehat{t}_2 \text{ (see (31))}. \tag{32}$$

Let $\widehat{C}_\lambda = \{t \in (\widehat{t}_2 + \delta, \widehat{t}_1] : \widehat{\varphi}_\lambda(tu) = 0\}$ and define

$$t_* = \begin{cases} \min \widehat{C}_\lambda & \text{if } \widehat{C}_\lambda \neq \emptyset, \\ 1 & \text{if } \widehat{C}_\lambda = \emptyset. \end{cases} \tag{33}$$

From (29), (32) and (33) we infer that

$$t_* > \widehat{t}_2 + \delta. \tag{34}$$

Since $\widehat{\varphi}_\lambda(t_* u) = 0$, from (31) we have

$$\widehat{\varphi}_\lambda(tu) < 0 \quad \text{for all } t \in (t_* - \widehat{\delta}, t_*),\ \widehat{\delta} < t_* - (\widehat{t}_2 + \delta) \text{ (see (34))}. \tag{35}$$

From (32), (35) and Bolzano's theorem, we see that there exists $\widetilde{t} \in (\widehat{t}_2 + \delta, t_* - \widehat{\delta})$ such that $\widehat{\varphi}_\lambda(\widetilde{t}u) = 0$, which contradicts (33). Hence $\widehat{t}_1 \leq \widehat{t}_2$. In fact from the definition of $\widehat{t}_1$ and (29), we see that $\widehat{t}_1 = \widehat{t}_2$.

Let $\widehat{t}(u) = \widehat{t}_1 = \widehat{t}_2$. Then

$$\widehat{\varphi}_\lambda(tu) < 0 \quad \text{for all } t \in (0, \widehat{t}(u)) \text{ and } \widehat{\varphi}_\lambda(tu) > 0 \text{ for all } t \in (\widehat{t}(u), 1].$$

For $\overline{B}_\rho = \{y \in W^{1,p}(\Omega) : \|y\| \leq \rho\}$, let $\widehat{\theta}_\lambda : \overline{B}_\rho \setminus \{0\} \to [0,1]$ be defined by

$$\widehat{\theta}_\lambda(u) = \begin{cases} 1 & \text{if } u \in \overline{B}_\rho \setminus \{0\},\ \widehat{\varphi}_\lambda(u) \leq 0, \\ \widehat{t}(u) & \text{if } u \in \overline{B}_\rho \setminus \{0\},\ \widehat{\varphi}_\lambda(u) > 0. \end{cases}$$

It is easy to see that $\widehat{\theta}_\lambda(\cdot)$ is continuous. Then we introduce the map $\mu_\lambda : \overline{B}_\rho \setminus \{0\} \to (\widehat{\varphi}_\lambda^0 \cap \overline{B}_\rho) \setminus \{0\}$ defined by

$$\mu_\lambda(u) = \begin{cases} u & \text{if } u \in \overline{B}_\rho \setminus \{0\},\ \widehat{\varphi}_\lambda(u) \leq 0, \\ \widehat{\theta}_\lambda(u)u & \text{if } u \in \overline{B}_\rho \setminus \{0\},\ \widehat{\varphi}_\lambda(u) > 0. \end{cases}$$

The continuity of $\widehat{\theta}_\lambda(\cdot)$ implies the continuity of $\mu_\lambda(\cdot)$. Moreover, we see that

$$\mu_\lambda\Big|_{(\widehat{\varphi}_\lambda^0 \cap \overline{B}_\rho)\setminus\{0\}} = \mathrm{id}\Big|_{(\widehat{\varphi}_\lambda^0 \cap \overline{B}_\rho)\setminus\{0\}}.$$

It follows that $(\widehat{\varphi}_\lambda^0 \cap \overline{B}_\rho) \setminus \{0\}$ is a retract of $\overline{B}_\rho \setminus \{0\}$. The set $\overline{B}_\rho \setminus \{0\}$ is contractible. Therefore $(\widehat{\varphi}_\lambda^0 \cap \overline{B}_\rho) \setminus \{0\}$ is contractible (see Gasiński-Papageorgiou [4], Problems 4.153 and 4.159). Also using the deformation

$$h(t,u) = (1-t)u \quad \text{for all } (t,u) \in [0,1] \times (\widehat{\varphi}_\lambda^0 \cap \overline{B}_\rho),$$

we see that $\widehat{\varphi}_\lambda^0 \cap \overline{B}_\rho$ is contractible too. Invoking Proposition 6.1.31, p. 385, of Papageorgiou-Rădulescu-Repovš [16], we have

$$\begin{aligned} & H_k(\widehat{\varphi}_\lambda^0 \cap \overline{B}_\rho, (\widehat{\varphi}_\lambda^0 \cap \overline{B}_\rho) \setminus \{0\}) = 0 \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow\ & C_k(\widehat{\varphi}_\lambda, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0. \end{aligned}$$

□

Now we can prove the existence of nodal solutions.

Proposition 6. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, H_0, $H(f)$ hold and $\lambda \in \mathcal{L}_+ \cap \mathcal{L}_-$, then problem (P_λ) has a nodal solution $y_\lambda \in [v_*^\lambda, u_*^\lambda] \cap C^1(\overline{\Omega})$.*

Proof. From (25), (26) and (3) we see that $\widehat{\varphi}_\lambda^+$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\widehat{u}_*^\lambda \in W^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_\lambda^+(\widehat{u}_*^\lambda) = \inf[\widehat{\varphi}_\lambda^+(u) : u \in W^{1,p}(\Omega)]. \tag{36}$$

As before, since $q < p$, we have that

$$\begin{aligned} & \widehat{\varphi}_\lambda^+(\widehat{u}_*^\lambda) < 0 = \widehat{\varphi}_\lambda^+(0), \\ \Rightarrow\ & \widehat{u}_*^\lambda \neq 0. \end{aligned} \tag{37}$$

From (36) it follows that

$$\widehat{u}_*^\lambda \in K_{\widehat{\varphi}_\lambda^+} = \{0, \widehat{u}_*^\lambda\} \quad \text{(see Proposition 4)},$$
$$\Rightarrow \quad \widehat{u}_*^\lambda = u_*^\lambda \in D_+ \quad \text{(see (37))}.$$

Note that

$$\widehat{\varphi}_\lambda\Big|_{C_+} = \widehat{\varphi}_\lambda^+\Big|_{C_+} \quad \text{(see (26))},$$
$$\Rightarrow \quad u_*^\lambda \in D_+ \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \widehat{\varphi}_\lambda,$$
$$\Rightarrow \quad u_*^\lambda \in D_+ \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \widehat{\varphi}_\lambda \tag{38}$$
$$\text{(see Papageorgiou-Rădulescu [13], Proposition 8).}$$

Similarly using this time the functional $\widehat{\varphi}_\lambda^-$, we show that

$$v_*^\lambda \in -D_+ \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \widehat{\varphi}_\lambda. \tag{39}$$

We may assume that $\widehat{\varphi}_\lambda(\widehat{v}_*^\lambda) \leq \widehat{\varphi}_\lambda(\widehat{u}_*^\lambda)$.

The reasoning is similar if the opposite inequality holds using this time (39) instead of (38). Also, on account of Proposition 4, we may assume that

$$K_{\widehat{\varphi}_\lambda} \text{ is finite.} \tag{40}$$

Otherwise we already have an infinity of smooth nodal solutions and so we are done.

Then from (38), (40) and invoking Proposition 5.7.6, p. 367, of Papageorgiou-Rădulescu-Repovš [16], we can find $\rho \in (0,1)$ small such that

$$\widehat{\varphi}_\lambda(v_*^\lambda) \leq \widehat{\varphi}_\lambda(u_*^\lambda) < \inf[\widehat{\varphi}_\lambda(u) : \|u - u_*^\lambda\| = \rho] = \widehat{m}_\lambda,\ \|v_*^\lambda - u_*^\lambda\| > \rho. \tag{41}$$

The functional $\widehat{\varphi}_\lambda(\cdot)$ is coercive (see (3) and (25)). Therefore

$$\widehat{\varphi}_\lambda \text{ satisfies the Palais-Smale condition.} \tag{42}$$

Then (41), (42) permit the use of the mountain pass theorem (see Papageorgiou-Rădulescu-Repovš [16], Theorem 5.4.6, p. 329). So, we can find $y_\lambda \in W^{1,p}(\Omega)$ such that

$$y_\lambda \in K_{\widehat{\varphi}_\lambda} \subseteq [v_*^\lambda, u_*^\lambda] \cap C^1(\overline{\Omega}) \text{ (see Proposition 4)}, \ \widehat{m}_\lambda \leq \widehat{\varphi}_\lambda(y_\lambda),$$
$$\Rightarrow \quad y_\lambda \notin \{u_*^\lambda, v_*^\lambda\} \quad \text{(see (41))}.$$

Since y_λ is a critical point of $\widehat{\varphi}_\lambda$ of mountain pass type, from Theorem 6.5.8, p. 431, of Papageorgiou-Rădulescu-Repovš [16], we have

$$C_1(\widehat{\varphi}_\lambda, y_\lambda) \neq 0. \tag{43}$$

On the other hand, from Proposition 5, we have

$$C_k(\widehat{\varphi}_\lambda, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{44}$$

Comparing (43) and (44), we see that

$$y_\lambda \neq 0,$$
$$\Rightarrow \quad y_\lambda \in [v_*^\lambda, u_*^\lambda] \cap C^1(\overline{\Omega}) \text{ is a nodal solution of } (P_\lambda).$$

□

Therefore we can state the following multiplicity theorem for problem (P_λ).

Theorem 1. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, H_0, $H(f)$ hold, then for all $\lambda > 0$ small ($\lambda \in \mathcal{L}_+ \cap \mathcal{L}_- \neq \emptyset$) problem (P_λ) has at least three nontrivial solutions $u_0 \in D_+$, $v_0 \in -D_+$ and $y_0 \in [v_0, u_0] \cap C^1(\overline{\Omega})$ nodal.*

References

[1] M. Fuchs and G. Li, *Variational inequalities for energy functionals with nonstandard growth conditions*, Abstr. Appl. Anal. **3** (1998), 41–64.

[2] L. Gasiński and N.S. Papageorgiou, *Nonlinear Analysis*, Ser. Math. Anal. Appl., **9**, Chapman and Hall/CRC Press, Boca Raton, Florida (2006).

[3] L. Gasiński and N.S. Papageorgiou, *Existence and multiplicity of solutions for Neumann p-Laplacian-type equations*, Adv. Nonlinear Stud., **8** (2008), 843–870.

[4] L. Gasiński and N.S. Papageorgiou, *Exercises in Analysis Part 2: Nonlinear Analysis*, Springer International Publishing, Switzerland (2016).

[5] L. Gasiński and N.S. Papageorgiou, *Positive solutions for the Robin p-Laplacian problem with competing nonlinearities*, Adv. Calc. Var., doi.org/10.1515/acv-2016-0039

[6] M. Guarnotta, S.A. Marano and N.S. Papageorgiou, *Multiple nodal solutions to a Robin problem with sign-changing potential and locally defined reaction*, Rend. Lincei Mat. Appl., to appear

[7] S. Hu and N.S. Papageorgiou, *Handbook of Multivalued Analysis. Vol. I. Theory*, Mathematics and its Applications, **419**, Kluwer Academic Publishers, Dordrecht, The Netherlands (1997).

[8] G. Lieberman, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations*, Comm. Partial Differential Equations, **16** (1991), 311–361.

[9] S. Marano and S.J.N. Mosconi, *Some recent results on the Dirichlet problem for (p,q)-Laplace equations*, Discrete Contin. Dyn. Syst. - Ser. S, **11** (2018), 279–291.

[10] D. Mugnai and N.S. Papageorgiou, *Resonant nonlinear Neumann problems with indefinite weight*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), Vol. XI (2012), 729-788.

[11] N.S. Papageorgiou and V.D. Rădulescu, *Coercive and noncoercive nonlinear Neumann problems with indefinite potential*, Forum Math., **28** (2016), 545–571.

[12] N.S. Papageorgiou and V.D. Rădulescu, *Multiple solutions with precise sign for parametric Robin problems*, J. Differential Equations, **256** (2014), 2449–2479.

[13] N.S. Papageorgiou and V.D. Rădulescu, *Nonlinear nonhomogeneous Robin problems with superlinear reaction*, Adv. Nonlinear Stud., **16** (2016), 737–764.

[14] N.S. Papageorgiou, V.D. Rădulescu and D. Repovš, *Positive solutions for perturbations of the Robin eigenvalue problem plus an indefinite potential*, Discrete Contin. Dyn. Syst. - Ser. A, **37** (2017), 2589–2618.

[15] N.S. Papageorgiou, V.D. Rădulescu and D. Repovš, *Nodal solutions for nonlinear nonhomogeneous Robin problems*, Rend. Lincei Mat. Appl., **29** (2018), 721–738.

[16] N.S. Papageorgiou, V.D. Rădulescu and D. Repovš, *Methods of Nonlinear Analysis and Boundary Value Problems*, Springer, Berlin (2019).

[17] N.S. Papageorgiou, P. Winkert, *Nonlinear Robin problems with reaction of arbitrary growth*, Ann. Mat. Pura Appl. **195** (2016), 1207–1235.

[18] P. Pucci and J. Serrin, *The Maximum Principle*, Birkhäuser, Basel (2007).

(N.S. Papageorgiou) Department of Mathematics, National Technical University, Zografou campus, 15780, Athens, Greece

E-mail address: `npapg@math.ntua.gr`

(C. Vetro) Department of Mathematics and Computer Science, University of Palermo, Via Archirafi 34, 90123, Palermo, Italy

E-mail address: `calogero.vetro@unipa.it`

(F. Vetro) Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam; Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

E-mail address: `francescavetro@tdtu.edu.vn`