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Further Properties of an Operator Commuting with an Injective Quasi-Nilpotent Operator

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Abstract. In (Aiena et al., Math. Proc. R. Irish Acad. 122A(2):101–116, 2022), it has been shown that a bounded linear operator $T \in L(X)$, defined on an infinite-dimensional complex Banach space X, for which there exists an injective quasi-nilpotent operator that commutes with it, has a very special structure of the spectrum. In this paper, we show that we have much more: if a such quasi-nilpotent operator does exist, then some of the spectra of T originating from B-Fredholm theory coalesce. Further, the spectral mapping theorem holds for all the B-Weyl spectra. Finally, the generalized version of Weyl type theorems hold for T assuming that T is of polaroid type. Our results apply to the operators that belong to the commutant of Volterra operators.

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1. Introduction

This paper concerns operators $T \in L(X)$ that belong to the commutant of an injective quasi-nilpotent operator Q. The spectral properties of such operators, from the point of view of the classical Fredholm theory, have been studied in [7]. In this paper, we consider the spectra originating from the B-Fredholm theory in the sense of Berkani et al., and we show, if a such operator Q does exist, that the upper B-Weyl spectrum $\sigma_{\mathrm{ubw}}(T)$ coincides with the left Drazin spectrum $\sigma_{\mathrm{ld}}(T)$, while the B-Weyl spectrum $\sigma_{\mathrm{bw}}(T)$ coincides with the Drazin spectrum $\sigma_{\mathrm{d}}(T)$. These spectral equalities have, as a consequence, that the B-Weyl spectrum, as well as the upper B-Weyl spectrum, obeys to the spectral mapping theorem. Dually, assuming that the dual Q^* of a quasi-nilpotent operator is injective, then the lower B-Weyl spectrum $\sigma_{\mathrm{lbw}}(T)$ coincides with the right Drazin spectrum $\sigma_{\mathrm{rd}}(T)$. In this

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case, both the lower B-Weyl spectrum and the B-Weyl spectrum satisfy the spectral mapping theorem.

The second section of this article concerns some generalized version of Weyl type theorems, as the generalized version of a-Weyl's theorem and the generalized version of property (ω). If T commutes with an injective quasi-nilpotent operator, then these generalized versions hold for T assuming that T is left polaroid, or that T is a-polaroid. Note that the generalized versions of Weyl type theorems are stronger than the classical versions, see [1, Chapter 5]. Another stronger variant of Weyl's theorem, the so-called S-Weyl's theorem, is also discussed in the last part.

Our results find a natural application to the operators which belong to the commutant of the Volterra V operator in $L^2[a,b]$, since, as it is well known, both V and its adjoint V' are quasi-nilpotent and injective. Since the Volterra operator is also compact, our results complement each other with the celebrate Lomonosov result [15] that V admits a non-trivial closed invariant subspace.

2. Preliminaries and Definitions

Let $L(X)$ denote the Banach algebra of all bounded linear operators acting on a complex Banach space X. If $T \in L(X)$, by $\alpha(T) := \dim \ker T$ and $\beta(T) := \operatorname{codim} T(X)$, we denote the *defects* of T. The class of all *upper semi-Fredholm operators* is defined by

$$\Phi_+(X) := \{T \in L(X) : \alpha(T) < \infty \text{ and } T(X) \text{ is closed}\},$$

while the class all *lower semi-Fredholm operators* is defined by

$$\Phi_-(X) := \{T \in L(X) : \beta(T) < \infty\}.$$

The class of all *semi-Fredholm* operators is defined by $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$. For every $T \in \Phi_\pm(X)$, the *index* of T is defined by $\operatorname{ind} T = \alpha(T) - \beta(T)$. The *upper semi-Fredholm spectrum* is defined by

$$\sigma_{\mathrm{usf}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_+(X)\},$$

and similarly, it is defined the *lower semi-Fredholm spectrum* $\sigma_{\mathrm{lsf}}(T)$. Recall that an operator $T \in L(X)$ is said to be *bounded below* if is injective and has closed range. The classical *approximate point spectrum* is defined by

$$\sigma_{\mathrm{ap}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\},$$

while the *surjectivity spectrum* is defined as

$$\sigma_{\mathrm{s}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not onto}\}.$$

If T^* denotes the *dual* of T, it is well known that $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{s}}(T^*)$ and $\sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{ap}}(T^*)$.

Denote by $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$ the class of all *Fredholm operators*. An operator $T \in L(X)$ is said to be a *Weyl operator* if $T \in \Phi(X)$ and $\operatorname{ind} T = 0$, $T \in L(X)$ is said to be *upper semi-Weyl* if $T \in \Phi_+(X)$ and $\operatorname{ind} T \leq 0$, while $T \in L(X)$ is said to be *lower semi-Weyl* if $T \in \Phi_-(X)$ and $\operatorname{ind} T \geq 0$. Denote by $\sigma_{\mathrm{w}}(T)$, $\sigma_{\mathrm{uw}}(T)$ and $\sigma_{\mathrm{lw}}(T)$, the *Weyl spectrum*, the *upper semi-Weyl spectrum*, and the *lower semi-Weyl spectrum*, respectively. Obviously, the inclusions

$$\sigma_{\mathrm{usf}}(T) \subseteq \sigma_{\mathrm{uw}}(T) \subseteq \sigma_{\mathrm{ap}}(T) \quad \text{and} \quad \sigma_{\mathrm{lsf}}(T) \subseteq \sigma_{\mathrm{lw}}(T) \subseteq \sigma_{\mathrm{s}}(T)$$

hold for every $T \in L(X)$, and there is a duality

$$\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{lw}}(T^*) \quad \text{and} \quad \sigma_{\mathrm{lw}}(T) = \sigma_{\mathrm{uw}}(T^*).$$

Recall that the *ascent* of $T \in L(X)$ is the smallest positive integer $p = p(T)$, whenever it exists, such that ker $T^p =$ ker T^{p+1}. If such p does not exist, we set $p(T) = \infty$. Analogously, the *descent* of T is defined to be the smallest integer $q = q(T)$, whenever it exists, such that $T^{q+1}(X) = T^q(X)$. If such q does not exist, we set $q(T) = \infty$. Note that if $p(T)$ and $q(T)$ are both finite, then $p(T) = q(T)$, see Chapter 1 of [1] Moreover, λ is a pole of the resolvent if and only if $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$; see [14, Proposition 50.2].

An operator $T \in L(X)$ is said to be *Browder* if $T \in \Phi(X)$ and $p(T) = q(T) < \infty$. $T \in L(X)$ is said to be *upper semi-Browder* if $T \in \Phi_+(X)$ and $p(T) < \infty$, while $T \in L(X)$ is said to be *lower semi-Browder* if $T \in \Phi_-(X)$ and $q(T) < \infty$. Every Browder operator is Weyl, and every upper semi-Browder (respectively, lower semi-Browder) operator is upper semi-Weyl (respectively, lower semi-Weyl); see [1, Chapter 3].

The *Browder spectrum*, the *upper semi-Browder spectrum*, and the *lower semi-Browder spectrum* are denoted by $\sigma_{\mathrm{b}}(T)$, $\sigma_{\mathrm{ub}}(T)$, and $\sigma_{\mathrm{lb}}(T)$, respectively. Note that if λ is a spectral point for which $\lambda I - T$ is Browder, then λ is an isolated point of $\sigma(T)$. Recall that $R \in L(X)$ is said to be a *Riesz operator* if $\lambda I - T \in \Phi(X)$ for each $\lambda \neq 0$. Quasi-nilpotent operators and compact operators are examples of Riesz operators. By a well-known result of Rakočević [16] (see also [3]), the Browder spectra are invariant under Riesz commuting perturbations R, that is

$$\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{ub}}(T+R), \quad \sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{lb}}(T+R), \quad \sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{b}}(T+R). \quad (1)$$

Semi-Fredholm operators have been generalized by Berkani [8,9] in the following way: for every $T \in L(X)$ and a nonnegative integer n, let us denote by $T_{[n]}$ the restriction of T to $T^n(X)$, viewed as a map from the space $T^n(X)$ into itself (we set $T_{[0]} = T$). $T \in L(X)$ is said to be *semi B-Fredholm* (resp. *B-Fredholm, upper semi B-Fredholm, lower semi B-Fredholm*), if, for some integer $n \geq 0$, the range $T^n(X)$ is closed and $T_{[n]}$ is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). In this case, $T_{[m]}$ is a semi-Fredholm operator for all $m \geq n$ [8] with the same index of $T_{[n]}$. This enables one to define the index of a semi B-Fredholm as ind $T =$ ind $T_{[n]}$. The *upper semi B-Fredholm spectrum* is defined

$$\sigma_{\mathrm{ubf}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Fredholm}\},$$

and analogously, it may be defined the *lower semi B-Fredholm spectrum* $\sigma_{\mathrm{lbf}}(T)$.

A bounded operator $T \in L(X)$ is said to be *B-Weyl* (respectively, *upper semi B-Weyl*, *lower semi B-Weyl*), if, for some integer $n \geq 0$, the range $T^n(X)$

is closed and $T_{[n]}$ is Weyl (respectively, upper semi-Weyl, lower semi-Weyl). The *B-Weyl spectrum* is defined by

$$\sigma_{\mathrm{bw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Weyl}\},$$

and analogously may be defined the *upper semi B-Weyl spectrum* $\sigma_{\mathrm{ubw}}(T)$ and the *lower semi B-Weyl spectrum* $\sigma_{\mathrm{lbw}}(T)$.

Recall that an operator $T \in L(X)$ is said to be *Drazin invertible* if $p(T) = q(T) < \infty$, i.e., 0 is a pole of the resolvent or T is invertible. An operator $T \in L(X)$ is said to be *left Drazin invertible* if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed. A scalar $\lambda \in \mathbb{C}$ is said to be a *left pole* if $\lambda I - T$ is left Drazin invertible and $\lambda \in \sigma_{\mathrm{ap}}(T)$. A left pole λ for which $\alpha(\lambda I - T) < \infty$ is said to have finite rank. Dually, $T \in L(X)$ is said to be *right Drazin invertible* if $q := q(T) < \infty$ and $T^q(X)$ is closed. A scalar $\lambda \in \mathbb{C}$ is said to be a *right pole* if $\lambda I - T$ is right Drazin invertible and $\lambda \in \sigma_{\mathrm{s}}(T)$.

The Drazin spectrum is defined by

$$\sigma_{\mathrm{d}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Drazin invertible}\},$$

and analogously are defined the *left Drazin* spectrum $\sigma_{\mathrm{ld}}(T)$ and the *right Drazin* spectrum $\sigma_{\mathrm{rd}}(T)$. It should be noted that there is a perfect duality, i.e., $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{rd}}(T^*)$ and $\sigma_{\mathrm{ld}}(T^*) = \sigma_{\mathrm{rd}}(T)$; see [1, Chapter 1]. The following inclusions hold for every operator $T \in L(X)$:

$$\sigma_{\mathrm{ubf}}(T) \subseteq \sigma_{\mathrm{ubw}}(T) \subseteq \sigma_{\mathrm{ld}}(T) \subseteq \sigma_{\mathrm{ap}}(T), \tag{2}$$

and

$$\sigma_{\mathrm{lbf}}(T) \subseteq \sigma_{\mathrm{lbw}}(T) \subseteq \sigma_{\mathrm{rd}}(T) \subseteq \sigma_{\mathrm{s}}(T); \tag{3}$$

see [1, Chapter 1]. The following lemma has been proved in [4, Lemma 3.5].

Lemma 2.1. *Suppose that $\rho_{uw}(T)$ is connected. Then, $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{w}}(T)$.*

A proof of the following theorem can be found in [1, Theorem 1.140].

Theorem 2.2. *Let $T \in L(X)$. Then, T is left Drazin invertible (respectively, right Drazin invertible, Drazin invertible) if and only if there exists $n \in \mathbb{N}$, such that $T^n(X)$ is closed and the restriction $T|T^n(X)$ is bounded below (respectively, onto, invertible).*

3. Injective Quasi-Nilpotent Operators

Recall that an operator $Q \in L(X)$ is said to be *quasi-nilpotent* if $\sigma(Q) = \{0\}$. In the sequel by *comm* (T), we denote the *commutant* of T. A proof of the following result may be found in [7].

Theorem 3.1. *Let $T \in L(X)$ and suppose that $Q \in comm(T)$ is a quasi-nilpotent operator.*

(i) *If Q is injective, then $\alpha(T) < \infty$ if and only if T is injective.*
(ii) *If the dual Q^* is injective, then $\beta(T) < \infty$ if and only if T is onto.*

In the sequel, we set

$$\mathbf{Q}_i(X) := \{T \in L(X) : \text{there exists an injective quasi-nilpotent operator } Q \in L(X) \text{ such that } TQ = QT\}.$$

Several examples of operators that commute with an injective quasi-nilpotent operator are given in [7]. Theorem 3.1 has some important consequences:

Theorem 3.2. *Let $T \in L(X)$ and $Q \in L(X)$ a quasi-nilpotent operator that commutes with T.*

(i) *If Q is injective, then*

$$\sigma_{\mathrm{usf}}(T) = \sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{ap}}(T) \quad \text{and} \quad \sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T) = \sigma(T). \tag{4}$$

(ii) *If Q^* is injective, then*

$$\sigma_{\mathrm{lsf}}(T) = \sigma_{\mathrm{lw}}(T) = \sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{s}}(T) \quad \text{and} \quad \sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T) = \sigma(T). \tag{5}$$

Proof. (i) Part (i) has been proved in [7, Corollary 3.7 and Theorem 3.8].

(ii) We have $\sigma_{\mathrm{lsf}}(T) \subseteq \sigma_{\mathrm{lw}}(T) \subseteq \sigma_{\mathrm{lb}}(T) \subseteq \sigma_{\mathrm{s}}(T)$, and so, to show the first equalities in (5), it suffices to prove the inclusion $\sigma_{\mathrm{s}}(T) \subseteq \sigma_{\mathrm{lsf}}(T)$. Let $\lambda \notin \sigma_{\mathrm{lsf}}(T)$ be arbitrary given. Then, $\beta(\lambda I - T) < \infty$. Since $T^*Q^* = Q^*T^*$, by Theorem 3.1, we have $\beta(\lambda I - T) = 0$, so $\lambda \notin \sigma_{\mathrm{s}}(T)$, and hence, $\sigma_{\mathrm{s}}(T) \subseteq \sigma_{\mathrm{lw}}(T)$.

The equalities $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T) = \sigma(T)$ may be proved in a similar way. □

An operator $T \in L(X)$, X a Banach space, is said to have *the single-valued extension property* at $\lambda_0 \in \mathbb{C}$, in short T has the SVEP at λ_0, if for every open disc $\mathbf{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathbf{D}_{\lambda_0} \to X$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0 \quad \text{for all } \lambda \in \mathbf{D}_{\lambda_0}$$

is the constant function $f \equiv 0$. T is said to have the SVEP if T has the SVEP for every $\lambda \in \mathbb{C}$. Evidently, both T and T^* have SVEP at the points $\lambda \notin \sigma(T)$.

Remark 3.3. Let $\lambda_0 \in \mathbb{C}$ and suppose that T has SVEP at the points λ of a punctured open disc $\mathbb{D}(\lambda_0, \varepsilon)\backslash\{\lambda_0\}$. Then, T has SVEP at λ_0. Indeed, suppose that $f : \mathbb{D}(\lambda_0, \varepsilon) \to X$ is an analytic function, such that $(\lambda I - T)f(\lambda) = 0$ holds for every $\lambda \in \mathbb{D}(\lambda_0, \varepsilon)$. Choose $\mu \in \mathbb{D}(\lambda_0, \varepsilon)\backslash\{\lambda_0\}$ and let $\mathbb{D}(\mu, \delta)$ be an open disc contained in $\mathbb{D}(\lambda_0, \varepsilon)\backslash\{\lambda_0\}$. The SVEP for T at μ entails $f(\lambda) = 0$ on $\mathbb{D}(\mu, \delta)$. Since f is continuous at λ_0, we then conclude that $f(\lambda_0) = 0$. Hence, $f \equiv 0$ on $\mathbb{D}(\lambda_0, \varepsilon)$, and thus, T has the SVEP at λ_0.

We now consider the Weyl spectra relative to the B-Fredholm theory.

Theorem 3.4. *Let $T \in L(X)$ and suppose that $Q \in L(X)$ is a quasi-nilpotent operator, such that $TQ = QT$.*

(i) *If Q is injective, then*

$$\sigma_{\mathrm{ubf}}(T) = \sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{ld}}(T) \quad \text{and} \quad \sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{d}}(T). \tag{6}$$

(ii) *If Q^* is injective, then*

$$\sigma_{\mathrm{lbf}}(T) = \sigma_{\mathrm{lbw}}(T) = \sigma_{\mathrm{rd}}(T) \quad \textit{and} \quad \sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{d}}(T). \tag{7}$$

Proof. (i) To show the first equality it suffices, by (2), to prove the inclusion $\sigma_{\mathrm{ld}}(T) \subseteq \sigma_{\mathrm{ubf}}(T)$. Let $\lambda \notin \sigma_{\mathrm{ubf}}(T)$ be arbitrarily given. There is no harm if we assume that $\lambda = 0$. Then, T is upper semi B-Fredholm, so $T^n(X)$ is closed for some $n \in \mathbb{N}$ and $T_{[n]} = T|T^n(X)$ is upper semi-Fredholm, and hence, $\alpha(T_{[n]}) < \infty$ and $T_{[n]}$ has a closed range. Now, from

$$Q(T^n(X)) = T^n(Q(X)) \subseteq T^n(X),$$

we see that $T^n(X)$ is invariant under Q, so we can consider the restriction $Q_{[n]} = Q|T^n(X)$, and, since T and Q commutes, we have the following:

$$T_{[n]}Q_{[n]} = Q_{[n]}T_{[n]}.$$

Clearly, $Q_{[n]}$ is injective and quasi-nilpotent. By Theorem 3.1, then $\alpha(T_{[n]}) = 0$. Since $T_{[n]}$ has closed range, then $T_{[n]}$ is bounded below, so, by Theorem 2.2, T is left Drazin invertible, i.e., $0 \notin \sigma_{\mathrm{ld}}(T)$. Therefore, the first equalities in (6) are proved.

The proof of the equality $\sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{d}}(T)$ is analogous: if $0 \notin \sigma_{\mathrm{bw}}(T)$, then T is semi B-Weyl, so $T^n(X)$ is closed for some $n \in \mathbb{N}$ and the restriction $T_{[n]}$ is Weyl, in particular $\alpha(T_{[n]}) = \beta(T_{[n]}) < \infty$. The restriction $Q_{[n]} = Q|T^n(X)$ is injective and commutes with $T_{[n]}$, so, always by Theorem 3.1, $\alpha(T_{[n]}) = \beta(T_{[n]}) = 0$, and hence, $T_{[n]}$ is invertible. By Theorem 2.2, then T is Drazin invertible, so $0 \notin \sigma_{\mathrm{d}}(T)$. Therefore, $\sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{d}}(T)$.

(ii) According the inclusions (3), to show the first equalities in (7), we need only to prove that $\sigma_{\mathrm{rd}}(T) \subseteq \sigma_{\mathrm{lbf}}(T)$. Let $\lambda_0 \notin \sigma_{\mathrm{lbf}}(T)$. Then, $\lambda_0 I - T$ is lower semi B-Fredholm, so, by [1, Theorem 1.117], there exists an open disc $\mathbb{D}_\varepsilon(\lambda_0)$ centered at λ_0 and radius $\varepsilon > 0$, such that $\lambda I - T$ is lower semi-Fredholm for all $\lambda \in \mathbb{D}_\varepsilon(\lambda_0)\backslash\{\lambda_0\}$. Since $\lambda I - T$ commutes with Q, by part (ii) of Theorem 3.2, then $\lambda I - T$ is lower semi-Browder, and hence, $q(\lambda I - T) < \infty$ for all $\lambda \in \mathbb{D}_\varepsilon(\lambda_0)\backslash\{\lambda_0\}$, and this implies, by [1, Theorem 2.65], that T^* has SVEP at every $\lambda \in \mathbb{D}_\varepsilon(\lambda_0)\backslash\{\lambda_0\}$. From Remark 3.3, we then conclude that T^* has SVEP also at λ_0. Since $\lambda_0 I - T$ has topological uniform descent (see Chapter 1 of [1] for details), then, by Theorem 2.98 of [1], $\lambda_0 I - T$ is right Drazin invertible, and hence, $\lambda_0 \notin \sigma_{\mathrm{rd}}(T)$. Therefore, $\sigma_{\mathrm{rd}}(T) \subseteq \sigma_{\mathrm{lbf}}(T)$.

The proof of the second equality in (7) is similar: we have only to prove the inclusion $\sigma_{\mathrm{d}}(T) \subseteq \sigma_{\mathrm{bw}}(T)$. Let $\lambda_0 \notin \sigma_{\mathrm{bw}}(T)$. Then, $\lambda_0 I - T$ is semi-B-Weyl and, by [1, Theorem 1.117], there exists an open disc $\mathbb{D}_\varepsilon(\lambda_0)$ centered at λ_0 and radius $\varepsilon > 0$, such that $\lambda I - T$ is Weyl for all $\lambda \in \mathbb{D}_\varepsilon(\lambda_0)\backslash\{\lambda_0\}$. From part (ii) of Theorem 3.2, it then follows that $\lambda I - T$ is Browder; hence, $p(\lambda I - T) = q(\lambda I - T) < \infty$ for all $\lambda \in \mathbb{D}_\varepsilon(\lambda_0)\backslash\{\lambda_0\}$. This implies, see [1, Theorem 2.65], that both T and T^* have SVEP at every $\lambda \in \mathbb{D}_\varepsilon(\lambda_0)\backslash\{\lambda_0\}$. From Remark 3.3, we then deduce that both T and T^* have SVEP at λ_0. Since $\lambda_0 I - T$ has topological uniform descent, from Theorem 2.97 and Theorem 2.98 of [1], we then conclude that $\lambda_0 I - T$ is both left and right Drazin invertible, i.e., Drazin invertible; hence, $\lambda_0 \notin \sigma_{\mathrm{d}}(T)$. □

Example 3.5. An important example of injective quasi-nilpotent operator is provided by the classical *Volterra operator* V, on the Banach space X, where $X := C[0,1]$, the space of all continuous functions on the closed interval $[0,1]$, or $X := L^2[0,1]$, the Hilbert space of all complex-valued square-integrable functions on the interval $[0,1]$. The operator V is defined by means

$$(Vf)(x) := \int_0^x f(t)\mathrm{d}t \quad \text{for all } f \in X \quad \text{and } x \in [0,1].$$

The class of operators which commute with the Volterra operators is large; for instance, examples of operators which commute with V have been studied in the framework of supercyclic operators [17]. Note that the adjoint of the Volterra operator V on $L^2[0,1]$ is given by

$$(V'f)(x) := \int_x^1 f(t)\mathrm{d}t.$$

Evidently, also V' is injective and quasi-nilpotent; moreover, V' commutes with the adjoint of every operator which belongs to the commutant of V.

For operators T defined on a Hilbert spaces, H is better to consider the adjoint T' instead of the dual T^*. We recall now the relationship between the Hilbert adjoint T' of an operator T defined on a Hilbert space and the dual T^*. By the *Frechét–Riesz representation theorem*, there exists a conjugated-linear isometry $U : H \to H^*$, H^* the dual of H, that associates to every $y \in H$ the linear form defined

$$f_y(x) := < x, y > \quad \text{for every} \;\; x \in H.$$

Moreover

$$(\bar{\lambda} I - T') = U^{-1}(\lambda I - T^*)U \quad \text{for every } \lambda \in \mathbb{C}.$$

Hence, for a Hilbert space operator T

$$T' \text{ is injective} \Leftrightarrow \; T^* \text{ is injective.}$$

Since the adjoint V' of the Volterra operator V in $L^2[0,1]$ is also injective and quasi-nilpotent, from Theorems 3.2 and 3.4, we then conclude the following:

Corollary 3.6. *The equalities (4), (5), (6), and (7) hold for every bounded linear operator* $T : L^2[0,1] \to L^2[0,1]$ *that belongs to the commutant of the Volterra operator* V.

Let $f \in \mathcal{H}(\sigma(T))$ be an analytic function defined on an open neighborhood U which contains the spectrum, and let $f(T)$ be defined by the classical functional calculus

$$f(T) := \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda I - T)^{-1}\, \mathrm{d}\lambda,$$

where Γ is a contour that surrounds $\sigma(T)$ in U. It is known that, in general, the spectral mapping theorem does not hold for $\sigma_{\mathrm{ubw}}(T)$ and $\sigma_{\mathrm{bw}}(T)$; indeed, we have only the inclusions

$$\sigma_{\mathrm{ubw}}(f(T)) \subseteq f(\sigma_{\mathrm{ubw}}(T)) \quad \text{and} \quad \sigma_{\mathrm{bw}}(f(T)) \subseteq f(\sigma_{\mathrm{bw}}(T));$$

see [1, Theorem 3.24], and these inclusions may be strict.

Let $\mathcal{H}_{\mathrm{nc}}(\sigma(T))$ denote the subset of $\mathcal{H}(\sigma(T))$ of all functions nonconstant on each component of its domain of definition.

Theorem 3.7. *Let $T \in L(X)$ and suppose that there exists a quasi-nilpotent operator Q that commutes with T.*

(i) *If Q is injective, then the spectral mapping theorem holds for $\sigma_{\mathrm{ubw}}(T)$ and $\sigma_{\mathrm{bw}}(T)$, i.e.,*

$$\sigma_{\mathrm{ubw}}(f(T)) = f(\sigma_{\mathrm{ubw}}(T)) \quad \textit{for all } f \in \mathcal{H}_{\mathrm{nc}}(\sigma(T)),$$

and

$$\sigma_{\mathrm{bw}}(f(T)) = f(\sigma_{\mathrm{bw}}(T)) \quad \textit{for all } f \in \mathcal{H}_{\mathrm{nc}}(\sigma(T)).$$

(ii) *If Q^* is injective, then the spectral mapping theorem holds for $\sigma_{\mathrm{lbw}}(T)$ and $\sigma_{\mathrm{bw}}(T)$, i.e.,*

$$\sigma_{\mathrm{lbw}}(f(T)) = f(\sigma_{\mathrm{lbw}}(T)) \quad \textit{for all } f \in \mathcal{H}_{\mathrm{nc}}(\sigma(T)),$$

and

$$\sigma_{\mathrm{bw}}(f(T)) = f(\sigma_{\mathrm{bw}}(T)) \quad \textit{for all } f \in \mathcal{H}_{\mathrm{nc}}(\sigma(T)).$$

Proof. (i) By Theorem 3.4, we know that $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{ld}}(T)$. The spectral mapping theorem holds for the left Drazin spectrum, since the set of all left Drazin invertible operators is a regularity; see [1, Theorem 3.109], hence

$$f(\sigma_{\mathrm{ubw}}(T)) = f(\sigma_{\mathrm{ld}}(T)) = \sigma_{\mathrm{ld}}(f(T)).$$

Let Q be an injective quasi-nilpotent operator Q which commutes with T. Evidently, if $\lambda \in \rho(T) := \mathbb{C}\backslash\sigma(T)$, then Q commutes also with $(\lambda I - T)^{-1}$, and consequently, Q commutes with $f(T)$. Therefore, always by Theorem 3.4, we have $\sigma_{\mathrm{ld}}(f(T)) = \sigma_{\mathrm{ubw}}(f(T))$, so the spectral mapping theorem holds for $\sigma_{\mathrm{ubw}}(T)$.

The spectral mapping theorem for the B-Weyl spectrum follows similarly, taking into account that, by Theorem 3.4, we have $\sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{d}}(T)$ and that the spectral mapping theorem holds for $\sigma_{\mathrm{d}}(T)$.

(ii) Evidently, Q^* commutes with T^*, and hence, with $f(T^*) = f(T)^*$, so by Theorem 3.4, we have $\sigma_{\mathrm{rd}}(f(T)) = \sigma_{\mathrm{lbw}}(f(T))$. The spectral mapping theorem also holds for the right Drazin spectrum, since the set of all right Drazin invertible operators is a regularity, see [1, Theorem 3.109], hence

$$f(\sigma_{\mathrm{lbw}}(T)) = f(\sigma_{\mathrm{rd}}(T)) = \sigma_{\mathrm{rd}}(f(T)) = \sigma_{\mathrm{lbw}}(f(T)).$$

The spectral mapping theorem for $\sigma_{\mathrm{bw}}(T)$ is proved similarly using the equality $\sigma_{\mathrm{d}}(f(T)) = \sigma_{\mathrm{bw}}(f(T))$, proved in part (ii) of Theorem 3.4. □

Corollary 3.8. *If a bounded linear operator $T : L^2[0,1] \to L^2[0,1])$ commutes with the Volterra operator V, then the spectral mapping theorem holds for all B-Weyl spectra.*

4. Weyl Type Theorems

An operator $T \in L(X)$ is said to verify *Weyl's theorem* if $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T)$, where $\pi_{00}(T) := \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(\lambda I - T) < \infty\}$. The operator $T \in L(X)$ is said to verify *a-Weyl's theorem* if $\sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi^a_{00}(T)$, where $\pi^a_{00}(T) := \{\lambda \in \operatorname{iso}\sigma_{\mathrm{ap}}(T) : 0 < \alpha(\lambda I - T) < \infty\}$, while $T \in L(X)$ is said to verify *property* (ω) if $\sigma_{\mathrm{ap}}(T)\setminus\sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$. Note that

$$\text{either } a\text{-Weyl's theorem or property } (\omega) \ \Rightarrow \text{Weyl' s theorem;}$$

see [1, Chapter 6]. An operator $T \in L(X)$ is said to verify *generalized Weyl's theorem* (shortly, (gWt)), if $\sigma(T)\setminus\sigma_{\mathrm{bw}}(T) = E(T)$, where $E(T) := \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(\lambda I - T)\}$. The operator $T \in L(X)$ is said to verify *generalized a-Weyl's theorem* (shortly, $(gaWt)$) if $\sigma_{\mathrm{ap}}(T)\setminus\sigma_{\mathrm{ubw}}(T) = E_a(T)$, where $E_a(T) := \{\lambda \in \operatorname{iso}\sigma_{\mathrm{ap}}(T) : 0 < \alpha(\lambda I - T)\}$, while $T \in L(X)$ is said to verify *generalized property* (ω) (shortly, $(g\omega)$) if $\sigma_{\mathrm{ap}}(T)\setminus\sigma_{\mathrm{ubw}}(T) = E(T)$. Note that $(gaWt)$ entails a-Weyl's theorem and generalized property (ω) entails property (ω). Furthermore

$$\text{either } (gaWt) \text{ or } (g\omega) \ \Rightarrow \ (gWt);$$

see [1, Chapter 6]. In [7], it has been proved that the existence of an injective quasi-nilpotent operator that commutes with T ensures that a-Weyl's theorem or property (ω) hold for T. It is a natural question if the generalized versions of Weyl type theorems hold for $T \in \mathbf{Q}_i(X)$. In the sequel, we shall prove that this is true under some additional conditions.

Let $\operatorname{iso} F$ denote the isolated points of $F \subseteq \mathbb{C}$. Recall that if Q is any quasi-nilpotent operator that commutes with T, then

$$\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{ap}}(T + Q) \quad \text{and} \quad \sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{s}}(T + Q); \tag{8}$$

see [1, Corollary 3.24]. Since $\sigma(T) = \sigma_{\mathrm{ap}}(T) \cup \sigma_{\mathrm{s}}(T)$, this easily implies that $\sigma(T) = \sigma(T + Q)$.

Theorem 4.1. *Let $T \in L(X)$ and suppose that Q is an injective quasi-nilpotent operator that commutes with T.*

- (i) *If $\operatorname{iso}\sigma(T) = \emptyset$, then both T and $T + Q$ satisfy generalized Weyl's theorem.*
- (ii) *If $\operatorname{iso}\sigma_{\mathrm{ap}}(T) = \emptyset$, then both T and $T + Q$ satisfy generalized a-Weyl's theorem and generalized property (ω).*

Proof.

- (i) We show first that $\sigma_{\mathrm{d}}(T) = \sigma(T)$. It suffices to prove that $\sigma(T) \subseteq \sigma_{\mathrm{d}}(T)$. Let $\lambda \notin \sigma_{\mathrm{d}}(T)$. Then, $\lambda I - T$ is Drazin invertible, and hence, $\lambda I - T$ is either invertible or λ is a pole of T. If λ is a pole, then λ is an isolated point of $\sigma(T)$, but this is impossible, since by assumption $\operatorname{iso}\sigma(T) = \emptyset$. Hence, $\lambda \notin \sigma(T)$, so $\sigma_{\mathrm{d}}(T) = \sigma(T)$. To show that T satisfies generalized Weyl's theorem, observe that, by Theorem 3.4, $\sigma(T)\setminus\sigma_{\mathrm{bw}}(T) = \sigma(T)\setminus\sigma_{\mathrm{d}}(T) = \emptyset$, and obviously, $E(T) = \emptyset$, since there are no isolated points in $\sigma(T)$. Since $T + Q$ commutes with Q and $\operatorname{iso}\sigma(T+Q) = \operatorname{iso}\sigma(T) = \emptyset$, the argument above proves that also $T+Q$ satisfies generalized Weyl's theorem.

(ii) We show first that $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{ap}}(T)$. Evidently, every bounded below operator is left Drazin invertible, so $\sigma_{\mathrm{ld}}(T) \subseteq \sigma_{\mathrm{ap}}(T)$. To prove the opposite inclusion, let $\lambda \notin \sigma_{\mathrm{ld}}(T)$ be arbitrary given. Then, $\lambda I - T$ is left Drazin invertible. There are two possibilities: $\lambda \in \sigma_{\mathrm{ap}}(T)$ or $\lambda \notin \sigma_{\mathrm{ap}}(T)$. If $\lambda \in \sigma_{\mathrm{ap}}(T)$, then λ is a left pole of T, and hence, by [1, Theorem 4.3], an isolated point of $\sigma_{\mathrm{ap}}(T)$, but this is impossible since $\mathrm{iso}\,\sigma_{\mathrm{ap}}(T) = \emptyset$. Hence, $\lambda \notin \sigma_{\mathrm{ap}}(T)$, so the equality $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{ap}}(T)$ holds. Now, by Theorem 3.4,

$$\sigma(T) \setminus \sigma_{\mathrm{bw}}(T) = \sigma(T) \setminus \sigma_{\mathrm{d}}(T) = \emptyset,$$

and obviously, $E_a(T) = \emptyset$, since there is no isolated point in $\sigma_{\mathrm{ap}}(T)$; hence, T satisfies generalized Weyl's theorem. Since $T + Q$ commutes with Q and

$$\mathrm{iso}\,\sigma_{\mathrm{ap}}(T + Q) = \mathrm{iso}\,\sigma_{\mathrm{ap}}(T) = \emptyset,$$

the argument above proves that $T + Q$ satisfies generalized a-Weyl's theorem.

To prove the generalized property (ω) for T, observe, as above, that $\sigma(T)\setminus\sigma_{\mathrm{bw}}(T) = \emptyset$. Furthermore, also $E(T) = \emptyset$, since every isolated point λ of the spectrum belongs to $\sigma_{\mathrm{ap}}(T)$; see [1, Theorem 1.12], and hence, $\lambda \in \mathrm{iso}\,\sigma_{\mathrm{ap}}(T)$ and this is impossible. Therefore, also generalized property (ω) holds for T.

Since $T + Q$ commutes with Q and $\mathrm{iso}\,\sigma_{\mathrm{ap}}(T + Q) = \emptyset$, then the previous reasoning shows that both generalized a-Weyl's theorem and generalized property (ω) hold for $T + Q$. □

An operator $T \in L(X)$ is said to be *polaroid* if every $\lambda \in \mathrm{iso}\,\sigma(T)$ is a pole of the resolvent. $T \in L(X)$ is said to be *left polaroid* if every $\lambda \in \mathrm{iso}\,\sigma_{\mathrm{ap}}(T)$ is left pole of the resolvent, while $T \in L(X)$ is said to be *a-polaroid* if every $\lambda \in \mathrm{iso}\,\sigma_{\mathrm{ap}}(T)$ is pole of the resolvent. Note that

$$T\ a\text{-polaroid} \Rightarrow T \text{ left polaroid} \Rightarrow T \text{ polaroid};$$

see [1, Corollary 4.13]. Recall that the spectral theorem holds for $\sigma_{\mathrm{ap}}(T)$.

Lemma 4.2. *Suppose that $T \in L(X)$ is polaroid (respectively, left polaroid, a-polaroid). Then, $f(T)$ is polaroid (respectively, left polaroid, a-polaroid) for every $f \in \mathcal{H}_{\mathrm{nc}}(\sigma(T))$.*

Proof. The case where T is polaroid is proved in [1, Theorem 4.19], while for the case where T is left polaroid; see [1, Remark 4.20]. Now, let T be a-polaroid and suppose that $\lambda_0 \in \mathrm{iso}\,\sigma_{\mathrm{ap}}(f(T)) = \mathrm{iso}\,f(\sigma_{\mathrm{ap}}(T))$. We show that λ_0 is a pole of the resolvent of $f(T)$. Let us first to show that $\lambda_0 \in f(\mathrm{iso}\,\sigma_{\mathrm{ap}}(T))$. Select $\mu_0 \in \sigma_{\mathrm{ap}}(T)$, such that $f(\mu_0) = \lambda_0$. Let Ω be the connected component of the domain of f which contains μ_0 and suppose that μ_0 is not isolated in $\sigma_{\mathrm{ap}}(T)$. Then, there exists a sequence $(\mu_n) \subset \sigma_{\mathrm{ap}}(T) \cap \Omega$ of distinct scalars, such that $\mu_n \to \mu_0$. The set $\Gamma := \{\mu_0, \mu_1, \mu_2, \dots\}$ is a compact subset of Ω, so, by the Principle of isolated zeros of analytic functions, the function f may assume the value $\lambda_0 = f(\mu_0)$ at only a finite number of points of Γ. Consequently, for n sufficiently large $f(\mu_n) \neq f(\mu_0) = \lambda_0$, and since $f(\mu_n) \to f(\mu_0) = \lambda_0$, it then follows that λ_0 is not an isolated point of $f(\sigma_{\rm ap}(T))$, a contradiction.

Hence, $\lambda_0 = f(\mu_0)$, with $\mu_0 \in \operatorname{iso} \sigma_{\rm ap}(T)$. Since T is a-polaroid, then μ_0 is a pole of the resolvent of T, and hence, by [1, Theorem 4.16], λ_0 is a pole of the resolvent of $f(T)$. Thus, $f(T)$ is a-polaroid. □

Theorem 4.3. *Let $T \in \mathbf{Q}_i(X)$. Then, we have the following:*

(i) *If T is polaroid, then $f(T)$ satisfies generalized Weyl's theorem for every $f \in \mathcal{H}_{\rm nc}(\sigma(T))$.*
(ii) *If T is left polaroid, then $f(T)$ satisfies generalized a Weyl's theorem theorem for every $f \in \mathcal{H}_{\rm nc}(\sigma(T))$.*
(iii) *If T is a-polaroid, then $f(T)$ satisfies generalized property (ω) for every $f \in \mathcal{H}_{\rm nc}(\sigma(T))$.*

Proof.

(i) Let T be polaroid. By Theorem 3.4, we have $\sigma_{\rm bw}(T) = \sigma_{\rm d}(T)$. Hence

$$\sigma(T)\backslash\sigma_{\rm bw}(T) = \sigma(T)\backslash\sigma_{\rm d}(T) = \Pi(T),$$

so every $\lambda \in \sigma(T)\backslash\sigma_{\rm bw}(T)$ is a pole of the resolvent, hence an isolated point of $\sigma(T)$ and an eigenvalue of T. Therefore, $\lambda \in E(T)$ and this show that $\sigma(T)\backslash\sigma_{\rm bw}(T) \subseteq E(T)$.
The opposite inclusion is also true, if $\lambda \in E(T)$ then λ is a pole, since T is polaroid, and so, by Theorem 3.2, $\lambda \in \sigma(T)\backslash\sigma_{\rm d}(T) = \sigma(T)\backslash\sigma_{\rm bw}(T)$, and hence, (gW) holds for T. By Lemma 4.2, also $f(T)$ is polaroid for every $f \in \mathcal{H}_{\rm nc}(\sigma(T))$, and since $f(T)$ commutes with Q, then (gW) holds for $f(T)$, by the first part of the proof.

(ii) Let T be left polaroid. If $\lambda \in E_a(T)$, then $\lambda \in \operatorname{iso} \sigma_{\rm ap}(T)$, and hence, λ is a left pole, since T is left polaroid; in particular, $\lambda I - T$ is left Drazin invertible.Taking into account Theorem 3.2

$$\lambda \in \sigma_{\rm ap}(T) \setminus \sigma_{\rm ld}(T) = \sigma_{\rm ap}(T) \setminus \sigma_{\rm ubw}(T),$$

so $E_a(T) \subseteq \sigma_{\rm ap}(T) \setminus \sigma_{\rm ubw}(T)$.
On the other hand, the opposite inclusion also holds. Indeed, if

$$\lambda \in \sigma_{\rm ap}(T)\backslash\sigma_{\rm ubw}(T) = \sigma_{\rm ap}(T)\backslash\sigma_{\rm ld}(T) = \Pi_a(T),$$

then $\lambda \in \operatorname{iso} \sigma_{\rm ap}(T)$. Furthermore, $\alpha(\lambda I - T) > 0$; otherwise, if were $\alpha(\lambda I - T) = 0$, we would have $p(\lambda I - T) = 0$, and hence, since $\lambda \notin \sigma_{\rm ld}(T)$, $(\lambda I - T)^{0+1}(X) = (\lambda I - T)(X)$ would be closed, contradicting the assumption that $\lambda \in \sigma_{\rm ap}(T)$. Therefore, $\lambda \in E_a(T)$, so $\sigma_{\rm ap}(T) \setminus \sigma_{\rm ubw}(T) = E_a(T)$; thus, generalized a-Weyl theorem holds for T.
As above, the same argument shows that generalized a-Weyl theorem holds for $f(T)$, since $f(T)$ commutes with Q and $f(T)$ is left polaroid, by Lemma 4.2.

(iii) Observe first that $E(T)$ is contained in $\sigma_{\rm ap}(T)\backslash\sigma_{\rm ubw}(T)$. Indeed, every isolated point of the spectrum belongs to its boundary and hence to $\sigma_{\rm ap}(T)$, by [1, Theorem 1.12]. Therefore, every $\lambda \in E(T)$ is an isolated point of $\sigma_{\rm ap}(T)$, and hence, a pole of the resolvent, since by assumption T is a-polaroid; in particular, λ is a left pole. By Theorem 3.4, then

$$\lambda \in \sigma_{\rm ap}(T) \setminus \sigma_{\rm ld}(T) = \sigma_{\rm ap}(T) \setminus \sigma_{\rm ubw}(T).$$

To show the opposite inclusion $\sigma_{\rm ap}(T)\setminus\sigma_{\rm ubw}(T) \subseteq E(T)$, observe that if $\lambda \in \sigma_{\rm ap}(T)\setminus\sigma_{\rm lbw}(T) = \sigma_{\rm ap}(T)\setminus\sigma_{\rm ld}(T)$, then λ is a left pole, hence an isolated point of $\sigma_{\rm ap}(T)$. The assumption that T is a-polaroid then entails that λ is a pole of T, hence an isolated point of $\sigma(T)$. Since λ is also an eigenvalue of T, we then conclude that $\lambda \in E(T)$, so property $(g\omega)$ holds for T. Since $f(T)$ is a-polaroid then, as above, we conclude that property $(g\omega)$ holds for $f(T)$. □

Corollary 4.4. *Let $T \in L(L^2[0,1])$ commute with the Volterra operator and $f \in \mathcal{H}_{\rm nc}(\sigma(T))$. If T is polaroid, then $f(T)$ satisfies generalized Weyl's theorem. If T is left polaroid, then generalized a-Weyl's theorem holds for $f(T)$. If T is a-polaroid, then generalized property (w) holds for $f(T)$.*

Denote by $\Pi(T)$ the set of all poles of T, i.e., $\Pi(T) := \sigma(T)\setminus\sigma_{\rm d}(T)$ and by $\Pi_a(T)$ the set of all left poles. i.e., $\Pi_a(T) := \sigma_{\rm ap}(T)\setminus\sigma_{\rm ld}(T)$. Note that $\Pi_a(T) \subseteq {\rm iso}\,\sigma_{\rm ap}(T)$ ([1, Theorem 4.3]. According to [10] and [11], an operator $T \in L(X)$ is said to satisfy *property* (gb) if $\sigma_{\rm ap}(T)\setminus\sigma_{ubw}(T) = \Pi(T)$.

Theorem 4.5. *Le $T \in \mathbf{Q}_i(X)$ be a-polaroid. Then, T satisfies property (gb).*

Proof. From Theorem 3.4, we know that

$$\sigma_{\rm ap}(T)\setminus\sigma_{\rm ubw}(T) = \sigma_{\rm ap}(T)\setminus\sigma_{\rm ld}(T) = \Pi_a(T).$$

Obviously, the inclusion $\Pi(T) \subseteq \Pi_a(T)$ holds for every operator. If $\lambda \in \Pi_a(T)$, then $\lambda \in {\rm iso}\,\sigma_{\rm ap}(T)$, and since T is a-polaroid, then $\lambda \in \Pi(T)$; hence, $\Pi_a(T) \subseteq \Pi(T)$, and thus, $\Pi_a(T) = \Pi(T)$. □

To introduce another stronger variant of Weyl's theorem, we introduce the following property.

Definition 4.6. An operator $T \in L(X)$ is said to verify *property (gaz)* if $\sigma(T)\setminus \sigma_{\rm ubw}(T) = \Pi_a(T)$.

Theorem 4.7. *Let $T \in \mathbf{Q}_i(X)$. Then, property (gaz) holds for T if and only if $\sigma_{\rm ap}(T) = \sigma(T)$.*

Proof. Suppose that property (gaz) holds for T. Then, $\sigma_{\rm ap}(T) = \sigma(T)$, by Theorem 3.2 of [5]. Conversely, suppose that $\sigma_{\rm ap}(T) = \sigma(T)$. Since $T \in \mathbf{Q}_i(X)$, then $\sigma_{\rm ubw}(T) = \sigma_{\rm ld}(T)$, by Theorem 3.2, so we have

$$\sigma(T)\setminus\sigma_{\rm ubw}(T) = \sigma_{\rm ap}(T)\setminus\sigma_{\rm ld}(T) = \Pi_a(T),$$

i.e., (gaz) holds for T. □

It should be noted that the equality $\sigma_{\rm ap}(T) = \sigma(T)$ holds if T^* has SVEP, and hence, if $T^\star$ has SVEP, then property (gaz) holds for T.

The following variant of Weyl type theorems has been introduced in [6]. An operator $T \in L(X)$ is said to satisfy *S-Weyl's theorem*, if $\sigma(T)\setminus\sigma_{\rm ubw}(T) = \pi_{00}(T)$. Note that S-Weyl's theorem holds for T if and only if T satisfies *property* (gaz) and the equality $\Pi_a(T) = \pi_{00}(T)$ holds; see [6, Theorem 3.6]. Furthermore, S-Weyl's theorem entails both a-Weyl's theorem and property (ω); see [6, Theorem 3.8].

Observe that for operators $T \in \mathbf{Q}_i(X)$, we have

$$\pi_{00}(T) = \pi_{00}^a(T) = \emptyset,$$

since, if λ belongs to one of these sets, then $0 < \alpha(\lambda I - T) < \infty$ and this is impossible by Theorem 3.1.

Corollary 4.8. *Let $T \in \mathbf{Q}_i(X)$ be such that $\sigma_{\mathrm{ap}}(T) = \sigma(T)$. Then, S-Weyl's theorem holds for T if and only if T has no left poles. In particular, if $\mathrm{iso}\,\sigma_{\mathrm{ap}}(T) = \emptyset$, then S-Weyl's theorem holds for T.*

Proof. By Theorem 4.3, if $\sigma_{\mathrm{ap}}(T) = \sigma(T)$, then T has property (gaz). Furthermore, as noted above, $\pi_{00}(T) = \emptyset$. Hence

$$\emptyset = \Pi_a(T) = \pi_{00}(T);$$

thus, S-Weyl's theorem holds for T. Conversely, if S-Weyl's theorem holds for T, then $\Pi_a(T) = \pi_{00}(T) = \emptyset$, so T has no left poles.

The last assertion is clear: every left pole of the resolvent of an operator is an isolated point of $\sigma_{\mathrm{ap}}(T)$; see [1, Theorem 4.3]. □

Theorem 4.9. *Suppose that $\sigma_{\mathrm{ap}}(T)$ has no hole and that $\sigma_{\mathrm{ap}}(T)$ has no isolated point. If Q is an injective quasi-nilpotent operator that commutes with T, then S-Weyl's theorem holds for T and $T + Q$.*

Proof. Since $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{uw}}(T)$, by Theorem 3.2, the set $\rho_{\mathrm{uw}}(T) := \mathbb{C}\backslash\sigma_{\mathrm{uw}}(T)$ is connected, so, by Lemma 2.1 and Theorem 3.2, we have

$$\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{w}}(T) = \sigma(T).$$

Therefore, T has property (gaz), by Theorem 4.3. To prove that S-Weyl's theorem holds for T, we need to prove that $\Pi_a(T) = \pi_{00}(T)$. As noted before, $\pi_{00}(T)$ is empty and also $\Pi_a(T) = \emptyset$, since $\Pi_a(T) \subseteq \mathrm{iso}\,\sigma_{\mathrm{ap}}(T)$, and thus, T satisfies S-Weyl's theorem. To show that S-Weyl's theorem holds also for $T + Q$, note that $T + Q$ commutes with Q, $\mathrm{iso}\,\sigma_{\mathrm{ap}}(T + Q) = \mathrm{iso}\,\sigma_{\mathrm{ap}}(T) = \emptyset$ and $\sigma_{\mathrm{ap}}(T + Q) = \sigma_{\mathrm{ap}}(T)$ has no hole. □

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