PARAMETER DEPENDENCE FOR THE POSITIVE SOLUTIONS OF NONLINEAR, NONHOMOGENEOUS ROBIN PROBLEMS

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Abstract. We consider a parametric nonlinear Robin problem driven by a nonlinear nonhomogeneous differential operator plus an indefinite potential. The reaction term is $(p-1)$-superlinear but need not satisfy the usual Ambrosetti-Rabinowitz condition. We look for positive solutions and prove a bifurcation-type result for the set of positive solutions as the parameter $\lambda > 0$ varies. Also we prove the existence of a minimal positive solution u_λ^* and determine the monotonicity and continuity properties of the map $\lambda \to u_\lambda^*$.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear parametric Robin problem

(P_λ)
$$\begin{cases} -\operatorname{div} a(z, \nabla u(z)) + \xi(z)u(z)^{p-1} = f_\lambda(z, u(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_a} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega,\ u \geq 0,\ 1 < p < \infty,\ \lambda > 0. \end{cases}$$

In this problem $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous map with $y \to a(z, y)$ monotone (hence maximal monotone too). The map $a(z, \cdot)$ satisfies certain other regularity and growth conditions listed in hypotheses $H(a)$ below. These hypotheses are general enough to incorporate in our framework many differential operators of interest such as the p-Laplacian and the (p, q)-Laplacian (sum of a p-Laplacian and of a q-Laplacian). The potential function $\xi(\cdot)$ is sign-changing. In the reaction term $f_\lambda(z, x)$, $\lambda > 0$ is a parameter and $(z, x, \lambda) \to f_\lambda(z, x)$ is Carathéodory, that is, for all $x \in \mathbb{R}$ and all $\lambda > 0$, $z \to f_\lambda(z, x)$ is measurable, while for a.a. $z \in \Omega$, $(x, \lambda) \to f_\lambda(z, x)$ is continuous. We assume that $f_\lambda(z, \cdot)$ exhibits $(p-1)$-superlinear growth near $+\infty$ but without satisfying the usual in such cases Ambrosetti-Rabinowitz condition (the AR-condition for short). On the other hand near zero, $f_\lambda(z, \cdot)$ has a concave term (that is, a term which is $(p-1)$-sublinear as $x \to 0^+$). So, we have a "concave-convex" problem, but without the two nonlinearities being decoupled and global. In the boundary condition, $\frac{\partial u}{\partial n_a}$ denotes the conormal derivative of u, defined by extension of the map

$$C^1(\overline{\Omega}) \ni u \to (a(z, \nabla u), n)_{\mathbb{R}^N},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The boundary coefficient $\beta(\cdot)$ satisfies $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

Our aim is to study the nonexistence, existence and multiplicity of the positive solutions as the parameter $\lambda > 0$ varies. In the past such studies were conducted primarily in the context of Dirichlet problems driven by the Laplacian or p-Laplacian. We refer to

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the works of Ambrosetti-Brezis-Cerami [3], García Azorero-Peral Alonso-Manfredi [9], Guo-Zhang [13] who deal with equations in which the potential function $\xi \equiv 0$ and the reaction term has the special form

$$f_\lambda(x) = \lambda x^{q-1} + x^{r-1} \quad \text{for all } x \geq 0 \quad \text{with} \quad 1 < q < p < r < p^*.$$

Marano-Papageorgiou [18] extended the aforementioned works to nonlinear equations driven by the Dirichlet p-Laplacian and a reaction term of the form

$$f_\lambda(z,x) = \lambda g(z,x) + h(z,x)$$

with $g(z,x)$ a $(p-1)$-sublinear Carathéodory function and $h(z,x)$ a $(p-1)$-superlinear Carathéodory function. The work of Marano-Papageorgiou [18] was extended by Papageorgiou-Rădulescu-Repovš [25] to semilinear Robin problems driven by the Laplacian plus an indefinite potential term and by Fragnelli-Mugnai-Papageorgiou [8] to nonlinear problems driven by the Neumann p-Laplacian plus an indefinite potential. In all the aforementioned works, the parameter enters into the equation by multiplying the concave term. We also mention the works of Aizicovici-Papageorgiou-Staicu [2], Cardinali-Papageorgiou-Rubbioni [5], Gasiński-Papageorgiou [12] and Motreanu-Tanaka [20]. In [2] the problem is Robin driven by a nonhomogeneous differential operator with zero potential term and $f_\lambda(z,\cdot)$ is strictly $(p-1)$-sublinear near $+\infty$ and near 0^+ (a geometry complementary to the one assumed here). In [5], the equation is nonlinear logistic of the superdiffusive type and the operator is the Neumann p-Laplacian with zero potential. Also, in [12], the equation is Dirichlet driven by the p-Laplacian with zero potential and the reaction term is $\lambda f(z,x)$ with $f(z,\cdot)$ being $(p-1)$-superlinear. Finally in [20] the authors deal with Dirichlet and Neumann problems driven by a nonhomogeneous differential operator and with a reaction with zeros.

Using variational methods based on the critical point theory, together with perturbation and truncation techniques and comparison arguments, we prove a bifurcation type result describing in a precise way the set of positive solutions of (P_λ) as the parameter $\lambda > 0$ varies. Also, we show that for every admissible parameter $\lambda > 0$, the problem (P_λ) admits a smallest positive solution u_λ^* and determine the monotonicity and continuity properties of the map $\lambda \to u_\lambda^*$.

2. Mathematical Background - Hypotheses

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X, \mathbb{R})$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following is true:

"Every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|)\varphi'(u_n) \to 0$ in X^* as $n \to +\infty$, admits a strongly convergent subsequence".

This is a compactness-type condition on φ. It leads to a deformation theorem from which one can derive the minimax theory for the critical values of φ. Prominent in that theory is the so-called "mountain pass theorem" of Ambrosetti-Rabinowitz [4]. Here we state it in a slightly more general form (see Motreanu-Motreanu-Papageorgiou [19], Theorem 5.40, p. 118).

Theorem 1. *If $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > \rho > 0$, $\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u - u_0\| = \rho\} = m_\rho$ and $c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$, then $c \geq m_\rho$ and c is a critical value of φ (that is, there exists $\widehat{u} \in X$ such that $\varphi(\widehat{u}) = c$, $\varphi'(\widehat{u}) = 0$).*

In the analysis of problem (P_λ), we will use the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the "boundary" Lebesgue spaces $L^q(\partial\Omega)$ $(1 \leq q \leq \infty)$.

By $\|\cdot\|$ we denote the norm of $W^{1,p}(\Omega)$ defined by

$$\|u\| = \left[\|u\|_p^p + \|\nabla u\|_p^p\right]^{1/p} \text{ for all } u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone given by

$$C_+ = \{u \in C^1(\overline{\Omega}) \,:\, u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$D_+ = \{u \in C_+ \,:\, u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

In fact the latter is the interior of C_+ also when $C^1(\overline{\Omega})$ is equipped with the relative $C(\overline{\Omega})$-topology.

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure on $\partial\Omega$, we can define in the usual way the boundary Lebesgue spaces $L^q(\partial\Omega)$ $(1 \leq q \leq \infty)$. From the theory of Sobolev spaces we know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, called the "trace map", such that

$$\gamma_0(u) = u\big|_{\partial\Omega} \quad \text{for all } u \in W^{1,p}(\Omega) \cap C^1(\overline{\Omega}).$$

We know that

$$\text{im } \gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega) \quad \left(\frac{1}{p} + \frac{1}{p'} = 1\right) \quad \text{and} \quad \ker \gamma_0 = W_0^{1,p}(\Omega).$$

The trace map is compact into $L^q(\partial\Omega)$ for all $q \in \left[1, \frac{(N-1)p}{N-p}\right)$ if $N > p$ and into $L^q(\partial\Omega)$ for all $1 \leq q < \infty$ if $N \leq p$. In what follows we drop the use of the map $\gamma_0(\cdot)$. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

Now we will introduce the hypotheses on the map $a(z,y)$. So, let $\theta \in C^1((0,+\infty))$ be a function such that

$$0 < \widehat{c}_0 \leq \frac{\theta'(t)t}{\theta(t)} \leq c_0 \quad \text{and} \quad c_1 t^{p-1} \leq \theta(t) \leq c_2(t^{\tau-1} + t^{p-1}) \tag{1}$$

for all $t > 0$, some $c_1, c_2 > 0$, $1 \leq \tau < p < \infty$.

Let $\mathbb{R}_+ = [0,+\infty)$. The hypotheses on the map $a(z,y)$ are the following:

$H(a)$: $a(z,y) = a_0(z,|y|)y$ with $a_0 \in C(\overline{\Omega} \times \mathbb{R}_+)$, $a_0(z,t)t > 0$ for all $z \in \overline{\Omega}$, all $t > 0$ and

- (i) $a_0 \in C^1(\overline{\Omega} \times (0,+\infty))$, for all $z \in \overline{\Omega}$ the function $t \to a_0(z,t)t$ is strictly increasing on $(0,+\infty)$ and $\lim\limits_{t\to 0^+} a_0(z,t)t = 0$;
- (ii) $|\nabla_y a(z,y)| \leq c_3 \dfrac{\theta(|y|)}{|y|}$ for all $z \in \overline{\Omega}$, all $y \in \mathbb{R}^N \setminus \{0\}$, some $c_3 > 0$;
- (iii) $(\nabla_y a(z,y)\xi, \xi)_{\mathbb{R}^N} \geq \dfrac{\theta(|y|)}{|y|}|\xi|^2$ for all $z \in \overline{\Omega}$, all $y, \xi \in \mathbb{R}^N$ with $y \neq 0$;
- (iv) there exists $\delta \in (0,1)$ such that $|\nabla_z a_0(z,t)| \leq c_4(1 + |\ln \delta|)a_0(z,t)$ for all $z \in \overline{\Omega}$, all $t \in [\delta, 1]$, some $c_4 > 0$;
- (v) if $G_0(z,t) = \int_0^t a_0(z,s)s\,ds$ for all $t > 0$, then $pG_0(z,t) - a_0(z,t)t \geq -\widetilde{\eta}$ for all $t > 0$, some $\widetilde{\eta} > 0$;
- (vi) there exists $q \in (1,p)$ such that for all $z \in \overline{\Omega}$

- $t \to G_0(z, t^{1/q})$ is convex,
- $\lim_{t\to 0^+} \dfrac{G_0(z,t)}{t^q} = 0$ uniformly for all $z \in \overline{\Omega}$,
- $\widetilde{c}t^p \le a_0(z,t)t^2 - qG_0(z,t)$ for all $z \in \overline{\Omega}$, all $t > 0$, some $\widetilde{c} > 0$.

Remark 1. Hypotheses $H(a)\,(i) \to (iv)$ allow us to use the nonlinear regularity theory of Lieberman [16] and the nonlinear maximum principle of Zhang [28]. Note that if $G_0(z,t) = \int_0^t a_0(z,s)s\,ds$ (see hypothesis $H(a)\,(v)$), then $G_0 \in C^1(\overline{\Omega} \times \mathbb{R}_+)$ and, for all $z \in \overline{\Omega}$, $G_0(z,\cdot)$ is strictly convex and strictly increasing. We set $G(z,y) = G_0(z,|y|)$ for all $(z,y) \in \overline{\Omega} \times \mathbb{R}^N$. Then $G \in C^1(\overline{\Omega} \times \mathbb{R}^N)$ and $G(z,\cdot)$ is convex. We have

$$\nabla_y G(z,y) = (G_0)'_t(z,|y|)\frac{y}{|y|} = a_0(z,|y|)y = a(z,y) \quad \text{for all } z \in \overline{\Omega},\ \text{all } y \in \mathbb{R}^N \setminus \{0\},$$
$$\nabla_y G(z,0) = 0.$$

Therefore $G(z,\cdot)$ is the primitive of $a(z,\cdot)$. The convexity of $G(z,\cdot)$ and the fact that $G(z,0) = 0$ for all $z \in \overline{\Omega}$, imply that

$$G(z,y) \le (a(z,y),y)_{\mathbb{R}^N} \quad \text{for all } z \in \overline{\Omega},\ \text{all } y \in \mathbb{R}^N. \tag{2}$$

The next lemma summarizes the main properties of the map $a(z,\cdot)$ and it is a consequence of (1) and of hypotheses $H(a)\,(i),(ii),(iii)$ (see also Papageorgiou-Rădulescu [22]).

Lemma 1. *If hypotheses $H(a)\,(i),(ii),(iii)$ hold, then*

(a) $a \in C(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N) \cap C^1(\overline{\Omega} \times \mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$ *for all $z \in \Omega$ the map $y \to a(z,y)$ is strictly monotone (thus maximal monotone too);*
(b) $|a(z,y)| \le c_5(t^{\tau-1} + |y|^{p-1})$ *for all $z \in \overline{\Omega}$, all $y \in \mathbb{R}^N$, some $c_5 > 0$;*
(c) $(a(z,y),y)_{\mathbb{R}^N} \ge \frac{c_1}{p-1}|y|^p$ *for all $z \in \overline{\Omega}$, all $y \in \mathbb{R}^N$.*

Using Lemma 1 and (2), we infer the following growth estimates for the primitive $G(z,y)$.

Corollary 1. *If hypotheses $H(a)\,(i),(ii),(iii)$ hold, then*

$$\frac{c_1}{p(p-1)}|y|^p \le G(z,y) \le c_6(1 + |y|^{p-1}) \text{ for all } z \in \overline{\Omega},\ \text{all } y \in \mathbb{R}^N,\ \text{some } c_6 > 0.$$

Example 1. Suppose that $\widehat{a} \in C^1(\overline{\Omega})$ satisfies

$$0 < \eta_0 \le \widehat{a}(z) \le \eta_1 \quad \text{and} \quad |\nabla \widehat{a}(z)| \le \eta_1 \quad \text{for all } z \in \overline{\Omega}.$$

We consider the following maps

$$a_1(z,y) = \widehat{a}(z)|y|^{p-2}y \quad 1 < p < \infty,$$
$$a_2(z,y) = |y|^{p-2}y + \widehat{a}(z)|y|^{q-2}y \quad 1 < q < p < \infty,$$
$$a_3(z,y) = \widehat{a}(z)[1 + |y|^2]^{\frac{p-2}{2}}y \quad 1 < p < \infty.$$

They satisfy hypotheses $H(a)$. Note that if $\widehat{a} \equiv 1$, then a_1 corresponds to the p-Laplacian and a_2 to the (p,q)-Laplacian.

Next let us introduce the hypotheses on the potential function $\xi(\cdot)$ and the boundary coefficient $\beta(\cdot)$:

H(ξ): $\xi \in L^\infty(\Omega)$.
H(β): $\beta \in C^{0,\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$ and $\beta(z) \ge 0$ for all $z \in \partial\Omega$.

Remark 2. When $\beta \equiv 0$, we recover the Neumann problem. So, our work unifies Robin and Neumann problems. In contrast, in [2] $\beta(z) > 0$ for all $z \in \partial\Omega$.

Suppose that $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$|f_0(z,x)| \le a_0(z)(1 + |x|^{p^*-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ with } a_0 \in L^\infty(\Omega).$$

Here p^* denotes the critical Sobolev exponent for $p \in (1, \infty)$ defined by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \le p. \end{cases}$$

We set $F_0(z,x) = \int_0^x f_0(z,s)ds$ and consider the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \int_\Omega G(z, \nabla u)dz + \frac{1}{p}\int_\Omega \xi(z)|u|^p dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_\Omega F_0(z,u)dz$$

for all $u \in W^{1,p}(\Omega)$.

From Papageorgiou-Rădulescu [24] (Proposition 8), we have:

Proposition 1. *If hypotheses $H(a)\,(i) \to (iv)$, $H(\xi)$, $H(\beta)$ hold and $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of $\varphi_0(\cdot)$, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \le \varphi_0(u_0 + h) \quad \text{for all } h \in C^1(\overline{\Omega}),\ \|h\|_{C^1(\overline{\Omega})} \le \rho_0,$$

then $u_0 \in C^{1,\mu}(\overline{\Omega})$ for some $\mu \in (0,1)$ and u_0 is a local $W^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \le \varphi_0(u_0 + h) \quad \text{for all } h \in W^{1,p}(\Omega) \quad \text{with} \quad \|h\| \le \rho_1.$$

Consider the operator $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by

$$\langle A(u), h\rangle = \int_\Omega (a(z,\nabla u), \nabla h)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W^{1,p}(\Omega).$$

From Gasiński-Papageorgiou [11] (Proposition 3.5), we have:

Proposition 2. *If hypotheses $H(a)\,(i),(ii),(iii)$ hold, then $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ is bounded (that is, it maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone too) and of type $(S)_+$, that is,*

$$\text{“}u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } \limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \le 0 \Rightarrow u_n \to u \text{ in } W^{1,p}(\Omega)\text{''}.$$

The next result is a variant of the strong comparison principle of Fragnelli-Mugnai-Papageorgiou [7] (Proposition 3). The present formulation is more suitable for our needs here.

In what follows by $\widehat{D}_+$ we denote the following open cone in $C^1(\overline{\Omega})$

$$\widehat{D}_+ = \left\{ u \in C^1(\overline{\Omega}) \,:\, u(z) > 0 \text{ for all } z \in \Omega, \quad \frac{\partial u}{\partial n}\bigg|_{\partial\Omega \cap u^{-1}(0)} < 0 \right\}.$$

Proposition 3. *If hypotheses $H(a)\,(i) \to (iv)$ hold, $\widehat{\xi} \in L^\infty(\Omega)$, $\widehat{\xi} \ge 0$ for a.a. $z \in \Omega$, h_1, $h_2 \in L^\infty(\Omega)$, $0 < c_7 \le h_2(z) - h_1(z)$ for a.a. $z \in \Omega$ and $u, v \in C^1(\overline{\Omega}) \setminus \{0\}$ satisfy $u \le v$ and*

$$-\,div\; a(z, \nabla u(z)) + \widehat{\xi}(z)|u(z)|^{p-2}u(z) = h_1(z) \quad \text{for a.a. } z \in \Omega,$$

$$-\ div\ a(z, \nabla v(z)) + \widehat{\xi}(z)|v(z)|^{p-2}v(z) = h_2(z) \quad for\ a.a.\ z \in \Omega,$$

then $v - u \in \widehat{D}_+$.

Proof. We have

$$\begin{aligned} &- \text{div } [a(z, \nabla v(z)) - a(z, \nabla u(z))] \\ &= h_2(z) - h_1(z) - \widehat{\xi}(z)[|v(z)|^{p-2}v(z) - |u(z)|^{p-2}u(z)] \text{ for a.a. } z \in \Omega. \end{aligned} \tag{3}$$

Let $a = (a_k)_{k=1}^N$ (a_k is the k^{th}-component function of $a(\cdot)$). The mean value theorem implies that

$$a_k(z, y) - a_k(z, y') = \sum_{i=1}^N \int_0^1 \frac{\partial}{\partial y_i} a_k(z, y' + t(y - y'))(y_i - y_i')dt$$

for all $y = (y_i)_{i=1}^N$, all $y' = (y_i')_{i=1}^N \in \mathbb{R}^N$ and all $k \in \{1, \dots, N\}$.

We introduce the following functions

$$\widehat{c}_{k,i}(z) = \int_0^1 \frac{\partial}{\partial y_i} a_k(z, \nabla u(z) + t(\nabla v(z) - \nabla u(z))(\nabla_i v(z) - \nabla_i u(z))dt$$

with $\nabla_i = \frac{\partial}{\partial z_i}$, for all $z \in \overline{\Omega}$, all $k \in \{1, \dots, N\}$. Then $\widehat{c}_{k,i}(\cdot) \in W^{1,\infty}(\Omega)$. Using these functions as coefficients, we introduce the following linear differential operator in divergence form

$$L(w) = -\text{div} \left(\sum_{i=1}^N \widehat{c}_{k,i}(z) \frac{\partial w}{\partial z_i} \right) = - \sum_{k,i=1}^N \frac{\partial}{\partial z_k} \left(\widehat{c}_{k,i}(z) \frac{\partial w}{\partial z_i} \right) \quad \text{for all } w \in H^1(\Omega).$$

Let $e = v - u \in C_+ \setminus \{0\}$. Then from (3) we have

$$L(e) = h_2(z) - h_1(z) - \widehat{\xi}(z)[|v(z)|^{p-2}v(z) - |u(z)|^{p-2}u(z)] \quad \text{for a.a. } z \in \Omega. \tag{4}$$

Suppose that for some $z_0 \in \Omega$ we have $u(z_0) = v(z_0)$. Then the uniform continuity of the map $x \to |x|^{p-2}x$ on $[-\|v\|_\infty, \|v\|_\infty]$ (the map is Hölder continuous if $1 < p < 2$ and locally Lipschitz if $p \geq 2$, see [9], inequalities (3.1)) and the fact that $\widehat{\xi} \in L^\infty(\Omega)$, imply that for $\delta > 0$ small we have

$$L(e)(z) \geq c_8 > 0 \quad \text{for a.a. } z \in B_\delta(z_0) = \{z \in \Omega : |z - z_0| < \delta\}.$$

Invoking the Harnack inequality (see Pucci-Serrin [26], Theorem 7.2.1, p. 163) we have $e(z) = (v - u)(z) > 0$ for all $z \in B_\delta(z_0)$, a contradiction since $u(z_0) = v(z_0)$ (alternatively, instead of Harnack's inequality, we can use the tangency principle of Pucci-Serrin [26], Theorem 2.5.2, p. 35 or Theorem 4 of Vázquez [27]). So, we have proved that

$$e(z) = (v - u)(z) > 0 \quad \text{for all } z \in \Omega.$$

Next let $\Sigma_0 = \{z \in \partial\Omega : e(z) = (v - u)(z) = 0\}$ and assume that $\Sigma_0 \neq \emptyset$ (otherwise we already have $e = v - u \in D_+ \subseteq \text{int } C_+$). Recall that $\partial\Omega$ is a C^2-manifold. Then given $z_0 \in \Sigma_0$, we can find $\rho > 0$ small and an ρ-ball B_ρ such that

$$B_\rho \subseteq \Omega \quad \text{and} \quad z_0 \in \partial\Omega \cap \partial B_\rho.$$

Choosing $\rho > 0$ small from (4) and since $u(z_0) = v(z_0)$ (recall $z_0 \in \Sigma_0$), we see that $L(\cdot)$ is strictly elliptic. So, from the strong maximum principle (see Theorem 4 of Vázquez [27] or Theorem 2.8.3, p. 43 of Pucci-Serrin [26]), we have

$$\begin{aligned} & \frac{\partial e}{\partial n}(z_0) = \frac{\partial (v-u)}{\partial n}(z_0) < 0, \\ \Rightarrow \quad & e = v - u \in \widehat{D}_+. \end{aligned}$$

□

Let $x \in \mathbb{R}$. We set $x^{\pm} = \max\{\pm x, 0\}$ and then given $u \in W^{1,p}(\Omega)$ we define $u^{\pm}(\cdot) = u(\cdot)^{\pm}$. We know that

$$u^{\pm} \in W^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Also given $\varphi \in C^1(X, \mathbb{R})$ (X being as before a Banach space), we define

$$K_\varphi = \{u \in X \, : \, \varphi'(u) = 0\}$$

the critical set of φ.

Finally we introduce the hypotheses on the reaction term $f_\lambda(z, x)$. In what follows $F_\lambda(z, x) = \int_0^x f_\lambda(z, s) ds$.

H(f): $f : \Omega \times \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ ($f(z, x, \lambda) = f_\lambda(z, x)$) is Carathéodory, for a.a. $z \in \Omega$, all $\lambda > 0$, $f_\lambda(z, 0) = 0$ and

(i) there exist $a_\lambda \in L^\infty(\Omega)$ and $r_\lambda \in (p, p^*)$ nondecreasing in $\lambda > 0$ such that

$$\begin{aligned} & \lambda \to \|a_\lambda\|_\infty \text{ is bounded on bounded sets of } (0, +\infty), \\ & |f_\lambda(z, x)| \le a_\lambda(z)(1 + x^{r_\lambda - 1}) \text{ for a.a. } z \in \Omega, \text{ all } x \ge 0, \\ & r_\lambda \to r_0 \in (p, p^*) \text{ as } \lambda \to 0^+; \end{aligned}$$

(ii) for every $\lambda > 0$, we have

$$\lim_{x \to +\infty} \frac{F_\lambda(z, x)}{x^p} = +\infty \quad \text{uniformly for a.a. } z \in \Omega;$$

(iii) for every $\lambda > 0$, there exists $\gamma_\lambda \in L^1(\Omega)_+$ such that

$$\lambda \to \|\gamma_\lambda\|_\infty \text{ is bounded on bounded sets of } (0, +\infty)$$

and if $d_\lambda(z, x) = f_\lambda(z, x)x - pF_\lambda(z, x)$, then

$$d_\lambda(z, x) \le d_\lambda(z, y) + \gamma_\lambda(z) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \le x \le y;$$

(iv) for every $\lambda > 0$, every $s > 0$, there exists $\theta_\lambda(s) > 0$ such that

$$\begin{aligned} & \theta_\lambda(s) \to +\infty \text{ as } \lambda \to +\infty, \\ & \inf[f_\lambda(z, x) \, : \, x \ge s] \ge \theta_\lambda(z) \text{ for a.a. } z \in \Omega; \end{aligned}$$

(v) for every $\lambda > 0$, there exist $q_\lambda \in [1, q)$, $\eta_\lambda > 0$ and $\delta_\lambda \in (0, 1]$ such that

$$\eta_\lambda x^{q_\lambda - 1} \le f_\lambda(z, x) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \le x \le \delta_\lambda$$

and there exist $r_* \in (r_\lambda, p^*)$ and $b_\lambda, b_1, b_2 > 0$ such that

$$\begin{aligned} & b_\lambda \to 0^+ \text{ as } \lambda \to 0^+, \\ & f_\lambda(z, x) \le b_\lambda x^{q_\lambda - 1} + b_1 x^{r_* - 1} - b_2 x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0; \end{aligned}$$

(vi) for every $\lambda > \lambda' > 0$, every $s > 0$, there exists $\eta_{\lambda,\lambda'}(s) > 0$ such that

$$f_\lambda(z, x) - f_{\lambda'}(z, x) \ge \eta_{\lambda,\lambda'}(s) \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge s;$$

(vii) for every $\lambda > 0$ and every $\rho > 0$, there exists $\widehat{\xi}_\lambda^\rho > 0$ such that, for a.a. $z \in \Omega$, the function

$$x \to f_\lambda(z,x) + \widehat{\xi}_\lambda^\rho x^{p-1}$$

is nondecreasing on $[0,\rho]$.

Remark 3. Since we are looking for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, without any loss of generality we assume that

$$f_\lambda(z,x) = 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \leq 0, \text{ all } \lambda > 0.$$

Hypotheses $H(f)\,(ii),(iii)$ imply that $f_\lambda(z,\cdot)$ is $(p-1)$-superlinear near $+\infty$. However, this superlinearity condition is not formulated using the AR-condition. Instead we employ a quasimonotonicity condition on $d_\lambda(z,\cdot)$ (see hypothesis $H(f)\,(iii)$). This way we can fit in our framework superlinear reactions with "slower" growth which fail to satisfy the AR-condition (see the Examples below). Recall that the AR-condition says that for all $\lambda > 0$, we can find $M_\lambda > 0$ such that

$$0 < pF_\lambda(z,x) \leq f_\lambda(z,x)x \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq M_\lambda, \tag{5a}$$

$$0 < \operatorname{ess\,inf}_\Omega F_\lambda(\cdot, M_\lambda) \tag{5b}$$

(see Ambrosetti-Rabinowitz [4] and Mugnai [21]).

The quasimonotonicity condition $H(f)\,(iii)$ is a slightly more general version of a condition used by Li-Yang [17]. It is satisfied if for all $\lambda > 0$, there exists $M_\lambda > 0$ such that for a.a. $z \in \Omega$

$$x \to \frac{f_\lambda(z,x)}{x^{p-1}} \quad \text{is nondecreasing on } [M_\lambda, +\infty),$$

or equivalently that

$$x \to d_\lambda(z,x) \quad \text{is nondecreasing on } [M_\lambda, +\infty).$$

For details, see Li-Yang [17]. Hypothesis $H(f)\,(v)$ implies the presence of a "concave" term near zero.

Example 2. Hypotheses $H(f)$ above incorporate in our setting the classical "concave-convex" nonlinearity encountered in the literature

$$f_\lambda^1(x) = \lambda[x^{q-1} + x^{r-1}] \quad \text{for all } x \geq 0, \text{ with } 1 < q < p < r < p^*.$$

This function clearly satisfies the AR-condtion (see (5a), (5b)). On the other hand the function

$$f_\lambda^2(x) \begin{cases} \lambda x^{q-1} - cx^{\theta-1} & \text{if } 0 \leq x \leq 1, \\ x^{p-1}\ln x + \lambda(1-c) & \text{if } 1 < x, \end{cases}$$

with $1 < q < \theta < p$, $0 \leq c < 1$ satisfies hypotheses $H(f)$ but not the AR-condition.

Finally we present a function satisfying hypothes $H(f)$, in which the parameter $\lambda > 0$ enters in a nonlinear nonmultiplicative way

$$f_\lambda^3(x) \begin{cases} x^{q-1} & \text{if } 0 \leq x \leq \rho_\lambda, \\ x^{p-1}\ln x + \mu_\lambda & \text{if } \rho_\lambda < x, \end{cases}$$

with $1 < q < p$, $\mu_\lambda = \rho_\lambda^{q-1}[1 - \rho_\lambda^{p-q}\ln \rho_\lambda]$, $\rho_\lambda \in (0,1]$, $\rho_\lambda \to 0^+$ as $\lambda \to 0^+$.

Since $q_\lambda < p < r^*$, by increasing appropriately b_λ (preserving the property that $b_\lambda \to 0^+$ as $\lambda \to 0^+$) and $b_1 > 0$, we can always assume that $b_2 > \|\xi\|_\infty$ (see hypothesis $H(\xi)$).

We introduce the following truncation-perturbation of $f_\lambda(z,\cdot)$:

$$\widehat{f}_\lambda(z,x) = \begin{cases} 0 & \text{if } x \le 0, \\ f_\lambda(z,x) + b_2 x^{p-1} & \text{if } x > 0. \end{cases} \tag{6}$$

This is a Charathéodory function. We set $\widehat{F}_\lambda(z,x) = \int_0^x \widehat{f}_\lambda(z,s)ds$ and consider the C^1-functional $\widehat{\varphi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda(u) = \int_\Omega G(z,\nabla u)dz + \frac{1}{p}\int_\Omega [\xi(z)+b_2]|u|^p dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_\Omega \widehat{F}_\lambda(z,u)dz$$

for all $u \in W^{1,p}(\Omega)$.

Proposition 4. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, $H(f)$ hold, then, for every $\lambda > 0$, $\widehat{\varphi}_\lambda$ satisfies the C-condition.*

Proof. Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ such that

$$|\widehat{\varphi}_\lambda(u_n)| \le M_1 \quad \text{for some } M_1 > 0, \text{ all } n \in \mathbb{N}, \tag{7}$$

$$(1 + \|u_n\|)\widehat{\varphi}'_\lambda(u_n) \to 0 \text{ in } W^{1,p}(\Omega)^* \text{ as } n \to +\infty. \tag{8}$$

From (8) we have

$$\langle \widehat{\varphi}'_\lambda(u_n), h\rangle \le \frac{\varepsilon_n\|h\|}{1+\|u_n\|}, \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } \varepsilon_n \to 0^+,$$

(9)

$$\Rightarrow \quad \left|\langle A(u_n), h\rangle + \int_\Omega [\xi(z)+b_2]|u_n|^{p-2}u_n\,h\,dz + \int_{\partial\Omega}\beta(z)|u_n|^{p-2}u_n\,h\,d\sigma - \int_\Omega \widehat{f}_\lambda(z,u_n)\,h\,dz\right|$$
$$\le \frac{\varepsilon_n\|h\|}{1+\|u_n\|}, \quad \text{for all } n \in \mathbb{N}.$$

In (9) we choose $h = -u_n^- \in W^{1,p}(\Omega)$ and obtain

$$\frac{c_1}{p-1}\|\nabla u_n^-\|_p^p + \int_\Omega [\xi(z)+b_2](u_n^-)^p\,dz \le \varepsilon_n \quad \text{for all } n \in \mathbb{N}$$

(see Lemma 1, hypothesis $H(\beta)$ and (6))

$$\Rightarrow \quad c_9\|u_n^-\|^p \le \varepsilon_n \quad \text{for some } c_9 > 0, \text{ all } n \in \mathbb{N}, \text{ (recall that } b_2 > \|\xi\|_\infty)$$

$$\Rightarrow \quad u_n^- \to 0 \quad \text{in } W^{1,p}. \tag{10}$$

Next in (9) we choose $h = u_n^+ \in W^{1,p}(\Omega)$. Then

(11)

$$-\int_\Omega (a(z,\nabla u_n^+),\nabla u_n^+)_{\mathbb{R}^N}dz - \int_\Omega \xi(z)(u_n^+)^p dz - \int_{\partial\Omega}\beta(z)(u_n^+)^p d\sigma + \int_\Omega f_\lambda(z,u_n^+)u_n^+ dz \le \varepsilon_n,$$

for all $n \in \mathbb{N}$ (see (6)).

On the other hand from (6), (7) and (10), we have

$$\int_\Omega pG(z,\nabla u_n^+)dz + \int_\Omega \xi(z)(u_n^+)^p dz + \int_{\partial\Omega}\beta(z)(u_n^+)^p d\sigma - \int_\Omega pF_\lambda(z,u_n^+)dz \le M_2, \tag{12}$$

for some $M_2 > 0$, all $n \in \mathbb{N}$.

We add (11) and (12). We obtain

$$\int_\Omega [pG(z, \nabla u_n^+) - (a(z, \nabla u_n^+), \nabla u_n^+)_{\mathbb{R}^N}]dz + \int_\Omega d_\lambda(z, u_n^+)dz \leq M_3 \quad \text{for some } M_3 > 0, \text{ all } n \in \mathbb{N}$$

(13)

$$\Rightarrow \quad \int_\Omega d_\lambda(z, u_n^+)dz \leq M_4 \quad \text{for some } M_4 > 0, \text{ all } n \in \mathbb{N} \text{ (see hypothesis } H(a)\,(v)).$$

Claim: $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ is bounded.

We argue by contradiction. So, suppose that the Claim is not true. Then we may assume that $\|u_n^+\| \to +\infty$ as $n \to +\infty$. Let $y_n = \dfrac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$ and so, by passing to a subsequence if necessary, we have

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^{r_\lambda}(\Omega) \text{ and in } L^p(\partial\Omega),\ y \geq 0. \tag{14}$$

First we assume that $y \neq 0$. Let $\Omega^* = \{z \in \Omega : y(z) > 0\}$. Then $|\Omega^*|_N > 0$ (here by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$). We have

$$u_n^+(z) \to +\infty \quad \text{for a.a. } z \in \Omega^*.$$

Then, on account of hypothesis $H(f)\,(ii)$, we have

$$\frac{F_\lambda(z, u_n^+(z))}{\|u_n^+\|^p} = \frac{F_\lambda(z, u_n^+(z))}{u_n^+(z)^p} y_n(z)^p \to +\infty \quad \text{for a.a. } z \in \Omega^*. \tag{15}$$

Hypotheses $H(f)\,(i), (ii)$ imply that we can find $c_{10} > 0$ such that

$$-c_{10} \leq F_\lambda(z, x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{16}$$

We have

$$\begin{aligned}\int_\Omega \frac{F_\lambda(z, u_n^+)}{\|u_n^+\|^p}dz &= \int_{\Omega^*} \frac{F_\lambda(z, u_n^+)}{(u_n^+)^p} y_n^p dz + \int_{\Omega\setminus\Omega^*} \frac{F_\lambda(z, u_n^+)}{\|u_n^+\|^p}dz \\ &\geq \int_{\Omega^*} \frac{F_\lambda(z, u_n^+)}{(u_n^+)^p} y_n^p dz - \frac{c_{10}}{\|u_n^+\|^p}|\Omega|_N \text{ for all } n \in \mathbb{N} \text{ (see (16))},\end{aligned}$$

$$\Rightarrow \int_\Omega \frac{F_\lambda(z, u_n^+)}{\|u_n^+\|^p}dz \to +\infty \quad \text{as } n \to +\infty \text{ (see (15) and use Fatou's lemma).} \tag{17}$$

Hypotheses $H(f)\,(iii)$ implies that

$$\begin{aligned} & d_\lambda(z, x) \geq -\gamma_\lambda(z) \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0, \\ \Rightarrow \quad & pF_\lambda(z, x) \leq f_\lambda(z, x)x + \gamma_\lambda(z) \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0.\end{aligned}$$

Then using (15) we have

$$\begin{aligned} & \int_\Omega p\frac{F_\lambda(z, u_n^+)}{\|u_n^+\|^p}dz \\ \leq & \int_\Omega \frac{f_\lambda(z, u_n^+)}{\|u_n^+\|^{p-1}} y_n dz + \frac{\|\gamma_\lambda\|_1}{\|u_n^+\|^p} \\ \leq & \ c_{11} + \frac{\|\gamma_\lambda\|_1}{\|u_n^+\|^p} \quad \text{(see (1) and hypotheses } H(a)\,(v), (vi))\end{aligned}$$

$$\leq M_5 \quad \text{for some } M_5 > 0, \text{ all } n \in \mathbb{N}. \tag{18}$$

Comparing (17) and (18), we have a contradiction.

Next suppose that $y = 0$. Consider the C^1-functional $\widetilde{\varphi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widetilde{\varphi}_\lambda(u) = \frac{c_1}{p(p-1)}\|\nabla u\|_p^p + \frac{1}{p}\int_\Omega [\xi(z) + b_2]|u|^p dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_\Omega \widehat{F}_\lambda(z, u)dz$$

for all $u \in W^{1,p}(\Omega)$. Evidently, $\widetilde{\varphi}_\lambda \leq \widehat{\varphi}_\lambda$ (see Corollary 1). For every $n \in \mathbb{N}$, let $t_n \in [0, 1]$ be such that

$$\widetilde{\varphi}_\lambda(t_n u_n) = \max[\widetilde{\varphi}_\lambda(t u_n) : 0 \leq t \leq 1]. \tag{19}$$

Let $\tau > 0$, $\eta_0 = \min\left\{\frac{c_1}{p-1}, 1\right\}$ and set $v_n = \left(\frac{\tau p}{\eta_0}\right)^{\frac{1}{p}} y_n$, $n \in \mathbb{N}$. From (14) and since $y = 0$, we have $v_n \to 0$ in $L^{r_\lambda}(\Omega)$ as $n \to +\infty$. Then from hypothesis $H(f)(i)$ and Krasnoselskii's theorem (see Gasiński-Papageorgiou [10], Theorem 3.4.4, p. 407), we have

$$\int_\Omega F_\lambda(z, v_n)dz \to 0 \quad \text{as } n \to +\infty.$$

Recall that $\|u_n^+\| \to +\infty$. So, we can find $n_0 \in \mathbb{N}$ such that

$$\left(\frac{\tau p}{\eta_0}\right)^{\frac{1}{p}} \frac{1}{\|u_n^+\|} \in (0, 1) \quad \text{for all } n \geq n_0.$$

Therefore for all $n \geq n_0$ we have

$$\begin{aligned}
&\widetilde{\varphi}_\lambda(t_n u_n^+) \geq \widetilde{\varphi}_\lambda(v_n) \quad (\text{ see (19)})\\
&= \frac{c_1}{p(p-1)}\|\nabla v_n\|_p^p + \frac{1}{p}\int_\Omega \xi(z) v_n^p dz + \frac{1}{p}\int_{\partial\Omega} \beta(z) v_n^p d\sigma - \int_\Omega F_\lambda(z, v_n)dz\\
&\qquad\qquad (\text{see (6) and note that } v_n \geq 0 \text{ for all } n \in \mathbb{N})\\
&\geq \frac{c_1}{p(p-1)}\|\nabla v_n\|_p^p + \frac{1}{p}\|v_n\|_p^p + \left[\frac{1}{p}\int_\Omega \xi(z) v_n^p dz - \frac{1}{p}\|v_n\|_p^p - \int_\Omega F_\lambda(z, v_n)dz\right]\\
&\geq \frac{\eta_0}{p}\|v_n\|_p^p + \mu_n
\end{aligned}$$

with $\mu_n = \dfrac{1}{p}\int_\Omega \xi(z) v_n^p dz - \dfrac{1}{p}\|v_n\|_p^p - \int_\Omega F_\lambda(z, v_n)dz$, $n \geq n_0$. It follows that

$$\widetilde{\varphi}_\lambda(t_n u_n^+) \geq \tau + \mu_n \quad \text{for all } n \geq n_0 \text{ (recall } \|y_n\| = 1 \text{ for all } n \in \mathbb{N}). \tag{20}$$

Evidently $\mu_n \to 0$ as $n \to +\infty$. Therefore from (20) we have

$$\widetilde{\varphi}_\lambda(t_n u_n^+) \geq \frac{\tau}{2} \quad \text{for all } n \geq n_1 \geq n_0. \tag{21}$$

Since $\tau > 0$ is arbitrary, from (21) we conclude that

$$\widetilde{\varphi}_\lambda(t_n u_n^+) \to +\infty \quad \text{as } n \to +\infty. \tag{22}$$

Note that

(23)

$\widetilde{\varphi}_\lambda(0) = 0$ and $\widetilde{\varphi}_\lambda(u_n^+) \leq M_1$ for all $n \in \mathbb{N}$ (see (7), (10) and recall that $\widetilde{\varphi}_\lambda \leq \widehat{\varphi}_\lambda$).

From (22) and (23) it follows that

$$t_n \in (0, 1) \quad \text{for all } n \geq n_2.$$

Hence for $n \geq n_2$ we have

$$\begin{aligned}
0 &= t_n \frac{d}{dt}\widetilde{\varphi}_\lambda(tu_n^+)\Big|_{t=t_n} \\
&= \langle \widetilde{\varphi}_\lambda'(t_n u_n^+), t_n u_n^+\rangle \quad \text{(by the chain rule)} \\
&= \frac{c_1}{p-1}\|\nabla(t_n u_n^+)\|_p^p + \int_\Omega \xi(z)(t_n u_n^+)^p dz + \int_{\partial\Omega} \beta(z)(t_n u_n^+)^p d\sigma \\
&\qquad - \int_\Omega f_\lambda(z, t_n u_n^+)(t_n u_n^+) dz \text{(see (6))},
\end{aligned}$$

$$\Rightarrow \quad p\widetilde{\varphi}_\lambda(t_n u_n^+) - \int_\Omega d_\lambda(z, t_n u_n^+) dz = 0 \text{ for all } n \geq n_2,$$

$$\Rightarrow \quad p\widetilde{\varphi}_\lambda(t_n u_n^+) - \int_\Omega d_\lambda(z, t_n u_n^+) dz - \|\gamma_\lambda\|_1 \leq 0 \text{ for all } n \geq n_2,$$

see hypothesis $H(f)\,(iii)$ and recall $t_n \in (0,1)$, $n \geq n_2$)

$$\Rightarrow \quad p\widetilde{\varphi}_\lambda(t_n u_n^+) \leq M_6 \quad \text{for some } M_6 \geq 0, \text{ all } n \geq n_2 \text{ (see (13))}. \tag{24}$$

Comparing (22) and (24) we have a contradiction. This proves the Claim. On account of the Claim, we may assume that

$$u_n^+ \xrightarrow{w} \widehat{u} \text{ in } W^{1,p}(\Omega) \text{ and } u_n^+ \to \widehat{u} \text{ in } L^{r_\lambda}(\Omega) \text{ and in } L^p(\partial\Omega). \tag{25}$$

From (9), (10) and (6), we have

(26)

$$\left|\langle A(u_n^+), h\rangle + \int_\Omega \xi(z)(u_n^+)^{p-1}\, h\, dz + \int_{\partial\Omega} \beta(z)(u_n^+)^{p-1}\, h\, d\sigma - \int_\Omega f_\lambda(z, u_n^+)\, h\, dz\right| \leq \varepsilon_n'\|h\|$$

for all $n \in \mathbb{N}$, all $h \in W^{1,p}(\Omega)$, with $\varepsilon_n' \to 0^+$.

In (26) we choose $h = u_n^+ - \widehat{u} \in W^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (25). Then

$$\begin{aligned}
&\lim_{n\to+\infty} \langle A(u_n^+), u_n^+ - \widehat{u}\rangle = 0, \\
\Rightarrow \quad & u_n^+ \to \widehat{u} \text{ in } W^{1,p}(\Omega) \quad \text{(see Proposition 2)}, \\
\Rightarrow \quad & u_n \to \widehat{u} \text{ in } W^{1,p}(\Omega) \quad \text{(see (10))}, \\
\Rightarrow \quad & \widehat{\varphi}_\lambda \text{ satisfies the } C\text{-condition.}
\end{aligned}$$

□

Next we show that for $\lambda > 0$ small the mountain pass geometry (see Theorem 1) is satisfied.

Proposition 5. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, $H(f)$ hold, then we can find $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ there exists $\rho_\lambda > 0$ for which we have*

$$\inf[\widehat{\varphi}_\lambda(u) \,:\, \|u\| = \rho_\lambda] = \widehat{m}_\lambda > 0.$$

Proof. Hypothesis $H(f)\,(v)$ implies that

$$F_\lambda(z,x) \leq \frac{b_\lambda}{q_\lambda}x^{q_\lambda} + \frac{b_1}{r_*}x^{r_*} - \frac{b_2}{p}x^p \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{27}$$

Then for all $u \in W^{1,p}(\Omega)$, we have

$$\widehat{\varphi}_\lambda(u) \geq \frac{c_1}{p(p-1)}\|\nabla u\|_p^p + \frac{1}{p}\int_\Omega [\xi(z) + b_2]|u|^p dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma$$

$$
\begin{aligned}
&\qquad\qquad -\frac{b_\lambda}{q_\lambda}\|u^+\|_{q_\lambda}^{q_\lambda} - \frac{b_1}{r_*}\|u^+\|_{r_*}^{r_*} \text{ (see (27) and (6))}\\
&\geq c_{12}\|u\|^p - c_{13}[b_\lambda\|u\|^{q_\lambda} + \|u\|^{r_*}] \quad \text{for some } c_{12}, c_{13} \text{ (recall } b_2 > \|\xi\|_\infty)\\
&= [c_{12} - c_{13}(b_\lambda\|u\|^{q_\lambda - p} + \|u\|^{r_* - p})]\|u\|^p. \qquad (28)
\end{aligned}
$$

Let $J_\lambda(t) = b_\lambda t^{q_\lambda - p} + t^{r_* - p}$ for all $t > 0$. Since $q_\lambda < p < r_*$, it follows that

$$J_\lambda(t) \to +\infty \quad \text{as } t \to 0^+ \text{ and } t \to +\infty.$$

So, we can find $t_0 \in (0, +\infty)$ such that

$$J_\lambda(t_0) = \inf_{t \geq 0} J_\lambda(t).$$

We have

$$
\begin{aligned}
& J'_\lambda(t_0) = 0,\\
\Rightarrow\quad & b_\lambda(p - q_\lambda)t_0^{q_\lambda - p - 1} = (r_* - p)t_0^{r_* - p - 1},\\
\Rightarrow\quad & t_0^{r_* - q_\lambda} = \frac{b_\lambda(p - q_\lambda)}{r_* - p},\\
\Rightarrow\quad & t_0 = \left(\frac{b_\lambda(p - q_\lambda)}{r_* - p}\right)^{\frac{1}{r_* - q_\lambda}}.
\end{aligned}
$$

Then we have

$$J_\lambda(t_0) = b_\lambda\left(\frac{r_* - p}{b_\lambda(p - q_\lambda)}\right)^{\frac{p - q_\lambda}{r_* - q_\lambda}} + \left(\frac{b_\lambda(p - q_\lambda)}{r_* - p}\right)^{\frac{r_* - p}{r_* - q_\lambda}}. \qquad (29)$$

Note that

$$\frac{p - q_\lambda}{r_* - q_\lambda} < 1 \quad \text{and} \quad b_\lambda \to 0^+ \text{ as } \lambda \to 0^+ \text{ (see hypothesis } H(f)\,(v)).$$

Then from (29) it follows that

$$J_\lambda(t_0) \to 0^+ \quad \text{as } \lambda \to 0^+.$$

Therefore we can find $\lambda_0 > 0$ such that

$$c_{13}J_\lambda(t_0) < c_{12} \quad \text{for all } \lambda \in (0, \lambda_0).$$

Using this in (28), we see that

$$\inf[\widehat{\varphi}_\lambda(u) : \|u\| = \rho_\lambda = t_0(\lambda)] = \widehat{m}_\lambda > 0 \quad \text{for all } \lambda \in (0, \lambda_0).$$

□

We introduce the following two sets:

$\mathcal{L} := \{\lambda > 0 : \text{problem } (P_\lambda) \text{ admits a positive solution}\}$,

S_λ is the set of positive solutions of problem (P_λ).

In the next result we will use the nonlinear strong maximum principle of Zhang [28] (nonlinear Hopf boundary point theorem). The result of Zhang [28] concerns nonlinear, nonhomogeneous differential operators with nonstandard growth (that is, $p(z)$-equations). Of course such a result includes as a special case, operators with standard balanced growth as is our operator here. In Zhang [28] the conditions on the map $a(z, y)$ are the "nonstandard" counterpart of the conditions used by Lieberman [15] in his first global nonlinear regularity results. Later Lieberman [16] extended those regularity results using the more general which as we already remarked, we use here (see hypotheses $H(a)\,(i),(ii),(iii)$). So, these conditions together with hypothesis $H(a)\,(iv)$ taken from Zhang [28] (see (7)), guarantee the validity of Theorems 1.1 and 1.2 of Zhang [28] in our setting. In fact a look at the proof of Theorem 1.1, p. 29 of [28] reveals that with very minor changes remains valid, under our conditions (see (1) in this paper). Therefore the result of Zhang [28] can be used here.

Proposition 6. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, $H(f)$ hold, then $S_\lambda \subseteq D_+$.*

Proof. Let $\lambda \in \mathcal{L}$ and $u_\lambda \in S_\lambda$. Then

$$\begin{aligned} &-\operatorname{div} a(z,\nabla u_\lambda(z)) + \xi(z)u_\lambda(z)^{p-1} = f_\lambda(z,u_\lambda(z)) \text{ for a.a. } z\in\Omega,\\ &\frac{\partial u}{\partial n_a} + \beta(z)u_\lambda^{p-1} = 0 \text{ on } \partial\Omega, \quad \text{(see Papageorgiou-Rădulescu [23])}. \end{aligned} \tag{30}$$

From (30) and Papageorgiou-Rădulescu [24] (Proposition 2.10), we have

$$u_\lambda \in L^\infty(\Omega).$$

Then the nonlinear regularity theory of Lieberman [16] implies that

$$u_\lambda \in C_+ \setminus \{0\}.$$

Let $\rho = \|u_\lambda\|_\infty$ and let $\widehat{\xi}_\lambda^\rho > 0$ be as postulated by hypothesis $H(f)\,(vii)$. From (30) we have

$$\begin{aligned} &\operatorname{div} a(z,\nabla u_\lambda(z)) \le [\|\xi\|_\infty + \widehat{\xi}_\lambda^\rho]u_\lambda(z)^{p-1} \quad \text{for a.a. } z\in\Omega\\ \Rightarrow\quad &u_\lambda \in D_+ \quad \text{(see Zhang [28], Theorem 1.2)}. \end{aligned}$$

Hence we conclude that $S_\lambda \subseteq D_+$. ∎

Next we prove a structural property of $\mathcal{L}$.

Proposition 7. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, $H(f)$ hold, then $(0,\lambda] \subseteq \mathcal{L}$ for each $\lambda \in \mathcal{L}$.*

Proof. Let $\tau \in (0,\lambda)$ and let $u_\lambda \in S_\lambda \subseteq D_+$ (see Proposition 6). Recall $b_2 > \|\xi\|_\infty$ and consider the following Carathéodory function

$$k_\tau(z,x) = \begin{cases} f_\tau(z,x) + b_2(x^+)^{p-1} & \text{if } x < u_\lambda(z),\\ f_\tau(z,u_\lambda(z)) + b_2u_\lambda(z)^{p-1} & \text{if } u_\lambda(z)\le x. \end{cases} \tag{31}$$

We set $K_\tau(z,x) = \int_0^x k_\tau(z,s)ds$ and let $\widehat{\psi}_\tau : W^{1,p}(\Omega)\to\mathbb{R}$ be the C^1-functional defined by

$$\widehat{\psi}_\tau(u) = \int_\Omega G(z,\nabla u)dz + \frac{1}{p}\int_\Omega[\xi(z)+b_2]|u|^p dz + \frac{1}{p}\int_{\partial\Omega}\beta(z)|u|^p d\sigma - \int_\Omega K_\tau(z,u)dz$$

for all $u \in W^{1,p}(\Omega)$. From (31) and since $b_2 > \|\xi\|_\infty$, we see that $\widehat{\psi}_\tau$ is coercive. Also, using the Sobolev embedding and the compactness of the trace map, we have that $\widehat{\psi}_\tau(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_\tau \in W^{1,p}(\Omega)$ such that

$$\widehat{\psi}_\tau(u_\tau) = \inf[\widehat{\psi}_\tau(u) : u \in W^{1,p}(\Omega)]. \tag{32}$$

Hypothesis $H(a)\,(vi)$ implies that we can find $\widehat{\delta} \in (0, \min\{\delta_\tau, \min_{\overline{\Omega}} u_\lambda, 1\})$ (recall that $u_\lambda \in D_+$) such that

$$G(z,y) \le |y|^q \quad \text{for all } y \in \mathbb{R}^N,\ |y| \le \widehat{\delta}. \tag{33}$$

Let $u \in D_+$ and choose $t \in (0,1)$ small such that

$$tu(z),\ t|\nabla u(z)| \in (0, \widehat{\delta}] \quad \text{for all } z \in \overline{\Omega}. \tag{34}$$

From (33), (34) and hypothesis $H(f)\,(v)$ (recall $\widehat{\delta} < \delta_\tau$), we have

$$\widehat{\psi}_\tau(tu) \le t^q c_{14} \|u\|^q + t^p \int_{\partial\Omega} \beta(z)|u|^p d\sigma - t^{q_\lambda} \eta_\lambda \|u\|_{q_\lambda}^{q_\lambda},$$

for some $c_{14} > 0$ (note that $|tu|^p \le |tu|^q$ since $q < p$, $\widehat{\delta} < 1$).

Recall that $1 \le q_\lambda < q$ and $t \in (0,1)$. So, by choosing $t \in (0,1)$ even smaller if necessary, we have

$$\begin{aligned} & \widehat{\psi}_\tau(tu) < 0, \\ \Rightarrow\ & \widehat{\psi}_\tau(u_\tau) < 0 = \widehat{\psi}_\tau(0) \quad \text{(see (32))}, \\ \Rightarrow\ & u_\tau \ne 0. \end{aligned}$$

From (32) we have

$$\begin{aligned} & \widehat{\psi}'_\tau(u_\tau) = 0, \\ \Rightarrow\ & \langle A(u_\tau), h\rangle + \int_\Omega [\xi(z) + b_2]|u_\tau|^{p-2} u_\tau h dz + \int_{\partial\Omega} \beta(z)|u_\tau|^{p-2} u_\tau h d\sigma = \int_\Omega k_\tau(z, u_\tau) h dz \end{aligned} \tag{35}$$

for all $h \in W^{1,p}(\Omega)$. In (35) we choose $h = -u_\tau^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & \frac{c_1}{p-1}\|\nabla u_\tau^-\|_p^p + \int_\Omega [\xi(z) + b_2](u_\tau^-)^p dz \le 0 \text{ (see hypothesis } H(\beta) \text{ and (31))}, \\ \Rightarrow\ & c_{15}\|u_\tau^-\|^p \le 0 \quad \text{for some } c_{15} > 0 \text{ (recall } b_2 > \|\xi\|_\infty), \\ \Rightarrow\ & u_\tau \ge 0, u_\tau \ne 0. \end{aligned} \tag{36}$$

Also, in (35) we choose $h = (u_\tau - u_\lambda)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & \langle A(u_\tau), (u_\tau - u_\lambda)^+\rangle + \int_\Omega [\xi(z) + b_2] u_\tau^{p-1} (u_\tau - u_\lambda)^+ dz + \int_{\partial\Omega} \beta(z) u_\tau^{p-1} (u_\tau - u_\lambda)^+ d\sigma \\ = & \int_\Omega [f_\tau(z, u_\lambda) + b_2 u_\lambda^{p-1}](u_\tau - u_\lambda)^+ dz \quad \text{(see (31))} \\ \le & \int_\Omega [f_\lambda(z, u_\lambda) + b_2 u_\lambda^{p-1}](u_\tau - u_\lambda)^+ dz \quad \text{(see hypothesis } H(f)\,(vi)) \\ = & \langle A(u_\lambda), (u_\tau - u_\lambda)^+\rangle + \int_\Omega [\xi(z) + b_2] u_\lambda^{p-1} (u_\tau - u_\lambda)^+ dz + \int_{\partial\Omega} \beta(z) u_\lambda^{p-1} (u_\tau - u_\lambda)^+ d\sigma \\ & \qquad \text{(since } u_\lambda \in S_\lambda), \\ \Rightarrow\ & u_\tau \le u_\lambda \quad \text{(recall } b_2 > \|\xi\|_\infty \text{ and see hypothesis } H(\beta)). \end{aligned}$$

Therefore, we have proved that

$$u_\tau \in [0, u_\lambda] = \{u \in W^{1,p}(\Omega) : 0 \le u(z) \le u_\lambda(z) \text{ for a.a. } z \in \Omega\}, \tag{37}$$

$u_\tau \neq 0$ (see (36)).

From (35), (37) and (31), we obtain

$$\langle A(u_\tau), h\rangle + \int_\Omega \xi(z) u_\tau^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_\tau^{p-1} h d\sigma = \int_\Omega f_\tau(z, u_\tau) h dz \quad \text{for all } h \in W^{1,p}(\Omega)$$

$$\Rightarrow \quad u_\tau \in S_\tau \subseteq D_+ \quad \text{and so } \tau \in \mathcal{L}.$$

Therefore $(0, \lambda] \subseteq \mathcal{L}$. ☐

The above proposition implies that $\mathcal{L}$ is an interval. Moreover, an interesting byproduct of the above proof, is the following monotonicity-type property for the solution multifunction $\lambda \to S_\lambda$:

(WM) "If $\lambda \in \mathcal{L}$, $\tau \in (0, \lambda]$ and $u_\lambda \in S_\lambda \subseteq D_+$, then $\tau \in \mathcal{L}$ and we can find $u_\tau \in S_\tau \subseteq D_+$ such that $u_\lambda - u_\tau \in C_+ \setminus \{0\}$."

With a little additional effort, we can have the following strong monotonicity property for the solution multifunction $\lambda \to S_\lambda$.

Proposition 8. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, $H(f)$ hold, $\lambda \in \mathcal{L}$, $\mu \in (0, \lambda)$ and $u_\lambda \in S_\lambda \subseteq D_+$, then $\mu \in \mathcal{L}$ and there exists $u_\mu \in S_\mu$ such that $u_\lambda - u_\mu \in \widehat{D}_+$.*

Proof. From Proposition 7 and the (WM)-property above, we already know that $\mu \in \mathcal{L}$ and we can find $u_\mu \in S_\mu \subseteq D_+$ such that

$$u_\lambda - u_\mu \in C_+ \setminus \{0\}. \tag{38}$$

Let $\rho = \|u_\lambda\|_\infty$ and let $\widehat{\xi}^\rho_\lambda, \widehat{\xi}^\rho_\mu > 0$ be as postulated by hypothesis $H(f)\,(vii)$. We set $\widetilde{\xi}^\rho_* = \max\{\widehat{\xi}^\rho_\lambda, \widehat{\xi}^\rho_\mu, \|\xi\|_\infty\}$ and $\widehat{\xi}^\rho_* > \widetilde{\xi}^\rho_*$. Evidently for a..a. $z \in \Omega$ we have that

$$x \to f_\lambda(z, x) + \widehat{\xi}^\rho_* x^{p-1} \quad \text{and} \quad x \to f_\mu(z, x) + \widehat{\xi}^\rho_* x^{p-1}$$

are nondecreasing on $[0, \rho]$.

We have

$$\begin{aligned}
&- \operatorname{div} a(z, \nabla u_\mu(z)) + [\xi(z) + \widehat{\xi}^\rho_*] u_\mu(z)^{p-1} \\
&= f_\mu(z, u_\mu(z)) + \widehat{\xi}^\rho_* u_\mu(z)^{p-1} \\
&= f_\lambda(z, u_\mu(z)) + \widehat{\xi}^\rho_* u_\mu(z)^{p-1} - [f_\lambda(z, u_\mu(z)) - f_\mu(z, u_\mu(z))] \\
&\leq f_\lambda(z, u_\mu(z)) + \widehat{\xi}^\rho_* u_\mu(z)^{p-1} - \eta_{\lambda,\mu}(s) \quad \text{with } s = \min\nolimits_{\overline{\Omega}} u_\mu \text{ (see hypothesis } H(f)\,(vi)) \\
&< f_\lambda(z, u_\lambda(z)) + \widehat{\xi}^\rho_* u_\lambda(z)^{p-1} \text{ (see (38) and recall } \eta_{\lambda,\mu}(s) > 0) \\
&- \operatorname{div} a(z, \nabla u_\lambda(z)) + [\xi(z) + \widehat{\xi}^\rho_*] u_\lambda(z)^{p-1} \quad \text{for a.a. } z \in \Omega \text{ (recall that } u_\lambda \in S_\lambda).
\end{aligned} \tag{39}$$

If $h_1(z) = f_\mu(z, u_\mu(z)) + \widehat{\xi}^\rho_* u_\mu(z)^{p-1}$, $h_2(z) = f_\lambda(z, u_\lambda(z)) + \widehat{\xi}^\rho_* u_\lambda(z)^{p-1}$, then

$$\begin{aligned}
&h_1, h_2 \in L^\infty(\Omega), \\
&0 < \eta_{\lambda,\mu}(s) \leq h_2(z) - h_1(z) \quad \text{for a.a. } z \in \Omega.
\end{aligned}$$

From (39) and Proposition 3, it follows that

$$u_\lambda - u_\mu \in \widehat{D}_+.$$

☐

Now we prove that $\mathcal{L} \neq \emptyset$ (that is, existence of admissible parameters).

Proposition 9. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, $H(f)$ hold, then $\mathcal{L} \neq \emptyset$.*

Proof. Let $\lambda_0 > 0$ be as in Proposition 5 and let $\lambda \in (0, \lambda_0)$. From Proposition 4 we know that

$$\widehat{\varphi}_\lambda \text{ satisfies the C-condition.} \tag{40}$$

Also from Proposition 2 we have

$$0 = \widehat{\varphi}_\lambda(0) < \inf[\widehat{\varphi}_\lambda(u) \,:\, u \in W^{1,p}(\Omega)] = \widehat{m}_\lambda. \tag{41}$$

If $u \in D_+$, then hypothesis $H(f)\,(iii)$ implies that

$$\widehat{\varphi}_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{42}$$

On account of (40), (41), (42) we can apply Theorem 1 (the mountain pass theorem) and find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$u_\lambda \in K_{\widehat{\varphi}_\lambda} \quad \text{and} \quad \widehat{m}_\lambda \leq \widehat{\varphi}_\lambda(u_\lambda). \tag{43}$$

From (43) and (41), we have

$$u_\lambda \neq 0$$

$$\langle A(u_\lambda), h\rangle + \int_\Omega [\xi(z) + b_2]|u_\lambda|^{p-2}u_\lambda h\,dz + \int_{\partial\Omega} \beta(z)|u_\lambda|^{p-2}u_\lambda h\,d\sigma = \int_\Omega \widehat{f}_\lambda(z, u_\lambda)h\,dz \tag{44}$$

for all $h \in W^{1,p}(\Omega)$.

In (44) we choose $h = -u_\lambda^- \in W^{1,p}(\Omega)$. Then

$$\frac{c_1}{p-1}\|\nabla u_\lambda^-\|_p^p + \int_\Omega [\xi(z) + b_2](u_\lambda^-)^p dz \leq 0 \text{ (see Lemma 1, hypothesis } H(\beta) \text{ and (6))},$$

$\Rightarrow \quad c_{16}\|u_\lambda^-\|^p \leq 0 \quad$ for some $c_{16} > 0$ (recall that $b_2 > \|\xi\|_\infty$),

$\Rightarrow \quad u_\lambda \geq 0, u_\lambda \neq 0.$

Then (44) becomes

$$\langle A(u_\lambda), h\rangle + \int_\Omega \xi(z)u_\lambda^{p-1}hdz + \int_{\partial\Omega} \beta(z)u_\lambda^{p-1}hd\sigma = \int_\Omega f_\lambda(z, u_\lambda)hdz$$

for all $h \in W^{1,p}(\Omega)$ (see (6))

$\Rightarrow \quad u_\lambda \in S_\lambda \subseteq D_+$ and $\lambda \in \mathcal{L}$, hence $\mathcal{L} \neq \emptyset$ (in fact $(0, \lambda_0) \subseteq \mathcal{L}$).

□

We set $\lambda^* = \sup \mathcal{L} > 0$.

Proposition 10. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, $H(f)$ hold, then $\lambda^* < \infty$.*

Proof. We fix $\mu > \|\xi\|_\infty$. We claim that we can find $\widehat{\lambda} > 0$ such that

$$f_{\widehat{\lambda}}(z, x) \geq \mu x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{45}$$

To see this, we fix $\lambda^0 > 0$. Hypotheses $H(f)\,(ii), (iii)$ imply that we can find $M_7 > 0$ such that

$$f_{\lambda^0}(z, x) \geq \mu x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq M_7. \tag{46}$$

Since $q_{\lambda^0} < q < p$, hypothesis $H(f)\,(v)$ implies that we can find $\delta_0 \in (0, \delta_{\lambda^0}]$ such that

$$f_{\lambda^0}(z, x) \geq \eta_{\lambda^0} x^{q_{\lambda^0}-1} \geq \mu x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta_0. \tag{47}$$

From hypothesis $H(f)\,(iv)$ we know

$$\theta_\lambda(\delta_0) \to +\infty \quad \text{as } \lambda \to +\infty.$$

Therefore we can find $\widehat{\lambda} > \lambda^0$ such that

$$\theta_{\widehat{\lambda}}(\delta_0) \geq \mu M_7^{p-1}.$$

Then hypothesis $H(f)\,(iv)$ implies that

$$f_{\widehat{\lambda}}(z,x) \geq \theta_{\widehat{\lambda}}(\delta_0) \geq \mu M_7^{p-1} \geq \mu x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } \delta_0 \leq x \leq M_7. \tag{48}$$

From (46), (47), (48) and since $\lambda \to f_\lambda(z,x)$ is increasing (see hypothesis $H(f)\,(vi)$), we infer that (45) holds.

Consider $\lambda > \widehat{\lambda}$ and suppose that $\lambda \in \mathcal{L}$. Then we can find $u_\lambda \in S_\lambda \subseteq D_+$. We set $0 < m_\lambda = \min_{\overline{\Omega}} u_\lambda$.

For $\delta > 0$ small, let $m_\lambda^\delta = m_\lambda + \delta$. For $\rho = \|u_\lambda\|_\infty$ let $\widehat{\xi}_\lambda^\rho > 0$ be as postulated by hypothesis $H(f)\,(vii)$. We can always assume that $\widehat{\xi}_\lambda^\rho > \|\xi\|_\infty$. Then

$$\begin{aligned}
&-\operatorname{div} a(z,\nabla m_\lambda^\delta) + [\xi(z) + \widehat{\xi}_\lambda^\rho](m_\lambda^\delta)^{p-1} \\
&= [\xi(z) + \widehat{\xi}_\lambda^\rho](m_\lambda^\delta)^{p-1} \\
&< [\mu + \widehat{\xi}_\lambda^\rho](m_\lambda^\delta)^{p-1} \quad (\text{recall that } \mu > \|\xi\|_\infty) \\
&\leq [\mu + \widehat{\xi}_\lambda^\rho] m_\lambda^{p-1} + \chi(\delta) \quad \text{with } \chi(\delta) \to 0^+ \text{ as } \delta \to 0^+ \\
&\leq f_{\widehat{\lambda}}(z,m_\lambda) + \widehat{\xi}_\lambda^\rho m_\lambda^{p-1} + \chi(\delta) \quad (\text{see (45)}) \\
&= f_\lambda(z,m_\lambda) + \widehat{\xi}_\lambda^\rho m_\lambda^{p-1} + [f_{\widehat{\lambda}}(z,m_\lambda) - f_\lambda(z,m_\lambda)] + \chi(\delta) \\
&\leq f_\lambda(z,m_\lambda) + \widehat{\xi}_\lambda^\rho m_\lambda^{p-1} - \eta_{\lambda,\widehat{\lambda}}(m_\lambda) + \chi(\delta) \quad (\text{see hypothesis } H(f)\,(vi)) \\
&\leq f_\lambda(z,u_\lambda(z)) + \widehat{\xi}_\lambda^\rho u_\lambda(z)^{p-1} - \eta_{\lambda,\widehat{\lambda}}(m_\lambda) + \chi(\delta) \\
&\qquad\qquad (\text{see hypothesis } H(f)\,(vii) \text{ and recall } m_\lambda = \min\nolimits_{\overline{\Omega}} u_\lambda) \\
&\leq f_\lambda(z,u_\lambda(z)) + \widehat{\xi}_\lambda^\rho u_\lambda(z)^{p-1} - \frac{1}{2}\eta_{\lambda,\widehat{\lambda}}(m_\lambda) \\
&\qquad\qquad \text{for } \delta > 0 \text{ small (recall } \chi(\delta) \to 0^+ \text{ as } \delta \to 0^+) \\
&< f_\lambda(z,u_\lambda(z)) + \widehat{\xi}_\lambda^\rho u_\lambda(z)^{p-1} \\
&= -\operatorname{div} a(z,\nabla u_\lambda(z)) + \widehat{\xi}_\lambda^\rho u_\lambda(z)^{p-1} \quad \text{for a.a. } z \in \Omega \text{ (since } u_\lambda \in S_\lambda). \qquad (49)
\end{aligned}$$

From (49) it follows that

$$m_\lambda^\delta \leq u_\lambda \quad \text{for } \delta > 0 \text{ small}.$$

This contradicts the fact that $m_\lambda = \min_{\overline{\Omega}} u_\lambda$. Therefore $\lambda \notin \mathcal{L}$ and we have

$$\lambda^* = \sup \mathcal{L} \leq \widehat{\lambda} < +\infty.$$

□

Combining Propositions 7 and 10, we have

$$(0,\lambda^*) \subseteq \mathcal{L} \subseteq (0,\lambda^*]. \tag{50}$$

Proposition 11. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, $H(f)$ hold and $\lambda \in (0,\lambda^*)$, then problem (P_λ) admits at least two positive solutions $u_\lambda, \widehat{u}_\lambda \in D_+$ and $u_\lambda \leq \widehat{u}_\lambda$.*

Proof. Let $\tau \in (\lambda, \lambda^*)$. From (50) we have that $\tau \in \mathcal{L}$ and so we can find $u_\tau \in S_\tau \subseteq D_+$. Then from Proposition 8 we know that we can find $u_\lambda \in S_\lambda \subseteq D_+$ such that

$$u_\tau - u_\lambda \in \text{int}\, C_+. \tag{51}$$

Using this $u_\lambda \in S_\lambda \subseteq D_+$ and $\mu > \|\xi\|_\infty$, we introduce the following truncation-perturbation of the reaction term for problem (P_λ):

$$e_\lambda(z,x) = \begin{cases} f_\lambda(z, u_\lambda(z)) + \mu u_\lambda(z)^{p-1} & \text{if } x \leq u_\lambda(z), \\ f_\lambda(z,x) + \mu x^{p-1} & \text{if } u_\lambda(z) < x. \end{cases} \tag{52}$$

This is a Carathédory function. We set $E_\lambda(z,x) = \int_0^x e_\lambda(z,s)ds$ and consider the C^1-functional $\widetilde{\varphi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widetilde{\varphi}_\lambda(u) = \int_\Omega G(z, \nabla u)dz + \frac{1}{p}\int_\Omega [\xi(z) + \mu]|u|^p dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_\Omega E_\lambda(z,u)dz$$

for all $u \in W^{1,p}(\Omega)$. From (52) it is clear that $e_\lambda(z,\cdot)$ has the same asymptotic behavior as $x \to +\infty$ with the function $\widehat{f}(z,\cdot)$. So, with minor modifications in the proof of Proposition 4 we show that

$$\widetilde{\varphi}_\lambda \text{ satisfies the } C\text{-condition.} \tag{53}$$

<u>Claim:</u> We may assume that $u_\lambda \in D_+$ is a local minimizer of $\widetilde{\varphi}_\lambda$. Consider the following truncation of $e_\lambda(z,\cdot)$:

$$\widehat{e}_\lambda(z,x) = \begin{cases} e_\lambda(z,x) & \text{if } x \leq u_\tau(z), \\ e_\lambda(z, u_\tau(z)) & \text{if } u_\tau(z) < x. \end{cases} \tag{54}$$

This is a Carathédory function. We set $\widehat{E}_\lambda(z,x) = \int_0^x \widehat{e}_\lambda(z,s)ds$ and consider the C^1-functional $\widetilde{\varphi}^*_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widetilde{\varphi}^*_\lambda(u) = \int_\Omega G(z, \nabla u)dz + \frac{1}{p}\int_\Omega [\xi(z) + \mu]|u|^p dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_\Omega \widehat{E}_\lambda(z,u)dz$$

for all $u \in W^{1,p}(\Omega)$. Evidently $\widetilde{\varphi}^*_\lambda$ is coercive (see (54), (52)) and sequentially weakly lower semicontinuous. So, we can find $u^*_\lambda \in W^{1,p}(\Omega)$ such that

$$\widetilde{\varphi}^*_\lambda(u^*_\lambda) = \inf[\widetilde{\varphi}^*_\lambda(u) \,:\, u \in W^{1,p}(\Omega)]. \tag{55}$$

From (55) we have

$$(\widetilde{\varphi}^*_\lambda)'(u^*_\lambda) = 0,$$

(56)

$$\Rightarrow \quad \langle A(u^*_\lambda), h\rangle + \int_\Omega (\xi(z) + \mu)|u^*_\lambda|^{p-2}u^*_\lambda h dz + \int_{\partial\Omega} \beta(z)|u^*_\lambda|^{p-2}u^*_\lambda h d\sigma = \int_\Omega \widehat{e}_\lambda(z, u^*_\lambda)h dz$$

for all $h \in W^{1,p}(\Omega)$.

In (56) first we choose $h = (u_\lambda - u^*_\lambda)^+ \in W^{1,p}(\Omega)$. Then

$$\langle A(u^*_\lambda), (u_\lambda - u^*_\lambda)^+\rangle + \int_\Omega [\xi(z) + \mu]|u^*_\lambda|^{p-2}u^*_\lambda (u_\lambda - u^*_\lambda)^+ dz$$
$$+ \int_{\partial\Omega} \beta(z)|u^*_\lambda|^{p-2}u^*_\lambda (u_\lambda - u^*_\lambda)^+ d\sigma$$

$$
\begin{aligned}
&= \int_\Omega [f_\lambda(z,u_\lambda)+\mu u_\lambda^{p-1}](u_\lambda-u_\lambda^*)^+dz \quad \text{(see (54), (52))} \\
&= \langle A(u_\lambda),(u_\lambda-u_\lambda^*)^+\rangle + \int_\Omega [\xi(z)+\mu]u_\lambda^{p-1}(u_\lambda-u_\lambda^*)^+dz \\
&\qquad\qquad + \int_{\partial\Omega}\beta(z)u_\lambda^{p-1}(u_\lambda-u_\lambda^*)^+d\sigma \text{ (since } u_\lambda\in S_\lambda),
\end{aligned}
$$

$$\Rightarrow \quad u_\lambda \le u_\lambda^*.$$

Also, in (56) we choose $h=(u_\lambda^*-u_\tau)^+\in W^{1,p}(\Omega)$. Then

$$
\begin{aligned}
&\langle A(u_\lambda^*),(u_\lambda^*-u_\tau)^+\rangle + \int_\Omega [\xi(z)+\mu](u_\lambda^*)^{p-1}(u_\lambda^*-u_\tau)^+dz \\
&\qquad\qquad + \int_{\partial\Omega}\beta(z)(u_\lambda^*)^{p-1}(u_\lambda^*-u_\tau)^+d\sigma \\
&= \int_\Omega [f_\lambda(z,u_\tau)+\mu u_\tau^{p-1}](u_\lambda^*-u_\tau)^+dz \quad \text{(see (54), (52) and recall } u_\lambda\le u_\tau) \\
&\le \int_\Omega [f_\tau(z,u_\tau)+\mu u_\tau^{p-1}](u_\lambda^*-u_\tau)^+dz \quad \text{(since } \lambda<\tau, \text{ see hypothesis } H(f)\,(vi)) \\
&= \langle A(u_\tau),(u_\lambda^*-u_\tau)^+\rangle + \int_\Omega [\xi(z)+\mu]u_\tau^{p-1}(u_\lambda^*-u_\tau)^+dz \\
&\qquad\qquad + \int_{\partial\Omega}\beta(z)u_\tau^{p-1}(u_\lambda^*-u_\tau)^+d\sigma \text{(since } u_\tau\in S_\tau),
\end{aligned}
$$

$$\Rightarrow \quad u_\lambda^* \le u_\tau.$$

These facts and the nonlinear regularity theory of Lieberman [16], imply that

$$u_\lambda^* = [u_\lambda,u_\tau]\cap D_+.$$

If $u_\lambda\neq u_\lambda^*$, then this is the desired second positive solution of (P_λ) (see (54), (52)). So, we have two positive solutions $u_\lambda,u_\lambda^*\in D_+$, $u_\lambda\le u_\lambda^*$ and we are done.

So, we assume that

$$u_\lambda^* = u_\lambda.$$

Note that

$$\widetilde{\varphi}_\lambda^*\big|_{[0,u_\tau]} = \widetilde{\varphi}_\lambda\big|_{[0,u_\tau]} \quad \text{(see (52), (54)).}$$

From (51) and (55), we see that

$$
\begin{aligned}
& u_\lambda \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \widetilde{\varphi}_\lambda \\
\Rightarrow\quad & u_\lambda \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \widetilde{\varphi}_\lambda.
\end{aligned}
$$

This proves the Claim. From the proof of the Claim, we have that

$$K_{\widetilde{\varphi}_\lambda}\subseteq [u_\lambda)\cap D_+ = \{u\in D_+ : u_\lambda(z)\le u(z) \text{ for all } z\in\overline{\Omega}\}. \tag{57}$$

From (57) we see that we may assume that

$$K_{\widetilde{\varphi}_\lambda} \text{ is finite.} \tag{58}$$

Otherwise we already have an infinity of positive smooth solutions of (P_λ) (see (52), (57)). So, we are done. On account of the Claim and (58), we can find $\rho\in(0,1)$ small such that

$$\widetilde{\varphi}_\lambda(u_\lambda) < \inf[\widetilde{\varphi}_\lambda(u) : \|u-u_\lambda\|=\rho] = \widetilde{m}_\lambda. \tag{59}$$

(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 29). Hypothesis $H(f)\,(ii)$ and (52) imply that

$$\widetilde{\varphi}_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{60}$$

Then (53), (59), (60) permit the use of Theorem 1 (the mountain pass theorem). So, we can find $\widehat{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\widehat{u}_\lambda \in K_{\widetilde{\varphi}_\lambda} \quad \text{and} \quad \widetilde{m}_\lambda \leq \widetilde{\varphi}_\lambda(\widehat{u}_\lambda). \tag{61}$$

From (57), (59), (61) we conclude that

$$\widehat{u}_\lambda \in D_+ \text{ is a second positive solution of } (P_\lambda),\ u_\lambda \neq \widehat{u}_\lambda,\ u_\lambda \leq \widehat{u}_\lambda.$$

□

Next we check the admissibility of the critical parameter $\lambda^* \in (0, +\infty)$ (that is, whether $\lambda^* \in \mathcal{L}$).

Proposition 12. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, $H(f)$ hold, then $\lambda^* \in \mathcal{L}$, that is, $\mathcal{L} = (0, \lambda^*]$.*

Proof. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq (0, \lambda^*)$ and assume that $\lambda \to (\lambda^*)^-$ as $n \to +\infty$. Let $u_n \in S_{\lambda_n} \subseteq D_+$ for all $n \in \mathbb{N}$. We have

$$\langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h d\sigma = \int_\Omega f_{\lambda_n}(z, u_n) h dz \tag{62}$$

for all $h \in W^{1,p}(\Omega)$, all $n \in \mathbb{N}$.

In (62) we choose $h = u_n \in W^{1,p}(\Omega)$. Then

$$-\int_\Omega (a(z, \nabla u_n), \nabla u_n)_{\mathbb{R}^N} dz - \int_\Omega \xi(z) u_n^p dz - \int_{\partial\Omega} \beta(z) u_n^p d\sigma + \int_\Omega f_{\lambda_n}(z, u_n) u_n dz = 0 \tag{63}$$

for all $n \in \mathbb{N}$. From the proof of Proposition 7, we know that we can assume that these solutions satisfy

$$\int_\Omega pG(z, \nabla u_n) dz + \int_\Omega \xi(z) u_n^p dz + \int_{\partial\Omega} \beta(z) u_n^p d\sigma - \int_\Omega pF_{\lambda_n}(z, u_n) dz < 0 \tag{64}$$

for all $n \in \mathbb{N}$. Adding (63), (64) and using hypothesis $H(a)\,(v)$, we obtain

$$\int_\Omega d_{\lambda_n}(z, u_n) dz \leq M_8 \quad \text{for some } M_8 > 0, \text{ all } n \in \mathbb{N}. \tag{65}$$

Using (65) and reasoning as in the Claim in the proof of Proposition 4, we show that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_* \text{ in } L^{r_{\lambda^*}}(\Omega) \text{ and in } L^p(\partial\Omega). \tag{66}$$

In (62) we choose $h = u_n - u_* \in W^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (66). Then

$$\lim_{n\to+\infty} \langle A(u_n), u_n - u_*\rangle = 0,$$

$$\Rightarrow \quad u_n \to u_* \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 2).} \tag{67}$$

Passing to the limit as $n \to +\infty$ in (62) and using (67), we obtain

$$\langle A(u_*), h\rangle + \int_\Omega \xi(z)u_*^{p-1}hdz + \int_{\partial\Omega}\beta(z)u_*^{p-1}hd\sigma = \int_\Omega f_{\lambda^*}(z,u_*)hdz \text{ for all } h\in W^{1,p}(\Omega)$$

$\Rightarrow u_*$ is a nonnegative solution of (P_{λ^*}).

We need to show that $u_*\neq 0$. Then $u_*\in S_{\lambda^*}\subseteq D_+$ and $\lambda^*\in\mathcal{L}$.

To this end, first note that

$$f_{\lambda_n}(z,x)\geq f_{\lambda_1}(z,x)\geq \eta_{\lambda_1}x^{q_{\lambda_1}-1}-c_{16}x^{r_{\lambda_1}-1} \quad \text{for a.a. } z\in\Omega, \text{ all } x\geq 0, \text{ some } c_{16}>0 \tag{68}$$

(see hypotheses $H(f)\,(i),(v)$).

Motivated by (68), we consider the following nonlinear auxiliary Robin problem

$$\begin{cases} -\operatorname{div} a(z,\nabla u(z)) + \xi^+(z)|u(z)|^{p-2}u(z) = \eta_{\lambda_1}|u(z)|^{q_{\lambda_1}-2}u(z) - c_{16}|u(z)|^{r_{\lambda_1}-2}u(z) & \text{in } \Omega,\\ \frac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{69}$$

If $\xi^+\equiv 0$ (that is $\xi\leq 0$ for a.a. $z\in\Omega$), then instead of ξ^+ we use any positive $L^\infty(\Omega)$-function.

We consider C^1-functional $\widehat{\theta}_+ : W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\widehat{\theta}_+(u) = \int_\Omega G(z,\nabla u)dz + \frac{1}{p}\int_\Omega \xi^+(z)|u|^p dz + \frac{1}{p}\int_{\partial\Omega}\beta(z)|u|^p d\sigma - \frac{\eta_{\lambda_1}}{q_{\lambda_1}}\|u^+\|_{q_{\lambda_1}}^{q_{\lambda_1}} + \frac{c_{16}}{r_{\lambda_1}}\|u^+\|_{r_{\lambda_1}}^{r_{\lambda_1}}$$

for all $u\in W^{1,p}(\Omega)$. Since $q_{\lambda_1}<p<r_{\lambda_1}$, the functional is coercive. It is also sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}\in W^{1,p}(\Omega)$ such that

$$\widehat{\theta}_+(\widetilde{u}) = \inf[\widehat{\theta}_+(u) \,:\, u\in W^{1,p}(\Omega)]. \tag{70}$$

As in the proof of Proposition 7, exploiting the fact that $q_{\lambda_1}<p<r_{\lambda_1}$, we have that

$$\begin{aligned} & \widehat{\theta}_+(\widetilde{u}) < 0 = \widehat{\theta}_+(0),\\ \Rightarrow\quad & \widetilde{u}\neq 0. \end{aligned}$$

From (70) we have

$$\begin{aligned} & \widehat{\theta}_+'(\widetilde{u}) = 0,\\ \Rightarrow\quad & \langle A(\widetilde{u}),h\rangle + \int_\Omega \xi^+(z)|\widetilde{u}|^{p-2}\widetilde{u}hdz + \int_{\partial\Omega}\beta(z)|\widetilde{u}|^{p-2}\widetilde{u}hd\sigma\\ & \qquad = \int_\Omega [\eta_{\lambda_1}(\widetilde{u}^+)^{q_{\lambda_1}-1} - c_{16}(\widetilde{u}^+)^{r_{\lambda_1}-1}]hdz \end{aligned} \tag{71}$$

for all $h\in W^{1,p}(\Omega)$.

Choosing $h=-\widetilde{u}^-\in W^{1,p}(\Omega)$ in (71), we obtain

$$\widetilde{u}\geq 0,\ \widetilde{u}\neq 0.$$

So, problem (69) admits a positive solution $\widetilde{u}$, which by the nonlinear regularity theory of Lieberman [16] and the nonlinear maximum principle of Zhang [28] belong in D_+ (that is, $\widetilde{u}\in D_+$).

We will show that this positive solution $\widetilde{u} \in D_+$ of (69) is in fact unique. For this purpose, we introduce the integral functional $j : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \int_\Omega G(z, \nabla u^{\frac{1}{q}})dz + \frac{1}{p}\int_\Omega \xi^+(z)u^{\frac{p}{q}}dz + \frac{1}{p}\int_{\partial\Omega}\beta(z)u^{\frac{p}{q}}d\sigma & \text{if } u \geq 0, u^{\frac{1}{q}} \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Using hypothesis $H(a)\,(vi)$ and Lemma 1 of Diaz-Saá [6], as in Papageorgiou-Rădulescu [22], we show that $j(\cdot)$ is convex (recall $q < p$, see hypothesis $H(a)\,(vi)$).

Suppose $\widetilde{v} \in W^{1,p}(\Omega)$ is another positive solution of (69). Again, we have

$$\widetilde{v} \in D_+.$$

Hence, if $\text{dom}\, j = \{u \in L^1(\Omega) : j(u) < +\infty\}$ (the effective domain of $j(\cdot)$, then for $|t|$ small we have for all $h \in C^1(\overline{\Omega})$

$$\widetilde{u}^q + th \in \text{dom}\, j \quad \text{and} \quad \widetilde{v}^q + th \in \text{dom}\, j.$$

Note that $j(\cdot)$ is Gâteaux differentiable at $\widetilde{u}^q$ and at $\widetilde{v}^q$ in the direction h and using the chain rule and the nonlinear Green's identity (see, for example, Gasiński-Papageorgiou [10], Theorem 2.4.53, p. 210), we have

$$j'(\widetilde{u}^q)(h) = \frac{1}{q}\int_\Omega \frac{-\text{div}\, a(z, \nabla\widetilde{u}) + \xi^+(z)\widetilde{u}^{p-1}}{\widetilde{u}^{q-1}}hdz = \int_\Omega \frac{\eta_{\lambda_1}\widetilde{u}^{q_{\lambda_1}-1} - c_{16}\widetilde{u}^{r_{\lambda_1}-1}}{\widetilde{u}^{q-1}}hdz$$
$$j'(\widetilde{v}^q)(h) = \frac{1}{q}\int_\Omega \frac{-\text{div}\, a(z, \nabla\widetilde{v}) + \xi^+(z)\widetilde{v}^{p-1}}{\widetilde{v}^{q-1}}hdz = \int_\Omega \frac{\eta_{\lambda_1}\widetilde{v}^{q_{\lambda_1}-1} - c_{16}\widetilde{v}^{r_{\lambda_1}-1}}{\widetilde{v}^{q-1}}hdz$$

for all $h \in C^1(\overline{\Omega})$.

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. Hence

$$0 \leq \int_\Omega \left[\eta_{\lambda_1}\left(\frac{1}{\widetilde{u}^{q-q_{\lambda_1}}} - \frac{1}{\widetilde{v}^{q-q_{\lambda_1}}}\right) - c_{16}\left(\widetilde{u}^{r_{\lambda_1}-q} - \widetilde{v}^{r_{\lambda_1}-q}\right)\right](\widetilde{u}^q - \widetilde{v}^q)dz$$
$$\Rightarrow \quad \widetilde{u} = \widetilde{v} \quad (\text{since } q_{\lambda_1} < q < p < r_{\lambda_1}).$$

This proves the uniqueness of the positive solution $\widetilde{u} \in D_+$ of (69).

Now let $n \in \mathbb{N}$. We will show that

$$\widetilde{u} \leq u \text{ for all } u \in S_{\lambda_n}. \tag{72}$$

Fix $u \in S_{\lambda_n}$ and consider the Carathéodory function $\tau(z, x)$ defined by

$$\tau(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ \eta_{\lambda_1}x^{q_{\lambda_1}-1} - c_{16}x^{r_{\lambda_1}-1} & \text{if } 0 \leq x \leq u(z), \\ \eta_{\lambda_1}u(z)^{q_{\lambda_1}-1} - c_{16}u(z)^{r_{\lambda_1}-1} & \text{if } u(z) < x. \end{cases} \tag{73}$$

We set $T(z, x) = \int_0^x \tau(z, s)ds$ and consider the C^1-functional $\theta_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\theta_0(u) = \int_\Omega G(z, \nabla u)dz + \frac{1}{p}\int_\Omega \xi^+(z)|u|^p dz + \frac{1}{p}\int_{\partial\Omega}\beta(z)|u|^p d\sigma - \int_\Omega \tau(z, u)dz$$

for all $u \in W^{1,p}(\Omega)$. Evidently $\theta_0(\cdot)$ is coercive (see (73)) and sequentially weakly lower semicontinuous. So, we can find $\widehat{u} \in W^{1,p}(\Omega)$ such that

$$\theta_0(\widehat{u}) = \inf[\theta_0(u) : u \in W^{1,p}(\Omega)] < 0 = \theta_0(0) \text{ (recall } q_{\lambda_1} < q < p < r_{\lambda_1})$$
$$\Rightarrow \quad \widehat{u} \neq 0 \quad \text{and} \quad \widehat{u} \in K_{\theta_0}.$$

We have

$$\begin{aligned}
&\theta_0'(\widehat{u}) = 0,\\
\Rightarrow\quad &\langle A(\widehat{u}), h\rangle + \int_\Omega \xi^+(z)|\widehat{u}|^{p-2}\widehat{u}hdz + \int_{\partial\Omega}\beta(z)|\widehat{u}|^{p-2}\widehat{u}hd\sigma = \int_\Omega \tau(z,\widehat{u})hdz
\end{aligned}$$

for all $h \in W^{1,p}(\Omega)$.

First we choose $h = -\widehat{u}^- \in W^{1,p}(\Omega)$ and obtain $\widehat{u} \geq 0$. Next we choose $h = (\widehat{u}-u)^+ \in W^{1,p}(\Omega)$. We have

$$\begin{aligned}
&\langle A(\widehat{u}), (\widehat{u}-u)^+\rangle + \int_\Omega \xi^+(z)\widehat{u}^{p-1}(\widehat{u}-u)^+dz + \int_{\partial\Omega}\beta(z)\widehat{u}^{p-1}(\widehat{u}-u)^+d\sigma\\
&= \int_\Omega [\eta_{\lambda_1}u^{q_{\lambda_1}-1} - c_{16}u^{r_{\lambda_1}-1}](\widehat{u}-u)^+dz \quad \text{(see (73))}\\
&\leq \int_\Omega f_{\lambda_n}(z,u)(\widehat{u}-u)^+dz\\
&= \langle A(u), (\widehat{u}-u)^+\rangle + \int_\Omega \xi(z)u^{p-1}(\widehat{u}-u)^+dz + \int_{\partial\Omega}\beta(z)u^{p-1}(\widehat{u}-u)^+d\sigma\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(since } u \in S_{\lambda_n}),\\
&\leq \langle A(u), (\widehat{u}-u)^+\rangle + \int_\Omega \xi^+(z)u^{p-1}(\widehat{u}-u)^+dz + \int_{\partial\Omega}\beta(z)u^{p-1}(\widehat{u}-u)^+d\sigma\\
\Rightarrow\quad &\widehat{u} \leq u.
\end{aligned}$$

From these observations and the nonlinear regularity theory, we have

$$\begin{aligned}
&\widehat{u} \in [0,u] \cap D_+\\
\Rightarrow\quad &\widehat{u} \text{ is a positive solution of (69)},\\
\Rightarrow\quad &\widehat{u} = \widetilde{u},\\
\Rightarrow\quad &\text{(72) holds.}
\end{aligned}$$

So, we have

$$\begin{aligned}
&\widetilde{u} \leq u_n \text{ for all } n \geq 1,\\
\Rightarrow\quad &\widetilde{u} \leq u_* \text{ (see (66))}\\
\Rightarrow\quad &u_* \in S_{\lambda^*} \quad \text{and so} \quad \lambda^* \in \mathcal{L} \text{ (that is, } \mathcal{L} = (0,\lambda^*]).
\end{aligned}$$

□

We summarize the work done in this section, with the following bifurcation-type result.

Theorem 2. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, $H(f)$ hold, then there exists $\lambda^* > 0$ such that*

(a) *for all $\lambda \in (0,\lambda^*)$ problem (P_λ) has at least two positive solutions*

$$u_\lambda, \widehat{u}_\lambda \in D_+, \quad u_\lambda \leq \widehat{u}_\lambda, \quad u_\lambda \neq \widehat{u}_\lambda;$$

(b) *for $\lambda = \lambda^*$ problem (P_λ) has at least one positive solution*

$$u_* \in D_+;$$

(c) *for $\lambda > \lambda^*$ problem (P_λ) has no positive solution.*

3. Minimal positive solutions

In this section we show that for every $\lambda \in \mathcal{L} \subseteq (0, \lambda^*]$ problem (P_λ) has a smallest positive solution $u_\lambda^* \in D_+$. We also study the monotonicity and continuity properties of the map $\lambda \to u_\lambda^*$.

Proposition 13. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, $H(f)$ hold, then for every $\lambda \in \mathcal{L} = (0, \lambda^*]$ problem (P_λ) admits a smallest positive solution $u_\lambda^* \in D_+$.*

Proof. We know that S_λ is downward directed, that is, given $u_1, u_2 \in S_\lambda$, we can find $u \in S_\lambda$ such that $u \leq u_1$, $u \leq u_2$ (see Papageorgiou-Rădulescu-Repovš [25], proof of Proposition 9). Then invoking Lemma 3.10, p.178, of Hu-Papageorgiou [14], we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq S_\lambda$ decreasing such that

$$\inf S_\lambda = \inf_{n\in\mathbb{N}} u_n.$$

From the proof of Proposition 12, we have

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega) \text{ is bounded}, \quad \widetilde{u} \leq u_n \text{ for all } n \in \mathbb{N}.$$

So we may assume that

$$u_n \xrightarrow{w} u_\lambda^* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_\lambda^* \text{ in } L^{r_\lambda}(\Omega) \text{ and in } L^p(\partial\Omega), \quad \widetilde{u} \leq u_\lambda^*. \tag{74}$$

For every $n \in \mathbb{N}$, we have

$$\langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h d\sigma = \int_\Omega f_\lambda(z, u_n) h dz \tag{75}$$

for all $h \in W^{1,p}(\Omega)$.

In (75) we choose $h = u_n - u_\lambda^* \in W^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (74). Then

$$\lim_{n\to+\infty} \langle A(u_n), u_n - u_\lambda^*\rangle = 0,$$

$$\Rightarrow \quad u_n \to u_\lambda^* \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 2).} \tag{76}$$

If in (75) we pass to the limit as $n \to +\infty$ and use (76), then

$$\langle A(u_\lambda^*), h\rangle + \int_\Omega \xi(z)(u_\lambda^*)^{p-1} h dz + \int_{\partial\Omega} \beta(z)(u_\lambda^*)^{p-1} h d\sigma = \int_\Omega f_\lambda(z, u_\lambda^*) h dz \quad \text{for all } h \in W^{1,p}(\Omega)$$

$$\widetilde{u} \leq u_\lambda^*.$$

Therefore $u_\lambda^* \in S_\lambda$ and $u_\lambda^* = \inf S_\lambda$. □

Consider the map $\widehat{e} : \mathcal{L} = (0, \lambda^*] \to C^1(\overline{\Omega})$ defined by

$$\widehat{e}(\lambda) = u_\lambda^*.$$

In the next proposition we establish the monotonicity and continuity properties of this map.

Proposition 14. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, $H(f)$ hold, then the map $\widehat{e} : \mathcal{L} = (0, \lambda^*] \to C_+$ defined above is*

- *strictly monotone in the sense that*

$$\lambda < \tau \in \mathcal{L} \Rightarrow u_\tau^* - u_\lambda^* \in \widehat{D}_+;$$

- $\widehat{e}(\cdot)$ *is left continuous, that is if* $\{\lambda_n, \lambda\}_{n\in\mathbb{N}} \subseteq \mathcal{L}$, *then*

$$\lambda_n \to \lambda^- \Rightarrow u^*_{\lambda_n} \to u^*_\lambda \text{ in } C^1(\overline{\Omega}).$$

Proof. Let $\tau \in \mathcal{L}$ and $\lambda < \tau$ (hence $\lambda \in \mathcal{L}$, see Proposition 7). On account of Proposition 8 we can find $u_\lambda \in S_\lambda \subseteq D_+$ such that

$$\begin{aligned} & u^*_\tau - u_\lambda \in \widehat{D}_+, \\ \Rightarrow \quad & u^*_\tau - u^*_\lambda \in \widehat{D}_+. \end{aligned}$$

This proves the strict monotonicity of $\widehat{e}(\cdot)$.

Next let $\{\lambda_n, \lambda\}_{n\in\mathbb{N}} \subseteq \mathcal{L}$ and assume that $\lambda_n \to \lambda^-$. From the first part of the proof we know that $\{u^*_{\lambda_n}\}_{n\in\mathbb{N}} \subseteq D_+$ is increasing. In addition $\{u^*_{\lambda_n}\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ is bounded. The nonlinear regularity theory of Lieberman [16] implies that there exist $\gamma \in (0,1)$ and $M_9 > 0$ such that

$$u^*_{\lambda_n} \in C^{1,\gamma}(\overline{\Omega}) \quad \text{and} \quad \|u^*_{\lambda_n}\|_{C^{1,\gamma}(\overline{\Omega})} \le M_9 \quad \text{for all } n \in \mathbb{N}.$$

The compact embedding of $C^{1,\gamma}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and the monotonicity of $\{u^*_{\lambda_n}\}_{n\in\mathbb{N}}$ imply that for the original sequence we have

$$u^*_{\lambda_n} \to \widetilde{u}^*_\lambda \text{ in } C^1(\overline{\Omega}). \tag{77}$$

Evidently $\widetilde{u}^*_\lambda \in S_\lambda$. Suppose that $\widetilde{u}^*_\lambda \neq u^*_\lambda$. Then we can find $z_0 \in \overline{\Omega}$ such that

$$\begin{aligned} & u^*_\lambda(z_0) < \widetilde{u}^*_\lambda(z_0) \\ \Rightarrow \quad & u^*_\lambda(z_0) < u^*_{\lambda_n}(z_0) \text{ for all } n \ge n_0 \text{ (see (77))}, \end{aligned}$$

a contradiction to the strict monotonicity of $\widehat{e}$. Therefore $\widetilde{u}^*_\lambda = u^*_\lambda$ and this proves the left continuity of the map $\widehat{e}(\cdot)$. □

We can state the following result which complements Theorem 2.

Theorem 3. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$, $H(f)$ hold, then for every $\lambda \in \mathcal{L} = (0, \lambda^*]$ problem (P_λ) admits a smallest positive solution $u^*_\lambda \in D_+$ and the map $\widehat{e} : \mathcal{L} = (0, \lambda^*] \to C^1(\overline{\Omega})$ is*

- *strictly monotone, that is $\lambda < \tau \in \mathcal{L} \Rightarrow u^*_\lambda - u^*_\tau \in \widehat{D}_+$;*
- *$\widehat{e}(\cdot)$ is left continuous, that is $\{\lambda_n, \lambda\}_{n\in\mathbb{N}} \subseteq \mathcal{L}$ and $\lambda_n \to \lambda^- \Rightarrow u^*_{\lambda_n} \to u^*_\lambda$ in $C^1(\overline{\Omega})$.*

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