Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education

Editors: Uffe Thomas Jankvist, Marja van den Heuvel-Panhuizen, Michiel Veldhuis
Organised by: Freudenthal Group & Freudenthal Institute, Utrecht University
Year: 2019

erme
european society for research in mathematics education

Editors

Uffe Thomas Jankvist, Danish School of Education, Aarhus University, Denmark.

Marja van den Heuvel-Panhuizen, Freudenthal Group & Freudenthal Institute, Utrecht University, Netherlands; Nord University, Norway.

Michiel Veldhuis, Freudenthal Group, Utrecht University, Netherlands; iPabo University of Applied Sciences, Amsterdam, Netherlands.

Editorial Board

Publisher

Freudenthal Group & Freudenthal Institute, Utrecht University, Netherlands and ERME

ISBN 978-90-73346-75-8

Recommended citation for the proceedings

Jankvist, U. T., Van den Heuvel-Panhuizen, M., & Veldhuis, M. (Eds.). (2019). *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education*. Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Recommended citation for single entries in the proceedings

Authors (2019). Title of paper/poster. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. xxxx-yyyy). Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Semiotic potential of a tractional machine: a first analysis

Michela Maschietto[1], Pietro Milici[2] and Dominique Tournès[3]

[1]Università degli Studi di Modena e Reggio Emilia, Italy; michela.maschietto@unimore.it

[2]Université de Bretagne Occidentale, Brest, France; p.milici@google.com

[3]Université de la Réunion, Saint-Denis, France; dominique.tournes@univ-reunion.fr

Tractional motion is an almost forgotten theory involving machines to geometrically solve inverse tangent problems. It was at the basis of Leibniz's development of calculus and interested many important mathematicians (e.g. Euler) mainly in the 17th and 18th centuries, before its disappearance because of the decrease of importance of geometric constructions. In this work, we analyse the exploration of a tractional motion machine using the framework of Theory of Semiotic Mediation. This is just a very first work in the perspective of introducing this kind of machine to students in order to mediate the meanings of transcendental curves and infinitesimal analysis.

Keywords: History of mathematics, tangent, mathematical machines, tractional motion, semiotic potential.

Introduction.

The use of historic machines is a well-established practice to mediate mathematical meaning (Maschietto & Bartolini Bussi, 2011). Specifically, many studies have been published about the construction of geometric curves as traces of both ideal and practical machines, but most of such studies focus on algebraic curves. In this work, we deal with an approach to transcendental curves with machines, that is *new*[1] in mathematics education. Such an approach, called "tractional motion," was so relevant to the conception of infinitesimal analysis (e.g. for Leibniz, who designed in 1693 a material instrument to practically solve quadrature problems). As deepened in Tournès (2009), tractional motion mixed theoretical developments and concrete constructions of artefacts for the resolution of the inverse tangent problem. Those artefacts could be interesting for STEM education. In our approach, before proposing them to students, the analysis of cognitive processes of exploration by experts is necessary. In the paper, we aim to present the first elements of this analysis for a particular tractional machine.

Brief history of tractional motion.

As clearly exposed in Bos (2001), in 17th century Descartes proposed a class of machines tracing algebraic curves as synthetic part of his method, and polynomial algebra as an analytical tool in order to simplify the problems. That determined a dualism between "geometric" and "mechanic" curves (in modern terms, algebraic and transcendental curves), being only the first one considered acceptable in the Cartesian setting. Soon after the spreading of Descartes' canon, mathematicians got habituated to polynomials, that became no longer an extremely important step for the problem solving, but directly the solution. In that period, there were only a few curves that were considered

[1]Such approach is not in use today, but it was already adopted in Italy by Giovanni Poleni in Padua (first half of the 18th century) and Ernesto Pascal in Naples (beginning of the 20th century).

“mechanic” (e.g. the quadratrix or the cycloid), but a new general method to generate curves beyond Cartesian limits was going to emerge. The main idea underlying behind such constructions was given by a problem proposed by the architect Claude Perrault in Paris in the 1670s. The problem was easily described: *if we move the end of a chain watch along a line slowly enough to avoid inertia, what curve does the watch describe*? Cartesian tools appeared powerless to answer.

In the case of Perrault’s watch, the motion is given by the traction of a cord: that gave the name to such constructions by “tractional motion.” The described curve (called “tractrix”) is constructed given its tangents: while the watch moves, the tangent to the obtained curve is given by the direction of the chain in traction. This is a new kind of tangent problem: instead of the classical tangent problem (given a curve, finding its tangent that satisfies certain properties), known at least since classical Greek period, this time the tangent properties are given, and the curve has to be determined as the solution. That’s the rise of “inverse tangent problems.”

But how can inverse tangent problems be concretely implemented? The reference to a heavy load having friction on a plane (as the watch) involved physical problems that should be kept away from the domain of pure geometry, e.g. the unwanted role of velocity (because of the inertia that the load acquires) or the non-perfect horizontality of the plane. It was missing a clear instrumental embodiment of the theoretical concept “driving the direction of a curve” (as a pair of compasses embeds the concept of equidistance): mathematicians as Huygens and Leibniz worked on it.

To sum up, after the habituation of mathematicians to consider polynomials as solutions, which brought to a loss of interest to machines for algebraic curves, at the end of 17th-century machines got a new importance; this time to justify the construction of transcendental curves (Bos, 1988).

In the 18th century, it was particularly interesting the role of Giovanni Poleni. Mathematician, physician, astronomer and also interested in engineering studies, Poleni (1729) proposed machines for the tractrix and the logarithmic curve, that were also built and used in cabinets for exhibitions and in teaching. In 1739, he obtained the creation, at the University of Padua, of a laboratory of experimental physics (“*Teatro di filosofia sperimentale*”) which was unanimously praised by his contemporaries. He took the opportunity to set up one of the first university courses based on laboratory experiments (previously, he had students coming to his home in his personal scientific cabinet). For this reason, Poleni got a particular attention to the design of his tractional machines: they were no longer only theoretical machines but also practical instruments guaranteeing an adequate usability and accuracy. In this period, curves like the tractrix or the logarithmic were well known and generally accepted, thus it was no longer necessary to justify their introduction as solutions to certain problems, but it became important to give them concreteness (something similar to the possible needs in today classrooms). The main component behind such machine was the introduction of the “wheel” to drive the direction of the curve (as in the front wheel of a bike). Poleni’s publication (1729) made a strong impression on the geometers of his time. Leonhard Euler (1736) read it and found in it an interesting mode of construction to address the differential equations that he could not integrate in finite form by algebraic methods. Among other results using tractional motion, Euler succeeded in integrating the famous Riccati equation, which had remained unsolved for a long time. From there, an epistolary dialogue continued between Euler and Poleni for more than four years, revealing interesting interactions between theoretical and practical concerns (Tournès 2009, p. 95-96): in particular, at Euler's request, Poleni gave the description of an instrument for drawing the tractrix of a circle.

Following on from Euler's work, Vincenzo Riccati (1752) published in Bologna a treatise in which he demonstrates that it is possible to exactly integrate any differential equation using tractional motion. From a theoretical point of view, Riccati's work is the culmination of the ancient tradition of problem-solving through the construction of curves. Descartes had shown the construction of algebraic curves by a simple continuous motion using linkages. Riccati, for his part, established that transcendental curves can also be constructed by a simple continuous motion from the differential equations that define them.

After the middle of the 18th century, the theory and the practice of the tractional motion got rapidly lost. The geometric paradigm was no longer dominant, and new mathematicians were interested in exploring only the analytical counterpart of infinitesimal analysis. However, in the late 19th and early 20th centuries, a new generation of mathematicians and engineers reinvented essentially the same solutions of Poleni, increasing just the complexity of these instruments (the "integraphs").

The tractional machine of our study.

One of the most interesting features of tractional motion is the deep link between its instrumental and theoretical components. Specifically, the possibility of introducing some transcendental constructions from history to classroom might be mediated by the exploration of a machine.

Out of the tractrix, the most famous construction easily given by tractional motion is for the logarithmic (or exponential) curve. As visible in Figure 1 (left and centre), for tracing this curve, a machine can implement the property of having a constant subtangent[2]. Poleni, who designed and build a machine tracing the exponential, also evinced that one tractional machine could generate both algebraic and transcendental curves. That was a fundamental idea while conceiving our machine: according to the notation of Figure 1, if we change the direction of the wheel in the machine for the exponential (i.e. we put the wheel direction perpendicular to the segment QS), the machine traces a parabola (Figure 1, right).

More specifically, according to Figure 2, the machine[3] proposed in this study is made up by a horizontal wooden table with two parallel fixed wooden guides in which a wooden frame can slide. A brass cylinder is forced to remain within the frame, free to rotate and to slide perpendicularly to the two fixed guides. Inside the cylinder, there is a wheel perpendicular to the plane. The cylinder rotation is determined by a rod that is constrained to pass through a pin fixed on the frame.

---

[2]In a Cartesian plane, given a curve C passing through the point (x,y), the subtangent to C in (x,y) is the segment between $(x,0)$ and the intersection of the line tangent to C in (x,y) with the x-axis. Referring to the Figure 1, the subtangent in S is the segment QP.

[3]Simulations of this machine are available online: https://www.geogebra.org/m/zrMyFGdd (for the parabola) and https://www.geogebra.org/m/LXugImiH (for the exponential). This machine was introduced by Milici & Di Paola (2012) and described with more details in Salvi & Milici (2013).

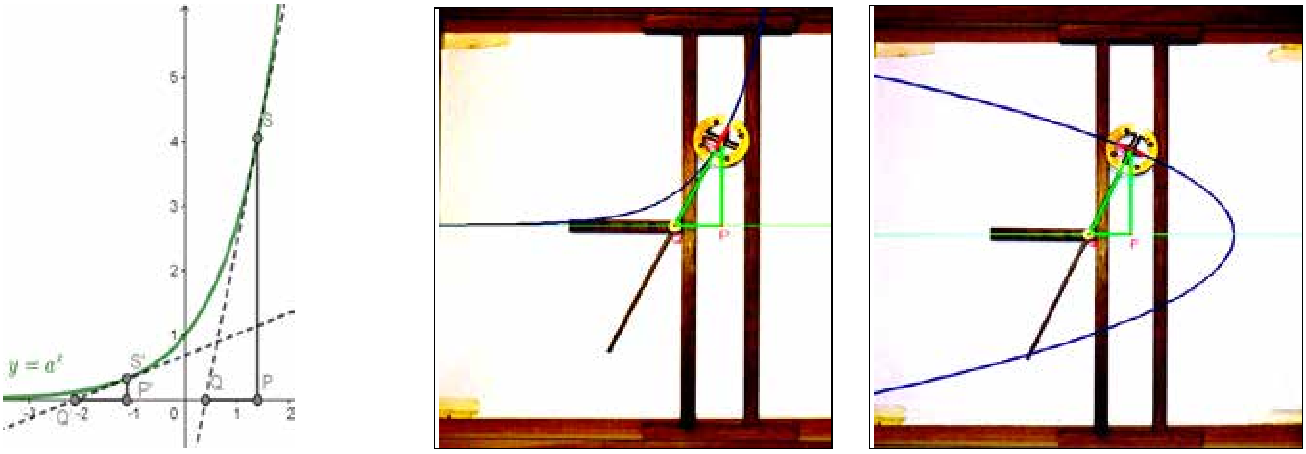
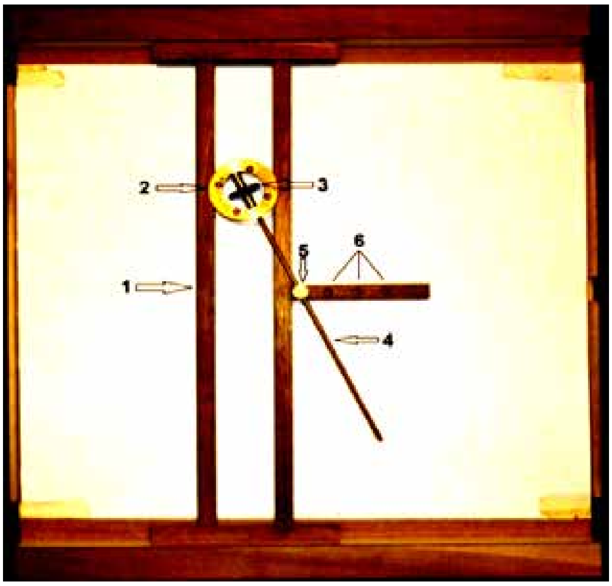


Figure 1:[Left] The exponential y=a^x has constant subtangent QP; [Centre] machine for exponentials; [Right] machine for parabolas



Components:

1. wood translating frame
2. brass hollow cylinder
3. tyred wheel that rotates inside the cylinder (2)
4. rod giving the direction of the cylinder (2)
5. pin with a hole where the rod (4) slides
6. holes where to put the pin (5)

Figure 2: The various components of the proposed machine

Artefacts in mathematics education.

Basing on previous works on the use of artefacts in mathematics education (for instance, the mathematical machines in Maschietto & Bartolini Bussi, 2011; www.mmlab.unimore.it and macchinematematiche.org), the analysis of the machine proposed in this paper is carried out within the framework of the Theory of Semiotic Mediation (TSM) developed by Bartolini Bussi & Mariotti (2008) in a Vygotskian perspective. In this, a teacher uses an artefact as a tool of semiotic mediation for mediating mathematical meaning to students by managing didactical cycles (composed of small group work, collective mathematical discussion and individual work). The first proposed activity is a small group work with the artefact, because its use solicits a certain semiotic activity (where the sings can be language, gestures, drawings, ...) that is the base for the teacher's intervention. The exploration of the chosen artefact related to the educational objectives is carried out following four questions (Bartolini Bussi, Garuti, Martignone & Maschietto, 2011, p.128): "How is the machine made?", "What does the machine make?", "Why does it make it?", "What could happen if ...?”. Within this framework, the choice and the educational use of an artefact is based on the analysis of its semiotic potential. In the TSM, the distinction between meanings emerging from the practice (use of artefact) and the mathematics knowledge evoked in the expert’s mind is very important. Mariotti & Maracci (2012) specify the notion of the semiotic potential of an artefact to make that distinction explicit:

> By semiotic potential of an artefact we mean the double semiotic link which may occur between an artefact and the personal meanings emerging from its use to accomplish a task and at the same time the mathematical meanings evoked by its use and recognizable as mathematics by an expert. (Mariotti & Maracci, 2012, p. 61)

Our research questions are about if and how we could use the machine as an instrument of semiotic mediation with secondary school and/or university students. As in other research on mathematical machines, we intend to propose first a material machine rather than its simulation in a DGS, because we assume the relevance of the exploration by the hand (allowing touching, decomposing the machine, perceiving the resistance to movement or not, ...). In order to answer to those questions, the first step is the analysis of the semiotic potential of the machine, considering the description of the machine by an expert that already knows it on the one hand, and the exploration of the machine by other subjects, experts in other kinds of machines or university students in mathematics. In this way, we aim to analyse what kind of semiotic activity can emerge from the exploration and use of the machine. In particular, we focus on: which are the processes of exploration of the machine? Do people identify the role and the meaning of the wheel? How?

Methods.

We have videotaped two exploration processes of the machine. We chose three people that knew the mathematical machines and the TSM, and were experts in mathematics but without any knowledge of tractional motion. The first exploration was carried out by two secondary teachers, following the structure of four questions that they knew very well. In the exploration, the researcher had to often put the attention on the wheel and its way of use, thus, for the second case, we began proposing only some components of the machine. The second exploration was performed by a postgraduate student in mathematics that studied mathematical machines during a university course. In this case, the machine has been disassembled and gradually reassembled by the researcher: first the cylinder with inside the rolling wheel (cf. Figure 2, (2) and (3)) is given free to move on the table; then the rod (cf. Figure 2,(4)) is added, in the two positions (parallel and perpendicular to the wheel direction); finally, the cylinder is put into the wood frame (cf. Figure 2, (1)).

Findings and discussion.

The analysis is based on videotapes and drawings produced by the people exploring the machine. More specifically, we analyse the part of the two explorations in which people detect the movement and constraints of the wheel and identify the link between wheel and the line tangent to the curve.

First exploration: Alberto and Bianca.

During the first moments of the exploration of the machine (corresponding to the first question, “how is it made?”), Bianca describes the machine but she does not seem to pay particular attention to the presence of the wheel, even if she considers it as a component of the machine. When the machine is then used to trace (intentionally, the ways of use are not suggested to her), this lack of attention is evident by how Bianca acts on the frame and on the wheel: sometimes the wheel rolls and sometimes it slips, tracing a curve or a straight line. The resistance to the slipping of the wheel is not noticed by Bianca, who raises doubts on how to use the machine until the researcher intervenes by asking it explicitly ("Don't you feel resistance?"), by specifying the principle of operation ("[the machine] works when the wheel turns") and by explaining the relationship between the resistance and the constraint of the machine ("you're forcing a constraint of the machine").

The machine doesn't seem to offer feedback sufficiently strong to actions that force the constraints; while forcing the wheel not to follow its direction, the resistance of the tyre is mixed with friction by other structural parts, such as the frame. This certainly is an important point to take into account for paying attention to the wheel as a new element, compared to other curve drawers (such as the ones using taut threads). To foster the identification of the role of the wheel in tracing the curve, the researcher asks to specify the physical constraints of the machine allowing that tracing; he also suggests a comparison with a tracer (which corresponds to the point of a pen) ("If, instead of the wheel, there had been a tracer, would it have been the same?"). The question corresponds to "What could happen if ...?" of our educational framework. However, it is a question that can be asked to expert people. Here are some excerpts of exchanges with Bianca, Alberto and the researcher.

Alberto: It would have been different.

Bianca: That is ...

Alberto: Then, the wheel...

Bianca: Yes, it would have been different.

Bianca: If there is a point... that is a wheel...

Alberto: It would slip, ... it would move randomly.

Researcher: And, so, what must the wheel do?

Bianca: It turns [laughter]

Bianca: It must not slip, because if it slips ...

Alberto: Turning without slipping.

Then, they turn their attention to the geometrical property of the machine. The relationship between the wheel and the tangent to the curve appears in terms of "direction."

Alberto: [The properties] of the tangent line, isn't it? But physically ... I don't know why it is physically ... that is, the small wheel gives the direction [*gesture by the hand*] that is not ... any direction to the curve.

Researcher: And then, what do we know ... having the small wheel this direction? ... at the drawn point.

Alberto: The curve must have that direction.

These excerpts show that the exploration is in discontinuity with the explorations of the curves drawers that Alberto and Bianca are familiar with. This seems to confirm the deep change quoted above from the epistemological point of view. The emergence of the link between the wheel and the tangent line to the curve is fostered by the researcher. However, some questions are not immediately understood by Alberto and Bianca, as they do not link the task and their personal meanings on curve and machine. The difference between rolling and crawling seems to emphasize such a change.

Second exploration: Corrado.

According to researchers' queries, Corrado moves the cylinder with inside the rolling wheel (cf. Figure 2, pieces 2 and 3) on the plane and defines some properties of the traced curve with consciousness ("It is not constrained to stay on a straight line or a circumference or a parabola or a branch of hyperbole, if we let it free it can go up and down in every place. ... It traces a continuous line, [...] can do edges but without jumps"). When asked to describe more precisely his interactions to make the machine move, he immediately defines the two degrees of freedom. In terms of the analysis of the semiotic potential, we could affirm that the task links the manipulation to Corrado's personal meanings of the curve.

Unlike Bianca, when Corrado is asked to explore the fully reassembled machine, he seems aware of the mechanical resistance to certain motions, and he is able to interpret it in relation with the constraints of the instrument. Specifically, when the wheel is put in the vertex of the parabola, Corrado notices that the machine "resists" when he tries to move along the direction of the axis of the parabola. To explain such behavior, he puts his attention on the idea of direction: "[the machine cannot move that way] because it cannot [with a gesture indicates to move along the axis direction], it is constrained here [indicates the pin, component 5 of Figure 2], it cannot invert, get comfortable [with a gesture of the hand: change the direction] and trace, it has to stay this way and it is unnatural to draw [gesture: in the direction of the axis], it cannot rotate this way [gesture of the rotation of the wheel perpendicularly to the direction of the wheel]. It [the wheel] draws only while rotating and in the direction of its rotation." Later on, he is asked to characterize the traced curve that he identified as a parabola. Corrado is able to find independently the role of the tangent for the wheel: "if we hypothesize that this [with the hand indicates the direction of the wheel] gives the direction of the tangent, the wheel is always perpendicular to this rod and so ... [he looks for a conjecture on the tangent perpendicular to something, but doesn't find any]."

Therefore, as a very early conclusion, we can suppose that disassembling the machine eases the understanding of the constraints of the machine. However, we cannot affirm that disassembling can also ease the characterization of the curve because Corrado tried to verify that the traced curve was a parabola focusing on the points traced on the plan and neglecting tangent properties. This corresponds to another aspect of the epistemological change that characterizes the machine.

Concluding remarks.

At our knowledge, this is the first empirical work aimed at analysing the semiotic potential of a tractional machine. As a first result, we can evince that the passage from algebraic to transcendental constructions, so groundbreaking from a historical and epistemological perspective, recalls also very different abilities in the exploration (the explorations were not trivial even for experts of mathematical machines). In particular, the relation between the wheel and the tangent did not appear immediately, but the idea of disassembling the machine in order to investigate simpler components appeared quite fruitful (even though it requires to be much more deepened). This suggests a possible task, but this is not enough sufficient for constructing worksheet for students. What done is only a very first step in this direction. We need other experimentations to clearly define the mathematical contents to be mediated (tangents, derivatives, integrals, differential equations) and the level of the exploration (high school, university).

References

Bartolini Bussi, M.G., Garuti, R., Martignone, F., & Maschietto, M. (2011). Tasks for teachers in the MMLAB-ER Project. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the IGPME* (Vol. 1, pp. 127–130). Ankara, Turkey: PME.

Bartolini Bussi, M.G., & Mariotti, M.A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English & al. (Eds.), *Handbook of International Research in Math. Educ.* (II edition, pp. 746–783). New York, NY: Routledge.

Bos, H.J.M. (1988). Tractional motion and the legitimation of transcendental curves. *Centaurus*, 31(1), 9–62.

Bos, H.J.M. (2001). *Redefining geometrical exactness: Descartes' transformation of the early modern concept of construction*. New York: Springer.

Euler, L. (1736). De constructione æquationum ope motus tractorii aliisque ad methodum tangentium inversam pertinentibus. *Commentarii academiæ scientiarum Petropolitanæ*, 8, 66–85.

Mariotti, M.A., & Maracci, M. (2012). Resources for the Teacher from a Semiotic Mediation Perspective. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *From Text to 'Lived' Resources*, Mathematics Teacher Education (Vol. 7, pp. 59–75). New York: Springer.

Maschietto, M., & Bartolini Bussi, M.G. (2011). Mathematical Machines: from History to the Mathematics Classroom. In P. Sullivan and O. Zavlasky (Eds.), *Constructing knowledge for teaching secondary mathematics: Tasks to enhance prospective and practicing teacher learning* (Vol. 6, pp. 227–245). New York: Springer.

Milici, P., & Di Paola, B. (2012). Geometrical–mechanical artefacts for managing tangent concept. *International Journal for Mathematics in Education*, 4, 486–492.

Poleni, J. (1729). *Epistolarum mathematicarum fasciculus*. Patavii: Typographia Seminarii.

Riccati, V. (1752). *De usu motus tractorii in constructione æquationum differentialium commentarius*. Bononiæ: Ex typographia Lælii a Vulpe.

Salvi, M., & Milici, P. (2013). Laboratorio di matematica in classe: due nuove macchine per problemi nel continuo e nel discreto. *Quaderni di Ricerca in Didattica (Mathematics)* 23, 15–24.

Tournès, D. (2009). *La construction tractionnelle des équations différentielles*. Paris: Blanchard.