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EFFECTIVE CONE OF THE BLOWUP OF THE SYMMETRIC PRODUCT OF A CURVE

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ABSTRACT. Let C be a smooth curve of genus $g \geq 1$ and let $C^{(2)}$ be its second symmetric product. In this note we prove that if C is very general, then the blowup of $C^{(2)}$ at a very general point has nonpolyhedral pseudo-effective cone. The strategy is to consider first the case of hyperelliptic curves and then to show that having polyhedral pseudo-effective cone is a closed property for families of surfaces.

INTRODUCTION

The study of the effective cone of the blow up $\tilde{S}$ of a projective surface S at a smooth point $x \in S$ is connected with the calculation of Seshadri constants. Deciding when the (pseudo)effective cone of $\tilde{S}$ is polyhedral is an open problem even when S is a toric surface. For instance, if the effective cone of the blowup of the weighted projective plane $\mathbb{P}(a,b,c)$ at a general point is not closed, then Nagata's conjecture holds for abc points in $\mathbb{P}^2$; see [4] and [8–11] for recent results on blowups of weighted projective planes. In [2] it has been shown that there exist toric surfaces whose blowup at a general point has nonpolyhedral pseudo-effective cone. This result allows one to deduce that the pseudo-effective cone of the Grothendieck–Knudsen moduli space $\bar{M}_{0,n}$ is not polyhedral for $n \geq 10$.

In this paper we focus on the second symmetric product $C^{(2)}$ of a positive genus curve C. In general, it is not known if the effective cone of these surfaces is open. This would be true if the Nagata conjecture holds, as shown in [3]. Our interest is in the blowup $\tilde{C}^{(2)}$ at a very general point $p \oplus q \in C^{(2)}$.

Theorem 1. *Let C be a very general curve of genus $g \geq 1$. Then the blowup of the symmetric product $C^{(2)}$ at a very general point has nonpolyhedral pseudo-effective cone.*

In order to prove the theorem we first show, in Proposition 1.3, that having polyhedral pseudo-effective cone is a closed property for families of surfaces and then we prove Theorem 2.

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Theorem 2. *Let C be a genus $g \geq 1$ hyperelliptic curve with hyperelliptic involution σ, let $p \in C$ and let $\tilde{C}^{(2)}$ be the blowup of $C^{(2)}$ at $p \oplus \sigma(p)$. If the class of $\sigma(p) - p$ is nontorsion in* $\operatorname{Pic}^0(C)$ *then* $\overline{\operatorname{Eff}}(\tilde{C}^{(2)})$ *is nonpolyhedral.*

When C is an elliptic curve, its symmetric product is the *Atyiah surface*. In this case in [7] it has been proved that if $q - p$ is nontorsion, then $\tilde{C}^{(2)}$ contains infinitely many negative curves. Therefore the pseudo-effective cone of $\tilde{C}^{(2)}$ is not polyhedral, and in [14] it is proved that the classes of the above mentioned curves (together with other two classes) indeed generate the pseudo-effective cone. Our main contribution here is a new description of the curves whose classes generate the pseudo-effective cone, which turn out to be hyperelliptic, as can be deduced from Theorem 4.8 and Proposition 4.9.

Our proof of Theorem 2 focuses on the quotient surface $\tilde{X}$ by the action of the hyperelliptic involution on both factors. We show that there is an irreducible curve B on $\tilde{X}$ having self intersection $B^2 = 0$, whose class spans an extremal ray of the pseudo-effective cone of $\tilde{X}$, so that the latter cannot be polyhedral by [2, Proposition 2.3]. We then apply Proposition 1.1 to the double cover $\tilde{C}^{(2)} \to \tilde{X}$ to conclude that the pseudo-effective cone of $\tilde{C}^{(2)}$ is not polyhedral.

The paper is structured as follows. In Section 1 we recall some definitions and we prove some preliminary results about the effective cone of projective surfaces. In Section 2 we study the symmetric product $C^{(2)}$ of a curve, with particular emphasis on the case C hyperelliptic. Section 3 is devoted to the proof of Theorem 1 and 2, while in Section 4 we prove some results in case $g(C) = 1$.

1. Preliminaries

Let k be an algebraically closed field of arbitrary characteristic. We recall some definitions (see, for example, [12,13]). If X is a normal projective irreducible variety over k, let $\operatorname{Cl}(X)$ be the divisor class group and let $\operatorname{Pic}(X)$ be the Picard group of X. As usual, we denote by $\sim$ the linear equivalence of divisors and by $\equiv$ the numerical equivalence. Recall that for Cartier divisors D_1, D_2, we have $D_1 \equiv D_2$ if and only if $D_1 \cdot C = D_2 \cdot C$, for any curve $C \subseteq X$. We let

$$\mathrm{N}^1(X) := \operatorname{Pic}(X)/\equiv$$

be the *Néron–Severi group*, i.e., the group of numerical equivalence classes of Cartier divisors on X. We denote by $\rho(X)$ the rank of $\mathrm{N}^1(X)$ and by $\mathrm{N}^1(X)_{\mathbb{R}} = \mathrm{N}^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$, $\mathrm{N}^1(X)_{\mathbb{Q}} = \mathrm{N}^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. We define the pseudo-effective cone

$$\overline{\operatorname{Eff}}(X) \subseteq \mathrm{N}^1(X)_{\mathbb{R}},$$

as the closure of the effective cone $\operatorname{Eff}(X)$, i.e., the convex cone generated by numerical classes of effective Cartier divisors [13, Definition 2.2.25]. We let $\operatorname{Nef}(X) \subseteq \mathrm{N}^1(X)_{\mathbb{R}}$ be the cone generated by the classes of *nef divisors*. Finally, given an ample class H, we define the *positive light cone* Q of X as follows:

$$Q := \{D \in \mathrm{N}^1(X)_{\mathbb{R}} : D^2 \geq 0 \text{ and } D \cdot H \geq 0\}.$$

Proposition 1.1. *Let $f\colon X \to Y$ be a finite surjective morphism of normal $\mathbb{Q}$-factorial projective varieties. If $\varrho(X) = \varrho(Y)$, then $f_*\colon \mathrm{N}^1(X)_{\mathbb{R}} \to \mathrm{N}^1(Y)_{\mathbb{R}}$ is an isomorphism such that $f_*(\operatorname{Eff}(X)) = \operatorname{Eff}(Y)$.*

Proof. Since Y is $\mathbb{Q}$-factorial, the image of $\mathrm{Pic}(Y)$ in the Néron–Severi group $\mathrm{N}^1(Y)$ has finite index. Over this subgroup the pullback is defined and the projection formula gives $f_* \circ f^* = n \cdot \mathrm{id}$, where $n = \deg(f)$. This, together with the hypothesis $\rho(X) = \rho(Y)$, imply that $f_*\colon \mathrm{N}^1(X)_\mathbb{R} \to \mathrm{N}^1(Y)_\mathbb{R}$ is an isomorphism whose inverse is $\frac{1}{n} f^*$. Then one concludes by the inclusions

$$f_*(\mathrm{Eff}(X)) \subseteq \mathrm{Eff}(Y) \text{ and } f^*(\mathrm{Eff}(Y)) \subseteq \mathrm{Eff}(X). \qquad \square$$

Proposition 1.2. *Let X be a normal $\mathbb{Q}$-factorial algebraic surface with $\varrho(X) \geq 3$ and positive light cone $Q \subseteq N^1(X)_\mathbb{R}$. Let $C_1, \ldots, C_n$ be irreducible curves of X. Then the following are equivalent:*

(1) $Q \subseteq \mathrm{Cone}([C_1], \ldots, [C_n])$;
(2) $\overline{\mathrm{Eff}}(X) = \mathrm{Cone}([C_1], \ldots, [C_n])$.

Moreover, if $\overline{\mathrm{Eff}}(X)$ is polyhedral then $\overline{\mathrm{Eff}}(X) = \mathrm{Eff}(X)$ holds and both cones are generated by classes of negative curves.

Proof. We prove (1) ⇒ (2). Let $\mathcal{C} \coloneqq \mathrm{Cone}([C_1], \ldots, [C_n])$. Since $\varrho(X) \geq 3$, the positive light cone Q is round, so that the extremal rays of $\mathcal{C}$ must lie outside it. In other words, $\mathcal{C}$ is generated by classes of irreducible negative curves. Now let $[D]$ be a divisor class which generates an extremal ray of the pseudo-effective cone $\overline{\mathrm{Eff}}(X)$. By [5, Lemma 6.2][1] $D^2 \leq 0$. Moreover D^2 can not be 0, since otherwise $[D] \in Q$ would not be an extremal ray of $\mathcal{C}$ and thus neither of $\overline{\mathrm{Eff}}(X)$, a contradiction. Then $D^2 < 0$ so that the hyperplane $D^\perp$ intersects Q along its interior. As a consequence, at least one of the C_i satisfies $D \cdot C_i < 0$. Thus any effective multiple of D contains C_i in its support, so that $[D] = [C_i]$ up to multiples.

The implication (2) ⇒ (1) is obvious. $\square$

Proposition 1.3. *Let $X \to B$ be a flat projective morphism of Noetherian schemes, whose very general fiber is a normal $\mathbb{Q}$-factorial surface. Assume that specialization induces an isometry within the Picard lattice of the general fiber and that of the special fiber X_0 over $0 \in B$. If the general fiber has polyhedral pseudo-effective cone, then the same holds for the special fiber.*

Proof. If the Picard rank is ≤ 2, then the pseudo-effective cone is polyhedral and there is nothing to prove. We then assume that the Picard rank is at least 3. By Proposition 1.2 the pseudo-effective cone of the general fiber is generated by finitely many classes of negative curves $C_1, \ldots, C_n$. By semicontinuity of cohomology dimension, each such curve C_i degenerates to a, possibly reducible, curve of X_0. Let $C_{i1}, \ldots, C_{ir_i}$ be the irreducible components of the degenerate curve. We claim that in the Néron–Severi space of the special fiber X_0 the following inclusions of cones hold

$$Q \subseteq \mathrm{Cone}([C_i] : 1 \leq i \leq n) \subseteq \mathrm{Cone}([C_{ij}] : 1 \leq i \leq n,\, 1 \leq j \leq r_i).$$

Indeed, by Proposition 1.2, the first inclusion holds true in the Néron–Severi space of the general fiber and, by the assumption on the Picard lattice of the special fiber, it holds as well on the Néron–Severi space of the special fiber. The second inclusion follows by the definition of the curves C_{ij}. Then, again by Proposition 1.2, one concludes that $\overline{\mathrm{Eff}}(X_0) = \mathrm{Cone}([C_{ij}] : 1 \leq i \leq n,\, 1 \leq j \leq r_i)$. $\square$

[1]The proof in [5] is for smooth surfaces but the argument works verbatim in our case.

2. Symmetric product of a curve

Given a genus $g \geq 1$ curve C, we denote by $C^{(2)}$ its second symmetric product, that is the quotient of $C \times C$ by the involution τ, defined by $(p,q) \mapsto (q,p)$, and we denote by $p \oplus q \in C^{(2)}$ the class of $(p,q) \in C \times C$.

From now on we assume that C is hyperelliptic, we fix a hyperelliptic involution σ and we denote by $p_1, \dots, p_{2g+2} \in C$ its fixed points. Observe that σ induces two commuting involutions σ_1, σ_2 on $C \times C$, each of which acts only on one coordinate. The group $G := \langle \sigma_1, \sigma_2, \tau \rangle$ is isomorphic to D_4, with center generated by the composition $\sigma_1 \cdot \sigma_2$, that we still denote by σ with abuse of notation. We have the following diagram of degree two quotient morphisms

$$\begin{array}{ccccc} C \times C & \longrightarrow & S & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 \\ \downarrow & & \downarrow & & \downarrow \\ C^{(2)} & \xrightarrow[\phi]{} & X & \xrightarrow[\psi]{} & \mathbb{P}^2, \end{array}$$

where each vertical map is the quotient by τ, the first orizontal map on each line is the quotient by σ, and the second one is the quotient by σ_1.

Remark 2.1. Let us consider the diagonal $\Delta_+ := \{p \oplus p \mid p \in C\}$ and the antidiagonal $\Delta_- := \{p \oplus \sigma(p) \mid p \in C\}$ in $C^{(2)}$. We set $C_\pm := \phi(\Delta_\pm) \subseteq X$ and $\Gamma := \psi(C_+) = \psi(C_-) \subseteq \mathbb{P}^2$. From the above diagram we see that Γ is the image of the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$ via the double cover defined by $([s_0 : s_1], [t_0 : t_1]) \mapsto [s_0 t_0 : s_1 t_1 : s_0 t_1 + s_1 t_0]$, so that it is the conic $\Gamma = V(x_3^2 - 4x_1 x_2) \subseteq \mathbb{P}^2$.

Given a point $p \in C$, consider the two curves $\{p\} \times C$ and $C \times \{p\}$ in $C \times C$. On $\mathbb{P}^1 \times \mathbb{P}^1$ they are mapped to two lines on the two different rulings, while on $C^{(2)}$ they are mapped to the curve $C_p := \{p \oplus q \mid q \in C\}$. We set $B_p := \phi(C_p) \subseteq X$ and $L_p := \psi(B_p) \subseteq \mathbb{P}^2$. Observe that B_p is isomorphic to C, while L_p is a line which is tangent to Γ at the image of $p \oplus p$ (equivalently, at the image of $\sigma(p) \oplus p$) in $\mathbb{P}^2$. We finally remark that given the curve $C_{\sigma(p)} := \{\sigma(p) \oplus q \mid q \in C\}$, we have $\phi(C_{\sigma(p)}) = \phi(C_p) = B_p \subseteq X$.

Proposition 2.2. *The surface X is a double cover of the plane, branched along the union of $2g+2$ lines, tangent to the conic Γ. It has $\binom{2g+2}{2}$ singular points, namely, the ordinary double points points $\phi(p_i \oplus p_j)$, for $1 \leq i < j \leq 2g+2$. The equation of X in the weighted projective space $\mathbb{P}(1,1,1,g+1)$ is*

$$x_4^2 + \prod_{i=1}^{2g+2} \ell_i = 0,$$

where $\ell_1, \dots, \ell_{2g+2} \in \mathbb{C}[x_1, x_2, x_3]$ are defining polynomials for the $2g+2$ lines.

Proof. The ramification divisor of $\psi : X \to \mathbb{P}^2$ consists of the images of points $(p,q) \in C \times C$ such that the orbit of (p,q) with respect to $\langle \sigma, \tau \rangle$ equals the orbit with respect to the whole group G, that is $(\sigma(p), q) \in \{(p,q), (q,p), (\sigma(p), \sigma(q)), (\sigma(q), \sigma(p))\}$. The latter condition holds if and only if either p or q is a fixed point for σ. Thus the ramification is the union of the curves $B_{p_1}, \dots, B_{p_{2g+2}}$, so that the branch divisor of ψ is the union of the $2g+2$ lines $L_{p_1}, \dots, L_{p_{2g+2}}$ which, by Remark 2.1, are tangent to Γ at the images of $p_j \oplus p_j$.

In order to get the equation given in the statement just observe that a double cover of $\mathbb{P}^2$ branched along a curve $V(g(x_1, x_2, x_3)) \subseteq \mathbb{P}^2$ of degree $2d$ can be described as the hypersurface $V(x_4^2+g(x_1,x_2,x_3))$ in the weighted projective space $\mathbb{P}(1,1,1,d)$. $\square$

Remark 2.3. Since X is a hypersurface of a weighted projective space, by [6, Theorem 4.2.2] we have that $q(X)=0$. In particular, X is a weak del Pezzo of degree 2 if $g=1$, it is a singular $K3$ when $g=2$, and it is of general type when $g\ge 3$.

Proposition 2.4. *Assume that C is a very general hyperelliptic curve of genus $g\ge 1$. Then both $C^{(2)}$ and X have Picard rank 2 and their effective cones are generated by the classes of the images of the diagonal and the antidiagonal. The intersection matrices of these curves in $C^{(2)}$ and of their images in X are*

$$\begin{pmatrix}4-4g & 2g+2\\ 2g+2 & 1-g\end{pmatrix} \qquad \textit{and} \qquad \begin{pmatrix}2-2g & 2g+2\\ 2g+2 & 2-2g\end{pmatrix},$$

respectively.

Proof. By [1, Chapter VIII, §5], $\mathrm{N}^1(C^{(2)})\simeq \mathbb{Z}\oplus \mathrm{N}^1(JC)$, so that [16, Proposition 3.4] implies that the Picard rank of $C^{(2)}$ is 2. As a consequence, the Picard rank of X is at most 2 and it is 2 because $N^1(X)$ contains two numerically independent classes. The diagonal Δ_+ and the antidiagonal Δ_- are both mapped to the conic Γ of $\mathbb{P}^2$, tangent to the $2g+2$ lines. Thus $C_+ + C_- = \phi(\Delta_+)+\phi(\Delta_-)$ is the pullback of Γ, so that $(C_+ + C_-)^2 = 8$. Since these two curves are numerically equivalent and intersect in $2g+2$ points, we obtain the second matrix. To get the first matrix it is enough to observe that the double cover $C^{(2)}\to X$ branches at C_-, which is the image of the antidiagonal. $\square$

3. Proof of Theorems 2 and 1

Proof of Theorem 2. Let us fix a point $p\in C$, such that the class of $\sigma(p)-p$ is nontorsion, and let $\tilde{C}^{(2)}\to C^{(2)}$ be the blowup at the point $p\oplus\sigma(p)\in\Delta_-\cap C_{\sigma(p)}$, with exceptional divisor E. First of all, observe that the point $p\oplus\sigma(p)$ is invariant for σ, so that the latter lifts to an involution on the blowup $\tilde{C}^{(2)}$ that, by abuse of notation, we denote by the same symbol σ. Let $\tilde{\phi}\colon\tilde{C}^{(2)}\to\tilde{X}:=\tilde{C}^{(2)}/\langle\sigma\rangle$ be the quotient morphism. The involution σ has two fixed points on the exceptional divisor E: the intersection point with the strict transform of Δ_-, and one isolated point x, so that $\tilde{\phi}(x)$ is a singular point of $\tilde{X}$. We have a birational morphism $\eta\colon\tilde{X}\to X$ which is the contraction of $\tilde{\phi}(E)$, having self-intersection $-1/2$ in $\tilde{X}$. The map η is a weighted blowup at the point $\phi(\sigma(p)\oplus p)$ and can also be described as follows.

Consider the blowup $X_1\to X$ at the point $\phi(p\oplus\sigma(p))$, with exceptional divisor E_1, and then the blowup $X_2\to X_1$, at the intersection point of E_1 with the strict transform of $B_p=\phi(C_p)=\phi(C_{\sigma(p)})$ (see Remark 2.1). Finally, contract the strict transform of E_1, which is now a (-2)-curve (its image gives the singular point $\tilde{\phi}(x)\in\tilde{X}$). We can summarize the above discussion in the following commutative diagrams:

$$\begin{array}{ccccc} \tilde{C}^{(2)} & \xrightarrow{\tilde{\phi}} & \tilde{X} & & \\ \downarrow & & \downarrow{\scriptstyle \eta} & & \\ C^{(2)} & \xrightarrow[\phi]{} & X & \xrightarrow[\psi]{} & \mathbb{P}^2 \end{array} \qquad\qquad \begin{array}{ccc} X_2 & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow{\scriptstyle \eta} \\ X_1 & \longrightarrow & X. \end{array}$$

We are going to show that the pseudo-effective cone of $\tilde{X}$ is not polyhedral. Observe that the strict transform of B_p in $\tilde{X}$ is isomorphic to B_p and hence to $C_{\sigma(p)}$ and to C. Therefore, by abuse of notation, we denote this strict transform by C. Since $\psi(B_p) \subseteq \mathbb{P}^2$ is the line L_p, tangent to the conic Γ, we have that $B_p^2 = 2$. By the description of $\eta: \tilde{X} \to X$, we are blowing up a point on B_p and then the same point on its strict transform. Therefore $C^2 = 0$, and we can write

$$C \sim (\psi \circ \eta)^*(L_p) - 2\tilde{\phi}(E). \tag{1}$$

Let us compute now the restriction $\mathcal{O}_C(C)$. Since L_p is a line in $\mathbb{P}^2$, the restriction of $(\psi \circ \eta)^*(L_p)$ to C is the g_2^1, so that it is equivalent to $\sigma(p) + p$. On the other hand, the restriction of $\tilde{\phi}(E)$ to C corresponds to the point we are blowing up in $\eta: \tilde{X} \to X$, i.e., to the image of $\sigma(p) \oplus p$ in X. Via the isomorphism $C_{\sigma(p)} \to C$, the point $\sigma(p) \oplus p$ corresponds to $p \in C$, so that we conclude that the restriction of $\tilde{\phi}(E)$ to C is p. Summing up, we obtain

$$\mathcal{O}_C(C) \simeq \mathcal{O}_C(\sigma(p) + p - 2p) = \mathcal{O}_C(\sigma(p) - p).$$

Since we are assuming that $\sigma(p) - p$ is nontorsion, we deduce that $\mathcal{O}_C(C)$ is nontorsion, so that $h^0(\mathcal{O}_C(nC)) = 0$ for any positive integer n. Moreover, $h^0(\mathcal{O}(C)) = 1$ implies $h^0(\mathcal{O}(nC)) = 1$ for any positive integer n. Since $q(\tilde{X}) = 0$, numerical and linear equivalence coincide, up to finite multiple. Thus the previous argument implies that the class of C spans an extremal ray of $\overline{\mathrm{Eff}}(\tilde{X})$. By [2, Proposition 2.3] we conclude that $\overline{\mathrm{Eff}}(\tilde{X})$ is not polyhedral, and by Proposition 1.1 the cone $\overline{\mathrm{Eff}}(\tilde{C}^{(2)})$ is not polyhedral either. □

Remark 3.1. In genus 2 the Abel–Jacobi map presents $C^{(2)}$ as the blowup of $\mathrm{Pic}^2 C$ in the point Ω that corresponds to the canonical class K_C of C, with exceptional divisor $\Delta_- \subseteq C^{(2)}$. So in this case we blow up $\mathrm{Pic}^2 C$ twice infinitely near at Ω. The map $C \to \mathrm{Pic}^2 C$ given by $x \mapsto [x + \sigma(p)]$ embeds C as a theta-divisor passing through Ω and with tangent direction $p + \sigma(p)$. So after the blowup the proper transform of C is a curve of self-intersection 0. The restriction of C to C will be $K_C - 2p$ (because we blow up the same point p of C twice), so it is $\sigma(p) - p$ as we claim in the proof of the general statement.

Remark 3.2. Assume that $\sigma(p) - p$ is not torsion, so that the pseudo-effective cone $\overline{\mathrm{Eff}}(\tilde{X})$ is not polyhedral. If $g = 1$ we are going to show that on $\tilde{X}$ there are infinitely many negative rays accumulating on C (see Proposition 4.9 and Figure 3). If $g > 1$, consider the intersection matrix of the classes $\Delta_+, \Delta_-, E_1, C, E$ on $\tilde{X}$:

$$\begin{pmatrix} 2-2g & 2+2g & 1 & 2 & 0 \\ 2+2g & -2g & 1 & 0 & 1 \\ 1 & 1 & \frac{1}{2} & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -\frac{1}{2} \end{pmatrix}.$$

We already know that C generates an extremal ray of $\overline{\mathrm{Eff}}(\tilde{X})$, and the same holds for the classes Δ_+, Δ_- and E, since they have negative self-intersection. In particular, $\overline{\mathrm{Eff}}(\tilde{X})$ (and hence also $\overline{\mathrm{Eff}}(\tilde{C}^{(2)})$) has a polyhedral part (see Figure 1).

Question 3.3. When C is hyperelliptic of genus $g > 1$, does $\tilde{C}^{(2)}$ have infinitely many negative curves?

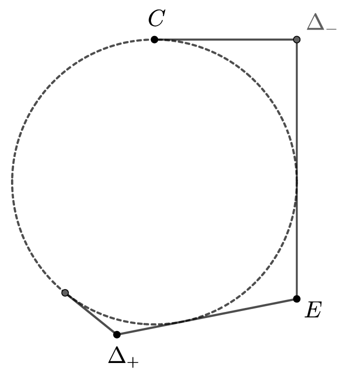

FIGURE 1. $\operatorname{Eff}(\tilde{X})$, when $g > 1$

Proof of Theorem 1. Let $\pi: \mathcal{X} \to B$ be a flat family whose general fiber is a general genus g curve C and whose special fiber over $0 \in B$ is a general hyperelliptic curve C_0. Passing to the symmetric product one gets a new flat family with basis B. Blowing up a section of the new family which cuts out $p \oplus \sigma(p)$ on $C_0^{(2)}$, with $\sigma(p) - p$ nontorsion, one concludes, by Theorem 2 and Proposition 1.3, that the pseudo-effective cone of the blowup $\tilde{C}^{(2)}$ is nonpolyhedral. □

We remark that when C and the point that we are blowing up are general, even if we know that $\overline{\operatorname{Eff}}(\tilde{C}^{(2)})$ is not polyhedral, we do not know any negative class. Therefore it is natural to ask Question 3.4.

Question 3.4. When C is general, does $\tilde{C}^{(2)}$ have infinitely many negative curves?

4. The genus one case

In this section we make the assumption that C has genus 1. In particular, we first show that in Theorem 1 the opposite implication also holds (see Theorem 4.8). Then we describe the rays of the pseudo-effective cone of $\tilde{X}$, both when it is polyhedral and when it is not (Proposition 4.9), and finally we give a planar model for the resolution Z of $\tilde{X}$.

Remark 4.1. When $g(C) = 1$, the symmetric product $C^{(2)}$ is a ruled surface whose fibers correspond to the g_1^2's of C. Observe that if we fix two points $p \neq q \in C$, they define a unique g_1^2, and hence a hyperelliptic involution σ. This implies that the antidiagonal $\Delta_- = \{r \oplus \sigma(r) \mid r \in C\}$ is indeed a fiber.

Let us recall Definition 4.2 from [2, §3].

Definition 4.2. An *elliptic pair* (C, X) consists of a projective rational surface X with log terminal singularities and an irreducible curve $C \subseteq X$, with arithmetic genus one, disjoint from the singular locus of X and such that $C^2 = 0$.

The elliptic pair (C, X) is a *minimal elliptic pair* if it does not contain irreducible curves E such that $K \cdot E < 0$ and $C \cdot E = 0$.

Consider the blowing-up $\tilde{C}^{(2)} \to C^{(2)}$ at the point $p \oplus q \in \Delta^-$, where $p \neq q$, or equivalently, at $p \oplus \sigma(p)$, where σ is the involution exchanging p and q. We denote by E the exceptional divisor and by $\tilde{C}_p \subseteq \tilde{C}^{(2)}$ the strict transform of the curve $C_p := \{p \oplus r \mid r \in C\} \subseteq C^{(2)}$. The involution σ induces an involution on $\tilde{C}^{(2)}$, that we still denote by σ, whose ramification is the strict transform $\tilde{\Delta}_- \subseteq \tilde{C}^{(2)}$ of Δ_-. We denote by $\tilde{\phi}: \tilde{C}^{(2)} \to \tilde{X} := \tilde{C}^{(2)}/\langle\sigma\rangle$ the quotient morphism. Since the curve $\tilde{\phi}(\tilde{C}_p)$ is isomorphic to C_p and hence to the curve C, by abuse of notation in what follows we will simply set $C := \tilde{\phi}(\tilde{C}_p)$.

Lemma 4.3. *The pair $(C, \tilde{X})$ is a minimal elliptic pair.*

Proof. The rationality of $\tilde{X}$ follows from Remark 2.3. From Proposition 2.2 we have that X has 6 ordinary double points, and none of them lies on $B_p := \phi(C_p) = \phi(C_q)$. Therefore they give rise to 6 ordinary double points of $\tilde{X}$, disjoint from the curve C. Moreover the involution on $\tilde{C}^{(2)}$ has 2 fixed points on E, but only one of them is isolated. Its image is the seventh ordinary double point of $\tilde{X}$ (which does not lie on C). This proves that $(C, \tilde{X})$ is an elliptic pair. By Proposition 4.7 we can compute $K_{\tilde{X}}^2 = 0$, so that by [2, Lemma 3.7] we conclude that $(C, \tilde{X})$ is minimal. □

Remark 4.4. Let us consider a minimal resolution $\pi: Z \to \tilde{X}$. Since $C \subseteq \tilde{X}$ does not pass through the singular points, we have an isomorphic copy of C in Z, that we still denote by C. Therefore (C, Z) is a smooth minimal elliptic pair and in particular, by [2, Theorem 3.8], the Picard rank of Z is 10.

Notation 4.5. Before stating our next results about $\tilde{X}$ and Z we need to fix some notation. First of all, we are going to denote by $L_i \subseteq \mathbb{P}^2$, $1 \leq i \leq 4$ the lines whose union is the branch locus of $X \to \mathbb{P}^2$, and by $E_i \subseteq \tilde{X}$ and $\bar{E}_i \subseteq Z$ the strict transforms of L_i on $\tilde{X}$ and Z, respectively. By abuse of notation we denote by E the image $\tilde{\varphi}(E) \subseteq \tilde{X}$ and by $\bar{E} \subseteq Z$ its strict transform. Analogously we denote simply Δ_- the curve $\tilde{\phi}(\tilde{\Delta}_-) \subseteq \tilde{X}$ and by $\bar{\Delta}_-$ its strict transform in Z. For any $1 \leq i < j \leq 4$, we denote by $\bar{E}_{ij} \subseteq Z$ the (-2)-curve over the singular point $p_{ij} := L_i \cap L_j \in \mathbb{P}^2$, while the (-2)-curve over the isolated singular point $\tilde{\phi}(x) \in \tilde{\phi}(E) \in \tilde{X}$ is denoted by $\bar{E}'$. Finally, for any (i, j, k) in $\{(1,2,3), (1,3,2), (2,3,1)\}$, we denote by $L_{(ij)(k4)} \subseteq \mathbb{P}^2$ the line through p_{ij} and p_{k4}. Observe that over any of these $L_{(ij)(k4)}$ we have two irreducible curves in X, say,

$$E_{(ij)(k4)} \text{ and } E'_{(ij)(k4)}.$$

We use the same notation for their strict transforms on $\tilde{X}$, while we denote by $\bar{E}_{(ij)(k4)}$ and $\bar{E}'_{(ij)(k4)}$ their strict transforms on Z.

We now recall that $\Gamma = V(x_3^2 - 4x_1x_2) \subseteq \mathbb{P}^2$, so that $\Delta_- = V(x_4 + 2x_1x_3 - 2x_2x_3)$ (it corresponds to one of the two irreducible components over Γ). Moreover, we can fix the tangent lines $L_1, \dots, L_4$ to be $V(x_1)$, $V(x_2)$, $V(x_1 + x_2 - x_3)$ and $V(x_1 + x_2 + x_3)$ respectively. Then, by Proposition 2.2,

$$X = V(x_4^2 - x_1x_2(x_1 + x_2 - x_3)(x_1 + x_2 + x_3)) \subseteq \mathbb{P}(1,1,1,2),$$

and $L_{(12)(34)} = V(x_1 + x_2)$, $L_{(13)(24)} = V(x_1 - x_2 + x_3)$, $L_{(23)(14)} = V(x_1 - x_2 - x_3)$. From these equations one can see that the 6 curves $E_{(ij)(kl)}$ and $E'_{(ij)(kl)}$ form a

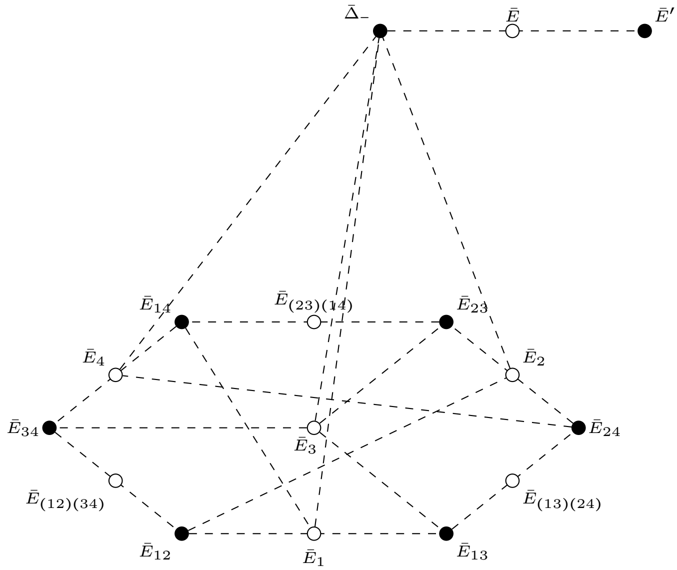

FIGURE 2. Intersection graph on Z

hexagon, and we can choose the labels in order to have

$$E_{(12)(34)} = V(x_1 + x_2, x_4 + x_2x_3),$$
$$E_{(13)(24)} = V(x_1 - x_2 + x_3, x_4 + 2x_2^2 - 2x_2x_3),$$
$$E_{(23)(14)} = V(x_1 - x_2 - x_3, x_4 - 2x_2^2 - 2x_2x_3).$$

It is now straightforward to check that these 3 curves are disjoint and do not meet Δ_-, so that the same holds for the strict transforms on $\tilde{X}$ and on Z (analogously, $E'_{(12)(34)}$, $E'_{(13)(24)}$ and $E'_{(23)(14)}$ are disjoint and do not meet Δ_+).

In Figure 2 we represent the intersection products of the negative curves described before. The black dots are the (-2)-curves while the white dots are the (-1)-curves. When two dots are connected, the two corresponding curves have intersection product 1, otherwise their product is 0.

Remark 4.6. The lattice $C^\perp$ in $\mathrm{Pic}(Z)$ is isomorphic to $\tilde{\mathbb{E}}_8$. Since the eight (-2)-curves described above are all disjoint, their classes span the sublattice $\mathbb{A}_1^8 \subseteq \tilde{\mathbb{E}}_8$.

Proposition 4.7. *On $\tilde{X}$ the following hold.*

(1) $\mathrm{Cl}(\tilde{X}) \simeq \mathbb{Z}^3 \oplus (\mathbb{Z}/2\mathbb{Z})^2$.
(2) $\mathrm{Cl}(\tilde{X})_{\mathrm{free}}$ *is generated by* E_1, $E_{(12)(34)}$ *and* E, *which have the following intersection matrix*

$$\begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}.$$

(3) $C \sim -K_{\tilde{X}} \sim 2E_1 - 2E$.
(4) $\Delta_- \sim 2E_{(12)(34)} - 2E$.

Proof. We prove (1). First of all, observe that the Picard rank of $\tilde{X}$ is 3 because we are contracting seven (-2)-curves of Z, which has Picard rank 10 (see Remark 4.4). Moreover the torsion part is of the form $(\mathbb{Z}/2\mathbb{Z})^s$, for some $0 \le s \le 7$, because the singularities of $\tilde{X}$ are ordinary double points. A basis for the Picard group of Z consists of the classes of the following curves: $\bar{E}_{12}$, $\bar{E}_{13}$, $\bar{E}_{14}$, $\bar{E}_1$, $\bar{E}_2$, $\bar{E}_3$, $\bar{E}_4$, $\bar{E}_{(12)(34)}$, $\bar{E}'$, $\bar{E}$ because the corresponding intersection matrix is unimodular of rank 10. This implies that $\mathrm{Cl}(\tilde{X})$ is generated by their images, i.e., E_1, E_2, E_3, E_4, $E_{(12)(34)}$, E (recall Notation 4.5). Observe that for any $i \neq j$, the class of $E_i - E_j$ is 2-torsion because $2E_i$ is pullback of a line of $\mathbb{P}^2$. Since the class of $E_1 + E_2 + E_3 + E_4$ is linearly equivalent to the pullback of a conic, it is divisible by 2, so that also the class of

$$(E_2 - E_1) + (E_2 - E_3) + (E_2 - E_4) = 4E_2 - (E_1 + E_2 + E_3 + E_4)$$

is divisible by 2, and in particular it is trivial. Therefore the class of E_4 is not needed to generate $\mathrm{Cl}(\tilde{X})$ and thus $s \le 2$. On the other hand $E_1 - E_2 \neq E_1 - E_3$ so that $s = 2$.

We prove (2). Since E is disjoint from E_1 and $E_{(12)(34)}$, we have $E_1 \cdot E = E_{(12)(34)} \cdot E = 0$. The self-intersection of E_1 is $1/2$ because $2E_1$ is the pullback of a line. The self-intersection of $E_{(12)(34)}$ is 0 because its pullback in Z is $1/2E_{12} + E_{(12)(34)} + 1/2E_{34}$, which has self-intersection 0. Similarly one shows that $E^2 = -1/2$ and that $E_{(12)(34)} \cdot E_1 = 1/2$.

We prove (3). The equivalence $C \sim 2E_1 - 2E$ follows from equation (1), since $L_p \subseteq \mathbb{P}^2$ is a line tangent to Γ. By the ramification formula,

$$K_X = \psi^* K_{\mathbb{P}^2} + R = \psi^*(-3L) + \psi^*(2L) = -\psi^*(L) = -2\eta(E_1).$$

Recall that the map $\eta\colon \tilde{X} \to X$ is obtained by blowing up twice (one time on the exceptional divisor) and then contracting the (-2)-curve. The contraction is crepant so that it does not affect the canonical class. From this we conclude that

$$K_{\tilde{X}} = \eta^* K_X + 2E = -2E_1 + 2E.$$

In order to prove (4) observe that on Z the divisors $2\bar{E}_{(12)(34)} + \bar{E}_{34} + \bar{E}_{12}$ and $\bar{\Delta}_- + 2\bar{E} + \bar{E}'$ have both self-intersection 0 and their intersection product is 0. By the Hodge index theorem it follows that the classes of these divisors must be proportional. Since both classes have intersection product 1 with some curve, they are primitive in $\mathrm{Pic}(Z)$. It follows that the two classes are equal, and one concludes by taking pushforward of these classes via $\pi\colon Z \to \tilde{X}$. □

Theorem 4.8. *With the notation above, the following are equivalent:*

(1) $\mathrm{Eff}(\tilde{C}^{(2)})$ *is rational polyhedral;*
(2) $\mathrm{Eff}(\tilde{X})$ *is rational polyhedral;*
(3) $\mathrm{Eff}(Z)$ *is rational polyhedral;*
(4) *the class of* $q - p$ *has order* $m < \infty$ *in* $\mathrm{Pic}^0(C)$*;*
(5) $\dim|-mK_Z| = 1$ *and* $\dim|-rK_Z| = 0$*, for* $0 \le r < m$*.*

Proof. By Proposition 2.4, $\rho(\tilde{X}) = \rho(\tilde{C}^2)$, so that the equivalence (1) ⇔ (2) follows from Proposition 1.1. Since $\tilde{X}$ has Du Val singularities, the equivalence (2) ⇔ (3) was proved in [2, Lemma 3.14]. We now prove the equivalence of (3) and (4). Since (C, Z) is an elliptic pair, by [2, §3], the effective cone of Z is rational polyhedral if and only if $C^\perp$ is generated by the kernel of $\mathrm{res}\colon C^\perp \to \mathrm{Pic}^0(C)$. In Remark 4.4 we have already seen that there are eight disjoint (-2)-curves in ker(res). Thus $C^\perp$ is spanned by elements of ker(res) if and only if there exists an integer $m > 1$ such that the multiple mC is in ker(res), that is if res(C) is of m-torsion. We conclude by observing that res$(C) = q - p$ (see the proof of Theorem 2).

Finally, from Proposition 4.7 $C \sim -K_{\tilde{X}}$, and since the curve C is disjoint from the singular points, also on Z we have $C \sim -K_Z$. The equivalence (4) $\Leftrightarrow$ (5) follows. □

We are now going to describe the extremal rays of $\mathrm{Eff}(\tilde{X})$, both when it is polyhedral and when it is not. We remark that by Proposition 1.1 we can identify $\mathrm{Eff}(\tilde{C}^{(2)})$ with $\mathrm{Eff}(\tilde{X})$.

Proposition 4.9. *Let $m > 1$ be the order of $q - p$ and let us consider the following classes of* $\mathrm{Cl}(\tilde{X})$*:*

$$D_n := 2n(n+1)E_1 - 2nE_{(12)(34)} + (1 - 2n^2)E, \quad n \in \mathbb{Z}_{\geq 0}.$$

Then the following hold:

(1) *If $m = \infty$, then* $\mathrm{Eff}(\tilde{X}) = \langle -K_{\tilde{X}}, \Delta_-, D_0, D_1, \dots, D_n, \dots \rangle$.

(2) *If $m < \infty$, then* $\mathrm{Eff}(\tilde{X}) = \langle -K_{\tilde{X}}, \Delta_-, D_0, D_1, \dots, D_{\lceil \frac{m}{2} \rceil - 1}, \Gamma_m \rangle$*, where*

$$\Gamma_m := mE_1 - E_{(12)(34)} + (1 - m)E.$$

(3) *Each D_n in* (1) *and* (2) *is the class of an irreducible rational curve of $\tilde{X}$.*

Proof. (1) If $m = \infty$, we already know that the effective cone of $\tilde{X}$ is not polyhedral. A direct calculation shows that $D_n^2 = -1/2$ and $D_n \cdot K_{\tilde{X}} = -1$, for any $n \geq 0$. The divisors $2E_1$ and $2E_{(12)(34)}$ are Cartier, while E is not Cartier. Since $\bar{E}'$ is the only (-2)-curve intersecting $\bar{E}$ and contracted by $\pi \colon Z \to \tilde{X}$, it follows that

$$R_n := \lfloor \pi^* D_n \rfloor = \pi^* D_n - \frac{1}{2}\bar{E}'$$

is a divisor with integer coefficients. Since $R_n^2 = R_n \cdot K_Z = -1$, by Riemann–Roch we conclude that R_n is linearly equivalent to an effective divisor. Moreover each R_n has nonnegative intersection product with all the (-2)-curves since $R_n \cdot \bar{E}_{ij} = 0$, $R_n \cdot \bar{E}' = 1$ and $R_n \cdot \bar{\Delta}_- = 2n + 1$. We claim that R_n is irreducible. Suppose that we can write $R_n = C_1 + N$, where C_1 is an irreducible (-1)-curve and N is a sum of (-2)-curves. The condition $R_n^2 = -1$ implies that either $R_n \cdot C_1 < 0$ or $R_n \cdot N < 0$, but the latter would imply that R_n has negative intersection with at least one (-2)-curve, a contradiction. Therefore $(C_1 + N) \cdot C_1 < 0$, so that $N \cdot C_1 = 0$, which implies that also $N^2 = 0$. Since the intersection form is negative semidefinite on the components of N, we deduce that N is indeed a multiple of $-K$. Therefore $R_n = C_1 - tK$, which gives $R_n^2 > -1$, again a contradiction. This proves the claim, and since $D_n = \pi_*(R_n)$, it is irreducible too.

Consider now the cone $\mathcal{C}$, generated by $-K_{\tilde{X}}$, Δ_- and D_n, for $n \geq 0$. The following matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}, \qquad \begin{pmatrix} -2 & 1 \\ 1 & -\frac{1}{2} \end{pmatrix}, \qquad \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

give the intersection form on the edges $\langle -K_{\tilde{X}}, \Delta_- \rangle$, $\langle \Delta_-, D_0 \rangle$ and $\langle D_n, D_{n+1} \rangle$ (for any $n \geq 0$) respectively. Since they are all negative semidefinite and the rays D_n accumulate on $-K_{\tilde{X}}$, we conclude that $\mathcal{C} = \mathrm{Eff}(\tilde{X})$, which proves (1).

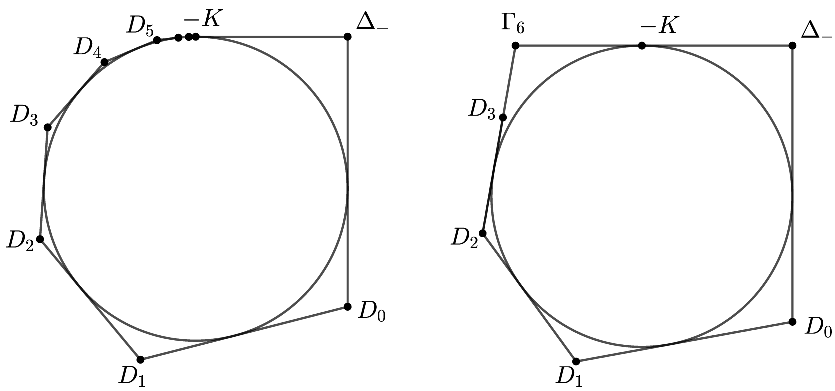

FIGURE 3. Eff $\tilde{X}$ for $m = \infty$ and $m = 6$

Let us prove (2). Observe that if $m < \infty$, then $|-mK_Z|$ defines an elliptic fibration which is *extremal* in the sense of Miranda–Persson [15]. According to [15, Theorem 4.1] the only extremal rational elliptic surface which contains eight disjoint (-2)-curves is $X_{11}(j)$, which has exactly two singular fibers of type I_0^*. Thus, as soon as $-mK_Z$ moves, two new (-2)-curves appear, each of which is the unique curve of multiplicity two in the fiber I_0^*. On $\tilde{X}$ one of these curves is disjoint from Δ_-, so that its self-intersection is 0, while the other one intersects Δ_- and has self-intersection $-1/2$. The class of the latter is

$$\Gamma_m := \frac{1}{2}(-mK_{\tilde{X}} - \Delta_-) = mE_1 - E_{(12)(34)} + (1-m)E,$$

and by the intersection matrix given in Proposition 4.7 we have that $\Gamma_m \cdot D_n = 1/2(m-1) - n$, which is nonnegative if and only if $n < \lceil \frac{m}{2} \rceil$.

Let us prove (3). Since $\tilde{X}$ has only rational double points, it follows that the resolution $Z \to \tilde{X}$ is crepant. Thus we have $-1 = K_{\tilde{X}} \cdot D_n = K_Z \cdot \tilde{D}_n$, where $\tilde{D}_n$ is the strict transform of D_n on Z. Being Z an anticanonical rational surface it follows that $\tilde{D}_n$ is the class of a (-1)-curve of Z. □

Remark 4.10. We remark that the irreducible negative curves on $\tilde{C}^{(2)}$, first found in [7], are *unexpected*, meaning with this that the expected dimension of the linear system is negative. The images of these curves in $\tilde{X}$ are the curves D_n of Proposition 4.9. They are still negative, of self-intersection $-\frac{1}{2}$, but we have seen that the round-down of the pullback $R_n = \lfloor \pi^* D_n \rfloor$ on Z is a (-1)-curve, and, in particular, it is expected.

Remark 4.11. Looking at Figure 2 we see that on Z there are 8 disjoint (-1)-curves, namely $\bar{E}_1, \dots, \bar{E}_4$, $\bar{E}_{(12)(34)}$, $\bar{E}_{(13)(24)}$, $\bar{E}_{(23)(14)}$, $\bar{E}$. If we contract all of them, $\bar{E}'$ becomes a (-1)-curve as well and if we contract also this one, we obtain a birational map $Z \to \mathbb{P}^2$. We denote by q_i, $q_{(ij)(kl)}$ and q the images of $\bar{E}_i$, $\bar{E}_{(ij)(kl)}$ and $\bar{E}'$ respectively. The image of $\bar{E}_{12}$ is a line l_{12}, passing through q_1, q_2 and $q_{(12)(34)}$, and anagolously l_{13} and l_{23}. The image of $\bar{\Delta}_-$ is a conic passing through $q_1, \dots, q_4$ and q.

Summarizing, we can describe Z as the blowup of $\mathbb{P}^2$ at a 0-dimensional scheme of length 9 that can be described as follows. We start from 4 general points $q_1, \dots, q_4$.

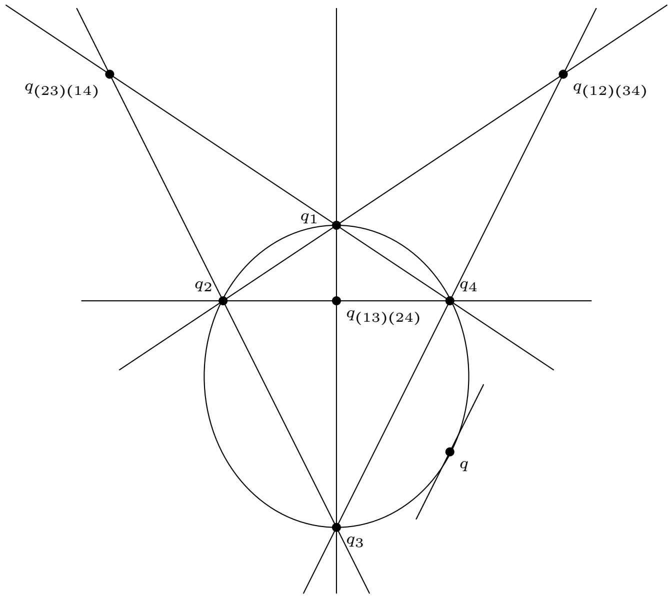

FIGURE 4. Points to blow up on $\mathbb{P}^2$ to obtain the surface Z.

We add the 3 points $q_{(ij)(kl)}$, each of which is the intersection of the pair of lines $l_{ij} = \langle q_i, q_j \rangle$ and $l_{kl} = \langle q_k, q_l \rangle$. Finally we fix a conic through $q_1, \ldots, q_4$ and we take a point q on it, together with the tangent direction to the conic at q (see Figure 4).

Therefore, for any $n > 0$, the curve D_n appearing in Proposition 4.9 corresponds to a plane curve. We can compute that its degree is $3n^2 + n$, intersecting $R_n = \pi^* D_n - \frac{1}{2}\bar{E}'$ (see the proof of Proposition 4.9) with the pullback of E_{12}, i.e., the class $\bar{E}_{12} + \bar{E}_1 + \bar{E}_2 + \bar{E}_{(12)(34)}$ in Z. In the same way we see that the multiplicity in q_1 is n^2, by taking the intersection with the pullback of E_1, namely $\bar{E}_1 + 1/2(\bar{E}_{12} + \bar{E}_{13} + \bar{E}_{14})$, and the same holds for q_2, q_3, q_4. Computing the intersection with $\bar{E}_{(12)(34)} + 1/2(\bar{E}_{12} + \bar{E}_{34})$ we have that the multiplicity in $q_{(12)(34)}$ is $n^2 + n$, and the same holds for $q_{(13)(24)}$ and $q_{(23)(14)}$. Finally, the curve has multiplicity n^2 at q, and multiplicity $n^2 - 1$ at the point infinitely near to q, in the direction of the conic.

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