PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Special Issue *ATMA2021 - Approximation: Theory, Methods, and Applications*, Volume 15 · 2022 · Pages 1–7

# Two positive solutions for a nonlinear Robin problem involving the discrete $p$—Laplacian

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**Abstract**

In this paper, combining variational methods and truncations techniques, the existence of at least two positive solutions for a nonlinear difference equation involving the discrete $p$-Laplacian with Robin boundary conditions is established.

## 1 Introduction

This paper deals with the following nonlinear discrete Robin boundary value problem

$$\begin{cases} -\Delta(\phi_p(\Delta u(k-1)))+q(k)\phi_p(u(k))=\lambda f_k(u(k)), & k\in[1,N],\\ u(0)=\Delta u(N)=0, \end{cases} \tag{$R_{\lambda,\underline{f}}$}$$

where $\lambda$ is a positive parameter, $N$ is a fixed positive integer, $[1,N]$ represents the discrete interval $\{1,...,N\}$, $q:[1,N]\to\mathbb{R}$, with $q(k)\geq 0$ for all $k\in[1,N]$, $f_k:\mathbb{R}\to\mathbb{R}$ are continuous functions, $\Delta u(k-1):=u(k)-u(k-1)$ and $-\Delta(\phi_p(\Delta u(k-1)))$ denote the forward difference and the discrete $p$—Laplacian operators where $\phi_p(s):=|s|^{p-2}s$, $1<p<+\infty$ , respectively, for all $k\in[1,N+1]$.

The functions $f_k$ are not, in general, the restrictions of a unique function of type $f:[1,N]\times\mathbb{R}\to\mathbb{R}$. Therefore the given problem turns out to be more general than the analogous continuous case, in which the nonlinear term is usually a function $f:[a,b]\times\mathbb{R}\to\mathbb{R}$. More information regarding the link between discrete problems and nonlinear differential boundary value problems can be found in [4, 23, 25, 31, 33, 34, 35]. For a general overview on difference equations and their applications, we mention [1, 28].

As one can see in [1, 3, 6, 9, 10, 26, 32] and the references therein, there is a big literature dedicated to difference equations with Dirichlet or Neumann boundary conditions. Instead, as far as we know, only few papers deal with problem of type ($R_{\lambda,\underline{f}}$). For instance, whenever $p=2$ and $q(k)=0$ for all $k\in[1,N]$, the existence of sign-changing and constant-sign solutions is established in [30], by applying invariant sets of descending flow and variational methods. While, when the parameter belongs to appropriate intervals, the existence of non-negative solutions is showed in [37], for problems with sign-changing weight reaction terms and by using the iterative method and Schauder's fixed point theorem. In [29], the critical point theory is also applied to get infinitely many positive solutions to ($R_{\lambda,\underline{f}}$).

The study of the existence and multiplicity of solutions for parameter-dependent discrete problems, approached through critical point theory, has been of great interest (see [11, 12, 13, 17, 18, 19, 21, 24, 36]). In particular, in [14] some classical results of variational methods have been suitably rewritten by taking full advantage of the characteristics of finite dimensional Banach spaces in order to obtain new and better performing results for discrete problems. More precisely, the existence of two solutions for nonlinear Dirichlet problem with the discrete $p$—Laplacian has been obtained in [22], see also [16] for $p=2$. Moreover, multiplicity results have been obtained in [8] for a Neuman problem and in [20] for a nonlinear algebraic systems. Such results are chiefly based on a recent result of Bonanno and D'Aguì [7] (Theorem 2.2), which gives the existence of at least two non trivial critical points for a certain class of functionals defined on a Banach space.

The paper is organized in the following way. Section 2 is devoted to the variational framework in order to study problem ($R_{\lambda,\underline{f}}$). In particular, two preliminary results of independent interest are given. One is Lemma 2.3 which gives a quantitative estimate on the interval of parameters such that the energy functional associated to our problem satisfies the Palais-Smale condition and is unbounded from below. In particular, it is proved, as in the above mentioned papers, but with a simpler proof, that in a finite dimensional Banach space the $p$—superlinearity at infinity of the reaction term is enough to our aim, without involving the so called Ambrosetti-Rabinowitz condition. The second one (Lemma 2.4) is dedicated to clarify the interaction between the variational and the truncation methods allowing us to obtain positive solutions without resorting to the strong discrete maximum principle. Section 3 is dedicated to the main results, where Theorem 3.1 gives, in a more general form, the existence of two positive solutions for the problem ($R_{\lambda,\underline{f}}$) is obtained. Finally, some corollaries and an example follow in order to show the applicability of the treatment.

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## 2 Mathematical Background

The variational framework in order to study problem ($R_{\lambda,\underline{f}}$) is the following finite dimensional Banach space

$$X = \{u : [0, N+1] \to \mathbb{R} : u(0) = \Delta u(N) = 0\},$$

endowed with the perturbed difference norm given by

$$\|u\| := \left( \sum_{k=1}^{N} |\Delta u(k-1)|^p + \sum_{k=1}^{N} q(k)|u(k)|^p \right)^{1/p} \quad \forall u \in X.$$

In the sequel, we will also need both the maximum norm and the classical Hölder norm, namely

$$\|u\|_\infty := \max_{k\in[1,N]} |u(k)|, \quad \|u\|_p = \left( \sum_{k=1}^{N} |u(k)|^p \right)^{\frac{1}{p}} \quad \forall u \in X.$$

Clearly, since $X$ is a finite dimensional Banach space, all the norms are equivalent. In particular, we have

$$N^{-1/p} 2^{(1-p)/p} \|u\|_p \le \|u\| \le (2^p + \|q\|_\infty)^{1/p} \|u\|_p, \tag{1}$$

and it is easy to show that, pointed $q := \min_{k\in[1,N]} q(k)$, one has

$$\|u\|_\infty \le \sigma \|u\|, \quad \forall u \in X, \tag{2}$$

where,

$$\sigma := \begin{cases} N^{\frac{p-1}{p}}, & \text{if} 0 \le q \le N^{1-p}, \\ q^{-1/p}, & \text{if} N^{1-p} \le q. \end{cases}$$

Furthermore, we introduce the following two functions

$$\Phi(u) := \frac{\|u\|^p}{p} \quad \text{and} \quad \Psi(u) := \sum_{k=1}^{N} F_k(u(k)), \quad \forall u \in X, \tag{3}$$

where $F_k(t) := \int_0^t f_k(\xi) d\xi$ for every $(k, t) \in [1, N] \times \mathbb{R}$. Clearly, $\Phi$ and $\Psi$ are two functionals of class $C^1(X, \mathbb{R})$ whose Gâteaux derivatives at the point $u \in X$ are given by

$$\Phi'(u)(v) = \sum_{k=1}^{N} \phi_p (\Delta u (k-1)) \Delta v (k-1) + q(k) |u(k)|^{p-2} u(k) v(k),$$

and

$$\Psi'(u)(v) = \sum_{k=1}^{N} f_k(u(k))v(k),$$

for all $v \in X$. On the other hand, one has

$$\sum_{k=1}^{N} \Delta(\phi_p(\Delta u(k-1)))v(k) =$$

$$\sum_{k=1}^{N} \big(\phi_p(\Delta u(k)) - \phi_p(\Delta u(k-1))\big) v(k) =$$

$$= -\sum_{k=1}^{N} \phi_p(\Delta u(k-1))\Delta v(k-1) + \phi_p(\Delta u(N))v(N) - \phi_p(\Delta u(0))v(0),$$

that is

$$-\sum_{k=1}^{N} \Delta(\phi_p(\Delta u(k-1)))v(k) = \sum_{k=1}^{N} \phi_p(\Delta u(k-1))\Delta v(k-1), \tag{4}$$

for all $u\ v, \in X$. This leads to the following lemma.

**Lemma 2.1.** *A vector $u \in X$ is a solution of problem ($R_{\lambda,\underline{f}}$) if and only if $u$ is a critical point of the function $I_\lambda = \Phi - \lambda\Psi$.*

Now, we recall the following classical definition.

Let $(X, \|\cdot\|)$ be a Banach space and let $I \in C^1(X, \mathbb{R})$. We say that $I$ satisfies the Palais-Smale condition, (in short $(PS)$—condition), if any sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that

1. $\{I(u_n)\}_{n\in\mathbb{N}}$ is bounded,
2. $\{I'(u_n)\}_{n\in\mathbb{N}}$ converges to 0 in $X^*$,

admits a subsequence which is convergent in $X$.

Our main tool is a two non-zero critical points theorem established in [7], that we recall here for the reader's convenience.

**Theorem 2.2.** *Let $X$ be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two functionals of class $C^1$ such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}, \tag{5}$$

*and, for each*

$$\lambda \in \Lambda = \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)} \right[,$$

*the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)$–condition and it is unbounded from below.*

*Then, for each $\lambda \in \Lambda$, the functional $I_\lambda$ admits at least two non-zero critical points $u_{\lambda,1}$, $u_{\lambda,2}$ such that $I(u_{\lambda,1}) < 0 < I(u_{\lambda,2})$.*

*Remark* 1. It is worth noticing that the previous result guaranties the existence of two non-zero critical points for an appropriate class of differentiable functionals. The main tools used in its proof are a local minimum theorem established in [15] and the powerful classical Ambrosetti-Rabinowitz theorem (see [5]).

Here and in the sequel, since we are interested in positive solutions of problem $(R_{\lambda,f})$, it is not restrictive to assume that

$$f_k(t) = f_k(0), \quad \forall t < 0, \forall k \in [1, N].$$

Moreover, we put $x^+ := \max\{x, 0\}$ and $x^- := \max\{-x, 0\}$ for all $x \in \mathbb{R}$.

The following lemma ensures that when the parameter $\lambda$ belongs to a suitable half-line related to the behaviour of the primitive $F$ at infinity, the energy functional $I_\lambda$ satisfies the $(PS)$–condition and it is also unbounded from below. Such Lemma is proved for $p = 2$ in [14], for $p \neq 2$ see also [22], [8].

For $k \in [1, N]$, set

$$L_\infty(k) := \liminf_{t\to+\infty} \frac{F_k(t)}{t^p}, \quad L_\infty := \min_{k\in[1,N]} L_\infty(k).$$

We have the following result.

**Lemma 2.3.** *Suppose $f_k(0) \geq 0$ for all $k \in [1, N]$. If $L_\infty > 0$, then $I_\lambda$ satisfies $(PS)$–condition and it is unbounded from below for all $\lambda \in \left] \dfrac{2^p + \|q\|_\infty}{pL_\infty}, +\infty \right[$.*

*Proof.* To show that $I_\lambda$ satisfies the $(PS)$–condition, we fix the following:

*Claim 1. If $\lim_{n\to+\infty} I'_\lambda(u_n) = 0$, then $\{u_n^-\}$ is bounded for every $\lambda > 0$. In particular, there exists $M_1 > 0$ such that $\|u_n^-\|_\infty \leq M_1$, for all $n \in \mathbb{N}$.*

In fact, one has

$$I'_\lambda(u_n)(-u_n^-) \quad = \quad \|u_n^-\|^p + \lambda \sum_{k=1}^{N} f_k(0) u_n^-(k) \geq \|u_n^-\|^p,$$

that is

$$\|u_n^-\|^p \leq -I'_\lambda(u_n)(u_n^-), \tag{6}$$

for all $n \in \mathbb{N}$. Now, from $\lim_{n\to+\infty} I'_\lambda(u_n) = 0$, that is $\lim_{n\to+\infty} \sup_{\|v\|\leq 1} I'_\lambda(u_n)(v) = 0$, one has $\lim_{n\to+\infty} \dfrac{I'_\lambda(u_n)(u_n^-)}{\|u_n^-\|} = 0$, for which, taking (6) into account, gives $\lim_{n\to+\infty} \|u_n^-\| = 0$. By now, it is evident as to obtain the estimate on the $\|u\|_\infty$ and our claim holds.

*Claim 2. If $\lim_{n\to+\infty} I_\lambda(u_n) = c$, then $\{u_n^+\}$ is bounded for all $\lambda > \dfrac{2^p + \|q\|_\infty}{pL_\infty}$.*

Taking into account that $L_\infty > 0$ there exists $l > 0$ such that $L_\infty > l > \dfrac{2^p + \|q\|_\infty}{p\lambda}$. Moreover, since $L_\infty(k) \geq L_\infty > l$ for each $k \in [1, N]$, there is $\delta > 0$ such that $F_k(t) \geq lt^p$, for all $t > \delta$ and for all $k \in [1, N]$. From this, it is not restrictive to assume that

$$F_k(u_n^+(k)) \geq l(u_n^+(k))^p, \tag{7}$$

for all $k \in [1, N]$ and for all $n \in \mathbb{N}$. Otherwise, we are done. Therefore, bearing in mind Claim 1 and (7) there exists $M_2 > 0$ such that

$$I_\lambda(u_n) \leq \left( \frac{2^p + \|q\|_\infty}{p} - \lambda l \right) \|u_n^+\|_p^p + \lambda M_2, \tag{8}$$

for all $n \in \mathbb{N}$. From this latter, since $\dfrac{2^p + \|q\|_\infty}{p} - \lambda l < 0$, our claim holds. On the contrary, if $\|u_n\|_p \to +\infty$ one would have $c = \lim_{n\to+\infty} I_\lambda(u_n) = -\infty$. Absurd.

Finally, from (8), it is easy to see that $I_\lambda$ is unbounded from below and this completes the proof. □

Analogously to the differential problems, next result shows as the truncation techniques allows us to have sign information on the non trivial critical points of the energy functional $I_\lambda$. More precisely, it ensures that a non trivial critical point of $I_\lambda$ is a positive solution of problem ($R_{\lambda,\underline{f}}$).

**Lemma 2.4.** *If $u \in X$ is a non trivial critical point of $I_\lambda$, then $u$ is a positive solution of problem ($R_{\lambda,\underline{f}}$), for every $\lambda > 0$.*

*Proof.* Fixed $\lambda > 0$ and let $u \in X$ a non trivial critical point of $I_\lambda$. Lemma 2.1 implies that $u$ is a non trivial solution of problem ($R_{\lambda,\underline{f}}$). In other words, we have that $u \in X$ satisfies the following condition

$$\sum_{k=1}^{N} \phi_p(\Delta u(k-1))\Delta v(k-1) + \sum_{k=1}^{N} q(k)\phi_p(u(k))v(k) = \lambda \sum_{k=1}^{N} f_k(u(k))v(k), \ \forall v \in X. \tag{9}$$

From this, taking as test function $v = -u^-$, with simple computations, we get

$$\|u^-\|^p = -\lambda \sum_{k=1}^{N} f_k(0)u^-(k) \leq 0,$$

which implies $\|u^-\| = 0$, that is $u$ is nonnegative. Moreover, arguing by contradiction, we show that $u$ is also a positive solution of problem ($R_{\lambda,\underline{f}}$). Suppose that $u(k) = 0$ for some $k \in [1,N]$. Being $u$ a solution of problem ($R_{\lambda,\underline{f}}$) we have

$$\phi_p(\Delta u(k-1)) - \phi_p(\Delta u(k)) = \lambda f_k(0) \geq 0,$$

which produces that

$$0 \geq -|u(k-1)|^{p-2}u(k-1) - |u(k+1)|^{p-2}u(k+1) \geq 0.$$

So, we have that $u(k-1) = u(k+1) = 0$. Hence, iterating this process, we get that $u(k) = 0$ for every $k \in [1,N]$, which contradicts that $u$ is nontrivial and this completes the proof. □

## 3 Main Results

In this section, we present some results on the existence of two positive solutions for problem ($R_{\lambda,\underline{f}}$). First, we put

$$Q = \sum_{k=1}^{N} q(k).$$

**Theorem 3.1.** *Let $f_k : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f_k(0) \geq 0$ for all $k \in [1,N]$. Assume also that there exist two positive constants $c$ and $d$ with $d < c$ such that*

$$\frac{\sum_{k=1}^{N} \max_{|\xi|\leq c} F_k(\xi)}{c^p} < \frac{1}{\sigma^p} \min\left\{ \frac{1}{(1+Q)} \frac{\sum_{k=1}^{N} F_k(d)}{d^p}, \frac{L_\infty}{2^p + \|q\|_\infty} \right\}, \tag{10}$$

*being $\sigma$ as given in (2). Then, for each $\lambda \in \bar{\Lambda}$ with*

$$\bar{\Lambda} = \left] \max\left\{ \frac{(1+Q)}{p} \frac{d^p}{\sum_{k=1}^{N} F_k(d)}, \frac{2^p + \|q\|_\infty}{pL_\infty} \right\}, \frac{1}{p\sigma^p} \frac{c^p}{\sum_{k=1}^{N} \max_{|\xi|\leq c} F_k(\xi)} \right[,$$

*the problem ($R_{\lambda,\underline{f}}$) admits at least two positive solutions.*

*Proof.* Our conclusions are proved in two main steps. First, we apply Theorem 2.2 to obtain two non-zero solutions for problem ($R_{\lambda,\underline{f}}$). Next, owing to Lemma 2.4 we complete the proof.

Put $\Phi$ and $\Psi$ as in (3). It is well known that $\Phi$ and $\Psi$ satisfy all regularity assumptions requested in Theorem 2.2 and that any non-zero critical point in $X$ of the functional $I_\lambda$ is precisely a nontrivial solution of problem ($R_{\lambda,\underline{f}}$). Clearly, $\inf_S \Phi = \Phi(0) = \Psi(0) = 0$. So, our end is to verify condition (5) of Theorem 2.2.

Fix $\lambda \in \bar{\Lambda}$ and observe that from (10) one has that $L_\infty > 0$ and $\bar{\Lambda}$ is non-degenerate. Then, by Lemma 2.3, the functional $I_\lambda$ satisfies the $(PS)$–condition for each $\lambda > \dfrac{2^p + \|q\|_\infty}{pL_\infty}$, and it is unbounded from below. Next, set $r = \dfrac{1}{p}\left(\dfrac{c}{\sigma}\right)^p$. Let be $u \in \Phi^{-1}(]-\infty, r])$; bearing in mind (2), one has

$$\|u\|_\infty \leq \sigma(pr)^{1/p} = c.$$

So,

$$\Psi(u) = \sum_{k=1}^{N} F_k(u(k)) \leq \sum_{k=1}^{N} \max_{|\xi|\leq c} F_k(\xi),$$

for all $u \in X$ such that $u \in \Phi^{-1}(]-\infty, r])$.
Hence,

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} \leq p\sigma^p \frac{\sum_{k=1}^{N} \max_{|\xi|\leq c} F_k(\xi)}{c^p}. \tag{11}$$

Now, define $\tilde{u} \in X$ be such that $\tilde{u}(k) = d$ for all $k \in [1, N+1]$. It is easy to see that

$$\Phi(\tilde{u}) = \frac{(1+Q)d^p}{p}, \tag{12}$$

and hence, one has

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} = \frac{p}{(1+Q)} \frac{\sum_{k=1}^{N} F_k(d)}{d^p}. \tag{13}$$

Therefore, from (11), (13) and assumption (10) one has

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}.$$

Moreover, since $0 < d < c$ and again thanks to (10), we obtain that

$$0 < d < \frac{c}{\sigma(1+Q)^{\frac{1}{p}}}. \tag{14}$$

Indeed, arguing by contradiction, if we assume that $d \geq \dfrac{c}{\sigma(1+Q)^{\frac{1}{p}}}$, we have

$$\frac{\sum_{k=1}^{N} \max_{|\xi|\leq c} F_k(\xi)}{c^p} \geq \frac{\sum_{k=1}^{N} F_k(d)}{c^p} \geq \frac{1}{(1+Q)\sigma^p} \frac{\sum_{k=1}^{N} F_k(d)}{d^p},$$

which contradicts (10). Hence by (12) and (14) we obtain that $0 < \Phi(\tilde{u}) < r$.

Therefore, Theorem 2.2 ensures that $I_\lambda$ admits at least two non-zero critical points and then, for all $\lambda \in \bar{\Lambda} \subset \Lambda$, Lemma 2.4 ensures that these are positive solutions of $(R_{\lambda,\underline{f}})$ and this completes the proof.

□

A consequence of Theorem 3.1 is the following result.

**Corollary 3.2.** *Let $f_k$ be a continuous function such that $f_k(0) \geq 0$, for every $k \in [1, N]$. Assume that*

$$\limsup_{t\to 0^+} \frac{F_k(t)}{t^p} = +\infty, \tag{15}$$

*and*

$$\lim_{t\to+\infty} \frac{F_k(t)}{t^p} = +\infty, \tag{16}$$

*for all $k \in [1, N]$, and put $\lambda^* = \dfrac{1}{p\sigma^p} \sup_{c>0} \dfrac{c^p}{\sum_{k=1}^{N} \max_{|\xi|\leq c} F_k(\xi)}$.*

*Then, for each $\lambda \in ]0, \lambda^*[$, the problem $(R_{\lambda,\underline{f}})$ admits at least two positive solutions.*

*Proof.* First, note that $L_\infty = +\infty$. Then, fix $\lambda \in ]0, \lambda^*[$ and $c > 0$ such that

$$\lambda < \frac{1}{p\sigma^p} \frac{c^p}{\sum_{k=1}^{N} \max_{|\xi|\leq c} F_k(\xi)}.$$

From (15) we have

$$\limsup_{t\to 0^+} \frac{\sum_{k=1}^{N} F_k(t)}{t^p} = +\infty,$$

then there is $d > 0$ with $d < c$ such that $\frac{p}{(1+Q)} \frac{\sum_{k=1}^{N} F_k(d)}{d^p} > \frac{1}{\lambda}$. Hence, Theorem 3.1 ensures the conclusion. □

**Corollary 3.3.** *Let $f_k$ be a continuous function satisfying (15) and (16) and let $\lambda^*$ as in Corollary 3.2. Then, for each $\lambda \in ]0, \lambda^*[$, the problem $(R_{\lambda,\underline{f}})$ admits at least two non trivial solutions.*

*Proof.* A careful reading of the proof of Theorem 3.1 allows to get the conclusion, at once. □

In order to show the applicability of the results, we give an example.

**Example 3.1.** Let $p = N = 2$ and consider problem $(R_{\lambda,\underline{f}})$ with $q(k) = 2k$ for $k = 1, 2$ and $\underline{f}(u) = (f_1(u), f_2(u))$ as nonlinear term, where

$$f_1(u) = e^u + 1\,,$$
$$f_2(u) = \frac{1 - u(1+3u) + u\ln(|u+1|)(-2 + 2(u-1)u + 3u(u+1)\ln(|u+1|))}{1+u}\,,$$

for all $u \in \mathbb{R}$. It follows that

$$F_1(t) = e^t + t - 1\,, \qquad F_2(t) = t\left(t^2 \ln^2|t+1| - t(\ln|t+1|+1) + 1\right) \qquad \forall t \geq 0\,.$$

Choosing, for instance, $c = 0.3$ and $d = 0.01$, all the assumptions of Theorem 3.1 are satisfied. Therefore, the problem

$$\begin{cases} -u(2) + 4u(1) = 10^{-1} f_1(u(1)) \\ 5u(2) - u(1) = 10^{-1} f_2(u(2)) \\ u(0) = \Delta u(2) = 0 \end{cases}$$

admits at least two positive solutions. We observe that in this case $\lambda = 10^{-1}$, and clearly it belongs to the interval of parameter which has the following form

$$\bar{\Lambda} = \left]\frac{7}{2} \frac{d^2}{F_1(d) + F_2(d)}, \frac{c^2}{F_1(d) + F_2(d)}\right[ \supseteq ]0.012, 0.107[\,.$$

## Acknowledgements

The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The paper is partially supported by PRIN 2017- Progetti di Ricerca di rilevante Interesse Nazionale,"Nonlinear Differential Problems via Variational, Topological and Set-valued Methods" (2017AYM8XW).

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