CONTINUOUS SPECTRUM FOR A TWO PHASE EIGENVALUE PROBLEM WITH AN INDEFINITE AND UNBOUNDED POTENTIAL

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ABSTRACT. We consider a two phase eigenvalue problem driven by the (p, q)-Laplacian plus an indefinite and unbounded potential, and Robin boundary condition. Using a modification of the Nehari manifold method, we show that there exists a nontrivial open interval $I \subseteq \mathbb{R}$ such that every $\lambda \in I$ is an eigenvalue with positive eigenfunctions. When we impose additional regularity conditions on the potential function and the boundary coefficient, we show that we have smooth eigenfunctions.

1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following two phase eigenvalue Robin problem

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u - \Delta_q u + \xi(z)|u|^{p-2}u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_{pq}} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega,\ \lambda \in \mathbb{R},\ 1 < q < p < +\infty. \end{cases}$$

For every $r \in (1, +\infty)$ by Δ_r we denote the r-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u) \quad \text{for all } u \in W^{1,r}(\Omega).$$

In problem (P_λ) we assume that $\xi \in L^s(\Omega)$ with $s > N/p$ if $1 < p \leq N$ and $s = 1$ if $p > N$. The potential function $\xi(\cdot)$ is in general indefinite (that is, sign changing). So, the left hand side in (P_λ) is a nonhomogeneous differential operator (the (p, q)-Laplacian) plus an indefinite and unbounded potential term. We say that $\lambda \in \mathbb{R}$ is an "eigenvalue" of (P_λ), if the problem admits a nontrivial weak solution. In the boundary condition, $\dfrac{\partial u}{\partial n_{pq}}$ denotes the conormal derivative corresponding to the differential operator of the problem. This directional derivative is interpreted using the nonlinear Green's identity, see Gasiński-Papageorgiou [7], p. 210. If $u \in C^1(\overline{\Omega})$, then

$$\frac{\partial u}{\partial n_{pq}} = (|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u, n)_{\mathbb{R}^N},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. Note that if we consider the map $a : \mathbb{R}^N \to \mathbb{R}^N$ defined by $a(y) = |y|^{p-2}y + |y|^{q-2}y$ for all $y \in \mathbb{R}^N$, then

$$\operatorname{div} a(\nabla u) = \Delta_p u + \Delta_q u \quad \text{for all } u \in W^{1,p}(\Omega).$$

Also the boundary coefficient $\beta(\cdot)$ is nonnegative and nonzero. In this paper we show that under appropriate conditions on $\xi(\cdot)$ and $\beta(\cdot)$, problem (P_λ) has a continuous

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spectrum, namely there is a whole nontrivial interval $I \subseteq \mathbb{R}$ such that for all $\lambda \in I$, problem (P_λ) has a nontrivial solution.

Eigenvalue problems for two phase equations were studied primarily in the context of Dirichlet equations with zero potential term. We refer to the works of Bhattacharaya-Emamizadeh-Farjudian [4], Chorfi-Rădulescu [6], Papageorgiou-Rădulescu-Repovš [14], Tanaka [17]. For Steklov problems with zero potential there is the recent work of Barbu-Moroşanu [2].

Two phase equations arise in many mathematical models of physical processes. In elasticity theory such equations characterize composites consisting of two different materials with distinct hardening exponents. Such anisotropic materials have energy functionals with unbalanced growth and were investigated by Marcellini [10] and Zhikov [18, 19]. Two phase equations arise also in other parts of mathematical physics. We mention the works of Bahrouni-Rădulescu-Repovš [1] (theory of transonic flows), Benci-D'Avenia-Fortunato-Pisani [3] (quantum physics) and Cherfils-Il'yasov [5] (reaction diffusion systems).

2. Mathematical Preliminaries

Throughout this paper, the following assumptions will be in effect about the potential function $\xi(\cdot)$ and the boundary coefficient $\beta(\cdot)$.

H_0: $\xi \in L^s(\Omega)$ with $s > N/p$ if $1 < p \leq N$, $s = 1$ if $N < p$, $\beta \in W^{1,\infty}(\partial\Omega)$, $\beta(z) \geq 0$ for all $z \in \partial\Omega$, $\beta \not\equiv 0$ and

$$\int_\Omega \xi(z)dz + \int_{\partial\Omega} \beta(z)d\sigma > 0.$$

Here by $\sigma(\cdot)$ we denote the $(N-1)$-dimensional Hausdorff measure (surface measure) on $\partial\Omega$. Also by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. We set

$$\lambda_0 = \frac{1}{|\Omega|_N}\left[\int_\Omega \xi(z)dz + \int_{\partial\Omega} \beta(z)d\sigma\right] > 0.$$

Let $\gamma_p : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\gamma_p(u) = \|\nabla u\|_p^p + \int_\Omega \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

Consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u + \xi(z)|u|^{p-2}u = \lambda|u|^{p-2}u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

In this case the conormal derivative is defined by

$$\frac{\partial u}{\partial n_p} = (|\nabla u|^{p-2}\nabla u, n)_{\mathbb{R}^N} \quad \text{for all } u \in C^1(\overline{\Omega}),$$

and the boundary condition is understood using the nonlinear Green's identity. We know (see Mugnai-Papageorgiou [11] (Neumann problems) and Papageorgiou-Rădulescu [12] (Robin problems)), that problem (1) has a smallest eigenvalue $\widehat{\lambda}_1 \in \mathbb{R}$ which has the following properties:

- $\widehat{\lambda}_1$ is isolated and simple;

$$\bullet \quad \widehat{\lambda}_1 = \inf\left[\frac{\gamma_p(u)}{\|u\|_p^p} : u \in W^{1,p}(\Omega),\, u \neq 0\right]. \tag{2}$$

The infimum in (2) is realized on the corresponding one dimensional eigenspace. Evidently, the elements of this eigenspace have fixed sign. By $\widehat{u}_1$ we denote the positive, L^p-normalized (that is, $\|\widehat{u}_1\|_p = 1$) eigenfunction corresponding to $\widehat{\lambda}_1$. Moreover, by the Harnack inequality (see Pucci-Serrin [16], p. 164), we have $\widehat{u}_1(z) > 0$ for a.a. $z \in \Omega$.

Since by hypothesis $\beta \not\equiv 0$, constant functions can not be eigenfunctions for $\widehat{\lambda}_1$ and so we have

$$\widehat{\lambda}_1 < \frac{1}{|\Omega|_N}\gamma_p(1) = \frac{1}{|\Omega|_N}\left[\int_\Omega \xi(z)dz + \int_{\partial\Omega}\beta(z)d\sigma\right] = \lambda_0. \tag{3}$$

Our approach will be based on a modification of the Nehari manifold method. So, we consider the energy functional $\varphi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ for problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|\nabla u\|_q^q - \frac{\lambda}{p}\|u\|_p^p \quad \text{for all } u \in W^{1,p}(\Omega).$$

We have $\varphi_\lambda \in C^1(W^{1,p}(\Omega))$. Then the Nehari manifold for this functional is defined by

$$N_\lambda = \left\{u \in W^{1,p}(\Omega) : \langle \varphi_\lambda'(u), u\rangle = 0\right\}.$$

However, since $W^{1,p}(\Omega)$ contains the constant functions (that is, $\mathbb{R} \subseteq W^{1,p}(\Omega)$), it is more convenient to work with the following subset of N_λ

$$\widehat{N}_\lambda = \left\{u \in N_\lambda : \int_\Omega \xi(z)|u|^{p-2}udz + \int_{\partial\Omega}\beta(z)|u|^{p-2}ud\sigma = \lambda\int_\Omega |u|^{p-2}udz\right\}.$$

For this reason we consider also the functional $\widehat{\vartheta}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\vartheta}_\lambda(u) = \int_\Omega \xi(z)|u|^{p-2}udz + \int_{\partial\Omega}\beta(z)|u|^{p-2}ud\sigma - \lambda\int_\Omega |u|^{p-2}udz \quad \text{for all } u \in W^{1,p}(\Omega).$$

We have $\widehat{\vartheta}_\lambda \in C(W^{1,p}(\Omega)) \cap C^1(W^{1,p}(\Omega) \setminus \{0\})$.

We will see that this restriction of the Nehari manifold is harmless since as we will eventually show $\inf_{N_\lambda}\varphi_\lambda = \inf_{\widehat{N}_\lambda}\varphi_\lambda$.

3. Continuous Spectrum

First we show the nonemptiness of $\widehat{N}_\lambda$.

Proposition 1. *If hypotheses H_0 hold and $\lambda \in (\widehat{\lambda}_1, \lambda_0)$ (see (3)), then $\widehat{N}_\lambda \neq \emptyset$.*

Proof. Consider the function $\tau_\lambda : [0, +\infty) \to \mathbb{R}$ defined by

$$\tau_\lambda(\eta) = \widehat{\vartheta}_\lambda(\widehat{u}_1 + \eta) \quad \text{for all } \eta \geq 0.$$

Evidently $\tau_\lambda(\cdot)$ is continuous. We have

$$\tau_\lambda(0) = \widehat{\vartheta}_\lambda(\widehat{u}_1) = \int_\Omega \xi(z)\widehat{u}_1^{p-1}dz + \int_{\partial\Omega}\beta(z)\widehat{u}_1^{p-1}d\sigma - \lambda\int_\Omega \widehat{u}_1^{p-1}dz. \tag{4}$$

Since $\widehat{u}_1$ is an eigenfunction of problem (1) corresponding to the eigenvalue $\widehat{\lambda}_1$, we have

$$\int_\Omega |\nabla\widehat{u}_1|^{p-2}(\nabla\widehat{u}_1, \nabla h)_{\mathbb{R}^N}dz + \int_\Omega \xi(z)\widehat{u}_1^{p-1}hdz + \int_{\partial\Omega}\beta(z)\widehat{u}_1^{p-1}hd\sigma = \lambda\int_\Omega \widehat{u}_1^{p-1}hdz$$

for all $h \in W^{1,p}(\Omega)$. We choose $h = 1$ and obtain

$$\int_\Omega \xi(z)\widehat{u}_1^{p-1}dz + \int_{\partial\Omega} \beta(z)\widehat{u}_1^{p-1}d\sigma = \widehat{\lambda}_1 \int_\Omega \widehat{u}_1^{p-1}dz. \tag{5}$$

Returning to (4) and using (5), we obtain

$$\tau_\lambda(0) = [\widehat{\lambda}_1 - \lambda]\int_\Omega \widehat{u}_1^{p-1}dz < 0 \quad \text{(recall that } \lambda > \widehat{\lambda}_1). \tag{6}$$

For $\eta > 0$, we have

$$\frac{\widehat{\vartheta}_\lambda(\widehat{u}_1+\eta)}{\eta^{p-1}} = \int_\Omega \xi(z)\frac{(\widehat{u}_1+\eta)^{p-1}}{\eta^{p-1}}dz + \int_{\partial\Omega}\beta(z)\frac{(\widehat{u}_1+\eta)^{p-1}}{\eta^{p-1}}d\sigma - \lambda\int_\Omega \frac{(\widehat{u}_1+\eta)^{p-1}}{\eta^{p-1}}dz,$$

$$\Rightarrow \quad \lim_{\eta\to+\infty}\frac{\widehat{\vartheta}_\lambda(\widehat{u}_1+\eta)}{\eta^{p-1}} = \int_\Omega \xi(z)dz + \int_{\partial\Omega}\beta(z)d\sigma - \lambda|\Omega|_N = [\lambda_0 - \lambda]|\Omega|_N > 0,$$

(7)

$$\Rightarrow \quad \widehat{\vartheta}_\lambda(\widehat{u}_1+\eta) \to +\infty \quad \text{as } \eta \to +\infty.$$

Then from (6), (7) and Bolzano's theorem, we infer that there is a smallest $\eta_0 > 0$ such that

$$\begin{aligned} &\tau_\lambda(\eta_0) = 0, \\ \Rightarrow \quad &\widehat{\vartheta}_\lambda(\widehat{u}_1+\eta_0) = 0, \end{aligned}$$

$$\Rightarrow \quad \int_\Omega \xi(z)(\widehat{u}_1+\eta_0)^{p-1}dz + \int_{\partial\Omega}\beta(z)(\widehat{u}_1+\eta_0)^{p-1}d\sigma = \lambda\int_\Omega(\widehat{u}_1+\eta_0)^{p-1}dz. \tag{8}$$

We consider the function $\widehat{t}: \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\widehat{t}(\eta) = \frac{1}{p}\int_\Omega \xi(z)(\widehat{u}_1+\eta)^p dz + \frac{1}{p}\int_{\partial\Omega}\beta(z)(\widehat{u}_1+\eta)^p d\sigma - \frac{\lambda}{p}\|\widehat{u}_1+\eta\|_p^p \quad \text{for all } \eta \geq 0.$$

Evidently $\widehat{t} \in C^1(\mathbb{R}_+)$ and

$$\begin{aligned} \widehat{t}\,'(\eta) &= \int_\Omega \xi(z)(\widehat{u}_1+\eta)^{p-1}dz + \int_{\partial\Omega}\beta(z)(\widehat{u}_1+\eta)^{p-1}d\sigma - \lambda\int_\Omega(\widehat{u}_1+\eta)^{p-1}dz \\ &= \widehat{\vartheta}_\lambda(\widehat{u}_1+\eta). \end{aligned}$$

Therefore we have

$$\begin{aligned} &\widehat{t}\,'(\eta) < 0 \quad \text{for all } \eta \in [0,\eta_0), \\ \Rightarrow \quad &\widehat{t}(\eta_0) < \widehat{t}(0), \\ \Rightarrow \quad &\frac{1}{p}\int_\Omega \xi(z)(\widehat{u}_1+\eta_0)^p dz + \frac{1}{p}\int_{\partial\Omega}\beta(z)(\widehat{u}_1+\eta_0)^p d\sigma - \frac{\lambda}{p}\|\widehat{u}_1+\eta_0\|_p^p \end{aligned}$$

$$< \widehat{t}(0) < [\widehat{\lambda}_1 - \lambda] < 0 \quad \text{(recall that } \|\widehat{u}_1\|_p = 1,\ \lambda > \widehat{\lambda}_1). \tag{9}$$

We set $y_0 = \widehat{u}_1 + \eta_0$. We consider ty_0 with $t > 0$ and we want to determine a $t > 0$ such that $ty_0 \in N_\lambda$. For this to be true, we must have

$$\begin{aligned} &\langle \gamma_p'(ty_0), ty_0\rangle + \int_\Omega |\nabla(ty_0)|^{q-2}(\nabla(ty_0),\nabla(ty_0))_{\mathbb{R}^N}dz = \lambda\|ty_0\|_p^p, \\ \Rightarrow \quad &t^p\gamma_p(y_0) + t^q\|\nabla y_0\|_q^q = \lambda t^p\|y_0\|_p^p, \\ \Rightarrow \quad &t^{p-q}[\lambda\|y_0\|_p^p - \gamma_p(y_0)] = \|\nabla y_0\|_q^q, \end{aligned}$$

$$\Rightarrow \quad t = t_\lambda = \left[\frac{\|\nabla y_0\|_q^q}{\lambda\|y_0\|_p^p - \gamma_p(y_0)}\right]^{\frac{1}{p-q}} > 0 \quad \text{(see (9))}.$$

Therefore for this particular $t_\lambda > 0$ we have $t_\lambda y_0 \in N_\lambda$. Also, on account of (8), we conclude that $t_\lambda y_0 \in \widehat{N}_\lambda \neq \emptyset$. □

Now we will minimize φ_λ on $\widehat{N}_\lambda$.

Proposition 2. *If hypotheses H_0 hold and $\lambda \in (\widehat{\lambda}_1, \lambda_0)$, then $\widehat{m}_\lambda = \inf\left[\varphi_\lambda(u) : u \in \widehat{N}_\lambda\right] \geq 0$.*

Proof. We will show that $\inf_{N_\lambda} \varphi_\lambda \geq 0$ and since $\widehat{N}_\lambda \subseteq N_\lambda$, we can conclude that $\widehat{m}_\lambda \geq 0$.

So, suppose that $u \in N_\lambda$. We have

$$\begin{aligned} & \langle \varphi_\lambda'(u), u\rangle = 0 \\ \Rightarrow \quad & \gamma_p(u) + \|\nabla u\|_q^q = \lambda\|u\|_p^p. \end{aligned} \tag{10}$$

Hence for $u \in N_\lambda$, we have

$$\begin{aligned} \varphi_\lambda(u) &= \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|\nabla u\|_q^q - \frac{\lambda}{p}\|u\|_p^p \\ &\geq \frac{1}{p}\left[\gamma_p(u) + \|\nabla u\|_q^q - \lambda\|u\|_p^p\right] \quad \text{(since } q < p) \\ &= 0 \quad \text{(see (10))}, \\ \Rightarrow \quad & \inf\left[\varphi_\lambda(u) : u \in N_\lambda\right] \geq 0, \\ \Rightarrow \quad & \widehat{m}_\lambda \geq 0. \end{aligned}$$

□

So, we have

$$0 \leq \widehat{m}_\lambda = \inf\left[\varphi_\lambda(u) : u \in \widehat{N}_\lambda\right]. \tag{11}$$

Next we examine the minimizing sequences for this problem.

Proposition 3. *If hypotheses H_0 hold and $\lambda \in (\widehat{\lambda}_1, \lambda_0)$, then every minimizing sequence of problem (11) is bounded in $W^{1,p}(\Omega)$.*

Proof. Let $\{u_n\}_{n\geq 1} \subseteq \widehat{N}_\lambda \subseteq W^{1,p}(\Omega)$ be a minimizing sequence for problem (11). We have

$$\varphi_\lambda(u_n) \downarrow \widehat{m}_\lambda \quad \text{as } n \to +\infty. \tag{12}$$

Arguing by contradiction, suppose that $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is not bounded. Then at least for a subsequence we have

$$\|u_n\| \to +\infty \quad \text{as } n \to +\infty. \tag{13}$$

We set $y_n = \dfrac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Since $u_n \in \widehat{N}_\lambda \subseteq N_\lambda$, we have

$$\begin{aligned} & \gamma_p(u_n) + \|\nabla u_n\|_q^q = \lambda\|u_n\|_p^p \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow \quad & \gamma_p(y_n) + \frac{1}{\|u_n\|^{p-q}}\|\nabla y_n\|_q^q = \lambda\|y_n\|_p^p \leq \lambda \quad \text{for all } n \in \mathbb{N} \text{ (note } \|y_n\| = 1), \\ \Rightarrow \quad & \gamma_p(y_n) \leq \lambda \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{14}$$

We assume $N \geq p$ (by the Sobolev embedding theorem the case $N < p$ is straightforward). Since $s > N/p$ (see hypothesis H_0), we have

$$s' < \left(\frac{N}{p}\right)' = \frac{N}{N-p} \quad \text{(recall for } r \in [1,+\infty),\ \frac{1}{r}+\frac{1}{r'}=1).$$

Hence the Sobolev embedding theorem guarantees that $|u|^p \in L^{s'}(\Omega)$. Then using Hölder's inequality, we have

$$\left|\int_\Omega \xi(z)|u|^p dz\right| \leq \|\xi\|_s \|u\|^p_{ps'}. \tag{15}$$

Since $ps' < \dfrac{Np}{N-p} = p^*$ (the critical Sobolev exponent), we have

$$W^{1,p}(\Omega) \hookrightarrow L^{ps'}(\Omega) \hookrightarrow L^p(\Omega)$$

and we know that the first embedding is compact (by the Sobolev embedding theorem). So, using Ehrling's inequality (see Papageorgiou-Winkert [15], p. 317), we see that given $\varepsilon > 0$, we can find $c(\varepsilon) > 0$ such that

$$\|u\|^p_{ps'} \leq \varepsilon\|u\|^p + c(\varepsilon)\|u\|^p_p. \tag{16}$$

Here by $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$ (recall $\|u\| = [\|u\|_p^p + \|\nabla u\|_p^p]^{1/p}$ for all $u \in W^{1,p}(\Omega)$). Returning to (15) and using (16), we obtain

$$\left|\int_\Omega \xi(z)|u|^p dz\right| \leq \|\xi\|_s \left[\varepsilon\|u\|^p + c(\varepsilon)\|u\|^p_p\right]. \tag{17}$$

Therefore we have

$$\begin{aligned}
&\|u\|^p - \int_\Omega \xi(z)|u|^p dz \\
&= \|u\|_p^p + \|\nabla u\|_p^p - \int_\Omega \xi(z)|u|^p dz \\
&\leq \|u\|_p^p + \|\nabla u\|_p^p + \varepsilon\|\xi\|_s\|u\|^p + c(\varepsilon)\|\xi\|_s\|u\|_p^p \quad \text{(see (17))}, \\
\Rightarrow \quad &(1-\varepsilon\|\xi\|_s)\|u\|^p \leq \|\nabla u\|_p^p + \int_\Omega \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma + [c(\varepsilon)\|\xi\|_s + 1]\|u\|_p^p \\
&\qquad \text{(since } \beta \geq 0\text{, see hypotheses } H_0).
\end{aligned}$$

Choosing $\varepsilon \in \left(0, \dfrac{1}{\|\xi\|_s}\right)$, we infer that

$$\|u\|^p \leq c_1\gamma_p(u) + c_2\|u\|_p^p \quad \text{for all } u \in W^{1,p}(\Omega), \text{ some } c_1, c_2 > 0. \tag{18}$$

Then from (14), (18) and since $\|y_n\|_p \leq 1$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\|y_n\|^p \leq c_1\lambda + c_2 \quad \text{for all } n \in \mathbb{N}, \\
\Rightarrow \quad &\{y_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}
\end{aligned}$$

So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{19}$$

From (12) we see that we can find $n_0 \in \mathbb{N}$ such that

$$\varphi_\lambda(u_n) \leq \widehat{m}_\lambda + 1 \quad \text{for all } n \geq n_0, \tag{20}$$

$$\Rightarrow \quad \frac{1}{p}\gamma_p(u_n) + \frac{1}{q}\|\nabla u_n\|_q^q - \frac{\lambda}{p}\|u_n\|_p^p \le \widehat{m}_\lambda + 1 \quad \text{for all } n \ge n_0,$$

$$\Rightarrow \quad \left[\frac{1}{q} - \frac{1}{p}\right]\|\nabla u_n\|_q^q \le \widehat{m}_\lambda + 1 \quad \text{for all } n \ge n_0 \text{ (see (10))},$$

$$\Rightarrow \quad \left[\frac{1}{q} - \frac{1}{p}\right]\|\nabla y_n\|_q^q \le \frac{1}{\|u_n\|^q}[\widehat{m}_\lambda + 1] \quad \text{for all } n \ge n_0,$$

$$\Rightarrow \quad y_n \to c \in \mathbb{R} \text{ in } W^{1,q}(\Omega) \text{ (see (19), (13) and recall } q < p). \tag{21}$$

We have

$$0 \le \varphi_\lambda(u_n) \le \widehat{m}_\lambda + 1 \quad \text{for all } n \ge n_0 \text{ (see (20) and Proposition 2)},$$

$$\Rightarrow \quad 0 \le \frac{1}{p}\gamma_p(y_n) + \frac{1}{q\|u_n\|^{p-q}}\|\nabla y_n\|_q^q - \frac{\lambda}{p}\|y_n\|_p^p \le \frac{1}{\|u_n\|^p}[\widehat{m}_\lambda + 1] \quad \text{for all } n \ge n_0,$$

$$\Rightarrow \quad \gamma_p(y_n) - \lambda\|y_n\|_p^p \to 0 \quad \text{as } n \to +\infty \text{ (see (21))}.$$

Then in the limit as $n \to +\infty$, we have

$$|c|^p\left[\int_\Omega \xi(z)dz + \int_{\partial\Omega}\beta(z)d\sigma - \lambda|\Omega|_N\right] = 0.$$

Since $\int_\Omega \xi(z)dz + \int_{\partial\Omega}\beta(z)d\sigma = \lambda_0|\Omega|_N > \lambda|\Omega|_N$ (recall that $\lambda < \lambda_0$), it follows that $c = 0$, which means that $y_n \to 0$ in $W^{1,p}(\Omega)$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \in \mathbb{N}$. Therefore we conclude that $\{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ is bounded. ☐

Proposition 4. *If hypotheses H_0 hold and $\lambda \in (\widehat{\lambda}_1, \lambda_0)$, then $\widehat{m}_\lambda > 0$.*

Proof. Arguing by contradiction, suppose that $\widehat{m}_\lambda = 0$. Let $\{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ be a minimizing sequence for problem (11). We have

$$\varphi_\lambda(u_n) \downarrow 0 \quad \text{as } n \to +\infty. \tag{22}$$

From Proposition 3 we know that $\{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ is bounded. Therefore we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{23}$$

Recall that since $u_n \in N_\lambda$, we have

$$\gamma_p(u_n) + \|\nabla u_n\|_q^q = \lambda\|u_n\|_p^p \quad \text{for all } n \in \mathbb{N}. \tag{24}$$

Hence we have

$$\varphi_\lambda(u_n) = \left[\frac{1}{q} - \frac{1}{p}\right]\|\nabla u_n\|_q^q \quad \text{for all } n \in \mathbb{N} \text{ (see (24))},$$

$$\Rightarrow \quad \|\nabla u_n\|_q \to 0 \quad \text{(see (22))},$$

$$\Rightarrow \quad u_n \to u = c \in \mathbb{R} \text{ in } W^{1,q}(\Omega) \text{ (see (23))}. \tag{25}$$

Also since $u_n \in \widehat{N}_\lambda$, we have

$$\int_\Omega \xi(z)|u_n|^{p-2}u_n dz + \int_{\partial\Omega}\beta(z)|u_n|^{p-2}u_n d\sigma = \lambda\int_\Omega |u_n|^{p-2}u_n dz \quad \text{for all } n \in \mathbb{N},$$

$$\Rightarrow \quad |c|^{p-2}c\left[\int_\Omega \xi(z)dz + \int_{\partial\Omega}\beta(z)d\sigma - \lambda|\Omega|_N\right] = 0 \quad \text{(see (25))},$$

$$\Rightarrow \quad c = 0 \quad (\text{recall } \lambda < \lambda_0).$$

Therefore $u = 0$.

Let $y_n = \dfrac{u_n}{\|u_n\|_p}$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} & \gamma_p(y_n) + \frac{1}{\|u_n\|_p^{p-q}} \|\nabla y_n\|_q^q = \lambda \|y_n\|_p^p = \lambda, \\ \Rightarrow \quad & \gamma_p(y_n) \leq \lambda \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow \quad & \{y_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded (see (18)).} \end{aligned}$$

So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{26}$$

Multiplying (24) with $\dfrac{1}{\|u_n\|_p^q}$, we obtain

$$\begin{aligned} & \|u_n\|_p^{p-q} \gamma_p(y_n) + \|\nabla y_n\|_q^q = \|u_n\|_p^{p-q} \|y_n\|_p^p, \\ \Rightarrow \quad & \|\nabla y_n\|_q \to 0 \quad \text{(see (23) and recall } u = 0), \\ \Rightarrow \quad & y_n \to c \in \mathbb{R} \text{ in } W^{1,q}(\Omega). \end{aligned} \tag{27}$$

Recall that

$$\begin{aligned} & \int_\Omega \xi(z)|u_n|^{p-2}u_n dz + \int_{\partial\Omega} \beta(z)|u_n|^{p-2}u_n d\sigma = \lambda \int_\Omega |u_n|^{p-2}u_n dz \\ & \qquad\qquad\qquad\qquad \text{for all } n \in \mathbb{N} \text{ (since } u_n \in \widehat{N}_\lambda), \\ \Rightarrow \quad & \int_\Omega \xi(z)|y_n|^{p-2}y_n dz + \int_{\partial\Omega} \beta(z)|y_n|^{p-2}y_n d\sigma = \lambda \int_\Omega |y_n|^{p-2}y_n dz \\ \Rightarrow \quad & |c|^{p-2}c \left[\int_\Omega \xi(z) dz + \int_{\partial\Omega} \beta(z) d\sigma - \lambda |\Omega|_N \right] = 0 \quad \text{(see (27))}, \\ \Rightarrow \quad & c = 0 \quad \text{(recall } \lambda < \lambda_0). \end{aligned} \tag{28}$$

Hence we have

$$y_n \to 0 \text{ in } L^p(\Omega) \text{ (see (26), (28))}$$

which contradicts the fact that $\|y_n\|_p = 1$ for all $n \in \mathbb{N}$. This contradiction proves that $\widehat{m}_\lambda > 0$. □

Proposition 5. *If hypotheses H_0 hold and $\lambda \in (\widehat{\lambda}_1, \lambda_0)$, then there exists $u_\lambda \in \widehat{N}_\lambda$ such that $\varphi_\lambda(u_\lambda) = \widehat{m}_\lambda$.*

Proof. We consider a minimizing sequence $\{u_n\}_{n\geq 1} \subseteq \widehat{N}_\lambda$. We have

$$\varphi_\lambda(u_n) \downarrow \widehat{m}_\lambda \quad \text{as } n \to +\infty.$$

From Proposition 3 we know that $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u_\lambda \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_\lambda \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{29}$$

Since $u_n \in \widehat{N}_\lambda$, we have

$$\gamma_p(u_n) + \|\nabla u_n\|_q^q = \lambda \|u_n\|_p^p \quad \text{for all } n \in \mathbb{N}, \tag{30}$$

and

$$\int_\Omega \xi(z)|u_n|^{p-2}u_n dz + \int_{\partial\Omega} \beta(z)|u_n|^{p-2}u_n d\sigma = \lambda \int_\Omega |u_n|^{p-2}u_n dz \quad \text{for all } n \in \mathbb{N}. \tag{31}$$

We use (18) in (30), so we obtain

$$\|u_n\|^p + c_1\|\nabla u_n\|_q^q \leq [\lambda c_1 + c_2]\|u_n\|_p^p. \tag{32}$$

If $u_\lambda = 0$, then from (29) and (32) we infer that $u_n \to 0$ in $W^{1,p}(\Omega)$.

Then we set $y_n = \dfrac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$, and reasoning as in the proof of Proposition 4 (see the last part of the proof after (26)), we reach a contradiction. Therefore $u_\lambda \neq 0$. Passing to the limit as $n \to +\infty$ in (31) and using (29), we obtain

$$\int_\Omega \xi(z)|u_\lambda|^{p-2}u_\lambda dz + \int_{\partial\Omega} \beta(z)|u_\lambda|^{p-2}u_\lambda d\sigma = \lambda \int_\Omega |u_\lambda|^{p-2}u_\lambda dz. \tag{33}$$

From (30) and since in a Banach space the norm functional is weakly lower semicontinuous, we obtain

$$\gamma_p(u_\lambda) + \|\nabla u_\lambda\|_q^q \leq \lambda\|u_\lambda\|_p^p.$$

Suppose that this inequality is strict, namely we have

$$\gamma_p(u_\lambda) + \|\nabla u_\lambda\|_q^q < \lambda\|u_\lambda\|_p^p. \tag{34}$$

As in the proof of Proposition 1, choosing

$$t = \left[\frac{\|\nabla u_\lambda\|_q^q}{\lambda\|u_\lambda\|_p^p - \gamma_p(u_\lambda)}\right]^{\frac{1}{p-q}} \in (0,1) \quad \text{(see (34))},$$

we have $tu_\lambda \in N_\lambda$. Moreover, since equality (31) is $(p-1)$-homogeneous, we also have that it is satisfied by tu_λ. Hence finally we can say that $tu_\lambda \in \widehat{N}_\lambda$.

Recall that

$$\varphi_\lambda(u_n) = \left[\frac{1}{q} - \frac{1}{p}\right]\|\nabla u_n\|_q^q \downarrow \widehat{m}_\lambda \quad \text{as } n \to +\infty.$$

Moreover from (29) it follows that

$$u_n \xrightarrow{w} u_\lambda \text{ in } W^{1,q}(\Omega).$$

Therefore in the limit as $n \to +\infty$, we obtain

$$\left[\frac{1}{q} - \frac{1}{p}\right]\|\nabla u_\lambda\|_q^q \leq \widehat{m}_\lambda. \tag{35}$$

Since $tu_\lambda \in \widehat{N}_\lambda$, we have

$$\begin{aligned} 0 < \widehat{m}_\lambda &\leq \varphi_\lambda(tu_\lambda) \quad \text{(see Proposition 4)} \\ &= t^q\left[\frac{1}{q} - \frac{1}{p}\right]\|\nabla u_\lambda\|_q^q \\ &\leq t^q\widehat{m}_\lambda \quad \text{(see (35))} \\ &< \widehat{m}_\lambda \quad \text{(since } 0 < t < 1), \end{aligned}$$

a contradiction. So, (34) is not true and we have

$$\gamma_p(u_\lambda) + \|\nabla u_\lambda\|_q^q = \lambda\|u_\lambda\|_p^p. \tag{36}$$

Then (33) and (36) imply that $u_\lambda \in \widehat{N}_\lambda$ and $\varphi_\lambda(u_\lambda) = \widehat{m}_\lambda$. □

We set

$$m_\lambda = \inf\left[\varphi_\lambda(u) : u \in N_\lambda\right].$$

Since $\widehat{N}_\lambda \subseteq N_\lambda$, we have $m_\lambda \leq \widehat{m}_\lambda$ and in principle we can have strict inequality. However, we will show that this can not happen and in fact we have $m_\lambda = \widehat{m}_\lambda$. So, the extra restriction introduced in the definition of $\widehat{N}_\lambda$ compared to the Nehari manifold N_λ, is harmless and does not have any effect in the constrained minimization.

Proposition 6. *If hypotheses H_0 hold and $\lambda \in (\widehat{\lambda}_1, \lambda_0)$, then $m_\lambda = \widehat{m}_\lambda$.*

Proof. We argue by contradiction. So, suppose that $m_\lambda < \widehat{m}_\lambda$. This means that we can find $u \in N_\lambda \setminus \widehat{N}_\lambda$ such that

$$\varphi_\lambda(u) < m_\lambda. \tag{37}$$

Since $u \in N_\lambda \setminus \widehat{N}_\lambda$, we have that $\widehat{\vartheta}_\lambda(u) \neq 0$. Then either $\widehat{\vartheta}_\lambda(u) > 0$ or $\widehat{\vartheta}_\lambda(u) < 0$.

Case 1: $\widehat{\vartheta}_\lambda(u) > 0$.

Reasoning as in the proof of Proposition 1, we can find a smallest $\eta_0 > 0$ such that

$$\widehat{\vartheta}_\lambda(u - \eta_0) = 0. \tag{38}$$

Then consider the C^1-functional $\mu_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\mu_\lambda(y) = \int_\Omega \xi(z)|y|^p dz + \int_{\partial\Omega} \beta(z)|y|^p d\sigma - \lambda\|y\|_p^p \quad \text{for all } y \in W^{1,p}(\Omega).$$

Exploiting the fact that $\lambda < \lambda_0$, we see that the function $\eta \to \mu_\lambda(u - \eta)$ is strictly decreasing on $[0, \eta_0]$. Therefore

$$\mu_\lambda(u - \eta_0) < \mu_\lambda(u). \tag{39}$$

We want to find $t > 0$ such that $t(u - \eta_0) \in N_\lambda$. So, we will have

$$\langle \varphi'_\lambda(t(u - \eta_0)), t(u - \eta_0)\rangle = 0.$$

On account of (39) and since $u \in N_\lambda$, we have

$$0 < \lambda\|u - \eta_0\|_p^p - \gamma_p(u - \eta_0).$$

So, if we choose

$$t = t_\lambda = \left[\frac{\|\nabla u\|_q^q}{\lambda\|u - \eta_0\|_p^p - \gamma_p(u - \eta_0)}\right]^{\frac{1}{p-q}},$$

then we have

$$t_\lambda(u - \eta_0) \in N_\lambda \quad \text{(see the proof of Proposition 1)}. \tag{40}$$

From (39) and since $\nabla u = \nabla(u - \eta_0)$, we have

$$\begin{aligned} t_\lambda &= \left[\frac{\|\nabla u\|_q^q}{-\mu_\lambda(u - \eta_0) + \|\nabla u\|_p^p}\right]^{\frac{1}{p-q}} \quad \text{(recall the definition of } \mu_\lambda(\cdot)) \\ &< \left[\frac{\|\nabla u\|_q^q}{-\mu_\lambda(u) + \|\nabla u\|_p^p}\right]^{\frac{1}{p-q}} \quad \text{(see (39))} \\ &= \left[\frac{\|\nabla u\|_q^q}{\lambda\|u\|_p^p - \gamma_p(u)}\right]^{\frac{1}{p-q}} \quad \text{(from the definition of } \mu_\lambda(\cdot)) \end{aligned}$$

$$= \left[\frac{\|\nabla u\|_q^q}{\|\nabla u\|_q^q}\right]^{\frac{1}{p-q}} = 1 \quad (\text{recall } u \in N_\lambda). \tag{41}$$

From (38) and the $(p-1)$-homogeneity of $\widehat{\vartheta}_\lambda(\cdot)$ we have

$$\begin{aligned} &\widehat{\vartheta}_\lambda(t_\lambda(u-\eta_0)) = 0, \\ \Rightarrow \quad & t_\lambda(u-\eta_0) \in \widehat{N}_\lambda \quad (\text{see (40)}). \end{aligned}$$

Therefore we have

$$\begin{aligned} \widehat{m}_\lambda &\le \varphi_\lambda(t(u-\eta_0)) \\ &= t_\lambda^q \left[\frac{1}{q} - \frac{1}{p}\right] \|\nabla u\|_q^q \\ &< \left[\frac{1}{q} - \frac{1}{p}\right] \|\nabla u\|_q^q \quad (\text{see (41)}) \\ &= \varphi_\lambda(u) < m_\lambda \quad (\text{see (37)}), \end{aligned}$$

a contradiction.

Case 2: $\widehat{\vartheta}_\lambda(u) < 0$.

This case is treated similarly. In the present case, we produce $\eta_0 > 0$ such that

$$\widehat{\vartheta}_\lambda(u+\eta_0) = 0.$$

Using $u+\eta_0$ we argue as in Case 1 above and again we reach a contradiction. Therefore we conclude that $m_\lambda = \widehat{m}_\lambda = \varphi_\lambda(u_\lambda)$. ☐

Consider the C^1-functional $k_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$k_\lambda(u) = \gamma_p(u) + \|\nabla u\|_q^q - \lambda \|u\|_p^p \quad \text{for all } u \in W^{1,p}(\Omega).$$

We have

$$m_\lambda = \inf\left[\varphi_\lambda(u) : k_\lambda(u) = 0\right]. \tag{42}$$

Then Proposition 6, (42) and the Lagrange multiplier rule (see Gasiński-Papageorgiou [8], Theorem 3.29), will imply that u_λ is an eigenfunction for the eigenvalue $\lambda \in (\widehat{\lambda}_1, \lambda_0)$.

Theorem 1. *If hypotheses H_0 hold, then every $\lambda \in (\widehat{\lambda}_1, \lambda_0)$ is an eigenvalue of problem (P_λ) with corresponding eigenfunction $u_\lambda \in W^{1,p}(\Omega) \setminus \{0\}$, $u_\lambda \ge 0$.*

Proof. According to Proposition 6, we have

$$m_\lambda = \widehat{m}_\lambda = \varphi_\lambda(u_\lambda). \tag{43}$$

First we show that

$$R(k_\lambda'(u_\lambda)) = \mathbb{R}. \tag{44}$$

To this end, let $\eta \in \mathbb{R}$. We need to show that there exists $h \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} p\Bigg[\int_\Omega |\nabla u_\lambda|^{p-2}(\nabla u_\lambda, \nabla h)_{\mathbb{R}^N} dz + \int_\Omega \xi(z)|u_\lambda|^{p-2}u_\lambda h dz + \int_{\partial\Omega} \beta(z)|u_\lambda|^{p-2}u_\lambda h d\sigma \\ - \lambda \int_\Omega |u_\lambda|^{p-2}u_\lambda h dz\Bigg] + q\int_\Omega |\nabla u_\lambda|^{q-2}(\nabla u_\lambda, \nabla h)_{\mathbb{R}^N} dz = \eta. \end{aligned} \tag{45}$$

In (45) we choose $h = \widehat{\mu} u_\lambda$ with $\widehat{\mu} \in \mathbb{R}$. Then

$$\widehat{\mu}\left[p\gamma_p(u_\lambda) - p\lambda\|u_\lambda\|_p^p\right] + \widehat{\mu} q \|\nabla u_\lambda\|_q^q = \eta,$$

$$\Rightarrow \quad \widehat{\mu}[q-p]\|\nabla u_\lambda\|_q^q = \eta \quad (\text{recall } u_\lambda \in \widehat{N}_\lambda \subseteq N_\lambda).$$

So, if we choose $\widehat{\mu} = \frac{\eta}{[q-p]\|\nabla u_\lambda\|_q^q} \in \mathbb{R}$, then for $h = \widehat{\mu} u_\lambda \in W^{1,p}(\Omega)$ we have

$$\langle k'_\lambda(u_\lambda), h\rangle = \eta.$$

Since $\eta \in \mathbb{R}$ is arbitrary, we conclude that (44) holds.

Applying the Lagrange multiplier rule, we can find $\tau \in \mathbb{R}$ such that

$$\varphi'_\lambda(u_\lambda) + \tau k'_\lambda(u_\lambda) = 0 \quad (\text{see } (43)).$$

If $\tau \neq 0$, then we have

$$\begin{aligned} & \langle \varphi'_\lambda(u_\lambda), u_\lambda\rangle + \tau\langle k'_\lambda(u_\lambda), u_\lambda\rangle = 0, \\ \Rightarrow \quad & \tau\langle k'_\lambda(u_\lambda), u_\lambda\rangle = 0 \quad (\text{since } u_\lambda \in \widehat{N}_\lambda \subseteq N_\lambda), \\ \Rightarrow \quad & \langle k'_\lambda(u_\lambda), u_\lambda\rangle = 0 \quad (\text{since } \tau \neq 0), \\ \Rightarrow \quad & [q-p]\|\nabla u_\lambda\|_q^q = 0. \end{aligned}$$

Since $q \neq p$, we have that $\|\nabla u_\lambda\|_q = 0$, hence $u_\lambda = c \in \mathbb{R}$. Recall that $u_\lambda \in \widehat{N}_\lambda$. Then

$$\begin{aligned} & |c|^{p-2}c\left[\int_\Omega \xi(z)dz + \int_{\partial\Omega} \beta(z)d\sigma - \lambda|\Omega|_N\right] = 0, \\ \Rightarrow \quad & c = 0 \quad (\text{since } \lambda < \lambda_0). \end{aligned}$$

Therefore $u_\lambda = 0$, which is a contradiction since $u_\lambda \in \widehat{N}_\lambda \subseteq N_\lambda$.

Clearly we can replace u_λ by $|u_\lambda|$ and so we can say that $u_\lambda \geq 0$. □

If we strengthen the conditions on the potential function $\xi(\cdot)$ and on the boundary coefficient $\beta(\cdot)$, we can improve the conclusion of Theorem 1.

The new hypotheses on $\xi(\cdot)$ and $\beta(\cdot)$ are the following:

H'_0: $\xi \in L^\infty(\Omega)$, $\beta \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$, $\beta(z) \geq 0$ for all $z \in \partial\Omega$, $\beta \not\equiv 0$ and

$$\int_\Omega \xi(z)dz + \int_{\partial\Omega} \beta(z)d\sigma > 0.$$

In what follows we use the Banach space $C^1(\overline{\Omega})$. This is an ordered Banach space with positive (order) cone $C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

Theorem 2. *If hypotheses H'_0 hold, then every $\lambda \in (\widehat{\lambda}_1, \lambda_0)$ is an eigenvalue of problem (P_λ) with corresponding eigenfunction $u_\lambda \in \operatorname{int} C_+$.*

Proof. From Theorem 1 we already know that every $\lambda \in (\widehat{\lambda}_1, \lambda_0)$ is an eigenvalue of (P_λ) with eigenfunction $u_\lambda \in W^{1,p}(\Omega)$, $u_\lambda \geq 0$. From Papageorgiou-Rădulescu [13], we have that $u_\lambda \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [9] implies that $u_\lambda \in C_+ \setminus \{0\}$.

We have

$$\begin{aligned} & \Delta_p u_\lambda + \Delta_q u_\lambda \leq [\|\xi\|_\infty + |\lambda|]u_\lambda^{p-1} \quad \text{for a.a. } z \in \Omega, \\ \Rightarrow \quad & u_\lambda \in \operatorname{int} C_+ \quad (\text{see Pucci-Serrin [16], pp. 111, 120}). \end{aligned}$$

□

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