AREA QUALITÀ, PROGRAMMAZIONE E SUPPORTO STRATEGICO
SETTORE STRATEGIA PER LA RICERCA
U. O. DOTTORATI

Dottorato di Ricerca in Matematica e Scienze Computazionali
Dipartimento di Matematica e Informatica
MAT/02

Divisoriality and m-canonical ideal for quadratic quotients of the Rees algebra

IL DOTTORE
FABIO FRIGENTI

IL COORDINATORE
MARIA CARMELA LOMBARDO

IL TUTOR
MARCO D'ANNA

IL CO TUTOR
CARMELO ANTONIO FINOCCHIARO

CICLO XXXV
ANNO CONSEGUIMENTO TITOLO 2024

UNIVERSITY OF PALERMO
PHD JOINT PROGRAM:
UNIVERSITY OF CATANIA - UNIVERSITY OF MESSINA
XXXV CYCLE

DOCTORAL THESIS

Divisoriality and m-canonical ideal for quadratic quotients of the Rees algebra

Author:
Frigenti Fabio

Supervisor:
Prof. Marco D'Anna

Co-Supervisor:
Prof. Carmelo Antonio Finocchiaro

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

in

Mathematics and Computational Sciences

February 12, 2024

Signed: 

Date: February 12, 2024

Dedicated to my wife and my daugther

UNIVERSITY OF PALERMO

Abstract

Department of Mathematics and Computer Sciences

Doctor of Philosophy

Divisoriality and m-canonical ideal for quadratic quotients of the Rees algebra

by Frigenti Fabio

Let R be a one-dimensional Cohen-Macaulay local ring and I an ideal of R. It is well known that both the classical construction of Nagata's idealization $R \ltimes I$ and the recent construction $R \bowtie I$, known as amalgamated duplication, are Gorenstein when I is a canonical ideal of R. This property holds also for a more general family of rings, the quadratic quotients of the Rees algebra associated to R with respect to an ideal I and the elements $a, b \in R$, defined in an attempt to provide a unified approach of the two construction above. Since for a one-dimensional Noetherian domain the Gorenstein property is equivalent to the divisorial property, our pourpose is to understand, in a more general setting, when a quadratic quotient $R(I)_{a,b}$ is divisorial when I is an m-canonical ideal of R.

Acknowledgements

I would like to express my deep and sincere gratitude to my research supervisor, Prof. Marco D'Anna and to my Co-Supervisor Prof. Carmelo Antonio Finocchiaro, for providing invaluable guidance throughout this research. Their friendship, empathy, sincerity and motivation have deeply inspired me. They have taught me the methodology to carry out the research and to present the research works as well as possible. It was a great privilege and honor to work and study under their guidance. I am extremely grateful for what they have offered me and for giving me the opportunity to become a teacher.

I am extremely grateful to my parents for their love, prayers, caring and sacrifices for educating and preparing me for my future. I am very much thankful to my wife and my daughter for their love, understanding and continuing support to complete this research work. Also I express my thanks to my sister, brother in laws for their constant support.

Contents

Introduction

Let R be a commutative ring with unit and M an R-module. M. Nagata, 1962, in an attempt to derive primary decomposition of modules from primary decomposition of ideals, introduced a sort of "ringification", i.e., put M inside a new commutative ring A in such a way that M becomes an ideal of this new ring. This construction is known as the idealization of M over R. More precisely, this ring, usually indicated by $R \ltimes M$, is the set $R \times M$ endowed with the ring structure whose addition is defined componentwise and whose multiplication is defined by setting:

$$(r_1, m_1)(r_2, m_2) := (r_1 r_2, r_1 m_2 + r_2 m_1),$$

for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$. This construction is very useful for producing examples and counterexamples in commutative ring theory and for reducing generalizing results from rings to modules, but the versatility of this ring is limited by the fact that $R \ltimes M$ is not reduced whenever $M \neq 0$; indeed M becomes an ideal whose square is zero.

A remarkable property of the idealization was discovered by I. Reiten, 1972, where it is proved that if R is a Cohen-Macaulay ring then $R \ltimes M$ is Gorenstein if and only if M is a canonical module of R.

More recently, D'Anna and Fontana, 2007a, introduced a new ring construction that behaves similarly to Nagata's idealization. Starting from a ring R and an R-submodule M of its total ring of fractions $Q(R)$ such that $M \cdot M \subseteq M$, they defined a new ring, called the *amalgamated duplication of* R *along* M, as the following subring

$$R \bowtie M := \{(r, r+m) \mid r \in R, m \in M\}$$

of $R \times Q(R)$. One of the main differences with respect to the idealization is that this new family of rings can contain reduced rings, in particular $R \bowtie M$ is always reduced when R is a domain. As for the idealization, M. D'Anna proved that when $E = I$ is a proper ideal of R and R is a local Cohen-Macaulay ring with canonical module ω_R, then $R \bowtie I$ is Gorenstein if and only if $I \cong \omega_R$ (see D'Anna, 2006). In the attempt to provide a unified approach, Barucci, D'Anna, and Strazzanti, 2015, introduced a new family of rings that generalize both the two construction recalled above: starting from a commutative ring R with identity, a nonzero proper ideal I of R and $a, b \in R$, consider the quotient of the Rees algebra $\mathcal{R}$ associated to R with respect I, by the contraction in $\mathcal{R}$ of the ideal of $R[t]$ generated by the polynomial $t^2 + at + b$. This new family of rings, denoted with $R(I)_{a,b}$, generalizes the Nagata's idealization, $R \ltimes I$, and the amalgamated duplication $R \bowtie I$. In fact when $a = b = 0$ we have that $R(I)_{0,0} \cong R \ltimes I$ and, when $a = -1, b = 0$, we have that $R(I)_{-1,0} \cong R \ltimes I$. Another remarkable fact about this construction is that the ring $R(I)_{a,b}$ inherits many properties of the ring R which do not depend on the choice of the elements a, b. For instance R and $R(I)_{a,b}$ share the same Krull dimension, $R(I)_{a,b}$ is Noetherian if and only if R is Noetherian, $R(I)_{a,b}$ is local if and only if R is local. Another interesting property which $R(I)_{a,b}$ inherits from R, in case R local Cohen-Macaulay, is the property to have a canonical reduction, i.e., the ring admits a canonical ideal that is a reduction of the maximal ideal (see Frigenti, 2019): in Theorem 3.2 it is shown that when R is a one-dimensional Cohen-Macaulay local ring and admits a canonical reduction J, then $R(I)_{a,b}$ has a canonical reduction for all $a, b \in R$ and for all I such that $J \subseteq I$.

The canonical ideal, introduced by Herzog and Kunz, 1971, is a particular fractional ideal ω of R which admits the duality property that $\omega : (\omega : I) = I$ for every regular ideal I of R. When R is a one-dimensional local Cohen-Macaulay ring and its integral closure $\overline{R}$ is finitely generated as R-module, Beweis 4. of Satz 3.6 of Herzog and Kunz, 1971, shows the existence of a canonical ideal ω of R such that $R \subseteq \omega \subseteq \overline{R}$. In this context Barucci, D'Anna, and Strazzanti, 2016, showed that $R(I)_{a,b}$ admits also a canonical ideal $\omega_{R(I)_{a,b}}$ such that $\omega_{R(I)_{a,b}} \cong \frac{1}{z}(\omega : I) + \frac{1}{z}\omega$; so in this particular case, the ring $R(I)_{a,b}$ inherits the property of R to have a canonical ideal. One of the most interesting consequences for a ring R that admits a canonical ideal I, in case R is a local one-dimensional Noetherian domain, is that $R(I)_{a,b}$ is Gorenstein if and only if I is a canonical ideal of R (see Corollary 3.3 of Barucci, D'Anna, and Strazzanti, 2015). Since for a one-dimensional Noetherian local domain R Theorem 6.3 of Bass, 1963, states that R is Gorenstein if and only if each non zero ideals of R is divisorial, it is natural to ask in a more general setting, not necessary Noetherian or local, if for a particular ideal I and a particular choice of $a, b \in R$, the domain $R(I)_{a,b}$ is divisorial, that is, every nonzero ideal of $R(I)_{a,b}$ is divisorial. Since the key duality property of the canonical ideal holds only in the one dimensional case, when we work in a more general situation, we need to work with another ideal that satisfies the property that $I : (I : J) = J$ for all fractional ideals J of R. This concept was introduced by Heinzer, Huckaba, and Papick, 1998, where the authors defined for an integral domain R the multiplicative canonical (briefly m-canonical), as an ideal I satisfying the above equality; moreover, in the one-dimensional Cohen-Macaulay case the notion of canonical ideal and m-canonical ideal coincide.

The aim of this thesis is to understand in which cases the domain $R(I)_{a,b}$ is divisorial, when I is an m-canonical ideal of R, trying to generalize what happens in the one-dimensional Noetherian local case. The class of divisorial domains is studied by Heinzer, 1968, in particular he characterized integrally closed divisorial domains as h-local Prüfer domains each of whose maximal ideals is finitely generated.

In the first chapter, we recall the construction of the quadratic quotients of the Rees algebra associated to R with respect an ideal I and $a, b \in R$ and we recall some useful results for our purposes. Moreover, with terminology used in Bazzoni and Salce, 1996, we recall the notion of reflexivity with respect an

R-submodule of $Q(R)$, and the notion of m-canonical ideal, together with its main properties. We focus our attention on a particular construction, introduced by Barucci et al., 2019, of domains with an m-canonical ideal and we propose another construction which provides a particular non-Noetherian domain with an m-canonical ideal. In the second chapter, we start by presenting a key isomorphism that links the I-reflexivity of each regular ideal of $R(I)_{a,b}$ to its reflexivity whit respect the $R(I)_{a,b}$-module $Hom_R(R(I)_{a,b}, I)$, providing the idea that starting with a domain R with an m-canonical ideal the quotient $R(I)_{a,b}$ have also an m-canonical ideal, under the hypothesis that each regular ideal of $R(I)_{a,b}$ is I-reflexive. Moreover, since in such a case $Hom_R(R(I)_{a,b}, I)$ is free as $R(I)_{a,b}$-module, $R(I)_{a,b}$ will be m-canonical of itself then divisorial. Thus, in case $R(I)_{a,b}$ is divisorial, we infer that $R(I)_{a,b}$ inherits form R the property to have an m-canonical ideal. Following this reasoning, and starting from the fact that in the one-dimensional Noetherian local case this always happens, we continue our work trying to understand which other classes of domains with an m-canonical ideal I produce a divisorial quotient $R(I)_{a,b}$. Since the h-local property is a necessary condition for a domain to be divisorial or to have an m-canonical ideal, we are interested to understand when $R(I)_{a,b}$ is h-local starting from a domain R which is also h-local. In the one-dimensional case, the h-local property is equivalent to the property to have a finite character and we can prove that R has this property if and only if $R(I)_{a,b}$ also has it. Moreover, when R is h-local and $a = 0$, we prove that $R(I)_{0,-b}$ is an h-local domain for all b for which $R(I)_{0,-b}$ is a domain. For dimensions higher than one and when the polynomial is $t^2 + at + b$ with $a \neq 0$, $R(I)_{a,b}$ can fail to inherit the h-local property from R and we present an appropriate counterexample. Known that the h-local property is assured in the one-dimensional case, the first class of domains which are of our interest are Dedekind domains, since for this kind of rings all principal ideals are m-canonical. In particular, when the domain R is a PID, we can prove that $R(I)_{a,b}$ is divisorial for all proper ideals I and for all $a, b \in R$ such that the polynomial $t^2 + at + b$ is irreducible in $Q(R)[t]$. When R is a Dedekind domain but not a PID we can prove the same result for quotients of type $R(I)_{0,-b}$ for all principal ideal I and for all $b \in R$ which is not a square in $Q(R)$. More generally, since the divisoriality is a local property, the same result holds for a generic quotient $R(I)_{0,-b}$, where R is a one dimensional Noetherian domain with an m-canonical ideal I and $b \in R$ a non-square in $Q(R)[t]$. After that, it is natural to focus our attention on the non-Noetherian case, in particular, since we have to work with a domain which have an m-canonical ideal, we study the case when we start from a valuation domain V with non-principal maximal ideal $\mathfrak{m}$; for a valuation domain, in fact, it is well known that the maximal ideal $\mathfrak{m}$ is always an m-canonical ideal. Fuchs and Salce, 2001, provided a useful necessary condition for a local domain with non-principal maximal ideal to be divisorial. This result could be used to provide, if there exists, an example of valuation domain V with non principal maximal ideal $\mathfrak{m}$ such that the quotient $V(\mathfrak{m})_{0,-b}$ is not divisorial for some $b \in V$. In the last part of the work, we present different opened problems. The first refers to a conjecture formulated by Matlis, 1968, then confirmed by Goeters, 1999, who suspected the existence of a divisorial domain with a maximal ideal which is not two-generated; to give an answer to this conjecture, we present an example of divisorial domain with a maximal ideal that could not be two-generated. Finally, in an attempt to find an example of non-Noetherian valuation domain with quotient $V(\mathfrak{m})_{a,b}$ not divisorial, we present a possible costruction of this type of rings.

Chapter 1

Some results about $R(I)_{a,b}$

1.1 Preliminaries

Throughout this thesis with ring R we mean a commutative ring with unit and we denote by $Q(R)$ its total ring of fractions. For M, N two R-modules, we will denote with

$$(M : N) := (M :_{Q(R)} N),$$

where $(M :_{Q(R)} N) := \{x \in Q(R) \mid xN \subseteq M\}$. In some example, with $\mathbb{Z}_p$ we indicate the quotient $\mathbb{Z}/p\mathbb{Z}$, where p is a prime element of $\mathbb{Z}$.

Definition 1.1. *An R-submodule F of $Q(R)$ is regular if $F \cap R$ is a regular ideal of R.*

Definition 1.2. *We say that a R-submodule F of $Q(R)$ is a fractional ideal of R if there exists a regular element $r \in R \setminus \{0\}$ such that $rF \subseteq R$. We say that a fractional ideal F is regular if $FQ(R) = Q(R)$. We denote with $\mathcal{F}(R)$ the set of fractional ideals of R and with $\mathcal{F}(R)^*$ the set of regular fractional ideals of R.*

Definition 1.3. *Let N be an R-module, we define the rank of N, $rk(N)$, the maximum number of elements of N which are R-independent.*

Let's start by recalling some properties of the *Nagata's idealization* (see Nagata, 1962 and Anderson and Winders, 2009), a classical construction which has had notable results in commutative algebra. Starting from a ring R and an R-module M, the idealization $R \ltimes M$ is a commutative ring with unit with componentwise addition and multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$. For every R-submodule N of M, then $0 \ltimes N$ is an ideal of $R \ltimes M$, in particular $0 \ltimes M$ is a nilpotent ideal of $R \ltimes M$ of index 2. For every ideal I of R and R-submodule N of M, $I \ltimes N$ is an ideal of $R \ltimes M$ if and only if $IM \subseteq N$ (Anderson and Winders, 2009, Theorem 3.1). The maximal ideals of $R \ltimes M$ have the form $\mathfrak{m} \ltimes M$, where $\mathfrak{m}$ is a maximal ideal of R, so $R \ltimes M$ is local if and only if R is local and have the same set of residue fields. The prime ideals of $R \ltimes M$ have the form $\mathfrak{p} \ltimes M$ where $\mathfrak{p}$ is a prime ideal of R. Moreover $ht(\mathfrak{p} \ltimes M) = ht(\mathfrak{p})$ and so $dim(R \ltimes M) = dim(R)$ (Anderson and Winders, 2009, Theorem 3.2). The set of zerodivisors of $R \ltimes M$ is $Z(R \ltimes M) = \{(r, m) \mid r \in Z(R) \cup Z(M)\}$, hence $S \ltimes M$ where $S = R \setminus (Z(R) \cup Z(M))$ is the set of regular elements (Anderson and Winders, 2009, Theorem 3.5). If S is a multiplicatively closed subset of R and N is an R-submodule of M, then $S \ltimes N$ is a multiplicatively closet subset of $R \ltimes M$ (Anderson and Winders, 2009, Theorem 3.8), moreover the localization $(R \ltimes M)_{S \ltimes N}$ is naturally isomorphic to $R_S \ltimes M_S$ and so the total ring of fraction $T(R \ltimes M)$ of $R \ltimes M$ is naturally isomorphic to $R_S \ltimes M_S$, where $S = R \backslash (Z(R) \cup Z(M))$ (Anderson and Winders, 2009, Theorem 4.1). $R \ltimes M$ is Noetherian, respectively Artinian, if and only if R is Noetherian, respectively Artinian, and M is finitely generated (Anderson and Winders, 2009, Theorem 4.8). If R is a Cohen-Macaulay ring, then $R \ltimes M$ is Gorenstein if and only if M is a canonical module of R (Reiten, 1972, Theorem 7).

A more recent construction that behaves similar to Nagata's idealization from several points of view, is the so called *amalgamated duplication* (see D'Anna and Fontana, 2007a and D'Anna and Fontana, 2007b). Starting from a ring R and an R-module M of $Q(R)$, this new family of rings denoted by $R \bowtie M$, is defined as the set $R \oplus M$ endowed with componentwise addition and multiplication defined by setting $(r_1, m_1)(r_2, m_2) := \{(r_1 r_2, r_1 m_2 + r_2 m_1 + m_2 m_1)\}$. This operations make $R \bowtie M$ a commutative ring with unit. In the particular case when $E = I$ is an ideal of R, the ring $R \bowtie I$ satisfies properties similar to those the Nagata's idealization. Indeed, by Corollary 3.3 of D'Anna and Fontana, 2007a $dim(R \bowtie I) = dim(R)$ and R is Noetherian if and only if $R \bowtie I$ is Noetherian. With regard to the prime spectrum of $R \bowtie I$, it is more complicated than the prime spectrum of the idealization, for a more accurate description we refer to D'Anna and Fontana, 2007a. As for idealization in the case R is Cohen-Macaulay, $R \bowtie I$ is Gorenstein in and only if I is a canonical ideal of R (D'Anna, 2006, Theorem 11).

1.1.1 m-canonical ideal

In this section we recall some results about m-canonical ideals, that we can find in Heinzer, Huckaba, and Papick, 1998 and Barucci et al., 2019. For the rest of section, we denote by R an integral domain.

Proposition 1.4. *The following properties hold.*
i) $\mathcal{F}(R)$ is closed under sum, intersection and multiplication.
ii) For $F \in \mathcal{F}(R)$ and $H \in \mathcal{F}(R)^*$, then $(F : H) \in \mathcal{F}(R)$.

Proposition 1.5. *Let I be an ideal of R and $F \in \mathcal{F}(R)^*$, then $(I : F)$ is isomorphic to $Hom_R(F, I)$.*

Given two R-modules F, H, we have a canonical homomorphism:

$$\rho_F : F \to \mathrm{Hom}_R(\mathrm{Hom}_R(F, H), H)$$

$$a \longmapsto \rho_F(a) : \mathrm{Hom}_R(F, H) \to H \quad \text{where} \;\; \rho_F(a)(f) = f(a)$$

for all $f \in \mathrm{Hom}_R(F, H)$ and $a \in F$.

Definition 1.6. *We say that an R-module F is H-*torsionless *if ρ_F is a monomorphism, H-*reflexive *if ρ_F is an isomorphism.*

Following Bazzoni and Salce, 1996, we recall the following notation:

Definition 1.7. *If H is a given R-submodule of $Q(R)$, we say that R is H-*divisorial *(respectively H-*reflexive*), if every H-torsionless $End_R(H)$-module of rank one (resp. of finite rank) is H-reflexive. R-reflexive and R-divisorial rings will be simply called "reflexive" and "divisorial" respectively.*

Remark 1.8. *If I is an ideal of R and $J \in \mathcal{F}(R)^*$, then the map ρ_J corresponds to the inclusion $J \subseteq (I : (I : J))$, so it is a monomorphism, therefore J is I-torsionless.*

Definition 1.9. *We say that the ideal I of R is a m-canonical ideal if each $J \in \mathcal{F}(R)^*$ is I-reflexive.*

In the following we recall several properties of the m-canonical ideal.

Lemma 1.10 (Lemma 2.2, Heinzer, Huckaba, and Papick, 1998)**.** *Let I be an m-canonical ideal of a domain R. Then:*

1) $(I : I) = R$.

2) *If I is a prime ideal, then I is a maximal ideal.*

3) $I : (I : J) = J$ *for each non zero fractional ideal J of R.*

Remark 1.11. *In order to verify that the ideal I is m-canonical, it is sufficient to test the I-reflexivity for all the ideals of R. In fact, every $J \in \mathcal{F}(R)^*$ is I-reflexive if and only if dJ is I-reflexive, for all $d \in R$ such that $dJ \subseteq R$.*

Proof. If $J \in \mathcal{F}(R)^*$ is I-reflexive, then $J = (I : (I : J))$. For every element $d \in R$, $dJ \in \mathcal{F}(R)^*$ and then $(I : dJ) \in \mathcal{F}(R)$ and $(I : (I : dJ)) \in \mathcal{F}(R)$; $dJ = d(I : (I : J)) = (I : (I : dJ))$. Conversely, if dJ is I-reflexive for all regular element $d \in R$ such that $dJ \subseteq R$, then $dJ = (I : (I : dJ)) = d(I : (I : J))$; since d is regular, then $J = (I : (I : J))$. □

Corollary 1.12. *I is m-canonical if and only if each ideal J of R is I-reflexive.*

Definition 1.13. *A domain R is said to have* finite character *if every non zero ideal of R is contained only in finitely many maximal ideals.*

Example 1.14. *Local and semilocal Dedekind domains have the finite character. On the other hand $k[x, y]$, with k a field, does not have the finite character.*

Definition 1.15. *A domain R is said to be h-*local *if it satisfies the following two conditions:*
1) R has finite character;
2) every non-zero prime ideal of R is contained in a unique maximal ideal.

Remark 1.16. *If R is a one-dimensional domain, since every nonzero prime ideal is maximal, then R is h-local if and only if R has finite character.*

A necessary condition for a domain R to have an m-canonical ideal is the following:

Proposition 1.17 (Proposition 2.4, Heinzer, Huckaba, and Papick, 1998)**.** *If R has an m-canonical ideal, then R is h-local.*

Proposition 1.18 (Corollary 3.4, Heinzer, Huckaba, and Papick, 1998)**.** *Let I be an ideal of R such that $(I : I) = R$. If J is a divisorial fractional ideal, then J is I-divisorial.*

Remark 1.19 (Remark 3.7, Heinzer, Huckaba, and Papick, 1998)**.** *From the last proposition, if D is a Dedekind domain, then each non zero ideal of D is m-canonical.*

Proposition 1.20 (Proposition 4.3, Heinzer, Huckaba, and Papick, 1998)**.** *Let R be a Noetherian domain. If R has an m-canonical ideal, then $dim(R) \leq 1$.*

Proposition 1.21 (Proposition 6.2, Heinzer, Huckaba, and Papick, 1998)**.** *Let $(R, \mathfrak{m})$ be a local integrally closed domain. Then, $\mathfrak{m}$ is an m-canonical ideal for R if and only if R is a valuation domain.*

Theorem 4.7 of Bazzoni and Salce, 1996 shows that R is I-divisorial, for a proper nonzero submodule I of $Q(R)$, if and only if the endomorphism ring S of I is h-local and every localization at maximal ideal $\mathfrak{m}$ of S is $I_{\mathfrak{m}}$-divisorial. Thus the converse of Proposition 5.5 of Heinzer, Huckaba, and Papick, 1998 holds true:

Proposition 1.22. *Let I be a non zero ideal of R such that $(I : I) = R$. Then I is an m-canonical ideal of R if and only if R is h-local and $I_{\mathfrak{m}}$ is an m-canonical ideal for $R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R.*

1.2 Quadratic quotients of the Rees algebra

In this chapter, we recall some helpful results for the next part of the work. All unproven results and definitions can be found in Barucci, D'Anna, and Strazzanti, 2015 and D'Anna and Strazzanti, 2017. Let $I \neq 0$ be a proper ideal of a ring R and let $a, b \in R$. We consider the Rees algebra associated to R and I, defined as the following subring of $R[t]$:

$$\mathcal{R} := \bigoplus_{n \geq 0} I^n t^n$$

Finally let $R(I)_{a,b}$ denote the factor ring of $\mathcal{R}$ modulo the contraction in $\mathcal{R}$ of the principal ideal of $R[t]$ generated by the monic polynomial $t^2 + at + b$, that is,

$$R(I)_{a,b} := \mathcal{R}/(\mathcal{R} \cap (t^2 + at + b)R[t]).$$

The ring $R(I)_{a,b}$ is also known as *the quadratic quotient of the Rees algebra associated to R with respect to I and the polynomial $t^2 + at + b$*. As an R-module, $R(I)_{a,b}$ is isomorphic to $R \oplus I$: more precisely, given a polynomial $g \in \mathcal{R}$, there is a unique pair $(r, i) \in R \oplus I$ such that $r + it$ is the representative of the equivalence class of g in $R(I)_{a,b}$ (Lemma 1.2 Barucci, D'Anna, and Strazzanti, 2015). With a small abuse of notation, we will identify $r + it$ with its equivalence class in $R(I)_{a,b}$. It easily follows from the definition that the multiplication of $R(I)_{a,b}$ is defined by

$$(r + it)(s + jt) = rs - bij + (rj + si - aij)t,$$

for every $r + it, s + jt \in R(I)_{a,b}$.

It is well known that for particular choices of a and b, we get particular rings constructions:

1) if $t^2 + at + b = (t - \alpha)^2$, with $\alpha \in R$, then $R(I)_{a,b}$ is isomorphic to $R \ltimes I$
2) if $t^2 + at + b = (t - \alpha)(t - \beta)$, and $(t - \alpha)$, $(t - \beta)$ are comaximal ideals of $R[t]$, then $R(I)_{a,b}$ is isomorphic to $R \bowtie I$.

Proposition 1.23 (Proposition 1.7, Barucci, D'Anna, and Strazzanti, 2015)**.** *If $Q(R)$ is the total ring of fractions of R, then the total ring of fractions of $R(I)_{a,b}$ is:*

$$Q(R(I)_{a,b}) = \left\{ \frac{r + it}{u} \mid r \in R, i \in I, u \text{ is a regular element of } R \right\}$$

As we can see in the following propositions, the ring $R(I)_{a,b}$ inherits many properties of the starting ring R, and do not depend on the choice of the ideal I or of the elements a, b.

Proposition 1.24 (Proposition 1.3, Barucci, D'Anna, and Strazzanti, 2015)**.** *For all ideal I and $a, b \in R$, the ring extensions*

$$R \subseteq R(I)_{a,b} \subseteq R[t]/(t^2 + at + b)$$

are integral, and thus the three rings have the same Krull dimension.

Proposition 1.25 (Proposition 1.11, Barucci, D'Anna, and Strazzanti, 2015). *The following conditions are equivalent:*
1) R is Noetherian;
2) $R(I)_{a,b}$ is Noetherian for all $a, b \in R$;
3) $R(I)_{a,b}$ is Noetherian for some $a, b \in R$.

Proposition 1.26 (Proposition 2.1, Barucci, D'Anna, and Strazzanti, 2015). *R is a local ring of maximal ideal $\mathfrak{m}$, if and only if $R(I)_{a,b}$ is a local ring. In this case, the maximal ideal of $R(I)_{a,b}$ is $\mathfrak{m} + It$.*

An interesting variation of quotient of Rees algebras was introduced by Licata, 2022 as follows. Given a ring R, consider an element $b \in R$ and a fractional ideal $I \subseteq Q(R)$ of R such that $bI^2 \subseteq R$. Then consider the following subring

$$R + It := \{\alpha + \beta t \mid \alpha \in R, \beta \in I\}$$

of the factor ring $\frac{Q(R)[t]}{(t^2-b)Q(R)[t]}$ (where, as before, $\alpha + \beta t$ is identified with its equivalence class in the quotient). In Licata, 2022 a complete characterization of when $R + It$ is an integral domain is provided and a number of results about quotient of Rees algebras have their canonical counterpart for rings of the type $R + It$, mutatis mutandis. It is worth noting that in case R is a Dedekind domain and $R + It$ is a domain, the integral closure of $R + It$ (in its quotient field) is $R + \widetilde{I}t$, where

$$\widetilde{I} := \{i \in Q(R) \mid bi^2 \in R\}.$$

Now, we want to give a deeper insight to the ideal structure of $R(I)_{a,b}$.

Lemma 1.27. *Given H, J ideals of R with $H \subseteq I$, $J + Ht$ is an ideal of $R(I)_{a,b}$ if and only if $bIH \subseteq J$ and $IJ \subseteq H$.*

Proof. $\Rightarrow$). Suppose that $J + Ht$ is an ideal of $R(I)_{a,b}$; in particular for all $h \in H, i \in I$, $(ht)(it) \in J + Ht$, then $-bih - aiht \in J + Ht$, so $bih \in J$. Moreover for all $i \in I, j \in J$, also $j(it) \in J + Ht$, thus $ij \in H$.
$\Leftarrow$). Suppose that $bIH \subseteq J$ and $IJ \subseteq H$, then

$$(r + it)(j + ht) = rj - bih + (rh + ij - iha)t \in J + Ht$$

for all $r \in R, i \in I, j \in J, h \in H$. This implies that $J + Ht$ is an ideal of $R(I)_{a,b}$. □

For an ideal E of $R(I)_{a,b}$,

$$A := \{r \in R \mid \exists i \in I \text{ such that } r + it \in E\}$$

$$B := \{i \in I \mid \exists r \in R \text{ such that } r + it \in E\}.$$

It is easy to check that both A and B are ideals of R and $B \subseteq I$.

Proposition 1.28. *With the notations above, $A + Bt$ is an ideal of $R(I)_{a,b}$.*

Proof. It is sufficient to prove that i) $bIB \subseteq A$ and ii) $IA \subseteq B$.
i) For all $\beta \in B$ there exists $r \in R$ such that $r + \beta t \in E$, moreover for all $i \in I$ we have $(r + \beta t)(it) = -ib\beta + (ri - ia\beta) \in E$ thus $ib\beta \in A$.
ii) For all $\alpha \in A$, there exists $j \in I$ such that $\alpha + jt \in E$, moreover for all $i \in I$ we have $(\alpha + jt)(it) = -bij + (\alpha i - aij)t \in E$. Since $\alpha i - aij \in B$ and $aij \in B$, it follows that $\alpha i \in B$. $\square$

Definition 1.29. *We say that an ideal E of $R(I)_{a,b}$ is homogeneous if*

$$E = A + Bt.$$

Lemma 1.30 (Lemma 1.1, D'Anna and Strazzanti, 2017)**.** *Let $\mathfrak{p}$ be a prime ideal of R and suppose that $t^2 + at + b = (t - \bar{\alpha}/\bar{\gamma})(t - \bar{\beta}/\bar{\gamma})$ in $Q(R/\mathfrak{p})[t]$. Let $\alpha, \beta, \gamma \in R$ such that their classes modulo $\mathfrak{p}$ are, respectively, $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$. Then, the two sets:*

$$\mathfrak{p}_1 := \{r + it \mid r \in R, i \in I, \gamma r + \alpha i \in \mathfrak{p}\},$$

$$\mathfrak{p}_2 := \{r + it \mid r \in R, i \in I, \gamma r + \beta i \in \mathfrak{p}\}$$

do not depend on the choice of α, β and γ and are prime ideals of $R(I)_{a,b}$. Moreover, $\mathfrak{p}_1 = \mathfrak{p}_2$ if and only if $(\alpha - \beta)I \subseteq \mathfrak{p}$.

Proposition 1.31 (Proposition 1.2, D'Anna and Strazzanti, 2017)**.** *Let $\mathfrak{p}$ be a prime ideal of R.*

1) *If $t^2 + at + b$ is irreducible in $Q(R/\mathfrak{p})[t]$, then the only prime ideal of $R(I)_{a,b}$ lying over $\mathfrak{p}$ is $\mathfrak{q} := \{p + it \mid p \in \mathfrak{p}, i \in I \cap \mathfrak{p}\}$.*

2) *If $t^2 + at + b = (t - \bar{\alpha}/\bar{\gamma})(t - \bar{\beta}/\bar{\gamma})$ in $Q(R/\mathfrak{p})[t]$, then the ideals $\mathfrak{p}_1, \mathfrak{p}_2$ defined in the previous lemma are the only prime ideals of $R(I)_{a,b}$ lying over $\mathfrak{p}$.*

Corollary 1.32 (Corollary 1.3, D'Anna and Strazzanti, 2017)**.** *$R(I)_{a,b}$ is an integral domain if and only if R is an integral domain and $t^2 + at + b$ is irreducible in $Q(R)[t]$.*

Remark 1.33. *As consequence of Proposition 1.2 of Barucci, D'Anna, and Strazzanti, 2016 if $\mathfrak{m} \in Max(R)$, there are at most two maximal ideals of $R(I)_{a,b}$ lying over $\mathfrak{m}$, precisely:*
1) If $t^2 + at + b$ is irreducible in $(R/m)[t]$, then the only maximal ideal lying over $\mathfrak{m}$ is $\mathfrak{m} + (\mathfrak{m} \cap I)t$.
2) If $t^2 + at + b = (t - \bar{\alpha})(t - \bar{\beta})$ in $(R/m)[t]$ and $(\alpha - \beta)I \subseteq \mathfrak{m}$, then the only maximal ideal lying over $\mathfrak{m}$ is $\mathfrak{m}_1 = \{r + it \mid r \in R, i \in I, r + \alpha i \in \mathfrak{m}\}$.
3) If $t^2 + at + b = (t - \bar{\alpha})(t - \bar{\beta})$ in $(D/\mathfrak{m})[t]$ but $(\alpha - \beta)I \nsubseteq \mathfrak{m}$, there are two distinct maximal ideals lying over $\mathfrak{m}$:

$$\mathfrak{m}_1 = \{r + it \mid r \in R, i \in I, r + \alpha i \in \mathfrak{m}\},$$

$$\mathfrak{m}_2 = \{r + it \mid r \in R, i \in I, r + \beta i \in \mathfrak{m}\}.$$

1.3 Example of non-Noetherian domain with an m-canonical ideal

Barucci et al., 2019 provided the following helpful construction of integral domain which have an m-canonical ideal. Let $K \subseteq L$ a finite field extension, $n > 1$ an integer and V be a valuation domain of the form $L + M$, where M is the maximal ideal of V. If we define $R := K + M^n$, since V/M^n is a finite dimensional K-vector space, we can choose $a_1, \ldots, a_m \in V$ such that $a_1 = 1$, $a_m \in M^{n-1} \backslash M^n$ and $\{\bar{a}_1, \ldots, \bar{a}_m\}$ is a K-basis for V/M^n. Then, Theorem 1.15 of Barucci et al., 2019 shows that $W = Ra_1 + \ldots + Ra_{m-1}$ is an m-canonical ideal of R.

Example 1.34. *Let $V := L[[X]] + YL((X))[[Y]]$ where $L := \mathbb{Q}(\sqrt{2})$ and X, Y are indeterminates over L. V is a valuation domain with maximal ideal $M := XV$. The ring $R := \mathbb{Q} + X^2V$ is a non-Noetherian domain with m-canonical ideal $I := X^2(\mathbb{Q}(\sqrt{2}) + X\mathbb{Q} + X^2V)$.*

Now we present an example of one-dimensional Noetherian domain with an m-canonical ideal. Let's start recalling a useful construction made by Gulliksen, 1974, we anticipate that this construction will be used in Example 3.7. Let k be a field and we consider $D := k[x_n \mid n \in \mathbb{N}]$. Consider the partition $F = \{A_1 = \{x_0, x_1\}, A_2 = \{x_2\}, \ldots, A_n = \{x_n\}, \ldots\}$ of $\{x_n \mid n \in \mathbb{N}\}$ and let $\mathfrak{p}_{A_i} = A_iD$. Finally, let $R := D_S$, where $S := D \backslash \bigcup_{A_i \in F} \mathfrak{p}_{A_i}$. Gulliksen, 1974 proved that $Max(R) = \{\mathfrak{m}_i := A_iR \mid A_i \in F\}$ and $dim(R) = 2$.

Proposition 1.35. *Let $R := D_S$ as before. We have that:*

1) R is an h-local domain;

2) $R_{\mathfrak{m}_i} \cong k(x_n \mid n \neq i)[x_i]_{(x_i)}$ for all $i \geq 2$;

3) $R_{\mathfrak{m}_1} \cong k(x_n \mid n \neq 0, 1)[x_0, x_1]_{(x_0, x_1)}$;

4) R is a Noetherian domain.

Proof. 1) The finite character follows easily by definition. Moreover, for all $i \geq 2$, $\mathfrak{m}_i$ contain only 0 as prime ideal thus every non zero prime ideal is contained in $\mathfrak{m}_1$.

2) For all $i \geq 2$ we have that $R_{\mathfrak{m}_i} = \left(D_{A_iD}\right)_S = D_{A_iD} = k(x_n \mid n \neq i)[x_i]_{(x_i)}$.

3) Similarly to point 2) $R_{\mathfrak{m}_1} = \left(D_{A_1D}\right)_S = D_{A_1D} = k(x_n \mid n \neq 0, 1)[x_0, x_1]_{(x_0, x_1)}$.

4) R is a Noetherian domain since is locally finite and locally Noetherian. □

Proposition 1.36. *With the notation above, the quotient $\tilde{R} := R/(x_0^3 - x_1^2)$, is a one-dimensional h-local Noetherian domain, with $Max(\tilde{R}) = \{\mathfrak{m}_i\tilde{R} \mid i \geq 1\}$, moreover the ideal $I = (\prod_{i \geq 2} x_i)\tilde{R}$ is an m-canonical ideal.*

Proof. Since:

$$\tilde{R}_{\mathfrak{m}_1\tilde{R}} \cong k(x_n \mid n \neq 0,1)[t^2,t^3]_{(t^2,t^3)} \text{ and } \tilde{R}_{\mathfrak{m}_i\tilde{R}} \cong k(x_n \mid n \neq i)[x_i]_{(x_i)}$$

for all $i \geq 2$, we have that:

1) $I\tilde{R}_{\mathfrak{m}_1\tilde{R}} \cong k(x_n \mid n \neq 0,1)[t^2,t^3]_{(t^2,t^3)}$ which is divisorial, so canonical of itself.

2) $I\tilde{R}_{\mathfrak{m}_i\tilde{R}} \cong x_i k(x_n \mid n \neq i)[x_i]_{(x_i)}$ for all $i \geq 2$, which is an m-canonical ideal, since it is the maximal ideal of a DVR.

3) $I : I = (\prod_{i\geq 2} x_i)\tilde{R} : (\prod_{i\geq 2} x_i)\tilde{R} = (\prod_{i\geq 2} x_i)(\prod_{i\geq 2} x_i)^{-1}(R : R) = R$.

By using Proposition 1.22, we can conclude that I is an m-canonical ideal for $\tilde{R}$, so $\tilde{R}$ is a Noetherian one-dimensional domain, with an m-canonical ideal.

□

Chapter 2

m-canonical ideal for $R(I)_{a,b}$

The aim of this chapter is to detect when $R(I)_{a,b}$ is a divisorial domain, in case R is a domain with an m-canonical ideal I.

Proposition 2.1. *Let R be a domain with m-canonical ideal I, and let E a regular fractional ideal of $R(I)_{a,b}$, then:*

$$Hom_R(E \otimes_{R(I)_{a,b}} R(I)_{a,b}, I) \cong Hom_{R(I)_{a,b}}(E, Hom_R(R(I)_{a,b}, I))$$

as R-module and $R(I)_{a,b}$-module.

Proof. We can observe that $R(I)_{a,b}$ is an $R(I)_{a,b}$-module, $E \otimes_{R(I)_{a,b}} R(I)_{a,b}$ is an $R(I)_{a,b}$-module and it is also an R-module with $r \cdot (x \otimes_{R(I)_{a,b}} y) = x \otimes_{R(I)_{a,b}} ry$ for all $x \in E, y \in R(I)_{a,b}$. We can give a structure of $R(I)_{a,b}$-module to $I^{R(I)_{a,b}} := \mathrm{Hom}_R(R(I)_{a,b}, I)$ with the external product:

$$(r + it) \circ f := f_{(r+it)} : R(I)_{a,b} \to I$$

by defining $f_{(r+it)}(s + jt) = f((r + it)(s + jt))$.
These conditions are sufficient to give the isomorphism of abelian groups:

$$\Psi : \mathrm{Hom}_R(E \otimes_{R(I)_{a,b}} R(I)_{a,b}, I) \to \mathrm{Hom}_{R(I)_{a,b}}(E, \mathrm{Hom}_R(R(I)_{a,b}, I))$$

$$f \longmapsto \Psi(f) : E \to \mathrm{Hom}_R(R(I)_{a,b}, I)$$
$$\Psi(f)(e)(r + it) := f(e \otimes (r + it))$$

Now, we observe that $\mathrm{Hom}_{R(I)_{a,b}}(E, \mathrm{Hom}_R(R(I)_{a,b}, I))$ is also an R-module, by defining the external product:

$$r \star f : E \to \mathrm{Hom}_R(R(I)_{a,b}, I)$$

$$e \longmapsto (r \star f)(e) : R(I)_{a,b} \to I$$
$$(r \star f)(e)(r + it) = rf(e)(r + it)$$

An easy verification of the linearity of the map shows that the isomorphism holds as R-module.
We can give, like for $I^{R(I)_{a,b}}$, the structure of $R(I)_{a,b}$-module to $\mathrm{Hom}_R(E \otimes_{R(I)_{a,b}} R(I)_{a,b}, I)$, by defining:

$$(r + it) \circ f := f_{(r+it)} : E \otimes_{R(I)_{a,b}} R(I)_{a,b} \to I$$

$$f_{(r+it)}(e\otimes(s+jt)) := f(e\otimes(r+it)(s+jt))$$

An easy verification of the linearity of the map shows that the isomorphism holds as $R(I)_{a,b}$-modules. □

Remark 2.2. *By defining $I^{R(I)_{a,b}} := Hom_R(R(I)_{a,b}, I)$ and with the previous notations we have:*

$$Hom_R(E,I) \cong Hom_{R(I)_{a,b}}(E, I^{R(I)_{a,b}}) \quad as \quad E\otimes_{R(I)_{a,b}} R(I)_{a,b} \cong E$$

and:

$$Hom_{R(I)_{a,b}}(Hom_{R(I)_{a,b}}(E, I^{R(I)_{a,b}}), I^{R(I)_{a,b}}) \cong Hom_R(Hom_{R(I)_{a,b}}(E, I^{R(I)_{a,b}}), I) \cong$$

$$\cong Hom_R(Hom_R(E,I),I)$$

as R-module and $R(I)_{a,b}$-modules.

Corollary 2.3. *With the previous notations, if each regular ideal of $R(I)_{a,b}$ is I-reflexive, i.e. $Hom_R(Hom_R(E,I),I) \cong E$, then*

$$E \cong Hom_R(Hom_R(E,I),I) \cong Hom_{R(I)_{a,b}}(Hom_{R(I)_{a,b}}(E, I^{R(I)_{a,b}}), I^{R(I)_{a,b}})$$

so $I^{R(I)_{a,b}}$ is an m-canonical ideal of $R(I)_{a,b}$.

Proposition 2.4. *If I is an ideal of R such that $(I : I) = R$, then $I^{R(I)_{a,b}}$ is a free $R(I)_{a,b}$-module of rank 1.*

Proof. Let $\varphi \in I^{R(I)_{a,b}}$, since $R(I)_{a,b} \cong R \oplus I$ as R-modules, φ is uniquely determined by $\varphi_{|R}$ and $\varphi_{|It}$. $\varphi_{|R} \in \mathrm{Hom}_R(R,I) \cong I$ thus $\varphi_{|R}$ is the multiplication by an element $i \in I$. Since $I \cong It$ as R-modules, $\varphi_{|It} \in \mathrm{Hom}_R(I,I) \cong R$, thus $\varphi_{|It}$ is the multiplication by an element $r \in R$. It is then easy to check that $I^{R(I)_{a,b}} = \{\eta_{(r,i)} \mid r \in R, i \in I\}$, where $\eta_{(r,i)} : R(I)_{a,b} \to I$ is defined by $\eta_{(r,i)}(s+jt) = rj + si$. If $\pi \in I^{R(I)_{a,b}}$ is such that $\pi(r+it) = i$, we claim that $\{\pi\}$ is a base of $I^{R(I)_{a,b}}$ as an $R(I)_{a,b}$-module. In order to prove this, fix $\eta_{(r,i)} \in I^{R(I)_{a,b}}$ and we look for $x+yt \in R(I)_{a,b}$ such that $\eta_{(r,i)} = (x+yt)\star\pi$. Since $i = \eta_{(r,i)}(1) = ((x+yt)\star\pi)(1) = \pi(x+yt) = y$, we have $y = i$. If we consider a regular element $k \in I$, $rk = \eta_{(r,i)}(kt) = ((x+it)\star\pi)(kt) = \pi(kt(x+it)) = \pi(-bki + (xk-aik)t) = xk - aik$. Since k is regular, the equality $rk = xk - aik$ implies $r = x - ai$, and thus $x = r + ai$. □

Proposition 2.5. *If E is an homogeneous ideal of $R(I)_{a,b}$, and I is an m-canonical ideal of R, then E is I-reflexive.*

Proof. It sufficies to note that

$$\mathrm{Hom}_R(\mathrm{Hom}_R(E,I),I) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(A\oplus B,I),I) \cong$$

$$\cong \mathrm{Hom}_R(\mathrm{Hom}_R(A,I)\oplus\mathrm{Hom}_R(B,I),I) \cong$$

$$\cong \mathrm{Hom}_R(\mathrm{Hom}_R(A,I),I)\oplus\mathrm{Hom}_R(\mathrm{Hom}_R(B,I),I) = A\oplus B \cong E$$

□

Let R be a ring with a m-canonical ideal I, and let E be a regular ideal of $R(I)_{a,b}$. The natural immersion of E in $R \oplus I$, induces on E a structure of R-submodule of R^2. We observe that E is I-torsionless, in fact:
E is I-torsionless if and only if ρ_E is injective, if and only if for all $x \in E \backslash \{0\}$ there exists $f \in \mathrm{Hom}_R(E, I)$ such that $\rho_E(x)(f) := f(x) \neq 0$. Let $x = x_1 + x_2 t \in E \backslash \{0\}$, $\pi_i : R^2 \to R$ the ith projection, and $y \in I$ such that $yx_i \neq 0$ for some $i = 1, 2$, we consider the following maps:

$$E \xrightarrow{i} R^2 \xrightarrow{\pi_i} R \xrightarrow{\cdot y} I$$

let $f \in \mathrm{Hom}_R(E, I)$ be the composition of the functions above, we have that $f(x) \neq 0$, thus E is I-torsionless.

Remark 2.6. *If N is an R-submodule of R^n, then $rk(N) \leq n$, since R-independent elements of $N \subseteq R^n$ are also R-independent as elements of R^n. In particular for every ideal N of $R(I)_{a,b}$, $rk(N) \leq 2$ as R-module.*

Lemma 2.7 (Fuchs and Salce, 2001, Chapter IV, Lemma 5.1)**.** *Let R be an integral domain and A a nonzero R-submodule of $Q(R)$. For a torsion-free R-module M of finite rank n, the following conditions are equivalent:*
(a) M is A-torsionless;
(b) $\mathrm{Hom}_R(M, A)$ has rank n;
(c) rank 1 torsion-free quotients of M are isomorphic to submodules of A;
(d) M can be embedded in A^n.

If R is an integral domain and E a regular ideal of $R(I)_{a,b}$. Since E is an R-module of finite rank, by Lemma 2.7, if $\mathrm{rk}(E) = 1$, E is isomorphic to a R-submodule of I so can be seen as a regular ideal of R. Since I is an m-canonical ideal, then E is I-reflexive.

Lemma 2.8 (Fuchs and Salce, 2001, Chapter IV, Proposition 5.2)**.** *Let R be a domain with m-canonical ideal I, let*

$$0 \to N \to M \to M/N \to 0$$

be an exact sequence of torsion-free R-module. If M is I-torsionless, then both N and M/N are I-torsionless.

Remark 2.9. *Given R a domain with an m-canonical ideal, for all E regular ideal of $R(I)_{a,b}$ of rank 2, which is in particular I-torsionless, it is possible to find a R-submodule L of E of rank 1 such that E/L is torsion-free of rank 1. As a matter of fact, we consider the embedding $R \hookrightarrow R(I)_{a,b}$ and we take the contraction $L := E^c$ in R; as E is a regular ideal, then $L \neq 0$. Since L is an R-submodule of E and $L \subseteq R$, L is torsion-free of rank 1. We claim that the quotient E/L is also torsion-free of rank 1. Indeed: for all non-zero $(r + it) + L \in E/L$, in particular $i \neq 0$, $\lambda((r + it) + L) = 0_{E/L} \iff \lambda r + \lambda it \in L \subseteq R$, in particular $\lambda i = 0$ this implies that $\lambda = 0$. This proves that E/L is torsion-free. Consider nonzero elements $\eta = (r + it) + L, \zeta = (s + jt) + L, \in E/L$, in particlar $i, j \neq 0$. Since $rk(I) = 1$*

as an R-module there exist nonzero $\lambda, \mu \in R$ such that $\lambda i + \mu j = 0$, for example $\lambda = j, \mu = -i$. Then $j((r+it)+L) + (-i)((s+jt)+L)$ is an element of E/L, and thus $jr + (-i)s \in L$ so $j\eta + (-i)\zeta = 0_{E/L}$ with $i, j \neq 0$, proving that η, ζ are linearly dependent. It follows that E/L has rank 1.

Remark 2.10. *We consider the following exact sequence:*

$$0 \to E^c \to E \to E/E^c \to 0.$$

By Proposition 2.8, both E^c and E/E^c are I-torsionless and by Lemma 2.7, both E and E/E^c can be embedded in R, so they are I-reflexive thus the maps ρ_{E^c} and ρ_{E/E^c} are isomorphisms. For simplicity, we denote with $(E^*)^* := Hom_R(Hom_R(E, I), I)$. Consider the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E^c & \longrightarrow & E & \xrightarrow{\pi} & E/E^c & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho_{E^c}} & & \downarrow{\scriptstyle \rho_E} & & \downarrow{\scriptstyle \rho_{E/E^c}} & & \\
0 & \longrightarrow & (E^{c*})^* & \longrightarrow & (E^*)^* & \longrightarrow & (E^*)^*/(E^{c*})^* & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

If this is a commutative diagram with exact rows, since π, ρ_{E^c} and ρ_{E/E^c} are surjective, then ρ_E is surjective and thus an isomorphism of R-modules. Since $R \subseteq R(I)_{a,b}$, the map ρ_E is $R(I)_{a,b}$-linear, thus is also an isomorphism of $R(I)_{a,b}$-modules.

Consider the following exact sequence:

$$0 \to E^c \xrightarrow{i} E \xrightarrow{\pi} E/E^c \to 0,$$

where E is an homogeneous ideal of $R(I)_{a,b}$. By applying the functor $\mathrm{Hom}_R(-, I)$ we obtain the sequence:

$$0 \to \mathrm{Hom}_R(E/E^c, I) \xrightarrow{\pi^*} \mathrm{Hom}_R(E, I) \xrightarrow{i^*} \mathrm{Hom}_R(E^c, I)$$

The map i^* is surjective. For all $\varphi \in \mathrm{Hom}_R(E^c, I)$ take the R-linear map $\psi : E \to I$ such that

$$\psi(r + jt) = \varphi(r) \text{ for every } r + jt \in E \subseteq R_{a,b}(I).$$

By definition $\varphi = i^*(\psi) = \psi i$. By applying again the functor $\mathrm{Hom}_R(-, I)$, the sequence

$$0 \to \mathrm{Hom}_R(\mathrm{Hom}_R(E^c, I), I) \xrightarrow{i^{**}} \mathrm{Hom}_R(\mathrm{Hom}_R(E, I), I) \xrightarrow{\pi^{**}} \mathrm{Hom}_R(\mathrm{Hom}_R(E/E^c, I), I)$$

is exact.

Proposition 2.11. *Let I be an ideal of a ring R. If each regular homegeneous ideal E of $R(I)_{a,b}$ is I-reflexive (in the sense that the map ρ_E is an isomorphism as R-modules and $R(I)_{a,b}$-modules) then I is an m-canonical ideal of R.*

Proof. Let H be a regular ideal of R, so $H + Ht$ is a regular ideal of $R(I)_{a,b}$. Let $H^* := \mathrm{Hom}_R(\mathrm{Hom}_R(H, I), I)$. By hypothesis, the map ρ_{H+HIt} is an isomorphism. Then by considering the short exact sequence

$$0 \to H \xrightarrow{i} H + HIt \xrightarrow{\pi} HIt \to 0$$

and dualizing it twice, we get the exact sequence

$$0 \to H^* \xrightarrow{i^{**}} (H + HIt)^* \xrightarrow{\pi^{**}} HIt^*$$

since i^* is surjective. If we consider the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H & \longrightarrow & H + HIt & \longrightarrow & HIt & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho_H} & & \downarrow{\scriptstyle \rho_{H+HIt}} & & \downarrow{\scriptstyle \rho_{HIt}} & & \\
0 & \longrightarrow & H^* & \xrightarrow{i^{**}} & (H + HIt)^* & \longrightarrow & (HIt)^* & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array},$$

since $HIt \cong HI$ as R-modules and since H is a regular ideal and in particular I-torsionless, by the five Lemma, ρ_H is an isomorphism of R-modules. □

The previous facts lead to the following Theorem.

Theorem 2.12. *Let R be a domain and I be an ideal of R such that $(I : I) = R$. We consider the following conditions:*
1) I is an m-canonical ideal for R;
2) $I^{R(I)_{a,b}}$ is an m-canonical ideal for $R(I)_{a,b}$ for all $a, b \in R$;
3) $I^{R(I)_{a,b}}$ is an m-canonical ideal for $R(I)_{a,b}$ for some $a, b \in R$;
4) $R(I)_{a,b}$ is an m-canonical ideal for $R(I)_{a,b}$ for some $a, b \in R$;
5) $R(I)_{a,b}$ is an m-canonical ideal for $R(I)_{a,b}$ for all $a, b \in R$.
We have that:

$$5) \iff 2) \implies 3) \iff 4) \implies 1)$$

Moreover, if every regular ideal E of $R(I)_{a,b}$ is I-reflexive, then all the previous conditions are equivalent. We recall that points 4) and 5) are equivalent to saying that $R(I)_{a,b}$ is a divisorial ring.

Proof. 5) $\iff$ 2) and 3) $\iff$ 4). Since $(I : I) = R$, by Proposition 2.4 $I^{R(I)_{a,b}}$ is a free $R(I)_{a,b}$-module of rank 1, that is $I^{R(I)_{a,b}} \cong R(I)_{a,b}$, for all $a, b \in R$.
2) $\implies$ 3). Trivial.
4) $\implies$ 1). If $R(I)_{a,b}$ is an m-canonical ideal for $R(I)_{a,b}$, for some $a, b \in R$, in

particular each regular ideal E of $R(I)_{a,b}$ is I-reflexive. By Proposition 2.11, I is an m-canonical ideal of R.
1) $\Longrightarrow$ 2). By Corollary 2.3, if I is an m-canonical ideal of R and every regular ideal E of $R(I)_{a,b}$ is I-reflexive, then $I^{R(I)_{a,b}}$ is an m-canonical ideal of $R(I)_{a,b}$. □

Remark 2.13. *If I is an m-canonical ideal of R, in particular:*

$$(I^{R(I)_{a,b}} : I^{R(I)_{a,b}}) = R(I)_{a,b}.$$

By definition, $R(I)_{a,b}$ is divisorial if and only if each $R(I)_{a,b}$-torsionless $R(I)_{a,b}$-module of rank 1 is $R(I)_{a,b}$-reflexive, that is if and only if every regular ideal of $R(I)_{a,b}$ is divisorial, i.e., for all regular ideals E of $R(I)_{a,b}$ we have

$$E \cong Hom_{R(I)_{a,b}}(Hom_{R(I)_{a,b}}(E, R(I)_{a,b}), R(I)_{a,b}) \cong$$

$$\cong Hom_{R(I)_{a,b}}(Hom_{R(I)_{a,b}}(E, I^{R(I)_{a,b}}), I^{R(I)_{a,b}}).$$

Since $Hom_{R(I)_{a,b}}(Hom_{R(I)_{a,b}}(E, I^{R(I)_{a,b}}), I^{R(I)_{a,b}}) \cong Hom_R(Hom_R(E, I), I)$, then every regular ideal E of $R(I)_{a,b}$ is I-reflexive if and only if $R(I)_{a,b}$ is divisorial.

Keeping in mind the Remark 2.13, by studying when $R(I)_{a,b}$ is divisorial, if we start from a domain R with an m-canonical ideal, we will find also the existence of an m-canonical ideal for $R(I)_{a,b}$. In the one-dimensional Noetherian local case, Theorem 2.12 yields the following corollary.

Theorem 2.14. *Let R be a one dimensional local Noetherian domain, I an ideal of R such that $(I : I) = R$. Then the following conditions are equivalent.*
1) I is an m-canonical ideal for R;
2) $I^{R(I)_{a,b}}$ is an m-canonical ideal for $R(I)_{a,b}$ for all $a, b \in R$;
3) $I^{R(I)_{a,b}}$ is an m-canonical ideal for $R(I)_{a,b}$ for some $a, b \in R$;
4) $R(I)_{a,b}$ is an m-canonical ideal for $R(I)_{a,b}$ for some $a, b \in R$;
5) $R(I)_{a,b}$ is an m-canonical ideal for $R(I)_{a,b}$ for all $a, b \in R$.

Proof. Since $(I : I) = R$, by Theorem 2.12 it is sufficient to prove that 1) $\Longrightarrow$ 5). As a consequence of Corollary 3.3 of Barucci, D'Anna, and Strazzanti, 2015, $R(I)_{a,b}$ is Gorenstein. Then by Theorem 6.3 of Bass, 1963 $R(I)_{a,b}$ is divisorial, so it is an m-canonical of itself. □

Chapter 3

Divisoriality of $R(I)_{a,b}$

In this chapter, we investigate the divisiorality property of $R(I)_{a,b}$, when we start from a domain R with an m-canonical ideal I. By studying this property and using Remark 3.13, we will be able to understand in which other cases, other than the one dimensional Noetherian local case, Theorem 2.12 holds.

3.1 h-local property for $R(I)_{a,b}$

Since the h-local property for a domain is a necessary condition to be divisorial, we first investigate when $R(I)_{a,b}$ inherits this property from R. We recall that in the one-dimensional case, the h-local property is equivalent to the property to have a finite character.

Lemma 3.1. *Let R be a ring, I an ideal of R and $a, b \in R$. Consider a nonzero element $\eta := r + it \in R(I)_{a,b}$. Then there exists a nonzero element $\zeta \in R(I)_{a,b}$ such that $\eta\zeta \in R$.*

Proof. If $i = 0$ then $\eta \in R$ and thus we can obviously take $\zeta = 1$. If $i \neq 0$ then the element $\zeta := r - ia - it \in R(I)_{a,b}$ is nonzero and an easy computation shows that $\eta\zeta \in R$. □

Corollary 3.2. *Let R and I be as before and let E be a regular ideal of $R(I)_{a,b}$. Then $R \cap E \neq 0$.*

Proof. Take a regular element $\eta \in E$. By the previous lemma, there exists a nonzero element $\zeta \in R(I)_{a,b}$ such that $\eta\zeta \in R \cap E$. Since η is regular, it follows that $\eta\zeta \neq 0$. □

Proposition 3.3. *If R is a one-dimensional domain with finite character, then $R(I)_{a,b}$ is a domain with finite character, for all I and for all a, b such that $R(I)_{a,b}$ is a domain.*

Proof. Let E be a non-zero ideal of $R(I)_{a,b}$ and suppose that E is contained in infinitely many maximal ideals, say $\{\mathfrak{m}_i \mid i \in H, |H| = \infty\}$. We consider the contraction E^c of E in R, thus $E^c \subseteq \mathfrak{m}_i^c$ for all $i \in H$, and we know that $\mathfrak{m}_i^c \in Max(R)$. Since R is h-local and $E^c \neq 0$ by the previous corollary, E^c is contained in only finitely many maximal ideals, so eventually the ideals $\mathfrak{m}_i^c$ must coincide, and they are in a finite number. But the maximal ideals of $R(I)_{a,b}$ lying over a maximal ideal of R are at most two, as showed in

Remark 1.33. Thus we get a contradiction and therefore E is contained in finitely many maximal ideals. $\square$

Now we study the particular case $a = 0$, but for rings of any dimension. If $\mathfrak{m} \in Max(R)$ and $\mathfrak{n}$ is a maximal ideal of $R(I)_{0,-b}$ lying over $\mathfrak{m}$, by Remark 1.33 only three cases can occur. We claim that the third case, for $R(I)_{0,-b}$, never occurs. In fact, if there exists $i \in I$, such that $(\alpha - \beta)i \notin \mathfrak{m}$, the following results hold true.

Proposition 3.4. *Let I be an ideal of R, $\mathfrak{m}$ be a maximal ideal of R and $\mathfrak{n} \in Max(R(I)_{0,-b})$ lying over $\mathfrak{m}$. Let $b \in R$ such that $t^2 - b = (t - \bar{\alpha})(t - \bar{\beta})$ in $(R/\mathfrak{m})[t]$ and $(\alpha - \beta)I \nsubseteq \mathfrak{m}$; let $i \in I$ be an element such that $(\alpha - \beta)i \notin \mathfrak{m}$. Then $s + jt \notin \mathfrak{n}$ if and only if $s \notin \mathfrak{m}$.*

Proof. First of all, we assume that $\mathfrak{n} = \{r + jt \mid r \in R, j \in I, r + \alpha j \in \mathfrak{m}\}$. Let us consider the element $-\beta i + it$, where i is the element fixed in the statement. Notice that $-\beta i + it \notin \mathfrak{n}$, in fact $-\beta i + \alpha i = (\alpha - \beta)i \notin \mathfrak{m}$.

If we take $s + jt \notin \mathfrak{n}$, we have $(s + jt)(-\beta i + it) \notin \mathfrak{n}$, but, since $(s + jt)(-\beta i + it) = -s\beta i + ijb + (si - \beta ji)t \notin \mathfrak{n}$, we obtain that $i(-s\beta + jb + \alpha s - \alpha\beta j) \notin \mathfrak{m}$, thus $s(\alpha - \beta) + j(b - \alpha\beta) \notin \mathfrak{m}$; since $\alpha - \beta \notin \mathfrak{m}$ and $b - \alpha\beta \in \mathfrak{m}$, it follows that $s \notin \mathfrak{m}$.

Conversely, take $s + jt \in \mathfrak{n}$; we need to prove that $s \in \mathfrak{m}$. Since we have $(s + jt)(-\beta i + it) = -s\beta i + ijb + (si - \beta ji)t \in \mathfrak{n}$, therefore $i(-s\beta + jb + \alpha s - \alpha\beta j) \in \mathfrak{m}$; then, since $i \notin \mathfrak{m}$, $s(\alpha - \beta) + j(b - \alpha\beta) \in \mathfrak{m}$ and since $\alpha - \beta \notin \mathfrak{m}$ and $b - \alpha\beta \in \mathfrak{m}$, we conclude that $s \in \mathfrak{m}$.

The same argument holds for $\mathfrak{n} = \{r + jt \mid r \in R, j \in I, r + \beta j \in \mathfrak{m}\}$, by using the key element $\alpha i - it \notin \mathfrak{n}$. $\square$

Corollary 3.5. *For $R(I)_{0,-b}$ the third case of Remark 1.33 never occurs.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of R such that $t^2 - b = (t - \bar{\alpha})(t - \bar{\beta})$ in $(R/\mathfrak{m})[t]$ and $(\alpha - \beta)I \nsubseteq \mathfrak{m}$. In this case we know that there are two distinct maximal ideals of $R(I)_{0,-b}$ lying over $\mathfrak{m}$ precisely: $\mathfrak{m}_1 = \{r + jt \mid r \in R, j \in I, r + \alpha j \in \mathfrak{m}\}$ and $\mathfrak{m}_2 = \{r + jt \mid r \in R, j \in I, r + \beta j \in \mathfrak{m}\}$. By Proposition 3.4 it follows that if $s + jt \in \mathfrak{m}_1$ then $s \in \mathfrak{m}$ and, since $s + \alpha j \in \mathfrak{m}$, also $j \in \mathfrak{m}$. From this fact we get $\mathfrak{m}_1 \subseteq \mathfrak{m} + (\mathfrak{m} \cap I)t$, which is a proper ideal of $R(I)_{0,-b}$, so from the maximality of $\mathfrak{m}_1$, the equality follows. Since the same fact holds for $\mathfrak{m}_2$, the maximal ideals lying over $\mathfrak{m}$ must coincide with $\mathfrak{m} + (\mathfrak{m} \cap I)t$. This is a contraddition. $\square$

Proposition 3.6. *If R is an h-local domain then $R(I)_{0,-b}$ is an h-local domain, for every b for which $R(I)_{0,-b}$ is a domain.*

Proof. As in Proposition 3.3, it easily follows that every ideal E of $R(I)_{0,-b}$ is contained in finitely many maximal ideals. To get thesis, it is sufficient to prove that each prime ideal is contained in a single maximal ideal. Let $\mathfrak{p} \in Spec(R(I)_{0,-b})$, by Corollary 3.2 $p^c \neq (0)$; if $\mathfrak{p} \subseteq \mathfrak{m}_1 \cap \mathfrak{m}_2$, then $\mathfrak{p}^c \subseteq \mathfrak{m}_1^c \cap \mathfrak{m}_2^c$; since R is $h-$local, $\mathfrak{m}_1^c = \mathfrak{m}_2^c$ and, by the previous corollary, $\mathfrak{m}_1 = \mathfrak{m}_2$. $\square$

When a is different from 0 it might happen that, even if R is h-local, $R(I)_{a,b}$ might not be h-local. In fact, from the condition $\mathfrak{m}_1^c = \mathfrak{m}_2^c$ it does not follow that $\mathfrak{m}_1 = \mathfrak{m}_2$: if $t^2 - b = (t - \bar{\alpha})(t - \bar{\beta})$ in $(R/\mathfrak{m})[t]$ but $(\alpha - \beta)I \not\subseteq \mathfrak{m}$, then $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are two distinct maximal ideals lying over $\mathfrak{m}$.

By using the same construction of Proposition 1.35, we provide an example of an h-local domain R such that the quotient $R(I)_{a,b}$, for a particular ideal I and particular elements $a, b \in R$, is not h-local.

Example 3.7. *We suppose that $D = \mathbb{Q}[X_1, X_2, X_3]$, $\mathcal{F} = \{A_1 = \{X_1, X_2\}, A_2 = \{X_3\}\}$, $\mathfrak{p}_1 = (X_1, X_2)D$ and $\mathfrak{p}_2 = X_3D$. If we consider $S = D\setminus(\mathfrak{p}_1 \cup \mathfrak{p}_2)$ and $R = D_S$, by Proposition 1.35 R is an h-local domain with $Max(R) = \{\mathfrak{p}_1 R, \mathfrak{p}_2 R\}$. If we choose $f(T) = T^2 - T + X_1$ and $I = X_3 R$, the domain $R(I)_{-1,X_1}$ is not h-local. In fact, since in $R/\mathfrak{p}_1 R$ $\overline{f(T)} = T^2 - T$ has two roots $\alpha = 0$ and $\beta = 1$, and $(\alpha - \beta)I = I \not\subseteq \mathfrak{p}_1 R$, then two distinct maximal ideals, say $\mathfrak{n}_1$ and $\mathfrak{n}_2$, lie over $\mathfrak{p}_1 R$. Finally the prime ideal $X_1 R \subsetneq \mathfrak{p}_1 R$ induces by the Lying-over Theorem a prime ideal $\tilde{\mathfrak{p}}$ of $R(I)_{-1,X_1}$, which is contained both in n_1 and n_2.*

3.2 The Noetherian case

In order to understand for which domains the conditions of Theorem 2.12 are true, we start from a Dedekind domain R. In this particular case, we know that all principal ideals of R are m-canonical ideals. We are interested in studying the quadratic quotients $R(I)_{0,-b}$ of the Rees algebra of the domain R with respect to a principal ideal I, especially when $R(I)_{0,-b}$ is not a Dedekind domain. We start with an example.

Example 3.8. *Let $R = \mathbb{Z}$, $Q(R) = \mathbb{Q}$. For every proper ideal I of $\mathbb{Z}$, $R(I)_{0,-2}$ is a domain, since $t^2 - 2$ is irreducibile in $\mathbb{Q}[t]$. Moreover, $\tilde{I} = \{i \in \mathbb{Q} \mid 2i^2 \in \mathbb{Z}\} = \mathbb{Z}$, $\overline{R(I)_{0,-2}} = \mathbb{Z} + \mathbb{Z}t \neq R(I)_{0,-2}$, so $R(I)_{0,-2}$ is not integrally closed then is not a Dedekind domain. If we pick $I = 2\mathbb{Z}$, we want to understand the form of the maximal ideals of $R(I)_{0,-2}$. In $\mathbb{Z}_2[t]$, the polynomial $t^2 - 2t$ has only the root 0 of multiplicity 2, so there exists only one prime ideal over $2\mathbb{Z}$, which is $2\mathbb{Z} + 2\mathbb{Z}t$.*

On the other hand, for $p \neq 0$, if $t^2 - 2$ is irreducible in $\mathbb{Z}_p[t]$, there is only one prime ideal over $p\mathbb{Z}$, we say $\mathfrak{m}_p := p\mathbb{Z} + 2p\mathbb{Z}t$. If $t^2 - 2$ is reducible in $\mathbb{Z}_p[t]$, with roots a, b such that $a + b = p$, then there are two prime ideals lying over $p\mathbb{Z}$, say $\mathfrak{n}_1 = \{r + it \mid r \in \mathbb{Z}, i \in 2\mathbb{Z}, r + ai \in p\mathbb{Z}\}$ and $\mathfrak{n}_2 = \{r + it \mid r \in \mathbb{Z}, i \in 2\mathbb{Z}, r + bi \in p\mathbb{Z}\}$. We can observe that in this case, since $r + ai \in p\mathbb{Z}$, there exists $z \in \mathbb{Z}$ such that $r + ai = pz$, so $r = pz - ai$ and $r + it = pz - ai + it = pz + 2i'(-a + t)$. Since $2(-a + t) \in \mathfrak{n}_1$ we obtain $\mathfrak{n}_1 = (p, -2a + 2t)\mathbb{Z}^*$. In the same way $\mathfrak{n}_2 = (p, -2b + 2t)\mathbb{Z}^*$. In both cases, each maximal ideal is either a principal maximal ideal or a 2−generated maximal ideal.

Proposition 3.9 (Theorem 3.9, Matlis, 1968)**.** *Let R be a Noetherian domain of dimension 1; if every maximal ideal can be generated by at most two elements, then R is reflexive(then divisorial).*

As conseguence of the last theorem, our domain $R(I)_{0,-2}$ is divisorial, thus $R(I)_{0,-2}$ is canonical of itself, even if it is not Dedekind.

The previous result on $\mathbb{Z}(I)_{0,-2}$ can be easily generalized if we start with any PID and any nonzero proper ideal.

Proposition 3.10. *Let R be a PID and let $t^2 + at + b \in R[t]$ be irreducible in $Q(R)[t]$; then for all $I = iR$ proper ideal of R, $R^* := R(I)_{a,b}$ is a divisorial domain.*

Proof. Let $\mathfrak{m} = mR \in Max(R)$; if $t^2 + at + b$ is irreducible in $(R/\mathfrak{m})[t]$, then there is only one prime ideal lying over $\mathfrak{m}$, that is the principal ideal $\mathfrak{m} + \mathfrak{m}It = mR^*$. On the other and, if $t^2 + at + b$ is reducible in $(R/\mathfrak{m})[t]$ and α, β are its roots, then there are two prime ideal lying over $\mathfrak{m}$:

$$\mathfrak{m}_1 = \{r + jt \mid r \in R, j \in I, r + \alpha j \in \mathfrak{m}\}$$

$$\mathfrak{m}_2 = \{r + jt \mid r \in R, j \in I, r + \beta j \in \mathfrak{m}\}$$

Since $r + \alpha j \in \mathfrak{m}$, it follows that $r + \alpha i d_1 = md_2$, for some $d_1, d_2 \in R$; hence $r = md_2 - \alpha i d_1$, and thus $r + jt = md_2 - \alpha i d_1 + i d_1 t = md_2 + d_1(-\alpha i + it)$. Since $m, -\alpha i + it \in \mathfrak{m}_1$, then $\mathfrak{m}_1 = (m, -\alpha i + it)R^*$; analogously $\mathfrak{m}_2 = (m, -\beta i + it)R^*$. In both cases, all the maximal ideal are principal or 2−generated, then by Theorem 3.9, R^* is divisorial. □

Now, we are interested to understand what happens, when we start from a domain R, which is Dedekind but not PID.

Let R be a Dedekind domain not PID, let $I = xR$ be a principal ideal, so an m-canonical ideal for R. We want to understand when, under the hypothesis that b is an element of R, which is not a square in $Q(R)$ and such that $R(xR)_{0,-b}$ is not integrally closed, the domain $R(xR)_{0,-b}$ is reflexive.

Proposition 3.11 (Theorem 4.7, Bazzoni and Salce, 1996). *Let R be a domain; the following facts are equivalent:*
1) R is divisorial.
2) R is h-local and $R_\mathfrak{m}$ is divisorial for all maximal ideal $\mathfrak{m}$.

In light of the last proposition, a good approach to our problem is to study the localizations of $R(xR)_{0,-b}$.

For all $\mathfrak{n} \in Max(R(xR)_{0,-b})$ lying over $\mathfrak{m} \in Max(R)$, we know that only two cases can occur:
1) if $t^2 - b$ is irreducibile in $(R/\mathfrak{m})[t]$, then $(R(xR)_{0,-b})_\mathfrak{n} \cong R_\mathfrak{m}(xR_\mathfrak{m})_{0,-b}$.
2) if $t^2 - b = (t - \bar{\alpha})(t - \bar{\beta})$ in $(R/\mathfrak{m})[t]$ and $(\alpha - \beta)I \subseteq \mathfrak{m}$ then also in this case $(R(xR)_{0,-b})_\mathfrak{n} \cong R_\mathfrak{m}(xR_\mathfrak{m})_{0,-b}$.

Proposition 3.12. *Let R be a Dedekind domain not PID, b an element of R which is not a square in $Q(R)$, $\mathfrak{m}$ a maximal ideal of R; then $R_\mathfrak{m}(xR_\mathfrak{m})_{0,-b}$ is reflexive, for every non zero $x \in R$.*

Proof. We can distinguish two cases.
1) If $x \in \mathfrak{m}$, then the ideal $xR_\mathfrak{m}$ is a proper ideal thus, since $R_\mathfrak{m}$ is a DVR, it follows the reflexivity by Proposition 3.2 .
2) If $x \notin \mathfrak{m}$, we have to study the reflexivity of $R_\mathfrak{m}(R_\mathfrak{m})_{0,-b} = \frac{R_\mathfrak{m}[t]}{(t^2-b)}$, so it is sufficient to prove that each maximal ideal can be at most 2-generated.

Let $k := R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$, $\tilde{R} := \frac{R_{\mathfrak{m}}[t]}{(t^2-b)}$ and $\mathfrak{m}R_{\mathfrak{m}} = mR_{\mathfrak{m}}$. Let us consider the projection

$$\pi : \tilde{R} \longrightarrow \frac{k[t]}{(t^2-b)}.$$

If $t^2 - b$ is irreducible then $\tilde{R}$ has a unique maximal ideal, which is $ker(\pi) = (t^2 - b, m)\tilde{R}$.

If $t^2 - b = (t-\alpha)(t-\beta)$, fix the ideal $(t-\alpha, m)\tilde{R}$. Since $\pi((t-\alpha, m)) = \overline{(t-\alpha)}$, we obtain $\tilde{R}/(t-\alpha, m)\tilde{R} \cong k$, so $(t-\alpha, m)\tilde{R}$ is a maximal ideal. In a similar way, also $(t-\beta, m)\tilde{R}$ is a maximal ideal.

Notice that the previous proposition is not in contradiction with Corollary 3.5, since in that case we are considering the quadratic quotient obtained using a proper ideal.

The previous proposition shows that, when we start from a Dedekind domain which is not PID, the quadratic quotient of the Rees algebra $R(xR)_{0,-b}$ is divisorial. This result is more general:

Theorem 3.13. *Let R be a one dimensional Noetherian domain with an m-canonical I and let b be an element of R which is not a square in $Q(R)$; then $R(I)_{0,-b}$ is divisorial.*

Proof. From Theorem 4.7 of Bazzoni and Salce, 1996, $R(I)_{0,-b}$ is divisorial if and only if $(R(I)_{0,-b})_{\mathfrak{m}}$ is divisorial, for every maximal ideal $\mathfrak{m}$. Since $(R(I)_{0,-b})_{\mathfrak{m}} = R_{\mathfrak{m}^c}(I_{\mathfrak{m}^c})_{0,-b}$, by Proposition 1.22 $I_{\mathfrak{m}}$ is an m-canonical ideal of $R_{\mathfrak{m}}$ and $R_{\mathfrak{m}}$ is a one dimensional local Noetherian domain, then $(R(I)_{0,-b})_{\mathfrak{m}}$ is Gorenstein so divisorial. □

3.3 The non-Noetherian case

In the non-Noetherian case we conjecture that Theorem 3.13 may not be true. We recall a remarkable result that could be used in an attempt to find a counterexample, when we work with a valuation domain with non principal maximal ideal. We know that if V is a valuation domain with maximal ideal $\mathfrak{m}$, then $\mathfrak{m}$ is an m-canonical ideal for V.

Proposition 3.14 (Proposition 8.2, Chapter XV Fuchs and Salce, 2001)**.** *Let R be a divisorial local domain with non-principal maximal ideal $\mathfrak{m}$, and let $R_1 = \mathfrak{m} : \mathfrak{m}$. Then one of the following mutually exclusive cases arises:*
1) R_1 is a local domain with maximal ideal $\mathfrak{m}_1 \supset \mathfrak{m}$; $\mathfrak{m}_1/\mathfrak{m}$ is simple both as an R_1-module and as an R-module; furthemore, $\mathfrak{m}_1^2 \subseteq \mathfrak{m}$ and $R : \mathfrak{m}_1 = \mathfrak{m}_1$.
2) R_1 is a valuation domain with maximal ideal $\mathfrak{m}$.
3) R_1 has exactly two maximal ideals, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ such that $\mathfrak{m}_1 \cap \mathfrak{m}_2 = \mathfrak{m}$; for $i = 1, 2$, $\mathfrak{m}_i/\mathfrak{m}$ is simple as an R_1-module and as an R-module, and $R : \mathfrak{m}_i = \mathfrak{m}_j$ for $i \neq j$; moreover, $R_1 = V_1 \cap V_2$ where the V_i are valuation domains with maximal ideals N_i, such that $\mathfrak{m} = N_1 \cap N_2$ (so R_1 is a Prüfer domain).

Lemma 3.15. *Let R be a ring, $\mathfrak{m}$ a regular maximal ideal of R and $a, b \in R$. Then:*

$$(\mathfrak{m} + \mathfrak{m}t :_{Q(R(\mathfrak{m})_{a,b})} \mathfrak{m} + \mathfrak{m}t) = (\mathfrak{m} :_{Q(R)} \mathfrak{m}) + (\mathfrak{m} :_{Q(R)} \mathfrak{m})t$$

Proof. Let $\frac{v+mt}{u} \in (\mathfrak{m} + \mathfrak{m}t : \mathfrak{m} + \mathfrak{m}t)$; then

$$\left(\frac{v+mt}{u}\right)(\mathfrak{m} + \mathfrak{m}t) \subseteq \mathfrak{m} + \mathfrak{m}t$$

In particular, for any $\tilde{m} \in \mathfrak{m}$,

$$\frac{v+mt}{u} \cdot \tilde{m} \in \mathfrak{m} + \mathfrak{m}t$$

so $\frac{v}{u} \cdot \tilde{m} \in \mathfrak{m}$ and $\frac{m}{u} \cdot \tilde{m} \in \mathfrak{m}$. Therefore $\frac{v+mt}{u} \in (\mathfrak{m} :_{Q(R)} \mathfrak{m}) + (\mathfrak{m}. :_{Q(R)} \mathfrak{m})t$.

Conversely, if $\alpha + \beta t \in (\mathfrak{m} :_{Q(R)} \mathfrak{m}) + (\mathfrak{m} :_{Q(R)} \mathfrak{m})t$, we can write it as $\frac{v}{u} + \frac{v'}{u}t = \frac{v+v't}{u}$. If we pick a regular element $x \in \mathfrak{m}$, then we have $\frac{v+v't}{u} = \frac{xv+xv't}{xu}$ that belongs to $Q(R(\mathfrak{m})_{a,b})$, by Proposition 1.23. Let $m_1 + m_2t \in \mathfrak{m}$:

$$\left(\frac{v}{u} + \frac{v'}{u}t\right) \cdot (m_1 + m_2t) = \left(\frac{v}{u}m_1 - \frac{v'}{u}m_2b\right) + \left(\frac{v}{u}m_2 + \frac{v'}{u}m_1 - \frac{v}{u}m_2a\right)t$$

Since $\frac{v}{u}, \frac{v'}{u} \in (\mathfrak{m} :_{Q(R)} \mathfrak{m})$ we get

$$\left(\frac{v}{u} + \frac{v'}{u}t\right) \cdot (m_1 + m_2t) \in \mathfrak{m} + \mathfrak{m}t,$$

so $\frac{v}{u} + \frac{v'}{u}t \in (\mathfrak{m} + \mathfrak{m}t :_{Q(R(\mathfrak{m})_{a,b})} \mathfrak{m} + \mathfrak{m}t)$.

Proposition 3.16. *Let V be a valuation domain with non principal maximal ideal $\mathfrak{m}$, and consider $V(\mathfrak{m})_{a,b}$, which is a local ring with maximal ideal $\mathfrak{m} + \mathfrak{m}t$. Then the following properties hold true:*
1) $(\mathfrak{m} + \mathfrak{m}t : \mathfrak{m} + \mathfrak{m}t) = V + Vt$;
2) if $b \in \mathfrak{m}$, $\mathfrak{m} + Vt$ is a maximal ideal for $V + Vt$;
3) if $a, b \in \mathfrak{m}$, $\mathfrak{m} + Vt$ is the unique maximal ideal of $V + Vt$.

Proof. 1) It follows from Lemma 3.15.

2) If I is an ideal of $V + Vt$, such that $I \supsetneq \mathfrak{m} + Vt$, then there exists $v_1 + v_2t \in I \backslash (\mathfrak{m} + Vt)$, so $v_1 \notin \mathfrak{m}$ and $v_1^{-1} \in V$. Since, in particular $Vt \subsetneq I$, $v_2t \in I$, so $v_1 \in I$. Thus I must contain a unit and $I = V + Vt$.

3) If $a, b \in \mathfrak{m}$, we prove that every element $v_1 + v_2t \notin \mathfrak{m} + Vt$ is a unit. In fact, since $v_1 \notin \mathfrak{m}$ we obtain $v_1^2 - av_1v_2 + bv_2^2 \notin \mathfrak{m}$, so the inverse of $v_1 + v_2t$ is $(v_1 - av_2)(v_1^2 - av_1v_2 + bv_2^2)^{-1} - v_2(v_1^2 - av_1v_2 + bv_2^2)^{-1}t$.

Remark 3.17. *If $a, b \in \mathfrak{m}$ and $V(\mathfrak{m})_{a,b}$ is a domain, then $V + Vt$ is a local domain with maximal ideal $\mathfrak{m} + Vt \supsetneq \mathfrak{m} + \mathfrak{m}t$ and $(\mathfrak{m} + Vt)/(\mathfrak{m} + \mathfrak{m}t)$ is simple as $V + Vt$ and $V(\mathfrak{m})_{a,b}$-module.*
Moreover $(\mathfrak{m} + Vt)^2 \subset \mathfrak{m} + \mathfrak{m}t$ and $(V(\mathfrak{m})_{a,b} : \mathfrak{m} + Vt) = \mathfrak{m} + Vt$; in fact, let $\frac{v+mt}{u} \in (V(\mathfrak{m})_{a,b} : \mathfrak{m} + Vt)$, so

$$\left(\frac{v+mt}{u}\right)(\mathfrak{m} + Vt) \subseteq V + \mathfrak{m}t;$$

if $\frac{m}{u} \notin V$, then $\frac{u}{m} \in \mathfrak{m}$ and in particular

$$\left(\frac{v+mt}{u}\right)\frac{u}{m} = \frac{v}{m} + t \in V + \mathfrak{m}t,$$

that is not possible; so $\frac{m}{u} \in V$.
If $\frac{v}{u} \notin \mathfrak{m}$, then $\frac{u}{v} \in V$and thus

$$\left(\frac{v+mt}{u}\right)\frac{u}{v}t = -\frac{bm}{v} + \left(1 - \frac{am}{v}\right)t \in V + \mathfrak{m}t;$$

since $\frac{u}{v} \in V$, $\frac{m}{u} \in V$ and $a \in \mathfrak{m}$, we get $\frac{am}{v} \in \mathfrak{m}$ and this is not possible since $1 \notin \mathfrak{m}$. Thus $\frac{v+mt}{u} \in \mathfrak{m} + Vt$. The other inclusion is straightforward.

Hence if $a, b \in \mathfrak{m}$, we are exactly in the first case of the Proposition 3.14, choosing as R the domain $V(\mathfrak{m})_{a,b}$ and as R_1 the local domain $V + Vt$.

Proposition 3.18. *Let V be a valuation domain, then $(V(\mathfrak{m})_{a,b} : \mathfrak{m} + \mathfrak{m}t) = V + Vt$ for all $a, b \in V$.*

Proof. If $\frac{v+mt}{u} \in (V(\mathfrak{m})_{a,b} : \mathfrak{m} + \mathfrak{m}t)$, then

$$\left(\frac{v+mt}{u}\right)(\mathfrak{m} + \mathfrak{m}t) \subseteq V + \mathfrak{m}t;$$

if $\frac{v}{u} \notin V$, then $\frac{u}{v} \in \mathfrak{m}$ and

$$\left(\frac{v+mt}{u}\right)\frac{u}{v} = 1 + \frac{m}{v}t \in V + \mathfrak{m}t$$

thus $\frac{m}{v} \in \mathfrak{m}$. Moreover,

$$\left(\frac{v+mt}{u}\right)\frac{u}{v}t = -\frac{bm}{v} + (1 - \frac{am}{v})t \in V + \mathfrak{m}t$$

this is not possible, since $1 \notin \mathfrak{m}$ and $\frac{am}{v} \in \mathfrak{m}$. In a similar way if $\frac{m}{u} \notin V$, then $\frac{u}{m} \in \mathfrak{m}$, so, in particular,

$$\left(\frac{v+mt}{u}\right)\frac{u}{m} = \frac{v}{m} + t \in V + \mathfrak{m}t$$

that is not possible, since $1 \notin \mathfrak{m}$. □

Proposition 3.19 (Proposition 2.2, Barucci, 2009). *Let $(R, \mathfrak{m})$ be a local ring having an m-canonical ideal. Then $l_R((R : \mathfrak{m})/R) = 1$ if and only if R is a divisorial ring.*

Corollary 3.20. *Let V be a valuation domain. If $V(\mathfrak{m})_{a,b}$ has an m-canonical ideal, then $V(\mathfrak{m})_{a,b}$ is a divisorial ring.*

Proof. Since $(V(\mathfrak{m})_{a,b} : \mathfrak{m} + \mathfrak{m}t) = V + Vt$, we get

$$l_{V(\mathfrak{m})_{a,b}}((V(\mathfrak{m})_{a,b} : \mathfrak{m} + \mathfrak{m}t)/V(\mathfrak{m})_{a,b}) = l_{V(\mathfrak{m})_{a,b}}((V + Vt)/(V + \mathfrak{m}t)) = 1$$

We refer to Example 3.36 as an attempt to prove that in the non-Noetherian case Theorem 3.13 may not be true.

3.4 An homological approach to our problem

Let R be a ring. We denote by $\mathcal{M}(R)$ the category of R-modules. Through this section we will follow the notations and we will recall some results of Hilton and Stammbach, 2013.

Definition 3.21. *Let $A, B \in \mathcal{M}(R)$; we call extension of A by B every short exact sequence of type*

$$0 \to B \to E \to A \to 0$$

We immediately observe that at least one extension exists

$$0 \to B \xrightarrow{i_B} A \oplus B \xrightarrow{\pi_A} A \to 0$$

Moreover, in this case $i_A : A \to A \oplus B$ and $\pi_B : A \oplus B \to B$ are maps such that $\pi_A i_A = 1_A$ e $\pi_B i_B = 1_B$.

Definition 3.22. *Let $\xi_1 := 0 \to B \to E_1 \to A \to 0$ and $\xi_2 := 0 \to B \to E_2 \to A \to 0$ be extensions of A by B; we define the following relation: $\xi_1 \sim \xi_2$ if and only if $E_1 \cong E_2$ as R-modules. This is an equivalence relation, and we denote by $E(A, B)$ the set of equivalence classes of extensions of A by B.*

Definition 3.23. *We say that $0 \to B \xrightarrow{i} E \xrightarrow{\pi} A \to 0$ is a* splitting exact extension *of A by B if it is equivalent to $0 \to B \to A \oplus B \to A \to 0$. In this case there exist $i' : A \to E$ and $\pi' : E \to B$ such that $\pi i' = 1_A$ e $\pi' i = 1_B$.*

Definition 3.24. *Consider the following diagram:*

$$\begin{array}{ccc} A \times_X B & \xrightarrow{\alpha} & A \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\varphi} \\ B & \xrightarrow{\psi} & X \end{array}$$

where $A \times_X B = \{(a, b) \in A \oplus B \mid \varphi(a) = \psi(b)\}$. This is a commutative diagram and we denote it by pull-back *of (φ, ψ).*

Lemma 3.25. *The square:*

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & A \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\varphi} \\ B & \xrightarrow{\psi} & X \end{array}$$

is a pull-back if and only if the sequence $0 \to Y \xrightarrow{\{\alpha,\beta\}} A \oplus B \xrightarrow{<\varphi,-\psi>} X \to 0$ is exact. Moreover, if this the case:
1) β induces an isomorphism between $\ker(\alpha)$ *and* $\ker(\psi)$.
2) If ψ è surjective, so is α.

Consider an R-modules homomorphism $\alpha : A' \to A$. Let $0 \to B \xrightarrow{k} E \xrightarrow{\nu} A \to 0$ be an element of $E(A,B)$ and take the pull-back (E^α, ν', ξ) of (α, ν)

$$\begin{array}{ccccccccc} & & & & E^\alpha & \overset{\nu'}{\dashrightarrow} & A' & & \\ & & & & \Big\downarrow{\scriptstyle \xi} & & \Big\downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & B & \xrightarrow{k} & E & \xrightarrow{\nu} & A & \longrightarrow & 0 \end{array} .$$

By Lemma 3.25, ξ induces an isomorphism between B and $\ker(\nu')$. Moreover, since ν is surjective, so is ν'. Then $0 \to B \to E^\alpha \to A' \to 0$ is an element of $E(A', B)$. The map $\alpha^* : E(A,B) \to E(A',B)$ which assigns the class of $0 \to B \to E \to A \to 0$ to the class of $0 \to B \to E^\alpha \to A' \to 0$ makes $E(-,B)$ a contravariant functor.

Definition 3.26. *Consider the following diagram*

$$\begin{array}{ccc} E & \xrightarrow{k} & A \\ \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \xi} \\ B & \xrightarrow{k'} & A \sqcup_E B \end{array}$$

where $A \sqcup_E B := (A \oplus B)/N$, *with* $N := \{(k(e), -\beta(e)) \mid e \in E\}$. *This is a commutative diagram and we denote it by* push-out *of* (k, β).

Lemma 3.27. *If the square*

$$\begin{array}{ccc} E & \xrightarrow{k} & A \\ \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \xi} \\ B & \xrightarrow{k'} & Y \end{array}$$

is a push-out of (k, β), *then:*
1) the map ξ *induces an isomorphism between* $\operatorname{coker}(k)$ *and* $\operatorname{coker}(k')$.
2) If k *is injective, so is* k'.

Let $\beta : B \to B'$ be an homomorphism of R-modules and let $0 \to B \xrightarrow{k} E \xrightarrow{\nu} A \to 0$ be an element of $E(A,B)$. Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{k} & E & \xrightarrow{\nu} & A & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \xi} & & & & \\ & & B' & \overset{k'}{\dashrightarrow} & E_\beta & & & & \end{array}$$

where (E_β, k', ξ) is the push-out of (β, k). By the previous lemma the sequence $0 \to B' \to E_\beta \to A \to 0$ is an element of $E(A, B')$. The map $\beta_* : E(A,B) \to E(A,B')$ which assigns the class of $0 \to B \to E \to A \to 0$ to the class of $0 \to B' \to E_\beta \to A \to 0$, makes $E(A,-)$ a covariant functor.

Proposition 3.28 (Theorem 1.4, Hilton and Stammbach, 2013). *For all $\alpha : A' \to A$ and for all $\beta : B \to B'$ the induced maps α^* and β_* are such that*

$$\alpha^* \beta_* = \beta_* \alpha^* : E(A,B) \to E(A',B').$$

Then $E(-,-)$ is a bifunctor from the category of R-modules to the category of sets. It is contravariant in the first variable and covariant in the second one.

3.4.1 The R-module $E(A,B)$

In this section we recall the operations that make $E(A,B)$ an R-module. We start by defining the sum, known as *Baer sum*, of two elements of $E(A,B)$, which will make it an abelian group.

Let $\xi_1 : 0 \to B \xrightarrow{i_1} E_1 \xrightarrow{\pi_1} A \to 0$ and $\xi_2 : 0 \to B \xrightarrow{i_2} E_2 \xrightarrow{\pi_2} A \to 0$ be two elements of $E(A,B)$. Consider the maps $\triangle_A : A \to A \oplus A$ such that $\triangle_A(a) = (a,a)$ and $\triangledown_B : B \oplus B \to B$ such that $\triangledown_B(b_1,b_2) = b_1 + b_2$. We define the sum operation on $E(A,B)$ by setting

$$\xi_1 + \xi_2 := \triangle_A^* \triangledown_{B*} (0 \to B \oplus B \xrightarrow{(i_1,i_2)} E_1 \oplus E_2 \xrightarrow{(\pi_1,\pi_2)} A \oplus A \to 0).$$

Firstly, we apply $\triangle_A^*$ by doing the pull-back of $((\pi_1,\pi_2), \triangle_A)$

$$\begin{array}{ccccccccc} & & & & ((E_1 \oplus E_2) \times_{A\oplus A} A) & \dashrightarrow & A & & \\ & & & & \vdots & & \downarrow \triangle_A & & \\ 0 & \longrightarrow & B \oplus B & \longrightarrow & E_1 \oplus E_1 & \xrightarrow[(\pi_1,\pi_2)]{} & A \oplus A & \longrightarrow & 0 \end{array}$$

Notice that $(E_1 \oplus E_2)^{\triangle_A} := ((E_1 \oplus E_2) \times_{A\oplus A} A) = \{((e_1,e_2),a) \mid (\pi_1(e_1),\pi_2(e_2)) = (a,a)\} = \{((e_1,e_2),a) \mid \pi_1(e_1) = \pi_2(e_2) = a\} \cong E_1 \times_A E_2$, since the map $E_1 \times_A E_2 \to (E_1 \oplus E_2)^{\triangle_A}$, which sends (e_1,e_2) to $((e_1,e_2),a)$, where a is the element of A such that $\pi_1(e_1) = \pi_2(e_2) = a$, is bijective. Thus we obtain

$$0 \to B \oplus B \to E_1 \times_A E_2 \to A \to 0$$

Now we apply $\triangledown_{B*}$ by considering the push-out

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B \oplus B & \longrightarrow & E_1 \times_A E_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow \triangledown_B & & \vdots & & & & \\ & & B & \dashrightarrow & \frac{(E_1 \times_A E_2) \oplus B}{C} & & & & \end{array}$$

where $C = \{((i_1(b_1), i_2(b_2)), -\triangledown_B(b_1,b_2)) \mid b_1, b_2 \in B\}$. An easy calculation shows that the map

$$E_1 \times_A E_2 \xrightarrow{\varphi} \frac{(E_1 \times_A E_2) \oplus B}{C}$$

$$(e_1, e_2) \mapsto \overline{((e_1, e_2), 0)}$$

is surjective. Moreover, since $\ker(\varphi) = \{(i_1(b), -i_2(b)) \mid b \in B\} := N$, we have

$$\frac{(E_1 \times_A E_2) \oplus B}{C} \cong \frac{E_1 \times_A E_2}{N}.$$

This shows that

$$\xi_1 + \xi_2 = \quad 0 \to B \to \frac{E_1 \times_A E_2}{N} \to A \to 0$$

is an element of $E(A, B)$, known as the Baer sum of the two classes of estensions. This operation gives to $E(A, B)$ the structure of abelian group; in particular, the zero element is the class of split extensions (see Corollary 3.4.5 of Weibel, 1994).

Theorem 3.29 (Corollary 3.4.5, Weibel, 1994). *$E(A, B)$ is isomorphic to $\mathrm{Ext}^1_R(A, B)$ as abelian groups.*

Since $\mathrm{Ext}^1_R(A, B)$ is in particular an R-module, the previous isomorphism of abelian groups induces to $E(A, B)$ the structure of R-module with the following external product. Let $r \in R$ and consider the map $r\cdot : A \to A$, which is the multiplication by r and let $\xi : 0 \to B \xrightarrow{i} E \xrightarrow{\pi} A \to 0$ be an element of $E(A, B)$; the external product is defined by setting $r \cdot \xi := (r\cdot)^*(\xi)$, which is the class of

$$0 \to B \to E^{r\cdot} \to A \to 0$$

where $E^{r\cdot}$ is the pull-back of $(\pi, r\cdot)$.

The following results, stated for $E(A, B)$ endowed with the structure of R-module, can be proved following the proofs given by Walker, 1964, in the case of abelian groups.

Definition 3.30. *Let A be an R-module and $r \in R$. We denote by*

$$A[r] := \{a \in A \mid ra = 0\}$$

Proposition 3.31 (Theorem 1, Walker, 1964). *An exact sequence $\xi : 0 \to B \xrightarrow{i} E \xrightarrow{\pi} A \to 0$ is a torsion element for $E(A, B)$ if and only if*

$$0 \to B/B[r] \to (B + rE)/B[r] \xrightarrow{\pi'} rA \to 0$$

is splitting exact for some $r \in R$, where $\pi'(x + B[r]) = \pi(x)$.

Proof. Let ξ be the class of $0 \to B \xrightarrow{i} E \xrightarrow{\pi} A \to 0$; if ξ is a torsion element of $E(A, B)$, there exists $r \in R$ such that the sequence $r \cdot \xi : 0 \to B \xrightarrow{j} E^{r\cdot} \xrightarrow{s} A \to 0$ is splitting exact, where $E^{r\cdot} = \{(e, a) \in E \oplus A \mid \pi(e) = a\}$, $j(b) = (i(b), 0)$ and $s((e, a)) = a$. Thus there exists $s' : A \to E^{r\cdot}$ such that $ss' = 1_A$. Consider $h : E^{r\cdot} \to E$ such that $h((e, a)) = e$. If $a \in A[r]$, then $s'(a) = (x, a)$ such that $rx = 0$ and $\pi(x) = ra = 0$, so $x = h(s'(a)) \in A[r]$. Let $g : rA \to E/B[r]$, such that $g(ra) = h(s'(a)) + B[r]$. Take $(e, a) \in E^{r\cdot}$, since $\pi(e) = ra \in rA$, then

$e \in B + rE$, so $g : rA \to (B + rE)/B[r]$. Finally, for all $a \in A$, $\pi'(g(ra)) = \pi'(h(s'(a)) + B[r]) = \pi(h(s'(a))) = ra$, then $\pi' g = 1_{rA}$, so the sequence is splitting exact.
Conversely, suppose that

$$0 \to B/B[r] \to (B + rE)/B[r] \xrightarrow{\pi'} rA \to 0$$

is splitting exact for some $r \in R$ and let $v : rA \to (B + rE)/B[r]$ such that $\pi' v = 1_{rA}$. Let $\xi : 0 \to B \xrightarrow{i} E \xrightarrow{\pi} A \to 0$ be an element of $E(A, B)$ and consider the class of $r^2 \cdot \xi$ represented by

$$0 \to B \xrightarrow{k} K \xrightarrow{u} A \to 0,$$

where $K = \{(e, a) \mid e \in E, a \in A, \pi(e) = r^2 a\}$ and k, u are the natural maps. Let $r' : (B + rE)/B[r] \to E$ such that $r'(x + B[r]) = rx$. Notice that $\pi(r'(x + B[r])) = r(\pi'(x + B[r]))$ for all $x \in (B + rE)/B[r]$, then $\pi(r'(v(ra))) = r(\pi'(v(ra))) = r(r(a)) = r^2 a$ for all $a \in A$. Consider the map $w : A \to K$ such that $w(a) = (r'(v(ra)), a)$, we have that $uw = 1_A$ then the sequence $0 \to B \xrightarrow{k} K \xrightarrow{u} A \to 0$ is splitting exact.

Corollary 3.32 (Theorem 3, Walker, 1964)**.** *Let A, B be two torsion-free R-modules. The sequence $\xi : 0 \to B \xrightarrow{i} E \xrightarrow{\pi} A \to 0$ is a torsion element for $E(A, B)$ if and only if the sequence*

$$0 \to B \to (B + rE) \to rA \to 0$$

is splitting exact, for some $r \in R$.

3.4.2 Homological approach for $R(I)_{a,b}$ divisoriality

Consider the same diagram of Remark 2.10

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E^c & \xrightarrow{i} & E & \xrightarrow{\pi} & E/E^c & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho_{E^c}} & & \downarrow{\scriptstyle \rho_E} & & \downarrow{\scriptstyle \rho_{E/E^c}} & & \\
0 & \longrightarrow & (E^{c*})^* & \xrightarrow{i^{**}} & (E^*)^* & \xrightarrow{\pi^{**}} & (E^*)^*/(E^{c*})^* & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}.$$

The hypothesis that every regular ideal E of $R(I)_{a,b}$ is I-reflexive in Theorem 2.12 is necessary since we are not sure about the exactness of the above diagram. In particular, the exactness holds if in the following sequence

$$0 \to Hom_R(E/E^c, I) \xrightarrow{\pi^*} Hom_R(E, I) \xrightarrow{i^*} Hom_R(E^c, I)$$

the map i^* was surjective. Following an argument similar to the proof of Bazzoni and Salce, 1996, Theorem 3.6, consider the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Hom_R(E/E^c, I) & \xrightarrow{\pi^*} & Hom_R(E, I) & \xrightarrow{i^*} & Hom_R(E^c, I) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Hom_R(E/E^c, Q) & \longrightarrow & Hom_R(E, Q) & \xrightarrow{v} & Hom_R(E^c, Q) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & tHom_R(E/E^c, Q/I) & \xrightarrow{u'} & tHom_R(E, Q/I) & \longrightarrow & tHom_R(E^c, Q/I) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The third line contains the torsion part of the corresponding Hom; C_1, C_2, C_3 are the cokernels of the respective columns.

Since v is surjective and u' is injective, by the Snake Lemma there exists $\tau : Hom_R(E^c, I) \to C_1$ such that the sequence

$$Hom_R(E/E^c, I) \to Hom_R(E, I) \to Hom_R(E^c, I) \xrightarrow{\tau} C_1 \to C_2 \to C_3$$

is exact. Since C_1 is a torsion submodule of $Ext^1_R(E/E^c, I)$, if the latter is torsion-free, then $C_1 = 0$, and thus i^* is surjective.

This moves our attention to trying to understand when $Ext^1_R(J, I)$ is torsion-free, when J is an ideal of a domain R with m-canonical ideal I. By Theorem 3.29 $Ext^1_R(J, I)$ is isomorphic to $E(J, I)$, then as consequence of Corollary 3.32 we have the following.

Corollary 3.33. *Let R be a domain with m-canonical ideal I. $\mathrm{Ext}^1_R(J, I)$ is torsion-free if and only if every sequence $0 \to I \to H \to J \to 0$ is splitting exact when there exists $r \in R$ such that $0 \to I \to I + rH \to rJ \to 0$ is splitting exact.*

In an attempt to understand when a sequence of the type $0 \to I \to H \to J \to 0$ is such as in Corollary 3.33, the next proposition shows that we need to investigate the case when H is not I-reflexive.

Proposition 3.34. *Let R be a domain with an m-canonical ideal I. An R-module H is I-riflexive if and only if every sequence of the type $0 \to I \to H \to J \to 0$, where J is an ideal of R, is splitting exact.*

Proof. Let H be a reflexive R-module. Consider the dual sequence

$$0 \to \mathrm{Hom}_R(J, I) \to \mathrm{Hom}_R(H, I) \to K \to 0$$

where K is the appropriate cokernel. Since $\text{rk}(\text{Hom}_R(H, I)) = \text{rk}(H) = 2$, then $\text{rk}(K) = 1$. Applying the functor $(-)^* := \text{Hom}_R(-, I)$, we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & H & \longrightarrow & J & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \rho_H} & & \downarrow{\scriptstyle \rho_J} & & \\ 0 & \longrightarrow & K^* & \longrightarrow & (H^*)^* & \xrightarrow{\beta} & (J^*)^* & & \end{array} .$$

By our assumption the maps ρ_J e ρ_H are isomorphisms, then β is surjective, so, induces an isomorphism Φ. Since $\text{Hom}_R(K, I)$ has rank 1, by Bazzoni and Salce, 1996, Lemma 3.1 K is I-torsionless, and then $K \cong \text{Hom}_R(\text{Hom}_R(K, I), I) \cong \text{Hom}_R(I, I) = R$. Since R is projective as R-module, the sequence

$$0 \to \text{Hom}_R(J, I) \to \text{Hom}_R(H, I) \to R \to 0$$

is splitting exact, then the I-dual

$$0 \to K^* \to (H^*)^* \to (J^*)^* \to 0$$

is also splitting exact. The isomorphisms make the sequence $0 \to I \to H \to J \to 0$ splitting exact.

Conversely, if $0 \to I \to H \to J \to 0$ is splitting exact, then $H \cong I \oplus J$, so, $\text{Hom}_R(\text{Hom}_R(H, I), I) \cong \text{Hom}_R(\text{Hom}_R(I \oplus J, I), I) \cong \text{Hom}_R(\text{Hom}_R(I, I), I) \oplus \text{Hom}_R(\text{Hom}_R(J, I), I) \cong I \oplus J \cong H$. □

3.5 Open problems

In remark (4), p. 22 of Matlis, 1968, the author formulates a conjecture by suspecting the existence of reflexive domains where not all maximal ideals are 2-generated. This suspicion it has been confirmed by Goeters, 1999 and Proposition 3.12 could be used to provide other examples.

Example 3.35. *$D = \mathbb{Z}[i\sqrt{5}]$, is a Dedekind domain not PID. Consider $I = 2\mathbb{Z}[i\sqrt{5}]$, $b = 2$, since $t^2 - 2$ is irreducible in $\mathbb{Q}(i\sqrt{5})$, then $\mathbb{Z}^* := \mathbb{Z}[i\sqrt{5}](2\mathbb{Z}[i\sqrt{5}])_{0,-2}$ is a domain. Notice that*

$$\tilde{I} = \{x \in \mathbb{Q}(i\sqrt{5}) \mid 2x^2 \in \mathbb{Z}[i\sqrt{5}]\} \supseteq \mathbb{Z}[i\sqrt{5}],$$

so $\overline{\mathbb{Z}^*} \supseteq \mathbb{Z}[i\sqrt{5}] + \mathbb{Z}[i\sqrt{5}]t \supset \mathbb{Z}^*$, then $\mathbb{Z}^*$ is not integrally closed. Consider the maximal ideal $\mathfrak{m} = (2, 1 + i\sqrt{5})D$; in $(\mathbb{Z}[i\sqrt{5}]/\mathfrak{m})[t]$ the polynomial $t^2 - 2$ has 0 as double root, so there exists exactly one prime ideal lying over $\mathfrak{m}$, which is

$$\tilde{\mathfrak{m}} := \mathfrak{m} + 2\mathbb{Z}[i\sqrt{5}]t = (2, 1 + i\sqrt{5}, 2t)\mathbb{Z}^*.$$

Since $2 = -(1 + i\sqrt{5})(1 - i\sqrt{5}) + 2t(2t)$, then $\tilde{\mathfrak{m}} = (1 + i\sqrt{5}, 2t)\mathbb{Z}^*$ is 2-generated.

If $\mathfrak{m} = (7, 3 + i\sqrt{5})D$, since $t^2 - 2 = (t - 3)(t - 4)$ in $(\mathbb{Z}[i\sqrt{5}]/\mathfrak{m})[t]$ there are two prime ideals lying over $\mathfrak{m}$:

$$\mathfrak{m}_1 = \{h_1 + 2h_2t \mid h_1 \in \mathbb{Z}[i\sqrt{5}], h_2 \in 2\mathbb{Z}[i\sqrt{5}], h_1 + 3(2h_2) \in (7, 3 + i\sqrt{5})D\}$$

$$\mathfrak{m}_2 = \{h_1 + 2h_2t \mid h_1 \in \mathbb{Z}[i\sqrt{5}], h_2 \in 2\mathbb{Z}[i\sqrt{5}], h_1 + 4(2h_2) \in (7, 3 + i\sqrt{5})D\}$$

Let $h_1 + 2h_2t$ an element of $\mathfrak{m}_1$. Since $h_1 + 3h_2 \in (7, 3 + i\sqrt{5})D$ we have that $h_1 + 3h_2 = 7l + (3 + i\sqrt{5})m$ for some $m \in \mathfrak{m}$, then $h_1 = 7l + (3 + i\sqrt{5})m - 3h_2$. Since $h_1 + h_2t = 7l + (3 + i\sqrt{5})m + (-3 + t)2h_2' = 7l + (3 + i\sqrt{5})m + (-6 + 2t)h_2'$, then $\mathfrak{m}_1 = (7, 3 + i\sqrt{5}, -6 + 2t)\mathbb{Z}^*$. We have the suspicion that $\mathfrak{m}_1$ is minimally generated by these three elements.

In order to find a counterexample in the non-Noetherian case of a domain R that admits an m-canonical ideal but have a quotient $R(I)_{a,b}$ not divisorial, we provide another construction that we conjecture it could be a solution of the problem.

Example 3.36. *Let K be a field and X an indeterminate on K. For all $n \in \mathbb{N}$ we define $Y_n := X^{\frac{1}{2^n}}$. We denote with $D_n := K[Y_n]$ and with $V_n : (D_n)_{Y_nD_n}$. The ring $V_\infty := \bigcup_{n\in\mathbb{N}} V_n$ is a valuation domain with non principal maximal ideal $\mathfrak{m}$ and $L := K(\{Y_n \mid n \in \mathbb{N}\})$ is its fractions field. If $K = \mathbb{Q}$, the polynomial $f(T) := T^2 + XT - 1 \in V_\infty[T]$ is irreducible in $L[T]$ but admits two distinct factors in $(V_\infty/\mathfrak{m})[T]$. So the ring $V_\infty(\mathfrak{m})_{X,-1}$ is a domain and $V_\infty + V_\infty T$ has exactly two distinct maximal ideal:*

$$\mathfrak{m}_1 := \{a + bT \mid a, b \in V_\infty \text{ and } a + b \in \mathfrak{m}\},$$

$$\mathfrak{m}_2 := \{a + bT \mid a, b \in V_\infty \text{ and } a - b \in \mathfrak{m}\}.$$

We claim that $\mathfrak{m}_1/(\mathfrak{m}+\mathfrak{m}T)$ is not simple as $V_\infty + V_\infty T$-module.

To prove this, we look for a principal ideal $(\lambda+\mu T) \subsetneq \mathfrak{m}_1/(\mathfrak{m}+\mathfrak{m}T)$, with $\lambda, \mu \in V_\infty/\mathfrak{m}$ and $\lambda+\mu \in \mathfrak{m}$, such that

$$(\alpha+\beta T)(\lambda+\mu T)+\mathfrak{m}+\mathfrak{m}T \subsetneq \mathfrak{m}_1$$

for all $\alpha+\beta T \in V_\infty + V_\infty T$.
Fix $(\lambda+\mu T) \subsetneq \mathfrak{m}_1/(\mathfrak{m}+\mathfrak{m}T)$, the question is: pick $a+bT \in \mathfrak{m}_1/(\mathfrak{m}+\mathfrak{m}T)$, with $a, b \in V_\infty/\mathfrak{m}$ and $a+b \in \mathfrak{m}$, is it possible to find $\alpha+\beta T \in V_\infty + V_\infty T$ such that

$$a+bT = (\alpha+\beta T)(\lambda+\mu T)+\sigma+\tau T$$

for any $\sigma, \tau \in \mathfrak{m}$? If such an element exists, it can be proved that:

$$\alpha = \frac{a\lambda - Xa\mu - \sigma\lambda + X\sigma\mu - \mu b + \mu\tau}{\lambda^2 - X\lambda\mu - \mu^2} \in V_\infty$$

$$\beta = \frac{\lambda b - \lambda\tau - a\mu + \mu\sigma}{\lambda^2 - X\lambda\mu - \mu^2} \in V_\infty$$

In particular, $\lambda b - \lambda\tau - a\mu + \mu\sigma \in (\lambda^2 - X\lambda\mu - \mu^2) = ((\lambda-\mu)(\lambda+\mu) - X\lambda\mu) \subseteq \mathfrak{m}$. If we chose $a, b, \lambda, \mu \in V_\infty$ such that $\lambda b - \mu a \notin \mathfrak{m}$, we have an absurd.

Bibliography

Anderson, D. and M. Winders (2009). "*Idealization of a Module*". In: Journal of Commutative Algebra 1.1, pp. 3–56.

Barucci, V. (2009). "*Local rings of minimal length*". In: Journal of Pure and Applied Algebra 213.6, pp. 991–996.

Barucci, V., M. D'Anna, and F. Strazzanti (2015). "*A family of quotients of the Rees algebra*". In: Commun. Algebra 43.1, pp. 130–142.

— (2016). "*Family of Gorenstein and almost Gorenstein rings*". In: Ark. Mat. 54.2, pp. 321–338.

Barucci, V. et al. (2019). "*m-Canonical Ideals in Integral Domains II*". In: Ideal Theoretic Methods in Commutative Algebra, pp. 89–108.

Bass, H. (1963). "*On the ubiquity of Gorenstein rings*". In: Math Z 82, pp. 8–28.

Bazzoni, S. and L. Salce (1996). "*Warfield Domains*". In: Journal of Algebra 185.3, pp. 836–868.

D'Anna, M. (2006). "*A construction of Gorenstein rings*". In: J. Algebra 185, pp. 507–519.

D'Anna, M. and M. Fontana (2007a). "*An amalgamated duplication of a ring along an ideal: the basic properties*". In: Journal of Algebra and its Applications 6.03, pp. 443–459.

— (2007b). "*The amalgamated duplication of a ring along a multiplicative-canonical ideal*". In: Ark. Mat. 45.2, pp. 241–252.

D'Anna, M. and F. Strazzanti (2017). "*New algebraic properties of quadratic quotients of the Rees algebra*". In: Journal of Algebra and Its Applications 18.

Frigenti, F. (2019). "*Canonical reduction for quadratic quotients of the Rees algebra*". In: Le Matematiche 74.2, pp. 397–409.

Fuchs, L. and L. Salce (2001). "*Modules over Non-Noetherian Domains*". In: Mathematical surveys and monographs 84.

Goeters, H.Pat (1999). "*Reflexive rings and subrings of a product of Dedekind domains*". In: Journal of Pure and Applied Algebra 137.3, pp. 237–252.

Gulliksen, Tor H. (1974). "*The Krull Ordinal, Coprof, and Noetherian Localizations of Large Polynomial Rings*". In: American Journal of Mathematics 96.2, pp. 324–339.

Heinzer, W. J., J. A. Huckaba, and I. J. Papick (1998). "*M-Canonical ideals in integral domains*". In: Communications in Algebra 26.9, pp. 3021–3043.

Heinzer, William (1968). "*Integral domains in which each non-zero ideal is divisorial*". In: *Mathematika* 15.2, 164–170.

Herzog, J. and E. Kunz (1971). "*Der kanonische Modul eines Cohen-Macaulay-Rings*". In: Lecture Notes in Math. 74.

Hilton, P. J. and U. Stammbach (2013). "*A Course in Homological Algebra*". In: Graduate Texts in Mathematics 1.

Licata, A. M. A. (2022). "*Double covering di curve e duplicazione di anelli e semigruppi*". In: Doctoral thesis.

Matlis, E. (1968). "*Reflexive domains*". In: Journal of Algebra 8.1, pp. 1–33.

Nagata, M. (1962). "*Local Rings*". In: Interscience Tracts in Pure and Applied Mathematics 13.

Reiten, I. (1972). "*The converse to a theorem of Sharp on Gorenstein modules*". In: Proc. Amer. Math. Soc. 32, pp. 417–420.

Walker, C. P. (1964). "*Properties of Ext and quasi-splitting of Abelian groups*". In: Acta Mathematica Academiae Scientiarum Hungaricae 15, 157–160.

Weibel, Charles A. (1994). An Introduction to Homological Algebra. Cambridge Studies in Advanced Mathematics. Cambridge University Press.