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Line element-less method (LEM) for arbitrarily shaped nonlocal nanoplates: exact and approximate analytical solutions

DI MATTEO Alberto[1, a *] and PIRROTTA Antonina[1,b]

[1]Dipartment of Engineering, University of Palermo, Viale delle Scienze Ed. 8, 90128, Palermo, Italy

[a]alberto.dimatteo@unipa.it, [b]antonina.pirrotta@unipa.it

* please mark the corresponding author with an asterisk

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Abstract. This paper presents an innovative procedure for the analysis of nonlocal plates with arbitrary shape and various boundary conditions. In this regard, the Eringen's nonlocal model is used to capture small length scale effects. The proposed procedure, referred to as Line Element-Less Method (LEM), is a completely meshfree approach requiring the evaluations of simple line integrals along the plate boundary parametric equation. Further, the deflection function is represented by a series expansion is terms of harmonic polynomials whose coefficients are found by performing variations of appropriately introduced functionals, leading to a linear system of algebraic. Notably, the proposed procedure yields approximate analytical solutions for general shapes and boundary conditions, and even exact solutions for some plate geometries.

Introduction

The mechanical behavior of most structures at the nanoscale is typically size dependent, and classical approaches of continuum mechanics cannot capture this peculiar characteristic. Therefore, more sophisticated continuum theories have been introduced, and several different models have been developed [1-4].

One of the most widely adopted is the nonlocal elasticity theory, firstly introduced by Eringen [5-6], in which the stress at some reference point is assumed to be a function of the strain field at every point in the body, and the size-effect feature is captured in the model through an additional material parameter generally referred to as “the nonlocal parameter”. This has paved the way for the application of Eringen's nonlocal elasticity theory in a plethora of studies involving the mechanical behavior of structural systems at the nanoscale, mostly related to nanobeams [7-9].

Recently, Eringen's nonlocal elasticity theory has been also used for the analysis on nanoplates. Initial contributions in [9-11] treated both isotropic linear elastic Kirchhoff plates as well as nonlinear plates considering higher-order shear deformation theory. Notably, most of these studies have focused on plates with simple rectangular shape. In this regard, the classical Navier's or Levy's approaches have been used in [10, 12]. Few other studies have focused on the analysis of nonlocal plates of different shapes [13], with most contributions mainly related to circular shapes [14, 15].

On this base, this study deals with the bending response of micro and nanoscale Kirchhoff plate, using Eringen's nonlocal theory, and considering arbitrary geometries and general boundary conditions. Specifically, the so-called Line Element-less Method (LEM) [16-25], is here extended to determine the deflection and bending moments of nonlocal plates subjected to transversal loads. Notably, the proposed procedure only requires the solution of simple line integrals of harmonic polynomials with unknown coefficients, along the boundary parametric equation and, eventually, the solution of a set of linear algebraic equations for these unknown terms. The LEM is completely

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element-free, since it does not require any discretization, be it in the domain or in the boundary, and it also differs from other so-called meshfree procedure since the expansion coefficients are not determined by collocation. Notably, this procedure yields approximate analytical solutions for generally shaped nonlocal plates, and even exact closed-form solution for some geometries and boundary conditions. These aspects clearly represent attractive features of the proposed procedure, especially with respect to other meshfree methods that are of numerical nature only. In this regard, several applications will be discussed, assessing the simplicity and accuracy of the considered approach.

Preliminary Concepts on Nonlocal Plate in Bending

Consider a homogeneous, isotropic, linear elastic Kirchhoff plate of arbitrary shape with contour Γ, domain Ω, and uniform thickness h, subjected to a transverse distributed load $q(x,y)$, and satisfying the Eringen's nonlocal model [5]. The plate is characterized by the modulus of elasticity E, Poisson's ratio ν, and nonlocal parameter $\lambda = (l\,e_0)^2 \geq 0$, where l is an internal characteristic length, whereas e_0 is the small length scale coefficient. Note that when $\lambda = 0$ the classical local Kirchhoff plate is obtained. The corresponding biharmonic governing differential equation for bending of the Eringen's nonlocal plate in terms of deflection function can be written as

$$D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=\left[q-\lambda\left(\frac{\partial^2 q}{\partial x^2}+\frac{\partial^2 q}{\partial y^2}\right)\right] \tag{1}$$

where $D = Eh^3 \big/ 12\left(1-\nu^2\right)$ is the plate flexural rigidity. Notably, as it can be seen in Eq. (1), the only difference with respect to classical plate differential equation (local model) stands in the additional term $\lambda\nabla^2 q(x,y)$ at the right-hand-side of Eq. (1), which is therefore an inherent effect due to the employed Eringen's nonlocal model.

Further, introduce the Marcus' moment sum $M(x,y)$, defined as [26, 27] $M(x,y) = \left(M_x + M_y\right)\big/\left(1+\nu\right)$, where $M_x(x,y)$ and $M_y(x,y)$ are the bending moments, Eq. (1) can be decomposed into two Poisson's equations as

$$\begin{aligned}&\nabla^2 M(x,y) = -q(x,y)\\ &D\nabla^2 w(x,y) = -\left(1-\lambda\nabla^2\right)M(x,y)\end{aligned} \tag{2.a, b}$$

Note that, in this manner, the solution of the plate problem Eq. (1) reduces to the integration, in sequence, of the two Poisson differential equations Eqs. (2), respectively, which is sometimes preferred depending upon the method of solution employed. As far as the boundary conditions (BCs) are concerned, denote as **n** the outward unit normal vector at a point of a generic curvilinear edge of the contour Γ. Thus, the BCs for the simply support curvilinear edge can be assumed as those of the classical local plate, that is $w(x,y)=0$ and $M_n(x,y)=0$, where $M_n(x,y) = n_x^2\,M_x + n_y^2\,M_y + 2\,n_x\,n_y\,M_{xy}$ denotes the normal bending moment applied at the edge, and n_x and n_y are the components of the unitary vector $\mathbf{n}$ along the x and y axes, respectively. Note that analogous expression can be found for different BCs in [25].

Line Element-Less Method for nonlocal plate analysis

In this section LEM approach is introduced for the analysis of nonlocal Kirchhoff plates of general shape, subjected to a transversal load $q(x,y)$. Specifically, based on the previous studies on the use of the LEM [16-25] and considering Eqs. (2), $M(x,y)$ and $w(x,y)$ can be expressed in terms of the so-called harmonic polynomials P_k and Q_k, generally defined recursively as.

$$P_k(x,y) = P_{k-1}x - Q_{k-1}y$$
$$Q_k(x,y) = Q_{k-1}x - P_{k-1}y \tag{3.a, b}$$

which are valid for $k>0$, and with $P_0 = 1$ and $Q_0 = 0$.

Based on the above relations, a solution of Eq. (2.a) can be obtained expressing the moment sum function in terms of harmonic polynomials, plus a particular solution of the Poisson equation Eq. (2.a), namely $M_c(x,y)$. That is

$$M(x,y) \cong \sum_{k=0}^{p} a_k P_k(x,y) + \sum_{k=1}^{p} b_k Q_k(x,y) + M_c(x,y) \tag{4}$$

where a_k and b_k are $(2p+1)$ unknown coefficients to be determined, and p is the truncation limit of the series expansion. Note that, for the typical case of a uniformly distributed load $q(x,y) = q_0$, the following expressions of $M_c(x,y) = -q_0(x^2+y^2)/4$. As far as the unknown coefficients in Eq. (4) are concerned, the $(2p+1)$ values of a_k and b_k can be determined appropriately imposing the specified BCs of the plate. In this context, it is convenient to address the case of polygonal plates with simply-supported edges, while the generalization to arbitrary shaped plates is reported in [25]. Specifically, in this case, the moment sum function must be zero along the entire contour of the polygonal plate; thus, $M_n(x,y) = M(x,y) = 0$ in Γ.

Therefore, the unknown coefficients (a_k, b_k) in Eq. (4) can be evaluated applying a minimization procedure on the closed contour path integral of the squared moment sum function; that is $\Psi(a_k, b_k) = \oint_\Gamma [M(x,y)]^2 \, d\gamma$. Thus, performing variations of the aforementioned functional with respect to (a_k, b_k), yields a linear algebraic system of $(2p+1)$ equations in the unknowns (a_k, b_k). In this manner, once the coefficients are determined, the moment sum function can be found through Eq. (4).

Next, the deflection function $w(x,y)$ can be obtained solving Eq. (2.b). In this regard, following a similar approach, a solution of this equation can be sought assuming $w(x,y)$ as the sum of harmonic polynomials, and a particular solution of the Poisson equation Eq. (2.b), namely $w_c(x,y)$; that is

$$w(x,y) \cong \sum_{k=0}^{m} c_k P_k(x,y) + \sum_{k=1}^{m} d_k Q_k(x,y) + w_c(x,y) \tag{5}$$

where c_k and d_k are $(2m+1)$ unknown coefficients to be determined, and m is the truncation limit of the series expansion. Again, note that the particular solution $w_c(x,y)$ can be evaluated applying the procedure in [25], considering the term at the right-hand side of Eq. (2.b).

As far as the unknown coefficients in Eq. (5) are concerned, the $(2m+1)$ values of c_k and d_k are determined appropriately imposing the BCs. In this regard, considering the case of a simply-supported plate, the coefficients (c_k, d_k) can be found minimizing the closed contour path integral of the squared deflection function; that is $\Theta(c_k, d_k) = \oint_{\Gamma}[w(x,y)]^2\, d\gamma$. Next, introducing Eq. (5) into the previous functional, and performing variation with respect to the unknown coefficients leads to an algebraic linear system in terms of the unknowns (c_k, d_k). Solution of the this set of $(2m+1)$ equation yields the sought deflection function of the plate $w(x,y)$ through Eq. (5).

Numerical Applications

In this section, the proposed LEM is applied to some nonlocal plate configurations, considering various shapes and boundary conditions. Specifically, the proposed method is employed for the analysis of different well-known examples, showing the ability of the approach to yield even exact solutions. In this regard, a triangular simply-supported plate and a simply-supported circular plate are considered.

Specifically, firstly consider the case of an equilateral triangular shaped plate with length side $2\sqrt{3}l$ under a uniformly distributed load q_0 and with simply-supported edges. Applying the previously described procedure, the obtained closed-form expression of the deflection function is

$$w(x,y) = \frac{q_0}{192\, l\, D}(l-y)\left[(2l-y)^2 - 3x^2\right]\left(4l^2 - x^2 - y^2 + 16\lambda\right) \tag{6}$$

Notably, Eq. (6) reverts to the classical solution of the Krichoff local plate for $\lambda = 0$. In this regard, deflection profile for $x = 0$ is shown in Fig. 1(a) for different values of nonlocal parameters.

Next, consider next the case of a circular plate of radius r under a uniformly distributed load q_0. Note that, in this case, the equilibrium equations of axisymmetric bending of circular nanoplates can be more simply written in polar coordinates as in [15]. Then, applying the previously described procedure, the closed-form expression of the deflection function is given as

$$w(x,y) = \frac{q_0\left(r^2 - x^2 - y^2\right)\left[-\left(x^2+y^2\right)(1+\nu) + r^2(5+\nu) + 8\lambda(3+\nu)\right]}{64\, D(1+\nu)} \tag{7}$$

Notably, again Eq. (7) reverts to the classical solution of the Krichoff local plate for $\lambda = 0$. In this regard, the deflection profile for $x = 0$ is given in Fig. 1(b) for different values of nonlocal parameter.

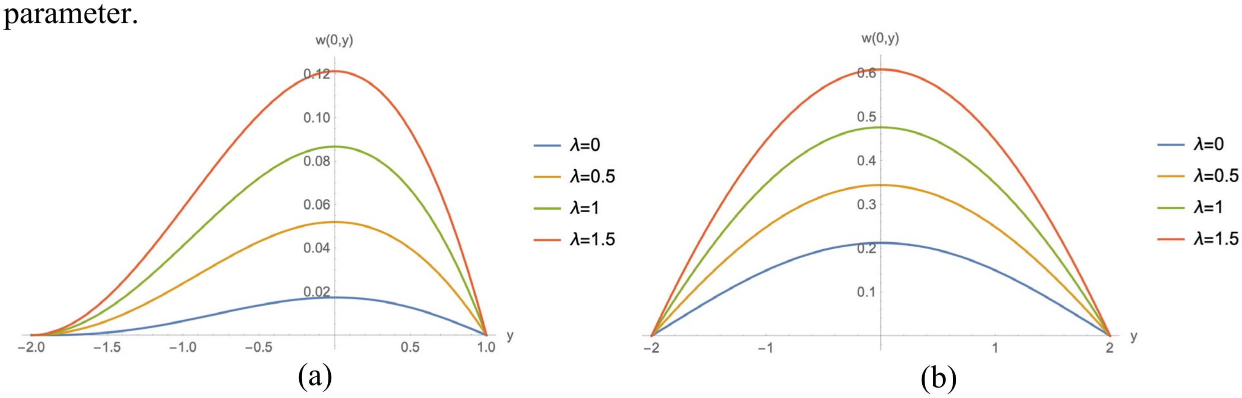

Fig. 1: Profile of $w(0,y)$ for different value of λ. (a) Simply-supported Triangular Plate; (b) Simply-supported Circular Plate.

Summary

In this paper, the so-called Line Element-Less Method (LEM) has been proposed for the analysis of nonlocal plates with arbitrary shape. Specifically, the classical Kirchhoff plate model has been assumed, employing the well-known Eringen's nonlocal elasticity theory to capture small length

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scale effects. The method is based on an expansion of the deflection function in terms of harmonic polynomials, whose expansion coefficients can be easily found solving a linear system of algebraic equations. Notably, the entire procedure proves to be entirely mesh-free, since it only requires the definition of simple line integrals which appear in appropriately introduced functionals. These functionals are employed to take into account the pertinent plate boundary conditions (BCs). It is worth mentioning that, the proposed approach yields approximate analytical solutions for general plate shapes and BCs, while exact closed form expressions of the deflection functions of nonlocal plates can be found for particular shapes.

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