COX RING OF THE GENERIC FIBER

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Abstract. Given a surjective morphism $\pi\colon X \to Y$ of normal varieties satisfying some regularity hypotheses we prove how to recover a Cox ring of the generic fiber of π from the Cox ring of X. As a corollary we show that in some cases it is also possible to recover the Cox ring of a very general fiber, and finally we give an application in the case of the blowing-up of a toric fiber space.

Introduction

Let X be a normal variety defined over an algebraically closed field $\mathbb{K}$ of characteristic zero. If the divisor class group $\mathrm{Cl}(X)$ of X is finitely generated and $\mathbb{K}[X]^* = \mathbb{K}^*$, i.e. the only global regular invertible functions of X are constants, the *Cox sheaf* of X can be defined as (see [2])

$$\mathcal{R} := \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{O}_X(D),$$

while its *Cox ring* $\mathcal{R}(X)$ is the ring of global sections $\Gamma(X, \mathcal{R})$. Given a morphism $\pi\colon X \to Y$ of normal varieties defined over $\mathbb{K}$, possible relations between the Cox rings of X and Y have been recently studied in many cases (see for instance [1, 3, 11, 12, 17]). On the contrary, to our knowledge there are no results concerning relations with the Cox ring of the fibers of π. Trying to fill this gap, in the present paper we consider the problem of determining the Cox ring of the generic fiber X_η (and in some cases also the Cox ring of the very general fiber) of π from the Cox ring of X and from the *vertical classes* of π, i.e. classes of divisors whose image in Y is not dense. Observe that since X_η is defined over a non closed field (isomorphic to the function field $\mathbb{K}(Y)$), we need to define a Cox ring for X_η following [6].

In order to describe our results let us denote by $\mathrm{Cl}_\pi(X)$ the subgroup of $\mathrm{Cl}(X)$ generated by classes of vertical divisors, or equivalently the kernel of the surjection $\mathrm{Cl}(X) \to \mathrm{Cl}(X_\eta)$, induced by the pull-back of the natural morphism $\imath\colon X_\eta \to X$. If we denote by $\mathcal{R}_\pi(X)$ the localization of $\mathcal{R}(X)$ by the multiplicative subsystem generated by the non-zero homogeneous elements $f \in \mathcal{R}(X)_w$, with $w \in \mathrm{Cl}_\pi(X)$, and by $\mathrm{Frac}_0(\mathcal{R}(X))$ the field of degree zero homogeneous fractions on $\mathcal{R}(X)$, the following holds.

Proposition 1. *The image of the homomorphism $\mathbb{K}(Y) \to \mathrm{Frac}_0(\mathcal{R}(X))$ induced by the pullback is $\mathcal{R}_\pi(X)_0$, the subset of degree zero homogeneous elements of $\mathcal{R}_\pi(X)$.*

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One consequence of the proposition above is that $\mathcal{R}_\pi(X)$ has a structure of $\mathbb{K}(Y)$-algebra. On the other hand, since the generic fiber X_η is defined over a field k, isomorphic to $\mathbb{K}(Y)$, following [6] we can construct a Cox ring $\mathcal{R}(X_\eta)$ which has the structure of $\mathbb{K}(Y)$-algebra too. Our main result is a description of the relation existing between these two algebras. In particular the latter turns out to be a quotient of the former, and the precise result is the content of the following.

Theorem 1. *Let $\pi\colon X \to Y$ be a proper surjective morphism of normal varieties having only constant invertible global sections, such that $\mathrm{Cl}(X)$ is finitely generated, $\mathrm{Cl}_\pi(X)$ torsion free, and the very general fiber of π is irreducible. Then there exists a Cox ring $\mathcal{R}(X_\eta)$ of the generic fiber X_η such that the canonical morphism $\imath\colon X_\eta \to X$ induces an isomorphism of $\mathrm{Cl}(X_\eta)$-graded $\mathbb{K}(Y)$-algebras*

$$\mathcal{R}_\pi(X)/\langle 1-u(w)\,:\, w\in \mathrm{Cl}_\pi(X)\rangle \to \mathcal{R}(X_\eta),$$

where $u\colon \mathrm{Cl}_\pi(X)\to \mathcal{R}_\pi(X)^*$ is any homomorphism satisfying $u(w)\in \mathcal{R}_\pi(X)^*_{-w}$ for each w.

Let us suppose in addition that the class group of the geometric generic fiber $X_\eta \times_k \bar{k}$ is isomorphic to $\mathrm{Cl}(X_\eta)$. We will show (see Corollary 5.2) that in this case it is possible to combine the theorem above with the results of [19] in order to recover the Cox ring of a very general fiber of π from the Cox ring of X. A direct consequence is that finite generation for the Cox ring of X implies finite generation for the Cox ring of the very general fiber. Applying these results to the blowing-up of a toric fiber space along a section, we will finally produce new examples of varieties with non-finitely generated Cox ring.

The paper is structured as follows. In Section 1 we first recall the definition of Cox sheaf and Cox ring for a variety defined over a closed field, and then, after remembering some facts about varieties defined over a (not necessarily closed) perfect field, following [6] we construct a Cox sheaf for such varieties. In Section 2 we collect some results about the generic fiber X_η of a proper surjective morphism $\pi\colon X\to Y$, whose very general fiber is irreducible. Section 3 contains the proof of Proposition 1 and some lemmas that we are going to use in Section 4, where we prove Theorem 1. In Section 5 we consider the very general fiber and in the last section we apply the results above to the blowing up of toric fiber spaces along a section.

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1. Preliminaries

In this section we first recall the definition of Cox sheaf and Cox ring in the case of a variety defined over an algebraically closed field (see [2]). Then, after recalling some known facts about algebraic varieties defined over a (non necessarily closed) perfect field we construct a suitable *Cox sheaf of type* λ for such varieties, according to [6, Definition 2.2].

1.1. Algebraically closed fields.

Construction 1.1. (see [2]) Let X be a normal variety defined over a closed field $\mathbb{K}$, such that $\mathbb{K}[X]^* = \mathbb{K}^*$ and $\mathrm{Cl}(X)$ is finitely generated. Let K be a finitely generated subgroup of $\mathrm{WDiv}(X)$ such that the class map $\mathrm{cl}\colon K \to \mathrm{Cl}(X)$ is onto. The *sheaf of divisorial algebras* associated to K and its global sections are

$$\mathcal{S} = \bigoplus_{D\in K} \mathcal{O}_X(D) \quad \text{and} \quad \Gamma(X,\mathcal{S}) = \bigoplus_{D\in K} \Gamma(X,\mathcal{O}_X(D))$$

respectively. In the rest of the paper, when we need to keep trace of the group K, we will use the notation $\mathcal{S}_K$ instead of $\mathcal{S}$. We will also denote by $\Gamma(X,\mathcal{S})_D$ the degree D part of the ring of global sections of $\mathcal{S}$. Let us consider now the kernel $K^0 \subseteq K$ of the class map and let $\mathcal{X} : K^0 \to \mathbb{K}(X)^*$ be a homomorphism of groups such that $\mathrm{div} \circ \mathcal{X} = \mathrm{id}$. Let $\mathcal{I}$ be the ideal sheaf of $\mathcal{S}$, locally generated by sections of the form $1 - \mathcal{X}(D)$, where $D \in K^0$. The quotient $\mathcal{R} := \mathcal{S}/\mathcal{I}$ turns out to be a sheaf (see [2, Lemma 1.4.3.5]) and it is called a *Cox sheaf* for X. The *Cox ring* $\mathcal{R}(X)$ of X can be defined as the ring of global sections of $\mathcal{R}(X)$, or equivalently

$$\mathcal{R}(X) = \frac{\Gamma(X,\mathcal{S})}{\Gamma(X,\mathcal{I})}. \tag{1.1}$$

1.2. **Non closed fields.** Let us recall now some facts about an algebraic variety X, defined over a perfect field k (see for instance [14, § A.1 and § A.2]). In what follows we will denote by $X_{\bar{k}}$ the base change of X over the algebraic closure $\bar{k}$ of k. From now on we assume that any variety X has only constant invertible global sections, i.e.

$$\bar{k}[X]^* = \bar{k}^*, \tag{1.2}$$

where $\bar{k}[X]$ denotes the ring of global sections of the structure sheaf of X. Let us denote by $G = \mathrm{Gal}(\bar{k}/k)$ the absolute Galois group of k and let

$$\mathrm{WDiv}(X) \quad := \quad \{D \in \mathrm{WDiv}(X_{\bar{k}}) : \sigma(D) = D \text{ for any } \sigma \in G\}$$

be the group of G-invariant Weil divisors of $X_{\bar{k}}$. We will denote by $\mathrm{PDiv}(X)$ the subgroup of $\mathrm{WDiv}(X)$ consisting of principal divisors of the form $\mathrm{div}(f)$, with $f \in k(X)$. By [14, Proposition A.2.2.10 (ii)] the equality $\mathrm{PDiv}(X) = \mathrm{WDiv}(X) \cap \mathrm{PDiv}(X_{\bar{k}})$ holds (observe that the hypothesis in the cited proposition asks for X to be projective, but it actually only makes use of the weaker condition (1.2)). Thus, if we denote by $\mathrm{Cl}(X)$ the quotient group $\mathrm{WDiv}(X)/\mathrm{PDiv}(X)$, we get inclusions

$$\mathrm{Cl}(X) \subseteq \mathrm{Cl}(X_{\bar{k}})^G \subseteq \mathrm{Cl}(X_{\bar{k}}). \tag{1.3}$$

Given a divisor $D \in \mathrm{WDiv}(X)$ and a Zariski open subset U of X, the space of sections $\mathcal{O}_{X_{\bar{k}}}(D)(U_{\bar{k}})$ is a $\bar{k}$ vector space acted by G, since both U and X are defined over k, and thus it is a G-module. Observe that a G-invariant element $f \in \mathcal{O}_{X_{\bar{k}}}(D)(U_{\bar{k}})^G$ is a rational function of $X_{\bar{k}}$, which is defined over k (see for instance [18, Exercise 1.12]). If we set

$$\mathcal{O}_X(D)(U) := \mathcal{O}_{X_{\bar{k}}}(D)(U_{\bar{k}})^G, \tag{1.4}$$

by [14, Proposition A.2.2.10 (i)] we have that the $\bar{k}$ vector space $\mathcal{O}_X(D)(U) \otimes_k \bar{k}$ is isomorphic to $\mathcal{O}_{X_{\bar{k}}}(D)(U_{\bar{k}})$.

We are now able to construct a sheaf $\mathcal{R}$ of $\mathcal{O}_X$-algebras which turns out to be a *Cox sheaf of type* λ according to [6, Definitions 2.2 and 3.3].

Construction 1.2. Let X be a variety defined over a perfect field k, satisfying (1.2). Let us suppose that $\mathrm{Cl}(X)$ is finitely generated and let K be a finitely generated subgroup of $\mathrm{WDiv}(X)$ whose image via the class map $\mathrm{cl}\colon K \to \mathrm{Cl}(X_{\bar{k}})$ is $\mathrm{Cl}(X)$. Let us consider the K-graded sheaf of $\mathcal{O}_X$-algebras

$$\mathcal{S} = \bigoplus_{D \in K} \mathcal{O}_X(D),$$

where $\mathcal{O}_X(D)$ is the sheaf defined in (1.4). Denote by $K^0 \subseteq K$ the kernel of the class map and let $\mathcal{X} : K^0 \to k(X)^*$ be a homomorphism of groups such that $\mathrm{div} \circ \mathcal{X} = \mathrm{id}$ (such a $\mathcal{X}$ exists again by [14, Proposition A.2.2.10 (ii)]). Let $\mathcal{I}$ be the ideal sheaf of $\mathcal{S}$ locally generated by sections of the form $1 - \mathcal{X}(D)$, where $D \in K^0$. Denote by $\mathcal{R}$ the presheaf $\mathcal{S}/\mathcal{I}$ and by $\pi\colon \mathcal{S} \to \mathcal{R}$ the quotient map.

Proposition 1.3. *The presheaf $\mathcal{R}$ defined above is a Cox sheaf of type λ (see [6, Definition 3.3]) where $\lambda\colon \mathrm{Cl}(X) \to \mathrm{Cl}(X_{\bar{k}})$ is the inclusion.*

Proof. By construction K/K^0 is isomorphic to the group $\mathrm{Cl}(X)$, so that grading over the former is equivalent to grading over the latter. Consider the sheaf of divisorial algebras

$$\bar{\mathcal{S}} = \bigoplus_{D \in K} \mathcal{O}_{X_{\bar{k}}}(D)$$

together with the G-invariant character $\mathcal{X}\colon K^0 \to k(X)^* \subseteq \bar{k}(X)^*$, where $G = \mathrm{Gal}(\bar{k}/k)$ as before, and let $\bar{\mathcal{I}}$ be the ideal sheaf of $\bar{\mathcal{S}}$ defined by $\mathcal{X}$. Let us denote by $\phi\colon X_{\bar{k}} \to X$ the base change map. According to the proof of [6, Proposition 3.19] the quotient sheaf $\bar{\mathcal{R}} = \bar{\mathcal{S}}/\bar{\mathcal{I}}$ is a G-equivariant Cox sheaf of type λ and the push forward of the sheaf of invariants $\phi_*\bar{\mathcal{R}}^G$ is a Cox sheaf of X of type λ. Given an open subset $U \subseteq X$ and a divisor $D \in K$ the following holds

$$\begin{aligned}(\phi_*\bar{\mathcal{R}}^G_{[D]})(U) &= (\bar{\mathcal{R}}_{[D]}(U_{\bar{k}}))^G\\ &= ((\bar{\mathcal{S}}/\bar{\mathcal{I}})_{[D]}(U_{\bar{k}}))^G\\ &\simeq (\bar{\mathcal{S}}_D(U_{\bar{k}})/\bar{\mathcal{I}}_D(U_{\bar{k}}))^G\\ &\simeq \mathcal{S}_D(U)/\mathcal{I}_D(U),\end{aligned}$$

where the first isomorphism is by [6, Construction 2.7], while the second one is by Lemma 1.5 and equality (1.4). Therefore $\mathcal{R} = \mathcal{S}/\mathcal{I}$ is isomorphic to $\phi_*\bar{\mathcal{R}}^G$, and in particular it is a Cox sheaf of type λ. □

Remark 1.4. We remark that by [6, Definition 3.3], the ring of global sections $\mathcal{R}(X) = \Gamma(X, \mathcal{S})/\Gamma(X, \mathcal{I})$ is a Cox ring of type λ, and in particular this implies that a Cox ring of type λ for $X_{\bar{k}}$ can be obtained from $\mathcal{R}(X)$ by a base change. In particular, if $\lambda = \mathrm{id}_{\mathrm{Cl}(X_{\bar{k}})}$, we get the usual Cox ring for $X_{\bar{k}}$.

Lemma 1.5. *Let L/k be a Galois extension of fields with Galois group G. Let $V_1 \subseteq V_2$ be k vector spaces and let $\overline{V}_i = V_i \otimes_k L$. Then $\overline{V}_1$ is G-invariant and the homomorphism*

$$j\colon V_2/V_1 \to (\overline{V}_2/\overline{V}_1)^G \qquad v + V_1 \mapsto v + \overline{V}_1$$

is an isomorphism.

Proof. By hypothesis there is a G-equivariant isomorphism $\overline{V}_2 \simeq \overline{V}_1 \oplus \overline{T}$ of G-invariant vector spaces, where T is obtained by completing a basis of V_1 to a basis of V_2. Since $\overline{V}_1 \cap V_2 = \overline{V}_1^G = V_1$, the map j is injective. To prove the surjectivity of j let $v+\overline{V}_1 \in (\overline{V}_2/\overline{V}_1)^G$, that is $gv+\overline{V}_1 = v+\overline{V}_1$ for any $g \in G$, or equivalently

$$gv - v \in \overline{V}_1.$$

Write $v = v_1 + t$ with $v_1 \in \overline{V}_1$ and $t \in \overline{T}$ and observe that $gv - v \in \overline{V}_1$ implies $gt - t \in \overline{V}_1 \cap \overline{T} = 0$, so that t is G-invariant, that is $t \in V_2$. Thus

$$v + \overline{V}_1 = t + \overline{V}_1 = j(t + V_1).$$

□

2. Divisors on the generic fiber

Let X and Y be normal varieties defined over an algebraically closed field $\mathbb{K}$ of characteristic zero and satisfying (1.2), and let η be the generic point of Y. In this section we are going to summarise some results about the generic fiber $X_\eta := X \times_Y \eta$ of a proper surjective morphism $\pi\colon X \to Y$, whose very general fiber is irreducible.

First of all observe that the morphism $\imath\colon X_\eta \to X$ induces a pullback isomorphism $\imath^*\colon \mathbb{K}(X) \to k(X_\eta)$, where $k = (\pi \circ \imath)^*(\mathbb{K}(Y)) \simeq \mathbb{K}(Y)$. We remark that the complementary of the smooth locus $X^{\rm sm}$ has codimension at least two in X and the same holds for the generic fiber of the restriction $\pi_{|X^{\rm sm}}$ in X_η. Therefore $\imath^*$ induces a surjective homomorphism

$$\mathrm{WDiv}(X) \to \mathrm{WDiv}(X_\eta). \tag{2.1}$$

In what follows, by abuse of notation, we will use the same symbol $\imath^*$ for the above homomorphism, and we will denote by $\mathrm{WDiv}_\pi(X)$ its kernel.

Proposition 2.1. *The following hold:*

(i) *the diagram*

$$\begin{array}{ccc} \mathbb{K}(X) & \xrightarrow{\imath^*} & k(X_\eta) \\ \downarrow{\scriptstyle \mathrm{div}} & & \downarrow{\scriptstyle \mathrm{div}} \\ \mathrm{WDiv}(X) & \xrightarrow{\imath^*} & \mathrm{WDiv}(X_\eta) \end{array}$$

is commutative;

(ii) *if $D \in \mathrm{WDiv}(X)$ is effective on an open subset $U \subseteq X$ then $\imath^*(D)$ is effective on the corresponding open subset $U_\eta \subseteq X_\eta$;*

(iii) *the group $\mathrm{WDiv}_\pi(X)$ is freely generated by the prime divisors that do not dominate Y;*

(iv) *the map (2.1) induces a surjective homomorphism $\mathrm{Cl}(X) \to \mathrm{Cl}(X_\eta)$, whose kernel $\mathrm{Cl}_\pi(X)$ is generated by the classes of divisors in $\mathrm{WDiv}_\pi(X)$;*

(v) *for any $D \in \mathrm{WDiv}(X)$ the pullback induces a map $\imath^*\colon \mathcal{O}_X(D) \to \imath_*\mathcal{O}_{X_\eta}(\imath^* D)$.*

Proof. Recall that the generic fiber X_η is limit of the family of open subsets $\pi^{-1}(V) \subseteq X$, where V varies through the open subsets of Y. Let us consider $V = \mathrm{Spec}(B)$, and let $U = \mathrm{Spec}(A)$ be an affine open subset of $\pi^{-1}(V)$. The morphism $\pi|_U\colon U \to V$ is induced by an injective homomorphism of $\mathbb{K}$-algebras, $B \to A$. Identifying B with a subalgebra of A we have that the affine open subset $U_\eta \subseteq X_\eta$, obtained by base change over U, is the spectrum of the localization $S_B^{-1}A$, whose multiplicative system is $S_B = B \setminus \{0\}$. The pullback

$$\imath^*\colon \mathcal{O}_X(U) \to \imath_*\mathcal{O}_{X_\eta}(U_\eta)$$

is thus defined on U by the injection $A \to S_B^{-1}A$. This shows that a prime divisor D defined by a prime ideal $\mathfrak{p} \subseteq A$ survives in the generic fiber if and only if $\mathfrak{p} \cap B = 0$, that is D has non-empty intersection with $\pi^{-1}(V)$. This proves (ii) and (iii). In order to prove (i) recall that the order of a rational function $f \in \mathbb{K}(X)$ at D is the length of the $\mathcal{O}_{X,\mathfrak{p}}$-module $\mathcal{O}_{X,\mathfrak{p}}/\langle f \rangle$, but the local rings $\mathcal{O}_{X,\mathfrak{p}}$ and $\mathcal{O}_{X_\eta,\mathfrak{p}}$ are isomorphic if $\mathfrak{p} \cap B = 0$. In order to prove (iv), observe first that the omomorphism

$$\mathrm{Cl}(X) \to \mathrm{Cl}(X_\eta), \qquad [D] \mapsto [\imath^*(D)]$$

is well defined by (i). Let us fix a divisor $D \in \mathrm{WDiv}(X)$ such that $[D]$ is in the kernel $\mathrm{Cl}_\pi(X)$ of the above map. By definition this implies that $\imath^*(D)$ is principal on X_η, so that we can write $\imath^*(D) = \mathrm{div}(g)$, with $g \in k(X_\eta)$. Since $\imath^* \colon \mathbb{K}(X) \to k(X_\eta)$ is an isomorphism we have $g = \imath^*(f)$, with $f \in \mathbb{K}(X)$. We conclude that

$$0 = \imath^*(D) - \mathrm{div}(g) = \imath^*(D - \mathrm{div}(f))$$

and in particular $D - \mathrm{div}(f)$ is a divisor of $\mathrm{WDiv}_\pi(X)$, linearly equivalent to D.

Finally, to prove (v) let $f \in \mathcal{O}_X(D)(U)$ so that the divisor $\mathrm{div}(f) + D$ is effective on U. Then

$$\mathrm{div}(\imath^*(f)) + \imath^*(D) = \imath^*(\mathrm{div}(f) + D)$$

is effective on U_η by (ii). □

In what follows we will refer to the elements of $\mathrm{WDiv}_\pi(X)$ as *vertical divisors* and similarly to the elements of $\mathrm{Cl}_\pi(X)$ as *vertical classes*.

Let us remark that every pull-back divisor in $\mathrm{WDiv}(X)$ is vertical. In the next lemma we are going to prove that in the case of principal divisor, also the converse is true. In order to do that, let us denote by $\mathrm{PDiv}_\pi(X)$ the subgroup of $\mathrm{WDiv}_\pi(X)$ consisting of the principal vertical divisors of X.

Lemma 2.2. *Under the hypotheses above, the equality* $\mathrm{PDiv}_\pi(X) = \pi^* \mathrm{PDiv}(Y)$ *holds.*

Proof. The inclusion $\pi^* \mathrm{PDiv}(Y) \subseteq \mathrm{PDiv}_\pi(X)$ is obvious. In order to prove the opposite inclusion, let $D \in \mathrm{PDiv}_\pi(X)$ be a principal vertical divisor and let $f \in \mathbb{K}(X)$ be a rational function such that $\mathrm{div}(f) = D$. By Proposition 2.1 we have

$$\mathrm{div}(\imath^*(f)) = \imath^*(D) = 0,$$

and thus $\imath^*(f)$ must be constant, being X_η complete by the properness hypothesis on π. In particular $\imath^*(f)$ is an element of $\bar{k} \cap k(X_\eta)$, where $\bar{k}$ is the algebraic closure of k. By [15, Example 2.1.12] the following equality

$$\bar{k} \cap k(X_\eta) = k$$

holds, so that $\imath^*(f) \in k$. In particular $f \in \pi^*(\mathbb{K}(Y))$ and thus $D = \mathrm{div}(f)$ lies in $\pi^* \mathrm{PDiv}(Y)$, which proves the second inclusion.

□

3. Proof of Proposition 1

In this section we are going to prove Proposition 1 together with some related results.

Let X be a normal variety defined over a closed field $\mathbb{K}$, such that $\mathrm{Cl}(X)$ is finitely generated and $\mathbb{K}[X]^* = \mathbb{K}^*$. Let K and $\mathcal{S}_K$ be as in Construction 1.1. In what follows, whenever we need to keep trace of the degree of an element in $\Gamma(X, \mathcal{S}_K)_D$, we will use the notation ft^D, where f is in the Riemann Roch space of

D. If we denote by $\mathrm{Frac}_0(\Gamma(X, \mathcal{S}_K))$ the field of fractions of homogeneous sections having the same degree, we have the following.

Lemma 3.1. *The map* $\mu_K\colon \mathrm{Frac}_0(\Gamma(X, \mathcal{S}_K)) \to \mathbb{K}(X)$, *defined by* $ft^D/gt^D \mapsto f/g$ *is an isomorphism.*

Proof. Since μ_K is a homomorphism of fields, we only need to show that it is surjective. To this purpose, let us fix $h \in \mathbb{K}(X)$. We can write $\mathrm{div}(h) = A - B$, with A and B effective divisors in $\mathrm{WDiv}(X)$ without common support. Since the class map $\mathrm{cl}\colon K \to \mathrm{Cl}(X)$ is surjective, there exist $D \in K$ and $g \in \Gamma(X, \mathcal{S}_K)_D$ such that $\mathrm{div}(g) + D = B$. The fraction hgt^D/gt^D is then an element in $\mathrm{Frac}_0(\Gamma(X, \mathcal{S}_K))$ whose image via μ_K is h. □

Let us suppose now that X and Y are normal varieties having only constant invertible global sections and let $\pi\colon X \to Y$ be a proper surjective morphism whose very general fiber is irreducible. Let K be a finitely generated subgroup of $\mathrm{WDiv}(X)$ such that the class map $K \to \mathrm{Cl}(X)$ is surjective. In order to prove Proposition 1 we are going to define the localization

$$\Gamma_\pi(X, \mathcal{S}_K) = S_\pi^{-1}\Gamma(X, \mathcal{S}_K),$$

where S_π is the multiplicative system consisting of the non-zero homogeneous elements whose degree $D \in K$ is such that $[D] \in \mathrm{Cl}_\pi(X)$.

Proof of Proposition 1. Let us denote by $\eta_K\colon \mathbb{K}(Y) \to \mathrm{Frac}_0(\Gamma(X, \mathcal{S}_K))$ the composition $\mu_K^{-1} \circ \pi^*$. We claim that it suffices to prove that

$$\mathrm{im}(\eta_K) = \Gamma_\pi(X, \mathcal{S}_K)_0.$$

Indeed, by (1.1), $\Gamma_\pi(X, \mathcal{S}_K)_0$ surjects onto $\mathcal{R}_\pi(X)_0$ and moreover the composition $\mathbb{K}(Y) \to \mathcal{R}_\pi(X)_0$ is injective because the domain is a field.

In order to prove the inclusion $\mathrm{im}(\eta_K) \subseteq \Gamma_\pi(X, \mathcal{S}_K)_0$, let $s \in \mathbb{K}(Y)$ and let $h := \pi^*(s) \in \mathbb{K}(X)$. Write $\mathrm{div}(h) = A - B$, with A and B vertical, effective, with no common support. Arguing as we did in the proof of Lemma 3.1 we have that

$$\eta_K(s) = \mu_K^{-1}(h) = hgt^D/gt^D$$

where $D \in K$ is linearly equivalent to the vertical divisor B, so that the above quotient is in $\Gamma_\pi(X, \mathcal{S}_K)_0$.

Viceversa let $ft^D/gt^D \in \Gamma_\pi(X, \mathcal{S}_K)_0$ be a degree zero homogeneous fraction. The divisor $\imath^*(\mathrm{div}(f) + D)$ is effective by Proposition 2.1 and it is principal, being $[D] \in \mathrm{Cl}_\pi(X)$. Since X_η is complete, $\imath^*(\mathrm{div}(f) + D) = 0$ or equivalently $\mathrm{div}(f) + D$ is vertical. In the same way one shows that also $\mathrm{div}(g) + D$ is vertical, so that their difference $\mathrm{div}(f/g)$ is vertical and principal. By Lemma 2.2 the latter divisor is a pullback so that $f/g = \pi^*(s)$ for some $s \in \mathbb{K}(Y)$, which proves the claim. □

Given K as before and the map $\imath^*\colon \mathrm{WDiv}(X) \to \mathrm{WDiv}(X_\eta)$, from now on for simplicity of notation we will set $K_\eta := \imath^*(K) \subseteq \mathrm{WDiv}(X_\eta)$. By Proposition 2.1 (v) we have a morphism of sheaves of divisorial algebras $\imath^*\colon \mathcal{S}_K \to \imath_*\mathcal{S}_{K_\eta}$ and passing to global sections we obtain a homomorphism of rings

$$\imath^*\colon \Gamma(X, \mathcal{S}_K) \to \Gamma(X_\eta, \mathcal{S}_{K_\eta}). \tag{3.1}$$

Remark 3.2. If the subgroup K does not contain vertical divisors, that is $K \cap \mathrm{WDiv}_\pi(X) = 0$, then the restriction of $\imath^*\colon \mathrm{WDiv}(X) \to \mathrm{WDiv}(X_\eta)$ gives an isomorphism between K and K_η. In this case the map $\imath^*$ defined in (3.1) is an injection since we have seen that $\imath^*$ induces also an isomorphism between the fields of rational functions.

Proposition 3.3. *If the subgroup K does not contain vertical divisors, then the map $\imath^*$ defined in (3.1) extends to an isomorphism of $\mathbb{K}(Y)$-algebras*

$$\imath^*\colon \Gamma_\pi(X, \mathcal{S}_K) \to \Gamma(X_\eta, \mathcal{S}_{K_\eta}), \qquad \frac{f}{g} \mapsto \frac{\imath^*(f)}{\imath^*(g)}.$$

Proof. By Remark 3.2 we already know that the map (3.1) is injective. We are now going to prove that the image of an element in the multiplicative system S_π is invertible. Let us fix $g \in S_\pi$, i.e. $g \in \Gamma(X, \mathcal{S}_K)_D$, and $[D] \in \mathrm{Cl}_\pi(X)$. Then $\imath^*(g) \in \Gamma(X_\eta, \mathcal{S}_{K_\eta})_{\imath^*(D)}$, where $\imath^*(D)$ is a principal divisor, being its class trivial. Thus

$$\imath^*(\mathrm{div}(g) + D) = \mathrm{div}(\imath^*(g)) + \imath^*(D) = 0,$$

where the first equality is by Lemma 2.1 and the second is due to the fact that the generic fiber X_η is complete, being π proper by hypothesis. In particular $\imath^*(g)$ is invertible with inverse $\imath^*(g^{-1}) \in \Gamma(X_\eta, \mathcal{S}_{K_\eta})_{\imath^*(-D)}$. This shows that the map defined in the statement is an injective homomorphism of $\mathbb{K}(Y)$-algebras.

In order to prove the surjectivity, it suffices to show that any homogeneous $s \in \Gamma(X_\eta, \mathcal{S}_{K_\eta})_{\imath^*(D)}$, with $D \in K$, is in the image. At the level of rational functions we have $s = \imath^*(f)$, where $f \in \mathbb{K}(X)$, with

$$\imath^*(\mathrm{div}(f) + D) = \mathrm{div}(\imath^*(f)) + \imath^*(D) = E_\eta, \quad \text{effective on } X_\eta.$$

If we denote by E the Zariski closure of E_η in X, we have that E is effective too and $\imath^*(E) = E_\eta$. Then the above formula implies that $\mathrm{div}(f) + D = E + V$, where $V \in \mathrm{WDiv}_\pi(X)$. Write $V = A - B$, with A and B effective. Let $B' \in K$ linearly equivalent to B and let $h \in \Gamma(X, \mathcal{S}_K)_{B'}$ be such that $\mathrm{div}(h) + B' = B$. By the equality

$$\mathrm{div}(fh) + D + B' = E + A$$

we deduce that $fh \in \Gamma(X, \mathcal{S}_K)_{D+B'}$ and thus $\frac{fh}{h} \in \Gamma_\pi(X, \mathcal{S}_K)$ is a preimage of s. □

4. Proof of Theorem 1

In this section we are going to give the proof of Theorem 1. In order to do that we first state and prove a couple of lemmas. Throughout the section $\pi\colon X \to Y$ will be a proper surjection of normal varieties whose very general fiber is irreducible. From now on we also suppose that the subgroup $K \subseteq \mathrm{WDiv}(X)$ does not contain vertical divisors, so that, by Lemma 3.2, it is isomorphic to K_η.

Lemma 4.1. *If we denote by K_η^0 the kernel of the surjection $K_\eta \to \mathrm{Cl}(X_\eta)$, we have an isomorphism between $K_\eta^0/\imath^*(K^0)$ and $\mathrm{Cl}_\pi(X)$.*

Proof. A diagram chasing in the following commutative diagram with exact rows and columns establishes the claimed isomorphism

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & \mathrm{Cl}_\pi(X) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K^0 & \longrightarrow & K & \longrightarrow & \mathrm{Cl}(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K^0_\eta & \longrightarrow & K_\eta & \longrightarrow & \mathrm{Cl}(X_\eta) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & K^0_\eta/\imath^*(K^0) & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

□

From now on we assume the group $\mathrm{Cl}_\pi(X)$ to be torsion-free. Let us consider the following set of characters (which are sections of the map $\mathrm{div}\colon \mathbb{K}(X) \to \mathrm{WDiv}(X)$)

$$\mathrm{SecDiv}(K^0, \mathbb{K}(X)^*) := \{\mathcal{X}\colon K^0 \to \mathbb{K}(X)^* \mid \mathrm{div} \circ \mathcal{X} = \mathrm{id}\},$$

and define $\mathrm{SecDiv}(K^0_\eta, k(X_\eta)^*)$ in a similar way. Let us define the map

$$\Phi\colon \mathrm{SecDiv}(K^0_\eta, k(X_\eta)^*) \to \mathrm{SecDiv}(K^0, \mathbb{K}(X)^*), \qquad \mathcal{X}_\eta \mapsto \imath^{*-1} \circ \mathcal{X}_\eta \circ \imath^*_{|K_0},$$

and observe that the group $\mathrm{Hom}(\mathrm{Cl}_\pi(X), k^*)$ acts on $\mathrm{SecDiv}(K^0_\eta, k(X_\eta)^*)$ by multiplication.

Lemma 4.2. *The following hold:*

(i) *Φ is the quotient map by the action of $\mathrm{Hom}(\mathrm{Cl}_\pi(X), k^*)$;*
(ii) *the pulback $\imath^*\colon \mathcal{S}_K \to \mathcal{S}_{K_\eta}$ maps the ideal sheaf $\mathcal{I}$ to $\mathcal{I}_\eta$.*

Proof. We prove (i). We first claim that Φ is surjective. Indeed, let us fix a character $\mathcal{X} \in \mathrm{SecDiv}(K^0, \mathbb{K}(X)^*)$. Observe that since $\imath^*_{|K_0}$ is an isomorphism on its image, there exists a unique homomorphism of groups $\varphi\colon \imath^*(K^0) \to k(X_\eta)^*$ which makes the following diagram commute

$$\begin{array}{ccc}
K^0 & \xrightarrow{\ \mathcal{X}\ } & \mathbb{K}(X)^* \\
\downarrow{\scriptstyle \imath^*_{|K_0}} & & \downarrow{\scriptstyle \imath^*} \\
\imath^*(K^0) & \xrightarrow{\ \varphi\ } & k(X_\eta)^*.
\end{array}$$

Therefore for any divisor $D \in K^0$ we have

$$\mathrm{div}(\varphi(\imath^*(D))) = \mathrm{div}(\imath^*(\mathcal{X}(D))) = \imath^*(\mathrm{div}(\mathcal{X}(D))) = \imath^*(D),$$

where the second equality follows from Proposition 2.1. Now, by Lemma 4.1 and the assumption that $\mathrm{Cl}_\pi(X)$ is torsion free, we deduce that the subgroup $\imath^*(K^0)$ is primitive in K^0_η, and in particular φ can be extended to a homomorphism $\mathcal{X}_\eta\colon K^0_\eta \to k(X_\eta)^*$. Moreover the extension can be chosen in such a way that the equality $\mathrm{div}\circ\mathcal{X}_\eta = \mathrm{id}$ holds, so that $\mathcal{X}_\eta$ is an element of $\mathrm{SecDiv}(K^0_\eta, k(X_\eta)^*)$. By construction, $\Phi(\mathcal{X}_\eta) = \mathcal{X}$, which proves the claim.

The map Φ is invariant for the group action. Indeed, given $\gamma \in \mathrm{Hom}(\mathrm{Cl}_\pi(X), k^*)$ and $\mathcal{X}_\eta \in \mathrm{SecDiv}(K^0_\eta, k(X_\eta)^*)$ we have $\Phi(\mathcal{X}_\eta) = \Phi(\gamma\cdot\mathcal{X}_\eta)$ because $\gamma([D]) = 1$ for any $D \in \imath^*(K^0)$. Since the group action is clearly free, in order to conclude the proof of (i) we only need to show that the group $\mathrm{Hom}(\mathrm{Cl}_\pi(X), k^*)$ acts transitively on the fibers of Φ. Given two elements $\mathcal{X}_\eta$ and $\mathcal{X}'_\eta$ in the same fiber of Φ, the homomorphism

$$K^0_\eta \to k(X_\eta)^*, \qquad D \mapsto \mathcal{X}_\eta(D)/\mathcal{X}'_\eta(D)$$

is trivial on $\imath^*(K^0)$, and thus by Lemma 4.1 it descends to a homomorphism $\gamma\colon \mathrm{Cl}_\pi(X) \to k(X_\eta)^*$. Since

$$\mathrm{div}(\mathcal{X}_\eta(D)/\mathcal{X}'_\eta(D)) = \mathrm{div}(\mathcal{X}_\eta(D)) - \mathrm{div}(\mathcal{X}'_\eta(D)) = D - D = 0$$

and X_η is complete, we deduce that $\mathcal{X}_\eta(D)/\mathcal{X}'_\eta(D) \in \bar{k}^*\cap k(X_\eta)^*$. Moreover, by [15, Example 2.1.12], we have that $\bar{k}^* \cap k(X_\eta)^* = k^*$, so that $\gamma \in \mathrm{Hom}(\mathrm{Cl}_\pi(X), k^*)$.

We now prove (ii). Given a character $\mathcal{X} \in \mathrm{SecDiv}(K^0, \mathbb{K}(X)^*)$, by Construction 1.1 we know that $\mathcal{I}$ is locally generated by the sections $1 - \mathcal{X}(D)$, for $D \in K^0$. By the surjectivity of Φ there exists a character $\mathcal{X}_\eta \in \mathrm{SecDiv}(K^0_\eta, k(X_\eta)^*)$ that makes the following diagram commute:

$$\begin{array}{ccc} K^0 & \xrightarrow{\ \mathcal{X}\ } & \mathbb{K}(X)^* \\ \Big\downarrow {\scriptstyle \imath^*_{|K_0}} & & \Big\downarrow {\scriptstyle \imath^*} \\ K^0_\eta & \xrightarrow{\ \mathcal{X}_\eta\ } & k(X_\eta)^*. \end{array}$$

Since $\imath^*(K_0) \subseteq K^0_\eta$ and, by Construction 1.2, $\mathcal{I}_\eta$ is locally generated by the sections $1 - \mathcal{X}_\eta(D')$, for $D' \in K^0_\eta$, we get the statement. ☐

Proof of Theorem 1. By Proposition 3.3, Lemma 4.2, the characterization of Cox rings in [2, Lemma 1.4.3.5] and [6, Construction 2.7], we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma_\pi(X,\mathcal{I}) & \longrightarrow & \Gamma_\pi(X,\mathcal{S}_K) & \longrightarrow & \mathcal{R}_\pi(X) & \longrightarrow & 0 \\ & & \Big\downarrow & & {\scriptstyle \imath^*}\Big\downarrow{\scriptstyle \simeq} & & \Big\downarrow{\scriptstyle \imath_R} & & \\ 0 & \longrightarrow & \Gamma(X_\eta,\mathcal{I}_\eta) & \longrightarrow & \Gamma(X_\eta,\mathcal{S}_{K_\eta}) & \longrightarrow & \mathcal{R}(X_\eta) & \longrightarrow & 0 \end{array}$$

where $\Gamma_\pi(X,\mathcal{I})$ is the localization of the ideal $\Gamma(X,\mathcal{I})$. In particular the morphism $\imath_R\colon \mathcal{R}_\pi(X) \to \mathcal{R}(X_\eta)$ is surjective, while the restriction of $\imath^*$ to $\Gamma_\pi(X,\mathcal{I})$ is injective.

Define the homomorphism of groups

$$u\colon \mathrm{Cl}_\pi(X) \to \mathcal{R}_\pi(X)^*, \qquad [D] \to {\imath^*}^{-1}(\mathcal{X}_\eta(D)) + \Gamma_\pi(X,\mathcal{I}), \tag{4.1}$$

where $D \in K^0_\eta$ and $\mathcal{X}_\eta(D) \in \Gamma(X_\eta, \mathcal{S}_{K_\eta})_{-D}$. Let us show that u is well defined. Indeed, if $D' \in K^0_\eta$ is another representative of the same class $[D]$, by Lemma 4.2

we have $D' - D = \imath^*(L)$ with $L \in K^0$, so that

$$\begin{aligned}
\imath^{*\,-1}(\mathcal{X}_\eta(D + i^*(L))) &= \imath^{*\,-1}(\mathcal{X}_\eta(D) \cdot \mathcal{X}_\eta(\imath^*(L))) \\
&= \imath^{*\,-1}(\mathcal{X}_\eta(D)) \cdot \mathcal{X}(L) \\
&\equiv \imath^{*\,-1}(\mathcal{X}_\eta(D)) \mod \Gamma_\pi(X, \mathcal{I}).
\end{aligned}$$

Observe that the homomorphism u satisfies the condition $u(w) \in \mathcal{R}_\pi(X)^*_{-w}$ for each $w \in \mathrm{Cl}_\pi(X)$. Moreover, given another morphism $u' \colon \mathrm{Cl}_\pi(X) \to \mathcal{R}_\pi(X)^*$ satisfying the same condition, reasoning as in the proof of Lemma 4.2 we see that $u'/u \in \mathrm{Hom}(\mathrm{Cl}_\pi(X), k^*)$, since $\mathcal{R}_\pi(X)^*_0 \simeq k^*$. Thus, by the same Lemma 4.2, u' can be defined as in (4.1) by taking the character $\mathcal{X}'_\eta = (u'/u) \cdot \mathcal{X}_\eta$. We can now describe the kernel of $\imath_R$ as follows

$$\begin{aligned}
\ker(\imath_R) &= \{s + \Gamma_\pi(X, \mathcal{I}) \,:\, \imath^*(s) \in \Gamma(X_\eta, \mathcal{I}_\eta)\} \\
&= \{\imath^{*\,-1}(s_\eta) + \Gamma_\pi(X, \mathcal{I}) \,:\, s_\eta \in \Gamma(X_\eta, \mathcal{I}_\eta)\} \\
&= \langle 1 - \imath^{*\,-1}(\mathcal{X}_\eta(D)) + \Gamma_\pi(X, \mathcal{I}) \,:\, D \in K^0_\eta \rangle \\
&= \langle 1 - u([D]) \,:\, [D] \in \mathrm{Cl}_\pi(X) \rangle,
\end{aligned}$$

which gives the claim.

□

Remark 4.3. Observe that if $\pi \colon X \to Y$ admits a rational section σ and $\mathrm{Cl}(Y)$ is torsion free, then $\mathrm{Cl}_\pi(X)$ is torsion free as well. Indeed we can identify $\mathrm{Cl}_\pi(X)$ with the quotient $\mathrm{WDiv}_\pi(X)/\,\mathrm{PDiv}_\pi(X)$. So let us take a divisor $V \in \mathrm{WDiv}_\pi(X)$ such that nV belongs to $\mathrm{PDiv}_\pi(X)$ for some integer $n > 1$. Since $\mathrm{PDiv}_\pi(X) = \pi^* \,\mathrm{PDiv}(Y)$ we can write $nV = \pi^* D$, with $D \in \mathrm{PDiv}(Y)$ and hence

$$D = (\pi \circ \sigma)^*(D) = \sigma^*(\pi^* D) = \sigma^*(nV) = n\sigma^*(V).$$

In particular $n\sigma^*(V) \in \mathrm{PDiv}(Y)$ and since we are assuming that $\mathrm{Cl}(Y)$ is torsion free we conclude that $\sigma^*(V) \in \mathrm{PDiv}(Y)$. By applying π^* to both sides of the equation above we deduce $nV = n\pi^*(\sigma^*(V))$ so that $V = \pi^*(\sigma^*(V))$ holds since $\mathrm{WDiv}(X)$ is free abelian. In particular we conclude that $V \in \pi^* \,\mathrm{PDiv}(Y) = \mathrm{PDiv}_\pi(X)$, which proves the statement.

5. Very general fibers

In this section we are going to apply the results of Theorem 1 in order to prove that, under an extra hypothesis, it is indeed possible to recover the Cox ring of a very general fiber of $\pi \colon X \to Y$ from the Cox ring of X and the vertical classes (see Corollary 5.2). In order to do that we need the following lemma.

Lemma 5.1. *Let X_i, with $i \in \{1, 2\}$ be a normal variety defined over an algebraically closed field k_i of characteristic 0. Assume that $\mathrm{Cl}(X_i)$ is finitely generated and that $k_i[X_i]^* = k_i^*$, for any $i \in \{1, 2\}$. If there is an isomorphism of fields $\varphi \colon k_2 \to k_1$ and an isomorphism of schemes $f \colon X_1 \to X_2$ such that the following diagram commutes*

$$\begin{array}{ccc}
X_1 & \xrightarrow{\;f\;} & X_2 \\
\downarrow & & \downarrow \\
\mathrm{Spec}(k_1) & \xrightarrow{\;\varphi^*\;} & \mathrm{Spec}(k_2)
\end{array}$$

then f induces an isomorphism of graded rings $f^*\colon \mathcal{R}(X_2) \to \mathcal{R}(X_1)$, such that $f^*|_{k_2} = \varphi$.

Proof. Observe that f induces the pullback isomorphism on the fields of rational functions $k_2(X_2) \to k_1(X_1)$. Given a prime divisor D of X_2, the restriction $D \cap X_2^\circ$ to the smooth locus X_2° of X_2 is a Cartier non-trivial divisor, because $X_2 \setminus X_2^\circ$ has codimension at least two by the normality of X_2. Since f is an isomorphism the pullback $f^*(D \cap X_2^\circ)$ is contained in the smooth locus X_1° of X_1 and it has a unique closure by the normality of X_1. By linearity the pullback map extends to an isomorphism $f^*\colon \operatorname{WDiv}(X_2) \to \operatorname{WDiv}(X_1)$ of the groups of Weil divisors, which maps principal divisors to principal divisors and thus gives also an isomorphism of divisor class groups $\operatorname{Cl}(X_2) \to \operatorname{Cl}(X_1)$. By the above discussion, given a Weil divisor D of X_2 and an open subset $U \subseteq X_2$, the pullback induces an isomorphism of Riemann-Roch spaces $\Gamma(U, \mathcal{O}_{X_2}(D)) \to \Gamma(f^{-1}(U), \mathcal{O}_{X_1}(f^*D))$. Thus, given a finitely generated subgroup $K \subseteq \operatorname{WDiv}(X_2)$ which surjects onto $\operatorname{Cl}(X_2)$, the pullback gives an isomorphism of sheaves of divisorial algebras $\mathcal{S}_K \to f_*\mathcal{S}_{f^*K}$, which induces an isomorphism of Cox sheaves. By taking global sections we get an isomorphism of Cox rings $f^*\colon \mathcal{R}(X_2) \to \mathcal{R}(X_1)$, Finally observe that the last isomorphism restricted to k_2 equals the restriction of the isomorphism $k_2(X_2) \to k_1(X_1)$ and thus it coincides with φ. □

Let us go back to a morphism $\pi\colon X \to Y$ satisfying the hypotheses of Theorem 1 and let X_η be the generic fiber of π. If we denote by $\bar{X}_\eta$ the base change $X_\eta \times_k \bar{k}$, we have the following.

Corollary 5.2. *Let $\pi\colon X \to Y$ satisfy the hypotheses of Theorem 1 and suppose in addition that the geometric divisor class group $\operatorname{Cl}(\bar{X}_\eta)$ is isomorphic to $\operatorname{Cl}(X_\eta)$. Then the Cox ring of a very general fiber of π is isomorphic (as a graded ring) to*

$$\mathcal{R}_\pi(X)/\langle 1 - u(w) \,:\, w \in \operatorname{Cl}_\pi(X)\rangle \otimes_k \bar{k}$$

where $u\colon \operatorname{Cl}_\pi(X) \to \mathcal{R}_\pi(X)^*$ is any homomorphism satisfying $u(w) \in \mathcal{R}_\pi(X)^*_{-w}$ for each w.

Proof. By [19, Lemma 2.1] there exists a subset $W \subseteq Y$ which is a countable intersection of non empty Zariski open subsets such that for each $b \in W$ there is an isomorphism of rings $\mathbb{K} \to \bar{k}$ which induces an isomorphism of schemes $X_b \to \bar{X}_\eta$. Therefore by Lemma 5.1 the Cox ring of the very general fiber X_b is isomorphic to the Cox ring $\mathcal{R}(\bar{X}_\eta)$ of the geometric generic fiber. The isomorphism between $\operatorname{Cl}(\bar{X}_\eta)$ and $\operatorname{Cl}(X_\eta)$ implies that the former can be generated by classes of divisors in $\operatorname{WDiv}_k(X_\eta)$. By Remark 1.4, the Cox ring $\mathcal{R}(\bar{X}_\eta)$ is obtained from $\mathcal{R}(X_\eta)$ by a base change, and hence we can conclude by means of Theorem 1. □

Remark 5.3. We remark that the isomorphism of the corollary above is not an isomorphism of graded algebras, since one of them is defined over $\mathbb{K}$ while the other one over $\bar{k}$.

Remark 5.4. Since by Corollary 5.2 the Cox ring of a very general fiber can be described as a quotient of a localization of the Cox ring of X, if the latter is finitely generated, then the former is finitely generated too. In particular, if we can construct a morphism $\pi\colon X \to Y$ satisfying the hypotheses of the above corollary and such that the Cox ring of a very general fiber is not finitely generated, then we can conclude that the Cox ring of X is not finitely generated.

6. Blowing-ups of toric fiber spaces

In this last section we apply our results to the blowing-up of a toric fiber space along a section, with the purpose of producing new examples of varieties with non-finitely generated Cox ring (see e.g. [7, 8, 13, 16]).

Construction 6.1. Let $\pi\colon X \to Y$ be a surjective toric morphism of normal toric varieties which has connected fibers and such that the induced homomorphism of tori $\pi|_{T_X}\colon T_X \to T_Y$ is surjective and $\mathrm{Cl}(Y)$ is torsion free. If we denote by $X_0 \subseteq X$ the Zariski closure of the kernel of $\pi|_{T_X}$, by [5, Eq. 3.3.6] we have that $\pi^{-1}(T_Y) \simeq X_0 \times T_Y$. Let $x_0 \in X_0$ be a general point and let $\phi\colon \tilde{X} \to X$ be the blowing-up of X along the Zariski closure of $\{x_0\} \times T_Y$ via the above isomorphism. Let $\tilde{X}_0$ be the preimage of X_0 via ϕ, so that the restriction $\phi_{|_{\tilde{X}_0}}\colon \tilde{X}_0 \to X_0$ is the blowing-up at x_0. We have a commutative diagram

$$\begin{array}{ccc} \tilde{X}_0 & \longrightarrow & \tilde{X} \\ \big\downarrow & & \big\downarrow{\scriptstyle \tilde{\pi}} \\ 1_{T_Y} & \longrightarrow & Y \end{array}$$

where the horizontal arrows are inclusions and $\tilde{\pi}$ denotes the composition $\pi \circ \phi$.

Proposition 6.2. *If the Cox ring of $\tilde{X}_0$ is not finitely generated then the same holds for the Cox ring of $\tilde{X}$.*

Proof. By Remark 5.4, it is enough to show that the toric morphism $\tilde{\pi}\colon \tilde{X} \to Y$ satisfies the hypotheses of Corollary 5.2. By construction, the varieties $\tilde{X}$ and Y are both normal, complete and $\tilde{\pi}$ is surjective. The divisor class group $\mathrm{Cl}(\tilde{X})$ is finitely generated, being X toric. All the fibers of $\tilde{\pi}$ over T_Y are isomorphic to $\tilde{X}_0$ because the point $x_0 \in X_0$ is general, and in particular they are connected and irreducible. We now claim that the group of vertical classes $\mathrm{Cl}_{\tilde{\pi}}(\tilde{X})$ is torsion free. Indeed, first of all observe that the restriction $\pi|_{T_X}\colon T_X \to T_Y$ is a surjective homomorphism of tori and thus it admits a section, which gives a rational section of $\pi\colon X \to Y$. Therefore, by Remark 4.3 (and the assumption $\mathrm{Cl}(Y)$ free), we deduce that $\mathrm{Cl}_\pi(X)$ is torsion-free. Moreover, if we denote by $\tilde{X}_\eta$ the generic fiber of $\tilde{\pi}$, we have that the exceptional divisor of ϕ restricts to the exceptional divisor of $\tilde{X}_\eta \to X_\eta$. This gives an isomorphism $\mathrm{Cl}_{\tilde{\pi}}(\tilde{X}) \simeq \mathrm{Cl}_\pi(X)$, and hence the claim. Finally the pullback homomorphism $\mathrm{Cl}(\tilde{X}) \to \mathrm{Cl}(\tilde{X}_0)$ is surjective by Lemma 6.3, and this implies that the divisor class group of the generic geometric fiber is isomorphic to the divisor class group of $\tilde{X}_0$. □

Lemma 6.3. *With the notation of Construction 6.1, the pullback homomorphism $\mathrm{Cl}(\tilde{X}) \to \mathrm{Cl}(\tilde{X}_0)$ is surjective.*

Proof. It suffices to show that the pullback $\mathrm{Cl}(X) \to \mathrm{Cl}(X_0)$ is surjective. To this end, let N_X and N_Y be the lattices of one-parameter subgroups of X and Y respectively and let Σ_X and Σ_Y be their defining fans. The morphism $\pi\colon X \to Y$ induces a surjective homomorphism $\alpha\colon N_X \to N_Y$ and, by [5, §3.3], the fiber X_0 is the toric variety whose defining fan is

$$\Sigma_0 := \{\sigma \in \Sigma_X \,:\, \sigma \subseteq \ker(\alpha)_{\mathbb{Q}}\}.$$

Each torus invariant prime divisor D_0 of X_0 corresponds to a one-dimensional cone $\tau_0 \in \Sigma_0$. Since Σ_0 is a subfan of Σ_X, τ_0 belongs also to Σ_X and thus D_0 is the restriction of a torus invariant prime divisor of X, which proves the statement. □

We conclude with an example involving weighted projective spaces.

Example 6.4. Let $\mathbb{P}(\mathbf{a})$ be a weighted projective space whose blowing-up at a general point has non-finitely generated Cox ring (see for instance [7,8,13,16]), and let $\Sigma_0 \subseteq (N_0)_{\mathbb{Q}}$ be a defining fan for $\mathbb{P}(\mathbf{a})$. Let $N := N_0 \oplus \mathbb{Z}$. Given $v \in N_0$ define the fan $\Sigma \subseteq N_{\mathbb{Q}}$ whose maximal cones are

$$\{\mathrm{cone}(\sigma, (0,1)), \mathrm{cone}(\sigma, (v, -1)) \,:\, \sigma \in \Sigma_0^{\max}\}.$$

The projection $N \to \mathbb{Z}$ induces a morphism $\pi\colon X(\Sigma) \to \mathbb{P}^1$, whose general fiber is $\mathbb{P}(\mathbf{a})$. Let $\tilde{X}$ be the blowing-up of $X(\Sigma)$ along the image of a rational section of π passing through a general point of X_0. By Proposition 6.2 we conclude that the Cox ring of $\tilde{X}$ is not finitely generated.

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