THE 2×2 UPPER TRIANGULAR MATRIX ALGEBRA AND ITS GENERALIZED POLYNOMIAL IDENTITIES

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Abstract. Let UT_2 be the algebra of 2×2 upper triangular matrices over a field F of characteristic zero. Here we study the generalized polynomial identities of UT_2, i.e., identical relations holding for UT_2 regarded as UT_2-algebra. We determine the generator of the T_{UT_2}-ideal of generalized polynomial identities of UT_2 and compute the exact values of the corresponding sequence of generalized codimensions. Moreover, we give a complete description of the space of multilinear generalized identities in n variables in the language of Young diagrams through the representation theory of the symmetric group S_n. Finally, we prove that, unlike in the ordinary case, the generalized variety of UT_2-algebras generated by UT_2 has no almost polynomial growth; nevertheless, we exhibit two distinct generalized varieties of almost polynomial growth.

1. Introduction

Let A be an associative algebra over a field F of characteristic zero, $F\langle X\rangle$ be the free algebra generated by the countable set $X = \{x_1, x_2, \ldots\}$ and W be a unitary associative algebra over F. Then A is called W-algebra if it has a structure of W-bimodule with some additional conditions. A generalized polynomial identity of A is a polynomial $f(x_1, \ldots, x_n)$ of the free W-algebra $W\langle X\rangle$ that vanishes under all substitutions of the elements of A. Roughly speaking, $f(x_1, \ldots, x_n)$ is a polynomial of $F\langle X\rangle$ with "coefficients" in W. Notice that such "coefficients" may appear also between two variables. Clearly these identities are a natural generalization of the ordinary polynomial ones arising when W coincides with F. The set of all generalized polynomial identities $\mathrm{GId}(A)$ is a T_W-ideal of $W\langle X\rangle$, i.e., an ideal stable by endomorphisms of $W\langle X\rangle$, and one of the main problems is to find a set of generators of such T_W-ideal.

The idea of generalized polynomial identities stems from the observation that sometimes when we study polynomials in matrix algebras, we want to focus on evaluations where certain variables are always replaced by specific elements. Therefore, it would be useful to have a theory that allows us to consider "polynomials" whose coefficients can be taken from an algebra, instead of from a field.

Generalized identities first appeared in 1965 in Amitsur's fundamental paper [1] on primitive rings satisfying generalized polynomial identities. In 1969, Martindale developed this idea further and applied it to prime rings [15]. Later, two generalizations were pursued: Martindale [16] and Rowen [20–22] investigated generalized polynomial identities involving involutions, while Kharchenko [10–12] explored generalized polynomial identities involving derivations and automorphisms. These two directions were further developed and studied by various authors (see [2] and its bibliography). In recent years, in case $W = A$ is finite dimensional and the bimodule action is the natural left and right multiplication, Gordienko in [8] proved the so-called Amitsur conjecture, i.e., the limit $\lim_{n\to+\infty} \sqrt[n]{gc_n(A)}$, where $gc_n(A)$, $n \geq 1$, is the generalized codimension sequence, exists and is a non-negative integer called the generalized PI-exponent of A. He also proved that the generalized exponent equals the ordinary one defined by mean of the ordinary codimension sequence $c_n(A)$. For what concern the general the problem of describing the concrete generalized identities of an algebra so far it has been achieved only for the algebra $M_n(F)$ of $n \times n$ full matrices for all $n \geq 1$ (see for example [4]).

The codimension sequence of an algebra was introduced by Regev in [19] and it measures the rate of growth of the multilinear polynomials lying in the corresponding T-ideal. In the same paper, Regev proved that if A satisfies a nontrivial polynomial identity, i.e., it is a PI-algebra, then its codimension sequence $c_n(A)$, $n \geq 1$, is exponentially bounded. Later Kemer in [13] showed that the variety generated by the algebra UT_2 of 2×2-upper triangular matrices is of almost polynomial growth, i.e., it has exponential growth of the codimensions but every proper subvariety has polynomial growth. Analogous results were proved in various settings such as varieties of group-graded algebras [23], algebras with derivation [6], special Jordan algebras [17]. It is worth mentioning that in the case of algebras with involution, Mishchenko and Valenti in [18] constructed out of UT_2 a suitable algebra generating a variety of almost polynomial growth.

Motivated by the above results, here we deal with the generalized polynomial identities of UT_2 and we investigate the growth of the generalized codimension sequence $gc_n(A)$ of any algebra A lying in the generalized variety generated by UT_2.

The paper is organized as follows. After a necessary background on the generalized identities involving basic definitions and preliminary settings given in Section 2, we describe in Section 3 the T-ideal of generalized identities of UT_2 as UT_2-algebra finding its generator. In Section 4 we study the space of multilinear generalized identities of UT_2 of degree n as a representation of the symmetric group S_n, decomposing its character into irreducibles by computing the corresponding multiplicities. Finally, in Section 5, we prove the main result of the paper, i.e., the generalized variety

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of UT_2-algebras generated by UT_2, $\mathrm{gvar}(UT_2)$, has no almost polynomial growth but we are able to construct inside $\mathrm{gvar}(UT_2)$ a subvarieties of almost polynomial growth. Moreover, we present another variety of UT_2-algebras of almost polynomial growth of the codimensions that is not contained in $\mathrm{gvar}(UT_2)$.

2. On generalized polynomial identities and W-algebras

Throughout this paper F will denote a field of characteristic zero and all the algebras will be associative and have F as their underlying field.

Given an algebra W, we say that an algebra A is a *W-algebra*, if A is a W-bimodule such that, for any $w \in W$, $a_1, a_2 \in A$,

$$w(a_1a_2) = (wa_1)a_2, \ (a_1a_2)w = a_1(a_2w), \ (a_1w)a_2 = a_1(wa_2). \tag{2.1}$$

When $W = F$, a W-algebra is just an F-algebra, that is an algebra over the field F. Clearly, W itself has a natural structure of W-algebra by taking the left and right W-actions to be the usual left and right multiplications of W. In general, this is not the only way to define a structure of W-algebra on W itself; in fact, there might exist different left and right W-actions on W itself that induces a structure of W-algebra (see for example Section 5).

For fixed W the class of W-algebras is a variety in the sense of universal algebra and is nontrivial since it contains W itself. Ideals of W-algebras (*W-ideals*) are understood to be invariant under the bimodule action of W, and homomorphisms $\varphi : A \to B$ between W-algebras A, B must satisfy $\varphi(wav) = w\varphi(a)v$ for $a \in A$, $w, v \in W$.

The variety of W-algebras contains the *free (associative) W-algebra* $W\langle X\rangle$, freely generated by the countably infinite set of variables $X := \{x_1, x_2, \dots\}$ which satisfies the following universal property: given a W-algebra A, any map $X \to A$ can be uniquely extended to a homomorphism of W-algebras $W\langle X\rangle \to A$.

We can give the following combinatorial description of $W\langle X\rangle$. First notice that it is not restrictive to assume that W is an unital algebra; in fact, if not, we can consider the unital algebra $W^+ = W + F1$ obtained from W by adding the unit element 1. So, given a basis $\mathcal{B}_W := \{w_i\}_{i\in\mathcal{I}}$ of W such that $w_0 = 1$, if we identify $x_i = 1x_i = x_i1$ for $i \geq 1$, then a basis of $W\langle X\rangle$ is the following

$$\mathcal{B}_{W\langle X\rangle} := \left\{ w_{i_0}x_{j_1}w_{i_1}x_{j_2}\cdots w_{i_{n-1}}x_{j_n}w_{i_n} \mid n \geq 1,\, j_1,\dots,j_n \geq 1,\, w_{i_0},\dots,w_{i_n} \in \mathcal{B}_W \right\}.$$

The multiplication of two elements $w_{i_0}x_{j_1}w_{i_1}x_{j_2}\cdots w_{i_{n-1}}x_{j_n}w_{i_n}$ and $w_{k_0}x_{l_1}w_{k_1}x_{l_2}\cdots w_{k_{m-1}}x_{l_m}w_{k_m}$ of $\mathcal{B}_{W\langle X\rangle}$ is given by first juxtaposition $w_{i_0}x_{j_1}w_{i_1}x_{j_2}\cdots w_{i_{n-1}}x_{j_n}w_{i_n}w_{k_0}x_{l_1}w_{k_1}x_{l_2}\cdots w_{k_{m-1}}x_{l_m}w_{k_m}$ and then expanding $w_{i_n}w_{k_0} = \sum_{p\in\mathcal{I}} \alpha_p w_p$, $\alpha_p \in F$. So, $W\langle X\rangle$ is also understood as some sort of non-commutative polynomials with coefficients in W. Clearly, the free W-algebra is endowed with a W-bimodule action that satisfies relations (2.1) determined by first juxtaposition

$$w_k(w_{i_0}x_{j_1}w_{i_1}x_{j_2}\cdots w_{i_{n-1}}x_{j_n}w_{i_n})w_l = w_kw_{i_0}x_{j_1}w_{i_1}x_{j_2}\cdots w_{i_{n-1}}x_{j_n}w_{i_n}w_l,$$

and then expanding $w_kw_{i_0}$ and $w_{i_n}w_l$ in the given basis $\mathcal{B}_W$ of W, for $w_k, w_l \in \mathcal{B}_W$ and $w_{i_0}x_{j_1}w_{i_1}x_{j_2}\cdots w_{i_{n-1}}x_{j_n}w_{i_n} \in \mathcal{B}_{W\langle X\rangle}$. The elements of the free W-algebra are called *generalized W-polynomials* or simply *generalized polynomials* when the role of W is clear. A *T_W-ideal* of the free W-algebra is an W-ideal which in addition is invariant under all algebra endomorphisms φ of $W\langle X\rangle$ such that $\varphi(wfv) = w\varphi(f)v$ for all $f \in W\langle X\rangle$ and $w, v \in W$; by the universal property, under the endomorphisms that we call *substitutions*, which send variables of $x_i \in X$ in elements of $W\langle X\rangle$.

Given a W-algebra A, a generalized polynomial $f(x_1, \dots, x_n) \in W\langle X\rangle$ is a *generalized W-identity*, or simply *generalized identity* if there is not ambiguity about W, of A if $f(a_1, \dots, a_n) = 0$ for any $a_1, \dots, a_n \in A$, i.e., f is in the kernel of every homomorphism from $W\langle X\rangle$ to A. We denote by $\mathrm{GId}_W(A)$, or simply $\mathrm{GId}(A)$ when ambiguity does not arise, the set of differential identities of A, which is a T_W-ideal of the free W-algebra. Remark that in case $W = F$, then we are dealing with the ordinary polynomial identities.

For $n \geq 1$, we denote by GP_n^W, or simply GP_n, the vector space of *multilinear generalized polynomials* with coefficient in W in the variables $x_1, \dots, x_n$, so that

$$GP_n := \mathrm{span}_F\{w_{i_0}x_{\sigma(1)}w_{i_1}x_{\sigma(2)}\cdots w_{i_{n-1}}x_{\sigma(n)}w_{i_n} \mid \sigma \in S_n, w_{i_0},\dots,w_{i_n} \in \mathcal{B}_W\},$$

where S_n denotes the symmetric group acting on $\{1, \dots, n\}$. As in the ordinary case, since F has characteristic zero, a Vandermonde argument and the standard linearization procedure show that the T_W-ideal $\mathrm{GId}(A)$ is completely determined by its multilinear generalized polynomials (see [5, Proposition 4.2.3]). We also consider the vector space

$$GP_n(A) := \frac{GP_n}{GP_n \cap \mathrm{GId}(A)},$$

and its dimension $gc_n(A) := \dim_F GP_n(A)$ is the nth *generalized codimension* of A. Remark that if W is a finite-dimensional algebra, then $gc_n(A)$ is finite for $n \geq 1$.

The symmetric group S_n acts naturally on the left on GP_n by permuting the variables: for $\sigma \in S_n$, $\sigma(wx_iv) = wx_{\sigma(i)}v$. Since $GP_n \cap \mathrm{GId}(A)$ is stable under this S_n-action, the space $GP_n(A)$ is a left S_n-module and its character, denoted by $g\chi_n(A)$, is called the *nth generalized cocharacter* of A. Also, since the characteristic of F is zero,

$$g\chi_n(A) = \sum_{\lambda\vdash n} m_\lambda\chi_\lambda,$$

where λ is a partition of n, χ_λ is the irreducible S_n-character associated to λ and $m_\lambda \geq 0$ is the corresponding multiplicity.

A variety of W-algebras generated by a W-algebra A is denoted by $\mathrm{gvar}_W(A)$, or simply $\mathrm{gvar}(A)$, and is called *generalized W-variety*, or simply *generalized variety*, and $\mathrm{GId}(\mathcal{V}) := \mathrm{GId}(A)$. The *growth* of $\mathcal{V} = \mathrm{gvar}(A)$ is the growth of the sequence $gc_n(\mathcal{V}) := gc_n(A)$, $n \geq 1$. We say that the generalized variety $\mathcal{V}$ has *polynomial growth* if $gc_n(\mathcal{V})$ is polynomially bounded and $\mathcal{V}$ has *almost polynomial growth* if $gc_n(\mathcal{V})$ is not polynomially bounded but every proper generalized subvariety of $\mathcal{V}$ has polynomial growth.

In the last part of this section our focus will be on generalized polynomials that are trivial. Recall that a generalized polynomial $f \in W\langle X\rangle$ is said *W-trivial*, or simply *trivial*, if $f = 0$; otherwise f is *W-nontrivial*, or simply *nontrivial*. Since determining whether a generalized polynomial is trivial or not is not always straightforward, we shall introduce some techniques and approaches that can help.

Let $f = f(x_1, \ldots, x_n) \in GP_n$ be a multilinear generalized W-polynomial in the variables $x_1, \ldots, x_n$. Given $\sigma \in S_n$, we denote by f_σ the sum of the monomials of f in which the variables occur exactly in the order $x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}$, and we call it a *generalized monomial* of f. If A is a W-algebra, then f is called *A-proper* if $f_\sigma \notin \mathrm{GId}(A)$ for some $\sigma \in S_n$. Clearly, if f is W-proper, then it is W-nontrivial. In general, the converse is not always true. Although, when $W = F$ "proper" and "nontrivial" are synonymous.

Now we shall focus on linear elements of $W\langle X\rangle$ in a single variable x, i.e., elements of $GP_1 = \mathrm{span}_F\{wxv \mid w, v \in \mathcal{B}_W\}$. Let us introduce the following notation. Let $\mathrm{End}_F(W)$ be the algebra of endomorphism of A with product $\circ$ given by the usual composition of function. Denote by $L, R : W \longrightarrow \mathrm{End}_F(W)$ the operators of left and right multiplications, i.e., for $w \in W$, the *left* (resp. *right*) *multiplication* by w is the endomorphism $L_w : W \longrightarrow W$ (resp. $R_w : W \longrightarrow W$) of W defined by

$$L_w(v) := wv \qquad \Big(\text{resp. } R_w(v) := vw\Big),$$

for all $v \in W$, and consider $L_W R_W := \mathrm{span}_F\{L_w \circ R_v \mid w, v \in \mathcal{B}_W\} \subseteq \mathrm{End}_F(W)$.

Lemma 2.1. *The linear map $\varphi : L_W R_W \longrightarrow GP_1$ given by*

$$\varphi(L_w \circ R_v) = wxv,$$

for any $w, v \in \mathcal{B}_W$, is an isomorphism.

Proof. Clearly, φ is surjective. Now, let $\sum_{i=1}^m \alpha_i L_{w_i} \circ R_{v_i} \in \ker\varphi$, where $w_i, v_i \in \mathcal{B}_W$ and $\alpha_i \in F$ for $1 \leq i \leq m$. Then $f = \sum_{i=1}^m \alpha_i w_i x v_i = 0$, i.e., f is a generalized polynomial W-trivial. As a consequence, if we consider W as a W-algebra with the natural the left and right W-actions defined by multiplication, then it follows that $\sum_{i=1}^m \alpha_i w_i a v_i = 0$ for all $a \in W$, i.e., $\sum_{i=1}^m \alpha_i L_{w_i} \circ R_{v_i} = 0$. Thus φ is also injective, as required. ☐

As a direct consequence of Lemma 2.1, we have the following criterion that establishes whether a linear generalized polynomial in one variable is trivial or not.

Proposition 2.2. *Let $f = \sum_{i=1}^m \alpha_i w_i x v_i \in GP_1$. Then f is W-trivial if and only if $\sum_{i=1}^m \alpha_i L_{w_i} \circ R_{v_i} = 0$.*

Corollary 2.3. *Let W be W-algebra with the left and right actions defined by multiplication and $f \in W\langle X\rangle$. If $f \in GP_1 \cap \mathrm{GId}(W)$, then f is W-trivial.*

Proof. Let $f = \sum_{i=1}^m \alpha_i w_i x v_i \in GP_1 \cap \mathrm{GId}(W)$, where $w_i, v_i \in \mathcal{B}_W$ and $\alpha_i \in F$ for $1 \leq i \leq m$. If $\alpha_i = 0$ for all $1 \leq i \leq m$, then f is W-trivial. So, let us assume that $\alpha_i \neq 0$ for some $1 \leq i \leq m$. Since $f \in \mathrm{GId}(W)$, $\sum_{i=1}^m \alpha_i w_i a v_i = 0$ for all $a \in W$, i.e., $\sum_{i=1}^m \alpha_i L_{w_i}\big(R_{v_i}(a)\big) = 0$ for all $a \in W$, and by Proposition 2.2 f is W-trivial. ☐

So, when we consider W as W-algebra with the natural left and right W-actions defined by multiplication, then there are no nonzero linear generalized identities in one variable. It is important to notice that in case we are considering W with the structure of W-algebra given by another action, then this result is not in general true (see Section 5).

In what follows we shall assume that $W = UT_2$, the algebra of 2×2 upper triangular matrices over F, i.e., we shall work in the class of UT_2-algebras. Also, we shall consider as a basis the set $\mathcal{B}_{UT_2} = \{1 := e_{11} + e_{22}, e_{22}, e_{12}\}$, where e_{ij}'s are the standard matrix units.

3. Generalized polynomial identities of UT_2

In this section we shall compute a basis for the T-ideal of generalized identities, and the corresponding codimension sequence, of UT_2 as UT_2-algebra with the left and right UT_2-actions given by the usual multiplication.

Let $[x_1, x_2] := x_1x_2 - x_2x_1$ be the *commutator* of x_1 and x_2. Also, in what follows we use $[x_1, x_2, \ldots, x_k]$ to denote a left normed commutator. A straightforward computation shows that the following polynomial is a generalized polynomial identity of UT_2 :

$$[x_1, x_2] - [x_1, x_2, e_{22}] \equiv 0. \tag{3.1}$$

Also, it is a UT_2-nontrivial polynomial since it is UT_2-proper. Next, we find some consequences that we will use to reach our goal.

Lemma 3.1. *The following polynomials are generalized identities of UT_2 and consequences of* (3.1)*:*

$$e_{22}[x_1, x_2];\ \ [x_1, x_2] - [x_1, x_2]e_{22};\ \ [x_1, x_2][x_3, x_4];\ \ [x_1, x_2]e_{12};\ \ e_{12}[x_1, x_2].$$

Proof. By acting on (3.1) by e_{22} from the right we get that $e_{22}[x_1,x_2]e_{22} \equiv 0$. By Proposition 2.2 $e_{22}xe_{22} = e_{22}x$, then it follows that $e_{22}[x_1,x_2] \equiv 0$. Moreover, as a consequence of (3.1) and $e_{22}[x_1,x_2] \equiv 0$, we obtain that $[x_1,x_2] - [x_1,x_2]e_{22} \equiv 0$.

Also, by multiplying $[x_1,x_2] - [x_1,x_2]e_{22} \equiv 0$ by $[x_3,x_4]$ on the right and by using $e_{22}[x_1,x_2] \equiv 0$ we get $[x_1,x_2][x_3,x_4] \equiv 0$.

Finally, the generalized identities $e_{12}[x_1,x_2] \equiv 0$ and $[x_1,x_2]e_{12} \equiv 0$ follow from $e_{22}[x_1,x_2] \equiv 0$ and $[x_1,x_2] - [x_1,x_2]e_{22} \equiv 0$, respectively, by acting by e_{12} respectively from the left and the right. □

We are now in a position to prove that the generalized polynomial (3.1) span $\mathrm{GId}(UT_2)$ as T-ideal.

Theorem 3.2. *Let UT_2 be the UT_2-algebra with the action given by the right and the left multiplication. Then* $\mathrm{GId}(UT_2)$ *is generated, as T_{UT_2}-ideal, by the following polynomial:*

$$[x_1,x_2] - [x_1,x_2,e_{22}].$$

Moreover, $gc_n(UT_2) = (n+2)2^{n-1} + 2$.

Proof. Let I be the T_{UT_2}-ideal generated by the above polynomials. It is clear that $I \subseteq \mathrm{GId}(UT_2)$. In order to prove the opposite inclusion, let w be a monomial of GP_n. If w does not contain any e_{22} or e_{12}, i.e., it is an ordinary monomial, then, since $[x_1,x_2][x_3,x_4] \in I$ (Lemma 3.1) and by applying the well-known reduction process modulo the ordinary polynomial identities of UT_2 (see for instance [7, Theorem 4.1.5]), w can be written as a linear combination of $x_1x_2\cdots x_n$ and

$$x_{l_1}\cdots x_{l_m}[x_k, x_{p_1}, \ldots, x_{p_{n-m-1}}],$$

where $0 \le m \le n-2$, $l_1 < \cdots < l_m$ and $k > p_1 < \cdots < p_{n-m-1}$.

Now, suppose that in w appears at least one e_{22}. By Proposition 2.2, $e_{22}xe_{22} = e_{22}x$, $e_{22}xe_{12} = 0$ and $e_{12}xe_{22} = e_{12}x$, then we may assume that w contains exactly one e_{22}. By the Poincarè-Birkhoff-Witt Theorem and by $e_{22}[x_1,x_2] \in I$ (Lemma 3.1), w can be written as a linear combination of $e_{22}x_1x_2\cdots x_n$ and polynomials of the type

$$x_{i_1}\cdots x_{i_r}c_1\cdots c_s,$$

where $i_1 < \cdots < i_r$ and $c_1,\ldots,c_s$ are left-normed commutators and just one of them contains e_{22}. Since $[x_1,x_2][x_3,x_4] \in I$ (Lemma 3.1), then $s = 2$ and one in between of the two commutators c_1, c_2 contains e_{22}. Now, since $e_{22}[x_1,x_2]$, $[x_1,x_2] - [x_1,x_2]e_{22} \in I$ (Lemma 3.1), we may assume that $s = 1$. Moreover, since $[x_1,x_2] - [x_1,x_2]e_{22} \in I$ (Lemma 3.1), then we can assume that e_{22} appears in the second position of the commutator (otherwise we can erase e_{22} and come back to the previous case of ordinary polynomials). Now, take the left-normed commutator $[x_k, e_{22}, x_{j_1}, \ldots, x_{j_s}]$ and notice that using the same reasoning as we did before, we may assume that $j_1 < \cdots < j_s$. Also, by the Jacobi identity $[x_2,e_{22},x_1] = [x_1,e_{22},x_2] - [x_1,x_2,e_{22}]$ it turns out that

$$[x_2,e_{22},x_1] \equiv [x_1,e_{22},x_2] - [x_1,x_2] \pmod{I}.$$

This implies that the left-normed commutator can be written as $[x_{i_1}, e_{22}, x_{i_2}, \ldots, x_{i_s}]$ where $i_1 < i_2 < \cdots < i_s$.

Finally, let w be a monomial of GP_n containing at least one e_{12}. Again by Proposition 2.2, $e_{12}xe_{12} = 0$, $e_{22}xe_{12} = 0$ and $e_{12}xe_{22} = e_{12}x$, then w must contain just one e_{12}. Moreover, since by Lemma 3.1 $e_{12}[x_1,x_2]$, $[x_1,x_2]e_{12} \in I$, all the variables on the left and on the right of e_{12} are ordered. Thus w can be written modulo I as

$$x_{i_1}\cdots x_{i_r}e_{12}x_{j_1}\cdots x_{j_{n-r}},$$

where $0 \le r \le n$, $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_{n-r}$.

By putting together all the previous remarks, we have proved that GP_n is generated modulo I by the polynomials:

$$\begin{aligned}
&x_1\cdots x_n;\\
&e_{22}x_1\cdots x_n;\\
&X_{12}^{(\mathcal{I})} = x_{i_1}\cdots x_{i_r}e_{12}x_{j_1}\cdots x_{j_{n-r}};\\
&X^{(\mathcal{L},k)} = x_{l_1}\cdots x_{l_s}[x_k, x_{m_1}, \ldots, x_{m_t}];\\
&X_{22}^{(\mathcal{P})} = x_{p_1}\cdots x_{p_u}[x_{q_1}, e_{22}, x_{q_2}, \ldots, x_{q_v}]
\end{aligned} \tag{3.2}$$

where $\mathcal{I} = \{i_1,\ldots,i_r\}$, $\mathcal{L} = \{l_1,\ldots,l_s\}$ and $\mathcal{P} = \{p_1,\ldots,p_u\}$ are subsets of $\{1,\ldots,n\}$, $i_1 < \cdots < i_r$, $j_1 < \cdots < j_{n-r}$, $l_1 < \cdots < l_s$, $k > m_1 < \cdots < m_t$, $p_1 < \cdots < p_u$, $q_1 < q_2 < \cdots < q_v$, $0 \le r \le n$, $t \ge 1$ and $v \ge 1$.

Next we prove that these elements are linearly independent modulo $\mathrm{GId}(UT_2)$. To this end, let

$$f = \alpha_1 x_1\cdots x_n + \alpha_2 e_{22}x_1\cdots x_n + \sum_{\mathcal{I}} \beta_{\mathcal{I}} X_{12}^{(\mathcal{I})} + \sum_{\mathcal{L},k} \gamma_{\mathcal{L},k} X^{(\mathcal{L},k)} + \sum_{\mathcal{P}} \delta_{\mathcal{P}} X_{22}^{(\mathcal{P})}$$

be a linear combination of the generalized polynomials (3.2) and suppose by contradiction that $f \neq 0$. We shall make suitable evaluations to prove that $f = 0$ and this will complete the proof.

First, if we evaluate $x_1 = \cdots = x_n = e_{11}$, then we get $\alpha_1 e_{11} + \beta_{\mathcal{I}} e_{12} = 0$, where $\mathcal{I} = \{1,\ldots,n\}$, thus $\alpha_1 = \beta_{\mathcal{I}} = 0$. Now let us make the evaluation $x_1 = \cdots = x_n = e_{22}$. In this case we get $\alpha_2 e_{22} + \beta_{\mathcal{I}'} e_{12} = 0$, where $\mathcal{I}' = \emptyset$, so $\alpha_2 = \beta_{\mathcal{I}'} = 0$. For a fixed $\mathcal{I} = \{i_1,\ldots,i_r\}$, we set $x_{i_1} = \cdots = x_{i_r} = e_{11}$ and $x_{j_1} = \cdots = x_{j_{n-r}} = e_{22}$ and we get $\beta_{\mathcal{I}} e_{12} = 0$ thus $\beta_{\mathcal{I}} = 0$. Now, for fixed $\mathcal{L} = \{l_1,\ldots,l_s\}$ and k, from the evaluation $x_{l_1} = \cdots = x_{l_s} = e_{11} + e_{22}$, $x_k = e_{12}$

and $x_{m_1}=\cdots=x_{m_t}=e_{22}$, we get $\gamma_{\mathcal{L},k}e_{12}=0$, thus $\gamma_{\mathcal{L},k}=0$. Here remark that all the polynomials of the type $X_{22}^{(\mathcal{P})}$ evaluate to zero since in $X^{(\mathcal{L},k)}$ it must be $k>m_1<\cdots<m_t$ whereas in $X_{22}^{(\mathcal{P})}$ it must be $q_1<q_2<\cdots<q_v$. Finally, for any fixed $\mathcal{P}=\{p_1,\dots,p_u\}$, we make the substitution $x_{p_1}=\cdots=x_{p_u}=e_{11}+e_{22}$, $x_{q_1}=e_{12}$ and $x_{q_2}=\cdots=x_{q_v}=e_{22}$ and we get $\delta_{\mathcal{P}}e_{12}=0$, that is $\delta_{\mathcal{P}}=0$. Therefore, all the scalars appearing in f are zero, i.e., $f=0$, a contradiction.

Thus the elements in (3.2) are linearly independent modulo $\mathrm{GId}(UT_2)$ and, since $GP_n\cap I\subseteq GP_n\cap\mathrm{GId}(UT_2)$, this proves that $\mathrm{GId}(UT_2)=I$ and the polynomials in (3.2) are a basis of GP_n modulo $GP_n\cap\mathrm{GId}(UT_2)$. Hence, by counting we get

$$\begin{aligned}gc_n(UT_2)&=2+\sum_{r=0}^{n}\binom{n}{r}+\sum_{r=2}^{n}\binom{n}{r}(r-1)+\sum_{r=0}^{n-1}\binom{n}{r}=2+\sum_{r=0}^{n}\binom{n}{r}+\sum_{r=1}^{n}r\binom{n}{r}-1-\sum_{r=0}^{n}\binom{n}{r}+2+\sum_{r=0}^{n}\binom{n}{r}-1\\&=(n+2)2^{n-1}+2.\end{aligned}$$

□

4. Generalized cocharacter sequence of UT_2

In this section, we shall determine the generalized cocharacter of UT_2 as UT_2-algebra where the action of UT_2 as bimodule over itself is the usual product of UT_2.

We shall start by proving some technical lemmas that give us a lower bound for the multiplicities m_λ of nth generalized UT_2-cocharacter of UT_2

$$g\chi_n(UT_2)=\sum_{\lambda\vdash n}m_\lambda\chi_\lambda. \tag{4.1}$$

To this end recall that any irreducible left S_n-module $W_\lambda\subseteq GP_n$ with character χ_λ can be generated as S_n-module by an element of the form $e_{T_\lambda}f$, for some $f\in W_\lambda$ and some tableau T_λ of shape λ. Here $e_{T_\lambda}=\sum_{\substack{\sigma\in R_{T_\lambda}\\ \tau\in C_{T_\lambda}}}(\operatorname{sgn}\tau)\sigma\tau$ is a minimal quasi-idempotent corresponding to T_λ, where R_{T_λ} and C_{T_λ} are the subgroups of S_n stabilizing the rows and columns of T_λ, respectively.

Lemma 4.1. *$m_{(n)}\geq 2n+3$ in (4.1).*

Proof. Let us consider the standard tableau

$$T_{(n)}=\boxed{1\mid 2\mid \cdots \mid n}\,,$$

and the following $2n+3$ generalized polynomials associated to it

$$a(x)=x^n, \tag{4.2}$$
$$a_{22}^{(0)}(x)=e_{22}x^n, \tag{4.3}$$
$$a_{22}^{(i)}(x)=x^{i-1}[x,e_{22}]x^{n-i},\quad 1\leq i\leq n, \tag{4.4}$$
$$a_{12}^{(j)}(x)=x^je_{12}x^{n-j},\quad 0\leq j\leq n. \tag{4.5}$$

These polynomials are obtained from the quasi-idempotents corresponding to the tableau $T_{(n)}$ by identifying all the elements. Clearly, the polynomials (4.2)–(4.5) do not vanish in UT_2. We claim that these generalized polynomials are linear independent modulo $\mathrm{GId}(UT_2)$. So, let $f\in\mathrm{GId}(UT_2)$ be a linear combination of such polynomials, i.e.,

$$f=\alpha a(x)+\sum_{i=0}^{n}\beta_ia_{22}^{(i)}(x)+\sum_{j=0}^{n}\gamma_ja_{12}^{(j)}(x).$$

First suppose that $\alpha\neq 0$ or $\gamma_n\neq 0$. Then, by making the evaluation $x=e_{11}$ one gets $\alpha e_{11}+\gamma_ne_{12}=0$. Hence, it follows that $\alpha=\gamma_n=0$, a contradiction.

Now assume that $\beta_0\neq 0$ or $\gamma_0\neq 0$. Then, if we consider the evaluation $x=e_{22}$, we obtain $\beta_0e_{22}+\gamma_0e_{12}=0$. Hence, it follows that $\beta_0=\gamma_0=0$, a contradiction.

Next, assume that there exists $\gamma_j\neq 0$ for some $1\leq j\leq n-1$. If we substitute $x=\delta e_{11}+e_{22}$ with $\delta\in F$, $\delta\neq 0$, we get $\sum_{j=1}^{n}\delta^j\gamma_j=0$. Since F is an infinite filed, we can choose $\delta_1,\dots,\delta_{n-1}\in F$ such that $\delta_i\neq 0$ and $\delta_i\neq\delta_j$ for $1\leq i\neq j\leq n-1$. Then we get the following homogeneous linear system of $n-1$ equations in the $n-1$ variables $\gamma_1,\dots,\gamma_{n-1}$

$$\sum_{j=1}^{n-1}\delta_k^j\gamma_j=0,\quad 1\leq k\leq n-1.$$

Since the matrix associated to the above system is a Vandermonde matrix, it follows that $\gamma_j=0$, for any $1\leq j\leq n-1$, a contradiction.

Finally, if $\beta_i \neq 0$ for some $1 \leq i \leq n$, then by making the substitution $x = \delta e_{11} + e_{22} + e_{12}$, we get that $\sum_{i=1}^{n} \delta^i \beta_i = 0$. Now, as above, one may choose distinct $\delta_1, \ldots, \delta_n \in F$ such that $\delta_i \neq 0$ for $1 \leq i \leq n$. Hence, we obtain the following linear system of n equations in the n variables $\beta_1, \ldots, \beta_n$

$$\sum_{i=1}^{n} \delta_k^i \beta_i = 0, \quad 1 \leq k \leq n.$$

Again, we obtained a linear system whose associated matrix is a Vandermonde matrix. Thus, it follows that $\beta_i = 0$ for any $1 \leq i \leq n$, a contradiction. Therefore the $2n+3$ generalized polynomials (4.2)–(4.5) are linearly independent modulo $\mathrm{GId}(UT_2)$.

Notice that the complete linearization of $a(x)$ is $e_{(n)}(x_1, \ldots, x_n) = e_{T_{(n)}}(x_1 \cdots x_n)$, and, for every $0 \leq i \leq n$, the complete linearization of $a_{22}^{(i)}(x)$ and $a_{12}^{(i)}(x)$ are the polynomials $e_{(n)}^{e_{22},i}(x_1, \ldots, x_n) = e_{T_{(n)}}(x_1 \cdots x_{i-1}[x_i, e_{22}]x_{i+1} \cdots x_n)$ and $e_{(n)}^{e_{12},i}(x_1, \ldots, x_n) = e_{T_{(n)}}(x_1 \cdots x_i e_{12} x_{i+1} \cdots x_n)$, respectively. Then it follows that the polynomials $e_{(n)}$, $e_{(n)}^{e_{22},i}$, $e_{(n)}^{e_{12},i}$ are also linearly independent modulo $\mathrm{Id}^{\varepsilon}(UT_2)$ and as a consequence $m_{(n)} \geq 2n+3$, as desired. ☐

Lemma 4.2. *If $p \geq 1$ and $q \geq 0$, then $m_{(p+q,p)} \geq 3(q+1)$ in* (4.1).

Proof. For any $0 \leq i \leq q$, let $T_{(p+q,p)}^{(i)}$ be the standard tableau

$i+1$	$i+2$	$\cdots$	$i+p-1$	$i+p$	1	$\cdots$	i	$i+2p+1$	$\cdots$	n
$i+p+2$	$i+p+3$	$\cdots$	$i+2p$	$i+p+1$						

,

and let associate to it the following generalized polynomials

$$b_{p,q}^{(i)}(x,y) = x^i \underbrace{\bar{x} \cdots \tilde{x}}_{p-1}[x,y] \underbrace{\bar{y} \cdots \tilde{y}}_{p-1} x^{q-i},$$

$$c_{p,q}^{(i)}(x,y) = x^i \underbrace{\bar{x} \cdots \tilde{x}}_{p-1}(x e_{12} y - y e_{12} x) \underbrace{\bar{y} \cdots \tilde{y}}_{p-1} x^{q-i},$$

$$d_{p,q}^{(i)}(x,y) = x^i \underbrace{\bar{x} \cdots \tilde{x}}_{p-1}(x e_{22} y - y e_{22} x) \underbrace{\bar{y} \cdots \tilde{y}}_{p-1} x^{q-i},$$

where the symbol $\bar{}$ or $\tilde{}$ means alternation on the corresponding variables. For any $1 \leq i \leq q$, these polynomials are obtained from the quasi-idempotents corresponding to the tableau $T_{(p+q,p)}^{(i)}$ by identifying all the elements in each row. Also, they are not generalized identities of UT_2.

Next, we shall show that the generalized polynomials $b_{p,q}^{(i)}(x,y)$, $c_{p,q}^{(i)}(x,y)$ $d_{p,q}^{(i)}(x,y)$, $0 \leq i \leq q$, are linear independent modulo $\mathrm{GId}(UT_2)$. To this end, let us consider

$$f = \sum_{i=0}^{q} \alpha_i b_{p,q}^{(i)}(x,y) + \sum_{i=0}^{q} \beta_i c_{p,q}^{(i)}(x,y) + \sum_{i=0}^{q} \gamma_i d_{p,q}^{(i)}(x,y) \in \mathrm{GId}(UT_2).$$

First, suppose that there exists $\beta_i \neq 0$ for some $0 \leq i \leq q$. By evaluating $x = \delta e_{11} + e_{22}$, with $\delta \in F$, $\delta \neq 0$, and $y = e_{11}$, we get $\sum_{i=0}^{q}(-1)^{p-1}\delta^i \beta_i = 0$. Since F is infinite, we may take $\delta_1, \ldots, \delta_{q+1} \in F$, where $\delta_j \neq 0$, $\delta_j \neq \delta_k$, for all $1 \leq j \neq k \leq q+1$. Thus, as in the proof of the previous lemma, we obtain the following homogeneous linear system of $q+1$ equations in the $q+1$ variables $\beta_0, \ldots, \beta_q$

$$\sum_{i=0}^{q} \delta_j^i \beta_i = 0, \quad 1 \leq j \leq q+1. \tag{4.6}$$

Since the matrix associated to the system above is a Vandermonde matrix, it follows that $\beta_i = 0$, for all $0 \leq i \leq q$.

Now if there exists $\alpha_i \neq 0$ for some $0 \leq i \leq q$, then we substitute $x = \delta e_{11} + e_{12} + e_{22}$, with $\delta \in F$, $\delta \neq 0$, and $y = e_{11}$, and we get $\sum_{i=0}^{q}(-1)^{p-1}\delta^i \alpha_i = 0$. Thus, as above, since F is infinite, we obtain a homogeneous linear system of $q+1$ equations in the $q+1$ variables $\alpha_0, \ldots, \alpha_q$ equivalent to (4.6). Therefore $\alpha_i = 0$ for all $0 \leq i \leq q$, a contradiction.

Finally, assume that there exists $\gamma_i \neq 0$ for some $0 \leq i \leq q$. By making the evaluation $x = \delta e_{11} + e_{12} + e_{22}$, with $\delta \in F$, $\delta \neq 0$, and $y = e_{22}$, we obtain $\sum_{i=0}^{q} \delta^i \gamma_i = 0$. Then, as above, we get a homogeneous linear system of $q+1$ equations in the $q+1$ variables $\gamma_0, \ldots, \gamma_q$ equivalent to (4.6). So, $\gamma_i = 0$ for all $0 \leq i \leq q$, a contradiction.

Thus, the $3(q+1)$ generalized polynomials $b_{p,q}^{(i)}(x,y)$, $c_{p,q}^{(i)}(x,y)$ $d_{p,q}^{(i)}(x,y)$, $0 \leq i \leq q$, are linearly independent modulo $\mathrm{GId}(UT_2)$ and, so, as in Lemma 4.1, $m_{(p+q,p)} \geq 3(q+1)$, as required. ☐

Lemma 4.3. *If $p \geq 1$ and $q \geq 0$, then $m_{(p+q,p,1)} \geq q+1$ in* (4.1).

Proof. For any $0 \leq i \leq q$, define $T_{(p+q,p,1)}^{(i)}$ to be the standard tableau

$i+p$	$i+1$	$\cdots$	$i+p$	1	$\cdots$	i	$i+2p+2$	$\cdots$	n
$i+p+1$	$i+p+3$	$\cdots$	$i+2p+1$						
$i+p+2$									

,

and associate to it the generalized polynomial

$$h^{(i)}_{p,q}(x,y,z) = x^i \underbrace{\hat{x}\cdots\tilde{x}}_{p-1}\bar{x}\bar{y}\bar{z}\underbrace{\hat{y}\cdots\tilde{y}}_{p-1}x^{q-i},$$

where the symbol ˆ or ˜ or ¯ means alternation on the corresponding variables. For any $1\le i\le q$ these generalized polynomials are obtained from the quasi-idempotents corresponding to the tableau $T^{(i)}_{(p+q,p,1)}$ by identifying all the elements in each row. Clearly, $h^{(i)}_{p,q}(x,y,z)$, $1\le i\le q$, do not belong to $\mathrm{GId}(UT_2)$. We claim that the $q+1$ generalized polynomials $h^{(i)}_{p,q}(x,y,z)$, $0\le i\le q$, are linear independent modulo $\mathrm{GId}(UT_2)$. If not, there exist $\alpha_0,\dots,\alpha_q\in F$ not all zero such that

$$\sum_{i=0}^{q}\alpha_i h^{(i)}_{p,q}(x,y,z)\in \mathrm{GId}(UT_2).$$

If we substitute $x=\beta e_{11}+e_{12}+e_{22}$, with $\beta\in F$, $\beta\neq 0$, $y=e_{11}$, and $z=e_{22}$, then we obtain $\sum_{i=0}^{q}\beta^i\alpha_i=0$, and again with a Vandermonde argument we get that $\alpha_i=0$ for all $0\le i\le q$, a contradiction.

Therefore the $q+1$ generalized polynomials $h^{(i)}_{p,q}(x,y,z)$, $0\le i\le q$, are linearly independent modulo $\mathrm{GId}(UT_2)$, as claimed. Again, as in Lemma 4.1, this implies that $m_{(p+q,p,1)}\ge q+1$. □

Next, we shall prove the main theorem of the section.

Theorem 4.4. *If $g\chi_n(UT_2)=\sum_{\lambda\vdash n}m_\lambda\chi_\lambda$ is the nth generalized cocharacter of UT_2, then*

$$m_\lambda=\begin{cases}2n+3, & \text{if } \lambda=(n)\\ 3(q+1), & \text{if } \lambda=(p+q,p)\\ q+1, & \text{if } \lambda=(p+q,p,1)\\ 0, & \text{in all other cases}\end{cases}.$$

Proof. Let $d_\lambda=\deg\chi_\lambda$ be the degree of χ_λ, $\lambda\vdash n$. Then $gc_n(UT_2)=\sum_{\lambda\vdash n}m_\lambda d_\lambda$, and by Lemmas 4.1, 4.2 and 4.3 we have that

$$gc_n(UT_2)\ge(2n+3)d_{(n)}+\sum_{\substack{1\le p\le\lfloor\frac{n}{2}\rfloor\\0\le q\le n-2p}}3(q+1)d_{(p+q,p)}+\sum_{\substack{1\le p\le\lfloor\frac{n-1}{2}\rfloor\\0\le q\le n-2p-1}}(q+1)d_{(p+q,p,1)}. \tag{4.7}$$

Thus, to complete the proof is enough to show the (4.7) is actually an equality. To this end, notice that for $n=2p+q$, by the hook formula (see for example [9, Theorem 2.3.21]) we have that

$$d_{(p+q,p)}=\frac{n!}{p!q!(p+q+1)\cdots(q+2)}=\binom{n}{p}\frac{n-2p+1}{n-p+1}.$$

Then, it follows that

$$\begin{aligned}\sum_{\substack{1\le p\le\lfloor\frac{n}{2}\rfloor\\0\le q\le n-2p}}(q+1)d_{(p+q,p)}&=(n+1)\sum_{p=1}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{p}-3\sum_{p=1}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{p}p+\sum_{p=1}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{p}\frac{p^2}{n-p+1}\\&=(n+1)\left(\sum_{p=1}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{p}+\sum_{p=n-\lfloor\frac{n}{2}\rfloor+1}^{n}\binom{n}{p}\right)-\sum_{p=1}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{p}p-\sum_{p=n-\lfloor\frac{n}{2}\rfloor+1}^{n}\binom{n}{p}p-2\sum_{p=1}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{p}p,\end{aligned}$$

where in the last equality we use that $\binom{n}{n-p+1}=\binom{n}{p}\frac{p}{n-p+1}$. Recall that $i\binom{i-1}{j-1}=j\binom{i}{j}$ and $\sum_{j=0}^{i}\binom{i}{j}=2^i$. Hence, if $n=2k$,

$$\sum_{\substack{1\le p\le k\\0\le q\le n-2p}}(q+1)d_{(p+q,p)}=(2k+1)(2^{2k}-1)-k2^{2k}-4k\sum_{p=1}^{k}\binom{2k-1}{p-1}=2^{2k}-2k-1.$$

In case $n=2k+1$,

$$\begin{aligned}\sum_{\substack{1\le p\le k\\0\le q\le n-2p}}(q+1)d_{(p+q,p)}=&(2k+2)\left(2^{2k+1}-1-\binom{2k+1}{k+1}\right)-(2k+1)2^{2k}+(k+1)\binom{2k+1}{k+1}-2(2k+1)\sum_{p=1}^{k}\binom{2k}{p-1}\\=&2^{2k+1}-2k-2.\end{aligned}$$

Thus, we have that

$$\sum_{\substack{1\le p\le\lfloor\frac{n}{2}\rfloor\\0\le q\le n-2p}}(q+1)d_{(p+q,p)}=2^n-n-1. \tag{4.8}$$

Now, for $n = 2p + q + 1$, by applying the hook formula again, we get that

$$d_{(p+q,p,1)} = \frac{n!}{(p-1)!q!(p+1)(p+q+2)(p+q)\cdots(q+2)} = \binom{n}{p+1}\frac{p(n-2p)}{n-p+1}.$$

Then, by recalling that $\binom{n}{p+1} = \binom{n}{p}\frac{n-p}{p+1}$, $\binom{n}{n-p+1} = \binom{n}{p}\frac{n}{n-p+1}$ and $\binom{n}{p+1} + \binom{n}{p} = \binom{n+1}{p+1}$, it follows

$$\sum_{\substack{1\leq p\leq \lfloor \frac{n-1}{2}\rfloor \\ 0\leq q\leq n-2p-1}} (q+1)d_{(p+q,p,1)} = \sum_{p=1}^{\lfloor \frac{n-1}{2}\rfloor} \left(\binom{n+1}{p+1}p - \binom{n}{p-1}\right)(n-2p).$$

Hence, with a similar computation as above, we obtain that

$$\sum_{\substack{1\leq p\leq \lfloor \frac{n-1}{2}\rfloor \\ 0\leq q\leq n-2p-1}} (q+1)d_{(p+q,p,1)} = (n-4)2^{n-1} + n + 2. \tag{4.9}$$

Thus, by (4.8), (4.9) and since $d_{(n)} = 1$, it follows that

$$(2n+3)d_{(n)} + 3\sum_{\substack{1\leq p\leq \lfloor \frac{n}{2}\rfloor \\ 0\leq q\leq n-2p}} (q+1)d_{(p+q,p)} + \sum_{\substack{1\leq p\leq \lfloor \frac{n-1}{2}\rfloor \\ 0\leq q\leq n-2p-1}} (q+1)d_{(p+q,p,1)}$$
$$= 2n + 3 + 3(2^n - n - 1) + (n-4)2^{n-1} + n + 2 = (n+2)2^{n-1} + 2.$$

Since by Theorem 3.2 $gc_n(UT_2) = (n+2)2^{n-1} + 2$, we get that 4.7 is actually an equality and we are done. □

5. On almost polynomial growth

In this section, we shall construct a variety of UT_2-algebras inside gvar(UT_2) of almost polynomial growth of the codimensions. We shall also present another variety of UT_2-algebras that have almost polynomial growth of the codimensions but it is not contained in gvar(UT_2).

Let us define on UT_2 a new structure as UT_2-bimodule in the following way: let $1 := e_{11} + e_{22}$ and e_{22} act by left and right multiplication as in the previous case and let

$$e_{12} \cdot a = a \cdot e_{12} = 0$$

for all $a \in UT_2$. It readily follows that this action defines UT_2 as a new UT_2-algebras that we will denote it by UT_2^D. Such a notation is justified by noticing that if we let D be the subalgebra of UT_2 spanned by e_{11} and e_{22}, then the above action is the natural generalization of the left and right multiplication of UT_2 by elements of D, i.e., we can also view UT_2 as a D-algebra.

Following step-by-step the lines of Theorem 3.2 and Theorem 4.4 with the necessary changes, we can prove the following results.

Theorem 5.1. *Let UT_2^D be the UT_2-algebra with the above action. Then* $\mathrm{GId}(UT_2^D)$ *is generated, as T_{UT_2}-ideal, by the following polynomials:*

$$e_{12}x; \quad xe_{12}; \quad [x_1, x_2] - [x_1, x_2, e_{22}].$$

Moreover, $gc_n(UT_2^D) = n2^{n-1} + 2$.

Theorem 5.2. *Let $g\chi_n(UT_2^D) = \sum_{\lambda\vdash n} m_\lambda\chi_\lambda$ be the nth generalized cocharacter of UT_2^D. Then*

$$m_\lambda = \begin{cases} n+2, & \text{if } \lambda = (n) \\ 2(q+1), & \text{if } \lambda = (p+q,p) \\ q+1, & \text{if } \lambda = (p+q,p,1) \\ 0, & \text{in all other cases} \end{cases}.$$

Remark that by Theorem 5.1 $UT_2^D \in$ gvar(UT_2) and gvar(UT_2^D) grows exponentially, then it immediately follows that

Corollary 5.3. *gvar(UT_2) does not have almost polynomial growth of the codimensions.*

Next, we shall prove that gvar(UT_2^D) is a variety of UT_2-algebras of almost polynomial growth. If $\mathcal{V}$ is a variety of UT_2-algebras, then for every $n \geq 1$, we write

$$g\chi_n(\mathcal{V}) = \sum_{\lambda\vdash n} m_\lambda^{\mathcal{V}}\chi_\lambda,$$

where $m_\lambda^{\mathcal{V}}$ denotes the multiplicity of irreducible character χ_λ in $g\chi_n(\mathcal{V})$.

Remark 5.4. *Recall that the $n+2$ linear independent generalized polynomials corresponding to the partition $\lambda = (n)$ are:*

$$\begin{aligned} a(x) &= x^n, \\ a_{22}^{(0)}(x) &= e_{22}x^n, \\ a_{22}^{(i)}(x) &= x^{i-1}[x, e_{22}]x^{n-i}, \quad 1 \le i \le n. \end{aligned}$$

The $2(q+1)$ linear independent generalized polynomials corresponding to the partition $\lambda = (p+q, p)$ are:

$$\begin{aligned} b_{p,q}^{(i)}(x,y) &= x^i \underbrace{\bar{x}\cdots\tilde{x}}_{p-1}[x,y]\underbrace{\bar{y}\cdots\tilde{y}}_{p-1}x^{q-i}, \quad 0 \le i \le q \\ d_{p,q}^{(i)}(x,y) &= x^i \underbrace{\bar{x}\cdots\tilde{x}}_{p-1}(xe_{22}y - ye_{22}x)\underbrace{\bar{y}\cdots\tilde{y}}_{p-1}x^{q-i}, \quad 0 \le i \le q. \end{aligned}$$

Finally, the $q+1$ linear independent generalized polynomials corresponding to the partition $\lambda = (p+q, p, 1)$ are:

$$h_{p,q}^{(i)}(x,y,z) = x^i \underbrace{\hat{x}\cdots\tilde{x}}_{p-1}\bar{x}\bar{y}\bar{z}\underbrace{\hat{y}\cdots\tilde{y}}_{p-1}x^{q-i}, \quad 0 \le i \le q.$$

Lemma 5.5. *Let $\mathcal{U}$ be a proper subvariety of $gvar(UT_2^D)$. Then there exist constants $M < N$ such that*

$$x^M y x^{N-M} + \sum_{i<M} \mu_i x^i y x^{N-i} \in \mathrm{GId}(\mathcal{U}),$$

for some $\mu_i \in F$.

Proof. Let $a(x)$, $a_{22}^{(i)}(x)$, $b_{p,q}^{(j)}(x,y)$, $d_{p,q}^{(j)}(x,y)$, and $h_{p,q}^{(j)}(x,y,z)$, $0 \le i \le n$, $0 \le j \le q$, be the polynomial of Remark 5.4. Since $\mathcal{U} \subsetneq \mathcal{V} = \mathrm{gvar}(UT_2^D)$, then there exists $\lambda \vdash n$ such that $m_\lambda^{\mathcal{U}} < m_\lambda^{\mathcal{V}}$. Thus by Theorem 5.2, it follows that either

$$\alpha_1 a(x) + \sum_{i=0}^{n} \alpha_2^{(i)} a_{22}^{(i)}(x) \in \mathrm{GId}(\mathcal{U}), \tag{5.1}$$

with $\alpha_1, \alpha_2^{(i)}$ not all zero, or

$$\sum_{i=0}^{q} \beta_i b_{p,q}^{(i)}(x,y) + \sum_{i=0}^{q} \delta_i d_{p,q}^{(i)}(x,y) \in \mathrm{GId}(\mathcal{U}), \tag{5.2}$$

with β_i, δ_i not all zero, or

$$\sum_{i=0}^{q} \eta_i h_{p,q}^{(i)}(x,y,z) \in \mathrm{GId}(\mathcal{U}), \tag{5.3}$$

with η_i not all zero. Suppose that (5.2) holds. Then

$$f(x,y) = \sum_{i=0}^{q} \beta_i x^i \underbrace{\bar{x}\cdots\tilde{x}}_{p-1}[x,y]\underbrace{\bar{y}\cdots\tilde{y}}_{p-1}x^{q-i} + \sum_{i=0}^{q} \delta_i x^i \underbrace{\bar{x}\cdots\tilde{x}}_{p-1}(xe_{22}y - ye_{22}x)\underbrace{\bar{y}\cdots\tilde{y}}_{p-1}x^{q-i} \in \mathrm{GId}(\mathcal{U}).$$

If we substitute in $f(x,y)$ the variable y with $y_1 + y_2$, we obtain that

$$\begin{aligned} f(x,y_1,y_2) = &\sum_{i=0}^{q} \beta_i x^i \underbrace{\bar{x}\cdots\tilde{x}}_{p-1}[x, y_1+y_2]\underbrace{\overline{(y_1+y_2)}\cdots\widetilde{(y_1+y_2)}}_{p-1}x^{q-i} \\ &+ \sum_{i=0}^{q} \delta_i x^i \underbrace{\bar{x}\cdots\tilde{x}}_{p-1}(xe_{22}(y_1+y_2) - (y_1+y_2)e_{22}x)\underbrace{\overline{(y_1+y_2)}\cdots\widetilde{(y_1+y_2)}}_{p-1}x^{q-i} \in \mathrm{GId}(\mathcal{U}). \end{aligned}$$

Now, let us consider in the polynomial $f(x,y_1,y_2)$ the component $f'(x,y_1,y_2)$ of degree 1 in y_2. By substituting in $f'(x,y_1,y_2)$ the variable y_1 with x^2 and y_2 with $[x,y]$, we get that

$$\sum_{i=0}^{q} \beta_i x^i \underbrace{\bar{x}\cdots\tilde{x}}_{p-1}[x,[x,y]]\underbrace{\overline{x^2}\cdots\widetilde{x^2}}_{p-1}x^{q-i} + \sum_{i=0}^{q} \delta_i x^i \underbrace{\bar{x}\cdots\tilde{x}}_{p-1}(xe_{22}[x,y] - [x,y]e_{22}x)\underbrace{\overline{x^2}\cdots\widetilde{x^2}}_{p-1}x^{q-i} \in \mathrm{GId}(\mathcal{U}).$$

Since $e_{22}[x,y]$, $[x,y] - [x,y]e_{22} \in \mathrm{GId}(UT_2^D) \subseteq \mathrm{GId}(\mathcal{U})$, it follows that

$$g(x,y) = \sum_{i=0}^{q} \beta_i \underbrace{\bar{x}\cdots\tilde{x}}_{p-1}x[x,y]\underbrace{\overline{x^2}\cdots\widetilde{x^2}}_{p-1}x^{q-i} - \sum_{i=0}^{q} \gamma_i x^i \underbrace{\bar{x}\cdots\tilde{x}}_{p-1}[x,y]x\underbrace{\overline{x^2}\cdots\widetilde{x^2}}_{p-1}x^{q-i} \in \mathrm{GId}(\mathcal{U}), \tag{5.4}$$

where $\gamma_i = \beta_i + \delta_i \in F$, $0 \le i \le q$.

Suppose first that $\beta_i \neq 0$ for some $0 \leq i \leq q$, and let $t = \max\{i \,|\, \beta_i \neq 0\}$ and $N' = \deg g(x,y)$. Since $g(x,y) \in \mathrm{GId}(\mathcal{U})$, we have that

$$\beta_t x^{t+2p-1}[x,y]x^{N'-2p-t-1} + \sum_{i<t+2p-1} \gamma_i' x^i [x,y] x^{N'-i-2} \in \mathrm{GId}(\mathcal{U}),$$

for some $\gamma_i' \in F$. Since $\beta_t \neq 0$, we get that

$$x^{t+2p} y x^{N'-2p-t-1} + \sum_{i<t+2p} \mu_i x^i y x^{N'-i-1} \in \mathrm{GId}(\mathcal{U}),$$

for some $\mu_i \in F$. Now, if we set $N = N'-1$ and $M = t+2p$, then it follows that

$$x^M y x^{N-M} + \sum_{i<M} \mu_i x^i y x^{N-i} \in \mathrm{GId}(\mathcal{U}),$$

for some $\mu_i \in F$, as required.

Assume now that in (5.4) $\beta_i = 0$ for all $1 \leq i \leq q$. Then $\gamma_i \neq 0$ for some $1 \leq i \leq q$. So, let $t = \max\{i \,|\, \gamma_i \neq 0\}$ and $N' = \deg g(x,y)$ in (5.4). Then

$$\gamma_t x^{t+2p-2}[x,y]x^{N'-2p-t} + \sum_{i<t+2p-2} \gamma_i' x^i [x,y] x^{N'-i-2} \in \mathrm{GId}(\mathcal{U}),$$

for some $\gamma_i' \in F$. Since $\gamma_t \neq 0$, we get that

$$x^{t+2p-1} y x^{N'-2p-t-1} + \sum_{i<t+2p-1} \mu_i x^i y x^{N'-i-1} \in \mathrm{GId}(\mathcal{U}),$$

for some $\mu_i \in F$. Now, if we set $N = N'-1$ and $M = t+2p-1$, then it follows that

$$x^M y x^{N-M} + \sum_{i<M} \mu_i x^i y x^{N-i} \in \mathrm{GId}(\mathcal{U}),$$

for same $\mu_i \in F$, as required.

Now, suppose that (5.1) holds. Then, we have that

$$\alpha_1 x^n + \alpha_1^{(0)} e_{22} x^n + \sum_{i=1}^{n} \alpha_2^{(i)} x^{i-1}[x, e_{22}] x^{n-i} \in \mathrm{GId}(\mathcal{U}). \tag{5.5}$$

Let us substitute x with $x_1 + x_2$ in (5.5), and consider the homogeneous component of degree 1 in x_2. Then in this homogeneous component, we substitute x_1 with x and x_2 with $[x,y]$. Thus, with similar computations as in the previous case, we reach the desired conclusion.

Finally, suppose that (5.3) holds in $\mathcal{U}$. By substituting in $h_{p,q}^{(i)}(x,y,z)$ the variable z with x^2, we obtain (5.2), and, by the first case, the proof is complete. ☐

Proposition 5.6. *Let $\mathcal{U}$ be a proper subvariety of $gvar(UT_2^D)$. Then there exists a constant $\bar{N}$ such that $m_\lambda^{\mathcal{U}} \leq \bar{N}$ for any $\lambda \vdash n$, $n \geq 1$.*

Proof. By Lemma 5.5, there exists N such that

$$x^M y x^{N-M} + \sum_{i<M} \mu_i x^i y x^{N-i} \in \mathrm{GId}(\mathcal{U}), \tag{5.6}$$

for some $\mu_i \in F$ and a suitable $M < N$. We shall prove that $m_\lambda^{\mathcal{U}} \leq 2N$ for all $\lambda \vdash n$. By Theorem 5.2 it is enough to consider the cases when either $\lambda = (n)$, or $\lambda = (p+q,p)$, or $\lambda = (p+q,p,1)$.

We prove the statement for $\lambda = (p+q,p)$. The other cases will follow with similar arguments.

If $q < N$ there is nothing to prove. So, let us assume that $q \geq N$ and consider the polynomials $b_{p,q}^{(i)}(x,y)$ and $d_{p,q}^{(i)}(x,y)$, $0 \leq i \leq q$, defined in Remark 5.4. Notice that from relation (5.6) it follows that

$$x^M \underbrace{\bar{x}\cdots\tilde{x}}_{p-1}[x,y]\underbrace{\bar{y}\cdots\tilde{y}}_{p-1} x^{N-M} \equiv \sum_{i<M} \mu_i x^i \underbrace{\bar{x}\cdots\tilde{x}}_{p-1}[x,y]\underbrace{\bar{y}\cdots\tilde{y}}_{p-1} x^{N-i} \pmod{\mathrm{GId}(\mathcal{U})}, \tag{5.7}$$

and

$$x^M \underbrace{\bar{x}\cdots\tilde{x}}_{p-1}(xe_{22}y - ye_{22}x)\underbrace{\bar{y}\cdots\tilde{y}}_{p-1} x^{N-M} \equiv \sum_{i<M} \mu_i x^i \underbrace{\bar{x}\cdots\tilde{x}}_{p-1}(xe_{22}y - ye_{22}x)\underbrace{\bar{y}\cdots\tilde{y}}_{p-1} x^{N-i} \pmod{\mathrm{GId}(\mathcal{U})}. \tag{5.8}$$

Hence, since $q \geq N$, we can apply the relation (5.7) to any polynomial $b_{p,q}^{(i)}(x,y)$ such that $i \geq M$, and, as a consequence, we get that

$$b_{p,q}^{(i)}(x,y) \equiv \sum_{j<M} b_{p,q}^{(j)}(x,y) \pmod{\mathrm{GId}(\mathcal{U})}.$$

Similarly, since $q \geq N$, we can apply the relation (5.8) to any polynomial $d^{(i)}_{p,q}(x,y)$ such that $i \geq M$, and we obtain that

$$d^{(i)}_{p,q}(x,y) \equiv \sum_{j<M} d^{(j)}_{p,q}(x,y) \pmod{\mathrm{GId}(\mathcal{U})}.$$

Therefore, it follows that $m^{\mathcal{U}}_\lambda \leq 2M \leq 2N = \bar{N}$, as required. ☐

Theorem 5.7. *The variety of UT_2-algebras generated by UT_2^D has almost polynomial growth.*

Proof. Let $\mathcal{U}$ be a proper subvariety of $\mathcal{V} = \mathrm{gvar}(UT_2^D)$. We shall prove that $\mathcal{U}$ has polynomial growth of the codimensions. By Lemma 5.5, there exist constant $M < N$ such that

$$x^M y x^{N-M} + \sum_{i<M} \mu_i x^i y x^{N-i} \in \mathrm{GId}(\mathcal{U})$$

for some $\mu_i \in F$. By a standard multilinearization process (see for instance [7, Theorem 1.3.8]), we get

$$\sum_{\sigma\in S_N} x_{\sigma(1)} \cdots x_{\sigma(M)} y x_{\sigma(M+1)} \cdots x_{\sigma(N)} + \sum_{i<M}\sum_{\sigma\in S_N} \mu_i x_{\sigma(1)} \cdots x_{\sigma(i)} y x_{\sigma(i+1)} \cdots x_{\sigma(N)} \in \mathrm{GId}(\mathcal{U})$$

where $x_1, \ldots, x_N$ are new variables.

In the previous identity, we substitute y by $[y_1, y_2]$, we multiply on the right by $z_1 \cdots z_M$ and we alternate x_i with z_i, for all $1 \leq i \leq M$. Since $[x_1, x_2][x_3, x_4] \in \mathrm{GId}(UT_2^D) \subseteq \mathrm{GId}(\mathcal{U})$, it follows that

$$\bar{x}_1 \cdots \tilde{x}_M [y_1, y_2] \bar{z}_1 \cdots \tilde{z}_M x_{M+1} \cdots x_N \in \mathrm{GId}(\mathcal{U}).$$

Now, we multiply on the left by $z_{M+1} \cdots z_N$ and we alternate x_j with z_j for all $M+1 \leq j \leq N$. It readily follows that

$$\bar{x}_1 \cdots \tilde{x}_N [y_1, y_2] \bar{z}_1 \cdots \tilde{z}_N \in \mathrm{GId}(\mathcal{U}). \tag{5.9}$$

Take the previous identity and substitute firstly y_1 by $y_1 e_{22}$ and, secondly, y_2 by $y_2 e_{22}$. We get

$$\bar{x}_1 \cdots \tilde{x}_N (y_1 e_{22} y_2 - y_2 y_1 e_{22}) \bar{z}_1 \cdots \tilde{z}_N \in \mathrm{GId}(\mathcal{U}),$$
$$\bar{x}_1 \cdots \tilde{x}_N (y_1 y_2 e_{22} - y_2 e_{22} y_1) \bar{z}_1 \cdots \tilde{z}_N \in \mathrm{GId}(\mathcal{U}).$$

Let us sum the previous identities and, since $[x_1, x_2] - [x_1, x_2]e_{22} \in \mathrm{GId}(UT_2^D) \subseteq \mathrm{GId}(\mathcal{U})$ and (5.9) holds, we obtain

$$\bar{x}_1 \cdots \tilde{x}_N (y_1 e_{22} y_2 - y_2 e_{22} y_1) \bar{z}_1 \cdots \tilde{z}_N \in \mathrm{GId}(\mathcal{U}).$$

By renaming the variables, we get

$$\bar{x}_1 \cdots \tilde{x}_N \hat{x}_{N+1} e_{22} \hat{z}_{N+1} \bar{z}_1 \cdots \tilde{z}_N \in \mathrm{GId}(\mathcal{U}). \tag{5.10}$$

The identities (5.9) and (5.10) tell us that the irreducible $S_{2(N+1)}$-character corresponding to the partition $\lambda = (N+1, N+1)$ participates into the $2(N+1)$th generalized cocharacter of $\mathcal{U}$ with a zero multiplicity, i.e., $m^{\mathcal{U}}_{(N+1,N+1)} = 0$.

Finally, take identity (5.9), multiply it on the right by y_{N+1} and alternate y_1, y_2 and y_{N+1}. By renaming as before the variable y_1 by x_{N+1} and y_2 by z_{N+1}, we get

$$\bar{x}_1 \cdots \tilde{x}_N \hat{x}_{N+1} \hat{y}_{N+1} \hat{z}_{N+1} \bar{z}_1 \cdots \tilde{z}_N \in \mathrm{GId}(\mathcal{U}).$$

Thus, as in the previous case, $m^{\mathcal{U}}_{(N+1,N+1,1)} = 0$.

Hence, if $\lambda \vdash n$ is such that $\lambda_2 \geq N+2$ then $m^{\mathcal{U}}_\lambda = 0$ or, equivalently, if χ_λ appears with a non-zero multiplicity in the generalized S_n-cocharacter of $\mathcal{U}$, then λ must contain at most $N+1$ boxes below the first row. Therefore

$$g\chi_n(\mathcal{U}) = \sum_{\substack{\lambda\vdash n \\ |\lambda|-\lambda_1\leq N+1}} m^{\mathcal{U}}_\lambda \chi_\lambda.$$

Recall that λ_i stands for the number of boxes of the ith row of λ.

Since $|\lambda| - \lambda_1 \leq N+1$, then $\lambda_1 \geq n - (N+1)$ and by the hook formula

$$d_\lambda = \chi_\lambda(1) \leq \frac{n!}{(n-(N+1))!} \leq n^{N+1}.$$

We are now in a position to reach the goal, in fact by the previous remark and by Proposition 5.6

$$gc_n(\mathcal{U}) = g\chi_n(\mathcal{U})(1) = \sum_{\substack{\lambda\vdash n \\ |\lambda|-\lambda_1\leq N+1}} m^{\mathcal{U}}_\lambda d_\lambda \leq \sum_{\substack{\lambda\vdash n \\ |\lambda|-\lambda_1\leq N+1}} \bar{N} n^{N+1} \leq (N+1)^2 N' n^{N+1},$$

since the number of partitions such that $|\lambda| - \lambda_1 \leq N+1$ is bounded by $(N+1)^2$. Therefore $gc_n(\mathcal{U})$ is polynomially bounded and we are done. ☐

Let us denote by UT_2^F the F-algebra UT_2 regarded as UT_2-algebra, i.e., UT_2^F has a structure of UT_2-bimodule where $1 := 1_{UT_2}$ acts by left and right by multiplication, $e_{22} \cdot a = a \cdot e_{22} = 0$ and $e_{12} \cdot a = a \cdot e_{12} = 0$ for all $a \in UT_2$. Clearly, from the definition of this new action it readily follows that $e_{22}x \equiv 0$, $xe_{22} \equiv 0$, $e_{12}x \equiv 0$ and $xe_{12} \equiv 0$ are generalize identities of UT_2^F. Thus, we are dealing with ordinary polynomial identities, and by the results in [3, 13, 14] we have the following.

Theorem 5.8. *Let UT_2^F be the UT_2-algebra with the above action. Then* $\mathrm{GId}(UT_2^F)$ *is generated, as T_{UT_2}-ideal, by the following polynomials:*

$$e_{22}x; \quad xe_{22}; \quad [x_1,x_2][x_3,x_4].$$

Moreover, $gc_n(UT_2^F) = 2^{n-1}(n-2)+2$.

Theorem 5.9. *Let $g\chi_n(UT_2^F) = \sum_{\lambda\vdash n} m_\lambda \chi_\lambda$ be the nth generalized cocharacter of UT_2^F. Then*

$$m_\lambda = \begin{cases} 1, & \text{if } \lambda=(n) \\ q+1, & \text{if } \lambda=(p+q,p) \text{ or } \lambda=(p+q,p,1) \\ 0, & \text{in all other cases} \end{cases}.$$

Theorem 5.10. *The variety of UT_2-algebras generated by UT_2^F has almost polynomial growth.*

Notice that from Theorems 3.2 and 5.8 it follows that $UT_2^F \notin \mathrm{gvar}(UT_2)$. Also, as a consequence of Theorems 5.1 and 5.8 we have that $\mathrm{GId}(UT_2^F) \nsubseteq \mathrm{GId}(UT_2^D)$ and $\mathrm{GId}(UT_2^D) \nsubseteq \mathrm{GId}(UT_2^F)$. Thus by Theorems 5.7 and 5.10 we have the following.

Corollary 5.11. *The algebras UT_2^F and UT_2^D generate two distinct varieties of UT_2-algebras of almost polynomial growth.*

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